Continuous inverse algebras with involution

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16th June 2004

Abstract

A large part of the theory of Banach *-algebras is developed and generalized to continuous inverse *-algebras (i.e. complex locally convex unital *-algebras with open unit group and continuous inversion) which are (Mackey) complete. If the involution is continuous, the closed unit ball with respect to the greatest C*-semi-norm is the closed convex hull of the unitary elements. (This is originally due to Palmer.) For hermitian continuous inverse *-algebras, we generalize characterizations due to Raĭkov, Pták, and Palmer, we prove the Shirali–Ford Theorem, and we show that closed subalgebras are equispectrally embedded.[1] [2]

Introduction

A complex continuous inverse algebra shares many important properties of Banach algebras if it is (sequentially) complete or satisfies a slightly weaker condition called Mackey completeness. For instance, every element has nonempty compact spectrum, and the holomorphic functional calculus works. The theory of commutative complete continuous inverse algebras was initiated by Waelbroeck [32, 33]. Non-commutative continuous inverse algebras are used in K-theory and non-commutative geometry [6, 8, 9, 22] and in the theory of pseudo-differential operators [12]. Recently, continuous inverse algebras have received renewed interest as the natural framework for the investigation of linear Lie groups of infinite dimension [11]. For example, they serve as coordinate domains for the infinite-dimensional analogues of the classical groups [18]. In order to study unitary groups, i.e. the invariance groups of (not necessarily positive definite) hermitian forms, one must equip the coordinate algebra with an involution. Moreover, this involution may be

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[1] *Mathematics Subject Classification (2000):* 46K05; 46H20, 46H40, 47L40, 46H30

[2] *Keywords:* continuous inverse algebra; involutive algebra; unitary semi-norm; hermitian algebra; symmetric algebra; Shirali–Ford Theorem

connected with the spectral properties of the algebra by certain conditions of positive definiteness. Typically, one would like elements of the form a^*a to have non-negative real spectrum. For instance, this condition is used for the Harish–Chandra decomposition of Lie groups of the type $\mathrm{U}(1,1;A)$, where A is a continuous inverse *-algebra [4]. The Harish–Chandra decomposition, in turn, yields one of the very few construction principles for irreducible unitary representations of these infinite-dimensional Lie groups.

The investigation of continuous inverse algebras with an involution is the subject of this article. It is remarkable how smoothly the theory of Banach algebras can be generalized to this context if one carefully chooses the line of attack. The deepest results, which have been announced in the abstract, are contained in Section 5 on unitary elements and Section 7 on algebras in which every self-adjoint element has real spectrum. The latter property implies the above positivity condition; for Banach algebras, this result is the Shirali–Ford Theorem. Section 7 also shows that Ψ^*-algebras in the sense of Gramsch [12, 5.1] are exactly semi-simple hermitian Fréchet continuous inverse algebras.

Section 1 contains elementary results such as the relation between the spectrum of an element and its spectrum with respect to a closed subalgebra. Section 2 provides the Gelfand homomorphism for commutative continuous inverse algebras. Section 3 introduces the greatest C^*-semi-norm and gives two results on automatic continuity. In Section 4, we report on the holomorphic functional calculus as developed by Glöckner [11], and we prove Ford's square root lemma. Section 6 contains the fundamental properties of positive linear functionals and includes another result on automatic continuity.

In my view, the most interesting continuous inverse *-algebras are those in which a unit element exists, the involution is continuous, and the multiplication is jointly and not just separately continuous. My attitude is to assume any of these three conditions whenever this makes the statement of a result shorter and clearer. However, this is often not the case, and then I have preferred the more general statement. In fact, since Jordan multiplication is continuous in any continuous inverse algebra, it turns out that joint continuity of the multiplication need not be assumed for any result in this article. Continuity of the involution is only needed in a very limited number of places. When a statement concerns unital algebras, it is usually not difficult to obtain a similar result for non-unital algebras by means of adjunction of a unit element.

1 Continuous inverse algebras

Let us first recall some algebraic concepts. A complex locally convex algebra is a complex associative algebra A with a locally convex Hausdorff vector space topology such that the algebra multiplication is separately continuous.

The latter condition means that $x \mapsto ax$ and $x \mapsto xa$ are continuous maps from A into itself whenever $a \in A$. Separate continuity is often more easy to prove than joint continuity, i.e. continuity of multiplication as a map from $A \times A$ into A, and it is sufficient for everything we will prove in this paper. We will sometimes consider Fréchet algebras, which are complex locally convex algebras in which the topology is completely metrizable. This implies that multiplication is jointly continuous (see Rudin [28, 2.17]). The *unitization* $A^1 = A + \mathbb{C} \cdot 1$ of a complex algebra A is defined as A if A has a unit element, and as the direct sum $A \oplus \mathbb{C}$ of locally convex vector spaces with the algebra multiplication defined by $(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$ if A does not have a unit element.

The adjunction of a unit element can be avoided by the concept of *quasi-multiplication*, which is also useful in some other respects. This is the binary operation defined on an algebra A by $a \circ b := a + b - ab$. Note that the equation $a \circ b = 1 - (1 - a)(1 - b)$ holds in A^1 and that $(A, \circ)$ is a monoid with neutral element $0 \in A$. An element $a \in A$ is called *quasi-invertible* if it is invertible with respect to this monoid structure. Every algebra (in fact, every ring) has a greatest ideal which consists of quasi-invertible elements. This ideal is called the *Jacobson radical* of the algebra A and denoted by $\mathrm{rad}(A)$. The algebra is called *semi-simple* if $\mathrm{rad}(A) = \{0\}$.

Let A be a complex algebra. The *spectrum* of an element $a \in A$ is the subset

$$\mathrm{Sp}(a) := \left\{\lambda \in \mathbb{C};\ \lambda \cdot 1 - a \notin (A^1)^\times\right\}$$

of $\mathbb{C}$. The *spectral radius* of $a \in A$ is

$$\rho(a) := \sup\{|\lambda|;\ \lambda \in \mathrm{Sp}(a)\} \in \mathbb{R}_0^+ \cup \{\pm\infty\}.$$

If $a, b \in A$ then $\mathrm{Sp}(ab) \cup \{0\} = \mathrm{Sp}(ba) \cup \{0\}$. Indeed, if $\lambda \in \mathbb{C}^\times \setminus \mathrm{Sp}(ab)$ then $\frac{1}{\lambda}(1 + b(\lambda - ab)^{-1}a)$ is the inverse of $\lambda - ba \in A^1$. In particular, the formula $\rho(ab) = \rho(ba)$ holds unless one product has spectrum $\{0\}$ and the other has empty spectrum (cf. Palmer [20, 2.2.1]). Let $a \in A$, and let f be a complex rational function without poles in $\mathrm{Sp}(a)$. Then one can form the element $f(a) \in A^1$, and the Spectral Mapping Theorem asserts that $\mathrm{Sp}(f(a)) = f(\mathrm{Sp}(a))$ unless $\mathrm{Sp}(a)$ is empty and f is a constant. For an elegant simple proof, see Palmer [20, 2.1.10].

1.1 Definition. A *continuous inverse algebra* is a complex locally convex algebra with unit in which the set of invertible elements is a neighbourhood of 1 and inversion is continuous at 1. A *continuous quasi-inverse algebra* is a complex locally convex algebra (with or without unit) such that the unitization A^1 is a continuous inverse algebra.

Our definition follows Turpin [31] in not requiring joint continuity of the multiplication. This is not because we are strongly interested in algebras with discontinuous multiplication, but because the following proposition yields continuity of the Jordan multiplication, which is all we need in this paper.

A short argument shows that a complex algebra with a locally convex vector space topology is a continuous quasi-inverse algebra if and only if the set of quasi-invertible elements is a neighbourhood of 0 and quasi-inversion is continuous at 0.

1.2 Proposition (Turpin [31]). *Let A be a continuous inverse algebra. Then $A^\times$ is an open subset of A, and inversion is a continuous map from $A^\times$ into itself. Jordan multiplication $(a,b) \mapsto ab+ba\colon A\times A \to A$ is continuous.*

In particular, the maps $(a,b)\mapsto aba\colon A\times A\to A$ and $a\mapsto a^n\colon A\to A$ for $n\in\mathbb{N}$ are continuous, and multiplication is continuous on every commutative subalgebra of A.

The following ingenious proof, which is due to Turpin [31], does not even use separate continuity of the multiplication, and indeed Turpin's definition does not include this condition. However, it will be used in Remark 1.3. Thus it is fundamental to the holomorphic functional calculus which will be developed in Section 4.

Proof. If $a\in A$ is sufficiently small then both $1+a$ and $1-a$ are invertible, and the formula $a^2 = 1-2\big((1+a)^{-1}+(1-a)^{-1}\big)^{-1}$ shows that the map $a\mapsto a^2$ is continuous at 0. This implies that the bilinear map

$$\mu\colon A\times A \longrightarrow A,\quad (a,b)\longmapsto ab+ba=(a+b)^2-a^2-b^2$$

is continuous at $(0,0)$. If $a\in A$ then the linear map $x\mapsto \mu(a,x)$ is continuous at 0 and hence continuous. We conclude that μ is continuous. By induction, this implies that the n-th power map $a\mapsto a^n$ is continuous on A for every $n\in\mathbb{N}$. Since $2aba=\mu(a,ab+ba)-\mu(a^2,b)$, we find that the map $(a,b)\mapsto aba$ is continuous as well.

Let $a\in A^\times$. If $x\in A$ is sufficiently small then $a^2+x=a(1+a^{-1}xa^{-1})a$ is invertible, and the map $x\mapsto (a^2+x)^{-1}$ is defined in a neighbourhood of 0 and continuous at 0. Hence the map $x\mapsto x^{-2}$ is defined in a neighbourhood of a and continuous at a. Finally, this proves that the map $x\mapsto x^{-1}=\frac{1}{2}\mu(x,x^{-2})$ is defined in a neighbourhood of a and continuous at a. □

1.3 Remark (Commutants). By many authors, the multiplication in a continuous inverse algebra is assumed to be jointly continuous. For instance, this is perfectly natural in the theory of linear Lie groups. We can often use these parts of the literature by working in commutative subalgebras. This is because the following observation leads to commutative subalgebras which are again continuous inverse algebras.

Let S be a subset of a continuous inverse algebra A. Then the commutant $S' := \{a\in A;\ \forall\, s\in S\colon as=sa\}$ of S is a subalgebra of A which satisfies

$$(S')^\times = A^\times \cap S'.$$

Indeed, it is easy to see that a^{-1} commutes with S whenever $a \in A^\times$ commutes with S. Since multiplication in A is separately continuous, the subalgebra S' is closed. In particular, it inherits the completeness properties of A.

We have proved that every commutant is a continuous inverse algebra. Note that this argument applies to any maximal commutative subalgebra because such a subalgebra equals its own commutant.

Also note that the double commutant of a commuting subset is commutative.

Continuous inverse Fréchet algebras admit a slightly different definition. Let A be a Fréchet algebra. We have noticed above that multiplication in A is jointly continuous. Quasi-inversion is a continuous map from the set of quasi-invertible elements into itself if and only if that set is a G_δ-set in A (Waelbroeck [34, VII, Prop. 2]). Hence A is a continuous quasi-inverse algebra if and only if the set of quasi-invertible elements is a neighbourhood of 0 in A. Indeed, this condition implies that the set of quasi-invertible elements is open and hence a G_δ-set in A because multiplication with an invertible element is a homeomorphism of A onto itself.

A complex locally convex algebra in which the topology can be described by a family of sub-multiplicative semi-norms is called locally multiplicatively convex, or locally m-convex for short. These algebras were introduced by Michael [16]. They are exactly the dense subalgebras of projective limits of Banach algebras. In particular, a Fréchet algebra is locally m-convex if and only if it is isomorphic to the projective limit of a *sequence* of Banach algebras. Many important examples of continuous inverse algebras are locally m-convex, although this is sometimes difficult to see; in other cases, the question is open (cf. Gramsch [13]). Turpin [31] proved that every commutative continuous inverse algebra is locally m-convex. However, this result does not extend to the non-commutative case. Indeed, Żelazko [35] constructed a continuous inverse Fréchet algebra which is not locally m-convex.

A *spectral semi-norm* on a complex algebra is a sub-multiplicative semi-norm which is greater than or equal to the spectral radius. In his two-volume monograph [20, 21], Palmer has generalized important parts of the theory of Banach algebras to algebras with a spectral semi-norm. By the following lemma, Palmer's results are immediately available for locally m-convex continuous inverse algebras (in particular, for commutative continuous inverse algebras).

1.4 Lemma (Locally m-convex algebras). *The following statements hold in every complex locally m-convex algebra A.*

(a) *Multiplication is continuous, and quasi-inversion is continuous on its domain.*

(b) *The algebra A is a continuous quasi-inverse algebra if and only if it admits a continuous spectral semi-norm.*

Statement (a) was already observed by Michael [16].

Proof. (a) Algebra multiplication is a bilinear map which is continuous at $(0,0)$ and hence continuous. In particular, multiplication in the monoid $(A, \circ)$ is continuous, so that it suffices to prove that quasi-inversion is continuous at $0 \in A$. Let σ be a sub-multiplicative semi-norm on A, and let a^q denote the quasi-inverse of a quasi-invertible element $a \in A$. Then $a^q = -a + a^q a$, whence $\sigma(a^q) \leq \sigma(a) + \sigma(a^q)\ \sigma(a)$. If $\sigma(a) < 1$, it follows that $\sigma(a^q) \leq \frac{\sigma(a)}{1-\sigma(a)}$. We conclude that quasi-inversion is continuous with respect to σ.

(b) Let σ be a continuous sub-multiplicative semi-norm on A such that $\rho \leq \sigma$. Since $a \in A$ is quasi-invertible if and only if $1 - a \in (A^1)^\times$, the open set $\{x \in A;\ \sigma(x) < 1\}$ consists of quasi-invertible elements, so that A is a continuous quasi-inverse algebra.

Conversely, assume that the set of quasi-invertible elements of A is open. Then there is a continuous sub-multiplicative semi-norm σ on A and a number $\varepsilon > 0$ such that $\{x \in A;\ \sigma(x) < \varepsilon\}$ consists of quasi-invertible elements. Let $x \in A$. If $\lambda \in \mathbb{C}$ satisfies $\sigma(x) < \varepsilon|\lambda|$ then $\lambda - x \in (A^1)^\times$. Hence $\rho(x) < |\lambda|$, which entails $\rho(x) \leq \varepsilon^{-1}\sigma(x)$. Since $\rho(x)^n = \rho(x^n) \leq \varepsilon^{-1}\sigma(x^n) \leq \varepsilon^{-1}\sigma(x)^n$ holds for all $n \in \mathbb{N}$, we conclude that $\rho(x) \leq \sigma(x)$. $\square$

From now on, we will consider the full class of continuous (quasi-) inverse algebras. The following result is fundamental for many others. This is one important reason why we include local convexity in our definition of continuous inverse algebras.

1.5 Lemma (Elementary properties of spectra). *In a continuous quasi-inverse algebra A, the following statements hold.*

(a) *Every element has non-empty compact spectrum.*

(b) *If $\Omega \subseteq \mathbb{C}$ is open then $A_\Omega := \{a \in A; \mathrm{Sp}(a) \subseteq \Omega\}$ is an open subset of A.*

(c) *If A is a skew field then A is topologically isomorphic to $\mathbb{C}$.*

Proof. (a) This follows from Liouville's Theorem. Since the spectrum of $a \in A$ with respect to A equals the spectrum with respect to the double commutant $\{a\}''$ and this is a continuous inverse algebra with continuous multiplication, we may refer to Glöckner [11, 4.3] for the details. As Glöckner himself observes [11, 4.15], the standing completeness hypothesis of [11, Section 4] is not used in the proof of this result.

(b) We may assume that A has a unit element. Let $a \in A$, and let $\Omega \subseteq \mathbb{C}$ be an open neighbourhood of $\mathrm{Sp}(a)$. First assume that $0 \in \Omega$. Then $K := \{0\} \cup \{\mu \in \mathbb{C}^\times;\ 1/\mu \notin \Omega\}$ is compact. If $\mu \in K$ then $1 - \mu a \in A^\times$. The map

$$K \times A \longrightarrow A\colon (\mu, b) \longmapsto (1-\mu a)^{-1} - \mu(1-\mu a)^{-1}\, b\, (1-\mu a)^{-1}$$

is continuous, and it maps $K \times \{0\}$ into $A^\times$. By compactness, there is a neighbourhood $U \subseteq A$ of 0 such that $K \times U$ is mapped into $A^\times$. We claim that $a + U \subseteq A_\Omega$. Let $b \in U$ and $\lambda \in \mathbb{C} \setminus \Omega$. Set $\mu := \frac{1}{\lambda} \in \Omega$. Then the calculation

$$\begin{aligned}\lambda - (a+b) &= (\lambda - a)\big((\lambda-a)^{-1} - (\lambda-a)^{-1}\, b\, (\lambda-a)^{-1}\big)(\lambda - a)\\ &= \frac{1}{\mu}\,(1-\mu a)\big((1-\mu a)^{-1} - \mu(1-\mu a)^{-1}\, b\, (1-\mu a)^{-1}\big)(1-\mu a)\end{aligned}$$

shows that $\lambda \notin \mathrm{Sp}(a+b)$, which proves our claim. If $0 \notin \Omega$ then we choose $\lambda \in \Omega$ and find that $A_\Omega = \lambda \cdot 1 + \{a \in A;\ \mathrm{Sp}(a) \in \Omega - \lambda\}$ is open in A.

(c) Assume that A is a skew field. Let $a \in A$. We may choose $\lambda \in \mathrm{Sp}(a)$. Then $\lambda - a \notin A^\times$, so that $\lambda - a = 0$, and we conclude that $a = \lambda \cdot 1$. (This generalization of the Gelfand–Mazur Theorem was first observed by Arens [2].) □

1.6 Lemma (The Neumann series). *Let A be a continuous quasi-inverse algebra. The following are equivalent for an element $a \in A$.*

(i) *The Neumann series $\sum_{n=1}^{\infty} a^n$ converges (its limit is $(1-a)^{-1} - 1$).*

(ii) $\lim_{n\to\infty} a^n = 0$.

(iii) $\rho(a) < 1$.

Proof. We may assume that A has a unit element. It is clear that (i) implies (ii). Lemma 1.5 shows that the balanced set $\{x \in A;\ \rho(x) < 1\}$ is an open zero-neighbourhood. In view of this fact, the implication (iii) $\Rightarrow$ (i) was proved by Glöckner [11, 3.3]. Finally, if $\lim_n a^n = 0$ then some $n \in \mathbb{N}$ satisfies $1 > \rho(a^n) = \rho(a)^n$, and we conclude that $\rho(a) < 1$. □

1.7 Proposition (Closed subalgebras). *Let A be a continuous inverse algebra, and let $B \subseteq A$ be a closed unital subalgebra. Then the following assertions hold:*

(a) *B is a continuous inverse algebra.*

(b) *The topological boundary $\partial B^\times$ of $B^\times$ with respect to B does not meet $A^\times$, and $B^\times$ is a union of connected components of $A^\times \cap B$.*

(c) *If* $b \in B$ *then* $\partial \operatorname{Sp}_B(b) \subseteq \operatorname{Sp}_A(b)$, *and* $\operatorname{Sp}_B(b)$ *is the union of* $\operatorname{Sp}_A(b)$ *and a (possibly empty) collection of bounded connected components of* $\mathbb{C} \setminus \operatorname{Sp}_A(b)$. *In particular, the spectral radii satisfy* $\rho_B(b) = \rho_A(b)$, *and if* $\operatorname{Sp}_A(b) \subseteq \mathbb{R}$ *then* $\operatorname{Sp}_B(b) = \operatorname{Sp}_A(b)$.

Proof. (a) The set $U := \{a \in A;\ \rho(a) < 1\}$ is open in A by Lemma 1.5. If $a \in U$ then Lemma 1.6 shows that the inverse of $1 - a$ is given by the Neumann series, i.e. $(1-a)^{-1} = \sum_{k=0}^{\infty} a^k$. This proves that $B \cap (1+U) \subseteq B^\times$.

(b) Assume that $b \in A^\times \cap \partial B^\times$. Let $V \subseteq A^\times$ be a neighbourhood of b^{-1}. Then V^{-1} is a neighbourhood of b and hence meets $B^\times$, which implies that V meets $B^\times$ as well. Thus b^{-1} belongs to the closure of $B^\times$ and hence to B. We conclude that $b \in B^\times$, which contradicts $b \in \partial B^\times$ because $B^\times$ is open in B.

This also implies the second part of assertion (b) by means of the following elementary topological observation (see Rudin [28, 10.16]): if W and W' are subsets of a topological space X with W open in X and contained in W' such that $W' \cap \partial W = \emptyset$ then $W' \subseteq W \cup (X \setminus \overline{W})$, and thus W is a union of connected components of W'.

(c) The remainder of the proof follows Rudin's treatment of the Banach algebra case [28, 10.18]. Choose $b \in B$. If $\lambda \in \partial \operatorname{Sp}_B(b)$ then $\lambda \cdot 1 - b \in \partial B^\times$, whence $\lambda \in \operatorname{Sp}_A(b)$ by part (b). Since $\mathbb{C} \setminus \operatorname{Sp}_B(b)$ is open in $\mathbb{C}$ and contained in $\mathbb{C} \setminus \operatorname{Sp}_A(b)$, the topological observation above shows that $\mathbb{C} \setminus \operatorname{Sp}_B(b)$ is the union of certain connected components of $\mathbb{C} \setminus \operatorname{Sp}_A(b)$, and the other connected components are contained in $\operatorname{Sp}_B(b)$. $\square$

1.8 Proposition (Semi-simple quotients). *Let A be a continuous quasi-inverse algebra, and let $I \subseteq A$ be an ideal such that A/I is semi-simple. Then I is closed in A, and* $\operatorname{rad}(A) \subseteq I$. *In particular, the Jacobson radical is a closed ideal of A.*

For Banach algebras, see Bonsall and Duncan [7, 25.10].

Proof. Let $a \in A$ be an element of the closure J of I. Since the set of quasi-invertible element of A is open, there is an element $b \in I$ such that $a - b$ is quasi-invertible. Hence $(a-b)+I = a+I$ is a quasi-invertible element of A/I. Thus the ideal J/I of A/I consists of quasi-invertible elements. As A/I is semi-simple, we conclude that $J = I$.

Similarly, the ideal $(\operatorname{rad}(A) + I)/I$ of A/I consists of quasi-invertible elements, so that $\operatorname{rad}(A) \subseteq I$.

The Jacobson radical is closed because $A/\operatorname{rad}(A)$ is semi-simple. $\square$

2 Applications of the Gelfand homomorphism

2.1 Definition. Let A be a complex algebra with unit.

(a) Define the *Gelfand spectrum* of A as $\Gamma_A := \mathrm{Hom}(A, \mathbb{C})$ with the topology of pointwise convergence on A. Note that $0 \notin \Gamma_A$ because we require homomorphisms to respect the unit elements.

(b) Each element $a \in A$ gives rise to a continuous function $\hat{a}\colon \Gamma_A \to \mathbb{C}$ by $\hat{a}(\chi) := \chi(a)$. The function $\hat{a}$ is called the *Gelfand transform* of a. The map $a \mapsto \hat{a}\colon A \to C(\Gamma_A)$, which is a homomorphism of unital algebras, is called the *Gelfand homomorphism* of the algebra A.

2.2 Theorem (The Gelfand homomorphism). *In a commutative continuous inverse algebra A, the following statements hold.*

(a) *Every element $a \in A$ satisfies*

$$\mathrm{Sp}(a) = \{\chi(a);\ \chi \in \Gamma_A\} = \hat{a}(\Gamma_A).$$

(b) *The Gelfand spectrum Γ_A is a compact Hausdorff space.*

(c) *The Gelfand homomorphism is continuous with respect to the topology of uniform convergence on $C(\Gamma_A)$. Its kernel is the Jacobson radical of A.*

(d) *Every element $a \in A$ satisfies $\rho(a) = \|\hat{a}\|_\infty$, so that the spectral radius is a continuous sub-multiplicative semi-norm on A with the Jacobson radical as its zero space.*

Proof. It is not hard to adapt the proof for Banach algebras (see, for instance, Rudin [28, 11.9]). The details can be found in [5]. □

2.3 Lemma. *Let A be a continuous quasi-inverse algebra. Then every algebra homomorphism $\chi\colon A \to \mathbb{C}$ is continuous.*

Proof. Let $\varepsilon > 0$. Choose a balanced 0-neighbourhood $U \subseteq A$ which consists of quasi-invertible elements. Then $\chi(U) \subseteq \mathbb{C}$ is a disc around 0 which consists of quasi-invertible elements and hence does not contain 1. The image of the 0-neighbourhood $\varepsilon U \subseteq A$ under χ is a disc around 0 of radius at most ε. We conclude that χ is continuous at 0 and hence continuous. □

The proof of the following result on automatic continuity depends on the Closed Graph Theorem. Therefore, we can only prove it for Fréchet algebras.

2.4 Proposition. *Let $\varphi\colon A \to B$ be an algebra homomorphism between continuous quasi-inverse Fréchet algebras. If B is commutative and semi-simple then φ is continuous.*

Proof. We adapt the well-known proof for Banach algebras (see Bonsall and Duncan [7, 17.8] or Rudin [28, 11.10]). Since the unitization of a semi-simple algebra is semi-simple (see Palmer [20, 4.3.3]), we may assume that B has a unit element, so that we can apply Theorem 2.2.

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A which converges to $a \in A$ such that the image sequence $(\varphi(a_n))_{n\in\mathbb{N}}$ converges to some element $b \in B$. By the Closed Graph Theorem [28, 2.15], it suffices to show that $b = \varphi(a)$. If $\chi \in \Gamma_B$ then $\chi \circ \varphi \in \mathrm{Hom}(A, \mathbb{C})$, and both χ and $\chi \circ \varphi$ are continuous. Hence for every $\chi \in \Gamma_B$,

$$\chi(b) = \chi\big(\lim_n \varphi(a_n)\big) = \lim_n \chi\big(\varphi(a_n)\big) = \chi\Big(\varphi\big(\lim_n a_n\big)\Big) = \chi\big(\varphi(a)\big).$$

By Theorem 2.2, the difference $b - \varphi(a)$ belongs to the Jacobson radical of B, which is the zero ideal by hypothesis. □

The Gelfand homomorphism has a wealth of consequences for non-commutative continuous inverse algebras. Many of these depend on the use of commutants as described in Remark 1.3.

2.5 Proposition. *Let A be a continuous quasi-inverse algebra, and let $a, b \in A$ be commuting elements. Then*

$$\mathrm{Sp}(a+b) \subseteq \mathrm{Sp}(a) + \mathrm{Sp}(b) \qquad \textit{and} \qquad \mathrm{Sp}(a \cdot b) \subseteq \mathrm{Sp}(a) \cdot \mathrm{Sp}(b).$$

Proof. Apply Theorem 2.2 to the commutative continuous inverse algebra $\{a, b\}'' \subseteq A^1$. This yields

$$\mathrm{Sp}(a+b) = \mathrm{im}\big((a+b)\widehat{\ }\big) = \mathrm{im}(\hat{a} + \hat{b}) \subseteq \mathrm{im}(\hat{a}) + \mathrm{im}(\hat{b}) = \mathrm{Sp}(a) + \mathrm{Sp}(b).$$

The analogous calculation holds for the product $a \cdot b$. □

3 C*-semi-norms

3.1 Definition. (a) A **-algebra* is an algebra A over $\mathbb{C}$ which carries a conjugate linear anti-multiplicative involution $a \mapsto a^* \colon A \to A$. If A does not have a unit element then the involution is extended to the unitization $A^1 = A \oplus \mathbb{C}$ by setting $(a, \lambda)^* := (a^*, \overline{\lambda})$.

The set of unitary elements of A^1 is $\mathrm{U}(A^1) := \big\{u \in (A^1)^\times;\ u^{-1} = u^*\big\}$. An element $a \in A$ is called *normal* if $a^*a = aa^*$, and *self-adjoint* if $a^* = a$. We often denote self-adjoint elements by the letter h, but we do not call them "hermitian" because this word is used for elements of a Banach algebra which have real numerical range. We occasionally write $\mathrm{Sym}(A)$ for the set of self-adjoint elements of A, which is a real vector subspace of A.

(b) A *continuous (quasi-) inverse *-algebra* is just a continuous (quasi-) inverse algebra which also is a *-algebra. Following the tradition in the theory of Banach algebras, we explicitly assume continuity of the involution when we need it.

For semi-simple commutative algebras, Proposition 2.4 implies the following result on automatic continuity.

3.2 Proposition. *The involution in a semi-simple commutative continuous quasi-inverse Fréchet *-algebra A is continuous.*

Proof. The opposite algebra A^{opp} is the real topological vector space A with the opposite complex structure $(\lambda, a) \mapsto \overline{\lambda} a$ and with the opposite algebra multiplication $(a, b) \mapsto ba$. It is a continuous quasi-inverse Fréchet algebra, and the involution is an algebra isomorphism from A^{opp} onto A. Such an isomorphism is continuous by Proposition 2.4. $\square$

3.3 Remark. In a semi-simple Banach *-algebra, the involution is always continuous (see Palmer [21, 11.1.1] or Bonsall and Duncan [7, 36.2]). Unfortunately, none of the proofs that I know can easily be generalized to non-commutative continuous inverse Fréchet *-algebras. However, positive results on automatic continuity are provided by Theorem 3.9 below.

Automatic continuity is a problem for which local m-convexity is profitable. Indeed, let A and B be locally m-convex continuous quasi-inverse Fréchet algebras, and assume that B is semi-simple. Then every surjective homomorphism from A onto B is automatically continuous. This is mentioned by Aupetit [3]. It can also be derived from Ransford's elegant treatment of the Banach algebra case [26], which is reproduced and suitably generalized by Palmer [20, 2.3.9]. As in the proof of Proposition 3.2, this result implies that the involution in a semi-simple locally m-convex continuous quasi-inverse Fréchet *-algebra is automatically continuous.

3.4 Definition. (a) A semi-norm σ on a complex *-algebra A is called a *C*-semi-norm* if $\sigma(a^*a) = \sigma(a)^2$ holds for all $a \in A$. Sebestyén's Theorem [29] (cf. Palmer [21, 9.5.14]) states that a C*-semi-norm σ is automatically sub-multiplicative, which means that σ satisfies $\sigma(ab) \le \sigma(a)\,\sigma(b)$ for all $a, b \in A$.

(b) Let A be a continuous quasi-inverse *-algebra. Then the *Raĭkov–Pták functional* on A is defined by

$$\tau\colon A \longrightarrow \mathbb{R}_0^+, \quad a \longmapsto \sqrt{\rho(a^*a)}\ .$$

The function τ will play a prominent role in the theory of hermitian continuous inverse *-algebras in Section 7. The name "Raĭkov–Pták functional" is suggested by Palmer [21] because τ appears implicitly in Raĭkov's work [25] and is explicitly used by Pták [23, 24].

3.5 Lemma (Elementary properties of τ). *Let A be a continuous quasi-inverse *-algebra.*

(a) *If $c \in A$ is a normal element then $\tau(c) \le \rho(c)$.*

(b) *If σ is a C*-semi-norm on A then $\sigma \leq \tau$.*

Proof. Assertion (a) follows from Proposition 2.5. To prove (b), consider the *-ideal $A_\sigma := \{a \in A;\ \sigma(a) = 0\}$ of A. Since σ induces a C*-norm on the quotient *-algebra $B := A/A_\sigma$, the completion C of B with respect to this norm is a C*-algebra. Hence every $a \in A$ satisfies $\sigma(a)^2 = \sigma(a^*a) = \rho_C(a^*a + A_\sigma) \leq \rho_A(a^*a) = \tau(a)^2$. $\square$

3.6 Proposition. *Let σ be a C*-semi-norm on a continuous quasi-inverse *-algebra A. Then* $\mathrm{rad}(A) \subseteq \{a \in A;\ \sigma(a) = 0\}$.

Note that the Jacobson radical of any *-algebra is a *-ideal.

Proof. Recall that $\mathrm{rad}(A)$ is the largest ideal of A which consists of quasi-invertible elements. Let $a \in \mathrm{rad}(A)$. Then $\lambda^{-1}a$ is quasi-invertible for every $\lambda \in \mathbb{C}^\times$, which means that $\lambda - a \in (A^1)^\times$. Therefore, the spectral radius vanishes on $\mathrm{rad}(A)$. Since $a^*a \in \mathrm{rad}(A)$, we conclude that $\sigma(a) \leq \tau(a) = \sqrt{\rho(a^*a)} = 0$. $\square$

3.7 Definition. Let A be a continuous quasi-inverse *-algebra. Lemma 3.5 implies that the supremum

$$\sigma_{\mathrm{C}^*}(a) := \sup\{\sigma(a);\ \sigma \text{ is a C}^*\text{-semi-norm on } A\}$$

is finite for every $a \in A$. Since σ_{C^*} is itself a C*-semi-norm on A, it is the greatest C*-semi-norm on A. It is sometimes called the *Gelfand–Naĭmark semi-norm* of A (Palmer [21]).

The zero space $I := \{a \in A;\ \sigma_{\mathrm{C}^*}(a) = 0\}$ of σ_{C^*} is a *-ideal of A. The completion $C^*(A)$ of A/I with respect to the C*-norm induced by σ_{C^*} is called the *enveloping C*-algebra* of A. It has the universal property that every *-homomorphism from A into a C*-algebra factors uniquely through the natural homomorphism $\pi\colon A \to C^*(A)$.

3.8 Proposition. *Let A be a continuous quasi-inverse *-algebra with continuous involution. Then every C*-semi-norm on A is continuous.*

Proof. Let σ be a C*-semi-norm on A. As in the proof of Lemma 3.5, let C be the completion of A/A_σ with respect to the C*-norm induced by σ. Lemma 1.5 yields a neighbourhood $U \subseteq A$ of 0 such that all $a \in U$ satisfy $\rho_A(a) < 1$. Proposition 1.2 and continuity of the involution yield a neighbourhood $V \subseteq A$ of 0 such that all $a \in V$ satisfy $a^*aa^* \in U$. If $a \in A$ then $\sigma(a)^4 = \sigma(a^*a)^2 = \sigma(a^*aa^*a) \leq \sigma(a^*aa^*)\,\sigma(a)$ by sub-multiplicativity of σ, whence $\sigma(a)^3 \leq \sigma(a^*aa^*)$. The spectral radius of a normal element of a C*-algebra equals its norm (see Rudin [28, 11.28]). Hence if $a \in A$ then

$$\sigma(a)^3 \leq \sigma(a^*aa^*) = \rho_C(a^*aa^* + A_\sigma) \leq \rho_A(a^*aa^*) < 1.$$

Therefore, the open unit ball in A with respect to σ is a neighbourhood of 0. We conclude that σ is continuous. $\square$

3.9 Theorem. *The following hold for a continuous quasi-inverse Fréchet *-algebra A:*

(a) *Every C*-semi-norm on A is continuous.*

(b) *If $\{a \in A;\ \sigma_{\mathrm{C}^*}(a) = 0\} = \{0\}$, i.e. if σ_{C^*} is a norm, then the involution of A is continuous.*

Proof. Let σ be a C*-semi-norm on A. We claim that the *-ideal $A_\sigma = \{a \in A;\ \sigma(a) = 0\}$ of A is closed. Set $B := A/A_\sigma$. The completion C of B with respect to the norm induced by σ is a C*-algebra. Let $\|\cdot\|$ be the norm on C induced by σ. Choose $x \in \operatorname{rad}(B)$. Then $\lambda^{-1}x$ is quasi-invertible for all $\lambda \in \mathbb{C}^\times$. This means that $\lambda - x$ is invertible in B and hence in C, so that $\rho_C(x) = 0$. Since $x^*x \in \operatorname{rad}(B)$, this shows that $\|x\|^2 = \|x^*x\| = \rho_C(x^*x) = 0$. We conclude that $\operatorname{rad}(B) = \{0\}$, whence the claim follows from Proposition 1.8. In particular, the quotient algebra B is a continuous quasi-inverse *-algebra, and it is a Fréchet space (see Rudin [28, 1.41]).

Now we will use the Closed Graph Theorem in order to prove that the involution on B is continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in B which converges to some element $y \in B$ such that $(x_n^*)_{n\in\mathbb{N}}$ converges to some element $z \in B$. Since

$$\|x_n - z^*\|^2 = \rho_C\big((x_n - z^*)^*(x_n - z^*)\big) \le \rho_B\big((x_n^* - z)(x_n - z^*)\big)$$

and $\lim_n (x_n^* - z)(x_n - z^*) = 0$, Lemma 1.5 shows that $\lim_n \|x_n - z^*\| = 0$. Similarly, the inequality $\|x_n - y\|^2 \le \rho_B\big((x_n^* - y^*)(x_n - y)\big)$ implies that $\lim_n \|x_n - y\| = 0$. Hence $y = z^*$, and the Closed Graph Theorem (see Rudin [28, 2.15]) yields that the involution on B is continuous. Proposition 3.8 shows that $\|\cdot\|$ is a continuous norm on B. This implies that σ is continuous.

Assume that σ_{C^*} is a norm. Set $\sigma := \sigma_{\mathrm{C}^*}$ in the above argument. Then $A_\sigma = 0$, so that $A \cong B$, and we have proved that the involution on A is continuous. □

4 The holomorphic functional calculus

4.1 Definition. (a) A sequence $(x_n)_{n\in\mathbb{N}}$ in a locally convex real vector space E is called a *Mackey–Cauchy sequence* if there is a net $(t_{m,n})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of positive real numbers which converges to 0 such that the set

$$\left\{\frac{x_m - x_n}{t_{m,n}};\ m, n \in \mathbb{N}\right\}$$

is a bounded subset of E. Every Mackey–Cauchy sequence is a Cauchy sequence.

(b) The locally convex real vector space E is called *Mackey complete* if every Mackey–Cauchy sequence in E converges. This holds if and only if every smooth curve $\alpha\colon [a,b] \to E$ (where $a, b \in \mathbb{R}$) has a Riemann integral $\int_a^b \alpha(t)\,dt$ in E (see Kriegl and Michor [15, 2.14]).

In a Mackey complete continuous inverse algebra, a holomorphic functional calculus can be based on integration along smooth contours. For algebras with continuous multiplication, this has been worked out by Glöckner [11], to whom the following theorem is essentially due. A functional calculus for a wider class of algebras was also sketched by Allan [1].

4.2 Theorem (Holomorphic functional calculus). *Let A be a Mackey complete continuous inverse algebra. For an open subset $\Omega \subseteq \mathbb{C}$, let $\mathcal{O}(\Omega)$ be the algebra of holomorphic functions on Ω, equipped with the locally convex topology of uniform convergence on compact subsets of Ω. Recall that $A_\Omega := \{a \in A;\ \mathrm{Sp}(a) \subseteq \Omega\}$ is an open subset of A.*

(a) *For each element $a \in A_\Omega$, there is a unique continuous homomorphism of unital algebras*

$$f \longmapsto f[a]\colon \mathcal{O}(\Omega) \longrightarrow A$$

which sends $\mathrm{id}_\Omega \in \mathcal{O}(\Omega)$ to $a \in A$.

(b) *The map*

$$(f, a) \longmapsto f[a]\colon \mathcal{O}(\Omega) \times A_\Omega \longrightarrow A$$

is continuous.

Proof. (a) Let $a \in A_\Omega$. Choose a smooth contour Γ surrounding $\mathrm{Sp}(a)$ in Ω, and set

$$f[a] := \frac{1}{2\pi i} \int_\Gamma f(\zeta)(\zeta - a)^{-1}\, d\zeta.$$

Glöckner [11, 4.7, 4.9, and 4.10] shows that this definition does not depend on the choice of Γ and yields a continuous homomorphism from $\mathcal{O}(\Omega)$ into the double commutant $\{a\}''$. The uniqueness of a continuous homomorphism with the required properties follows from Runge's Theorem (see Rudin [27, 13.9]), which states that the rational functions form a dense subset of $\mathcal{O}(\Omega)$.

(b) Let $(f, a) \in \mathcal{O}(\Omega) \times A_\Omega$. Choose an open neighbourhood $\Omega' \subseteq \Omega$ of $\mathrm{Sp}(a)$ which is relatively compact in Ω, and let Γ be a smooth contour which surrounds the closure of Ω' in Ω. Let $l > 0$ be the length of Γ (counting multiplicities). Let σ be a continuous semi-norm on A, and let $\varepsilon > 0$. By compactness of $\mathrm{im}(\Gamma) \subseteq \mathbb{C}$, there is a neighbourhood $U \subseteq A_{\Omega'}$ of a such that all $(\zeta, b) \in (\mathrm{im}(\Gamma), U)$ satisfy $\sigma\big((\zeta - b)^{-1} - (\zeta - a)^{-1}\big) < \varepsilon$. If $b \in U$ and

$g \in \mathcal{O}(\Omega)$ then

$$\begin{aligned}
\sigma(g[b]-f[a]) &= \frac{1}{2\pi}\,\sigma\left(\int_\Gamma g(\zeta)(\zeta-b)^{-1}\,d\zeta - \int_\Gamma f(\zeta)(\zeta-a)^{-1}\,d\zeta\right)\\
&= \frac{1}{2\pi}\,\sigma\biggl(\int_\Gamma g(\zeta)\bigl((\zeta-b)^{-1}-(\zeta-a)^{-1}\bigr)\,d\zeta\\
&\qquad\qquad + \int_\Gamma \bigl(g(\zeta)-f(\zeta)\bigr)(\zeta-a)^{-1}\,d\zeta\biggr)\\
&\le \frac{l}{2\pi}\left(\varepsilon \sup_{\zeta\in\mathrm{im}(\Gamma)}|g(\zeta)| + \sup_{\zeta\in\mathrm{im}(\Gamma)}|g(\zeta)-f(\zeta)|\cdot \sup_{\zeta\in\mathrm{im}(\Gamma)}\sigma\bigl((\zeta-a)^{-1}\bigr)\right)
\end{aligned}$$

This expression becomes arbitrarily small if ε is sufficiently small and g is sufficiently close to f. We conclude that the map $(g,b)\mapsto g[b]$ is continuous at (f,a). □

4.3 Remark (Spectral Mapping Theorem). Many familiar properties of the holomorphic functional calculus on a Banach algebra as well as their proofs carry over to a Mackey complete continuous inverse algebra A. For instance, let $\Omega \subseteq \mathbb{C}$ be an open subset, and choose $f \in \mathcal{O}(\Omega)$ and $a \in A_\Omega$. Then the Spectral Mapping Theorem $\mathrm{Sp}(f[a]) = f(\mathrm{Sp}(a))$ can be proved as in the Banach algebra case (see Rudin [28, 10.28]; cf. Glöckner [11, 4.12]). By continuity of the functional calculus, if f has a power series expansion $f(\zeta) = \sum_{n=0}^\infty \alpha_n(\zeta-\zeta_0)^n$ on Ω then the series $\sum_{n=0}^\infty \alpha_n(a-\zeta_0\cdot 1)^n$ converges to $f[a]$, cf. [11, 4.11].

4.4 Corollary (Composition). *Let A be a Mackey complete continuous inverse algebra. Let $\Omega \subseteq \mathbb{C}$ and $\Omega' \subseteq \mathbb{C}$ be open subsets, choose $f \in \mathcal{O}(\Omega)$ and $g \in \mathcal{O}(\Omega')$, and suppose that $f(\Omega) \subseteq \Omega'$. Then $(g\circ f)[a] = g[f[a]]$ holds for every $a \in A_\Omega$.*

Proof. Fix $a \in A_\Omega$ and $f \in \mathcal{O}(\Omega)$ such that $f(\Omega) \subseteq \Omega'$. The two maps from $\mathcal{O}(\Omega')$ to A which are given by $g \mapsto (g\circ f)[a]$ and by $g \mapsto g[f[a]]$ are continuous unital homomorphisms which map $\mathrm{id}_{\Omega'}$ to $f[a]$. By uniqueness of the continuous holomorphic functional calculus, they are equal.

Glöckner [11, 4.13] refers to Rudin [28, 10.29] for an alternative proof. □

4.5 Remark (Homomorphisms; real analytic functions). Runge's Theorem (see Rudin [27, 13.9]), which was used in the proof of Theorem 4.2, has many applications to the holomorphic functional calculus on a Mackey complete continuous inverse algebra A, of which we record three. Let $\Omega \subseteq \mathbb{C}$ be an open subset, and choose $f \in \mathcal{O}(\Omega)$ and $a \in A_\Omega$.

Let $\varphi\colon A \to B$ be a continuous unital homomorphism into a Mackey complete continuous inverse algebra B. Approximating f by rational functions, we find that $\varphi(f[a]) = f[\varphi(a)]$.

Every element of A which commutes with a also commutes with $f[a]$. In other words, $f[a]$ belongs to the double commutant $\{a\}''$ of a in A.

Assume that A is a Mackey complete continuous inverse *-algebra with continuous involution. Let $\Omega^* \subseteq \mathbb{C}$ be the image of Ω under complex conjugation. For $f \in \mathcal{O}(\Omega)$, define $f^* \in \mathcal{O}(\Omega^*)$ by $f^*(\zeta) := \overline{f(\overline{\zeta})}$. For every element $a \in A_\Omega$, the map

$$f \longmapsto \big(f^*[a^*]\big)^* : \mathcal{O}(\Omega) \longrightarrow A$$

is a continuous homomorphism of unital algebras which sends $\mathrm{id}_\Omega \in \mathcal{O}(\Omega)$ to $a \in A$. By uniqueness of the holomorphic functional calculus, the equation $f^*[a^*]^* = f[a]$ holds for every $f \in \mathcal{O}(\Omega)$ and every $a \in A_\Omega$. In particular, assume that Ω is connected and equals Ω^*. Let $f \in \mathcal{O}(\Omega)$ be a function which takes real values on $\Omega \cap \mathbb{R}$. Then f^* coincides with f on $\Omega \cap \mathbb{R}$. By the Identity Theorem [27, Corollary of Theorem 10.18], the functions f and f^* coincide on their domain Ω. Therefore, every element $a \in A_\Omega$ satisfies $f[a^*] = f[a]^*$.

4.6 Proposition (Square roots). *Let A be a Mackey complete continuous inverse algebra. Then every element $a \in A$ with $\mathrm{Sp}(a) \cap \,]-\infty, 0] = \emptyset$ has a unique square root with spectrum contained in the open right half plane. This square root belongs to the double commutant of a.*

Proof. Set $\Omega := \mathbb{C} \setminus \,]-\infty, 0]$. Let $f \in \mathcal{O}(\Omega)$ be the principal branch of the complex square root function, so that $f(1) = 1$ and $f(\zeta)^2 = \zeta$ for every $\zeta \in \Omega$. Define $b := f[a]$ by the holomorphic functional calculus. Then $b^2 = a$, the spectrum of b is contained in the open right half-plane, and b belongs to the double commutant of a.

Let $b_1 \in A$ be a square root of a with spectrum contained in the open right half plane. Then b_1 commutes with ${b_1}^2 = a$ and hence with b. Proposition 2.5 shows that $\mathrm{Sp}(b + b_1)$ is contained in the open right half-plane. In particular, the element $b + b_1$ is invertible. Since

$$0 = b^2 - {b_1}^2 = (b + b_1)(b - b_1),$$

this implies that $b = b_1$. In other words, the element b is the unique square root of a with spectrum contained in the open right half plane. □

4.7 Corollary (Self-adjoint square roots). *Let A be a Mackey complete continuous inverse *-algebra. Then every self-adjoint element $h \in A$ with $\mathrm{Sp}(h) \cap \,]-\infty, 0] = \emptyset$ has a unique square root with spectrum contained in the open right half plane. This square root is self-adjoint and belongs to the double commutant of h.*

Note that we need not assume continuity of the involution.

Proof. Proposition 4.6 yields a unique element $k \in A$ such that $k^2 = h$ and $\mathrm{Im}(\mathrm{Sp}(k)) > 0$. Since these two properties also hold for k^*, uniqueness implies that k is self-adjoint. The double commutant property also follows from Proposition 4.6. $\square$

5 The unitary semi-norm

5.1 Proposition ($\mathrm{U}(A)$ spans A). *Let A be a Mackey complete continuous inverse *-algebra, and let $h \in A$ be a self-adjoint element with spectral radius $\rho(h) < 1$. Then there is a unitary element $u \in \mathrm{U}(A)$ such that $h = \frac{1}{2}(u+u^*)$. In particular, the algebra A is the linear span of its unitary elements.*

Proof. The rational function $\zeta \mapsto \frac{1+\zeta}{1-\zeta}$ maps the open unit disc onto the right half plane. By the Spectral Mapping Theorem, the spectrum of the element $(1+h)(1-h)^{-1}$ is contained in the open right half plane. Let $k \in A$ be a self-adjoint element such that $k^2 = (1+h)(1-h)^{-1}$, and set $u := (k+i)(k-i)^{-1}$. Then u is a unitary element of A, and we calculate

$$\begin{aligned}
u + u^* &= (k+i)(k-i)^{-1} + (k+i)^{-1}(k-i) \\
&= \big((k+i)^2 + (k-i)^2\big)(k+i)^{-1}(k-i)^{-1} \\
&= (k^2 + 2ik - 1 + k^2 - 2ik - 1)\big((k-i)(k+i)\big)^{-1} \\
&= 2(k^2-1)(k^2+1)^{-1} \\
&= 2\big((1+h)(1-h)^{-1} - 1\big)\big((1+h)(1-h)^{-1}+1\big)^{-1} \\
&= 2\big((1+h)(1-h)^{-1} - 1\big)(1-h) \\
&\qquad \cdot (1-h)^{-1}\big((1+h)(1-h)^{-1}+1\big)^{-1} \\
&= 2\big((1+h)-(1-h)\big)\big((1+h)+(1-h)\big)^{-1} \\
&= 2h.
\end{aligned}$$

This proves the proposition. $\square$

5.2 Remark. We have chosen a proof which only needs the existence of square roots in a rather weak sense. The calculation becomes shorter if we exploit the full force of Corollary 4.7, cf. Bonsall and Duncan [7, 12.14]. Indeed, the double commutant of $1-h^2$ contains a self-adjoint element k such that $k^2 = 1-h^2$. Since h and k commute, the element $u := h + ik \in A$ is unitary, and it satisfies $h = \frac{1}{2}(u+u^*)$.

5.3 Definition. Let A be a Mackey complete continuous inverse *-algebra. The convex hull $\mathrm{conv}(\mathrm{U}(A))$ is *-invariant, balanced, and closed under multiplication, and Proposition 5.1 shows that it is absorbing. Therefore, the Minkowski functional of $\mathrm{conv}(\mathrm{U}(A))$,

$$\sigma_{\mathrm{U}} \colon A \longrightarrow \mathbb{R}_0^+, \quad a \longmapsto \inf\left\{\lambda \in \mathbb{R}^+;\ \tfrac{1}{\lambda}\, a \in \mathrm{conv}(\mathrm{U}(A))\right\},$$

is a *-invariant sub-multiplicative semi-norm on A. The semi-norm σ_{U} is called the *unitary semi-norm* of A.

If A is a Mackey complete continuous quasi-inverse *-algebra then the unitary semi-norm σ_{U} of A is defined as the restriction of the unitary semi-norm of the unitization $A^1 = A + \mathbb{C} \cdot 1$.

5.4 Lemma. *Let A be a Mackey complete continuous inverse *-algebra. Then for every self-adjoint element $h \in A$, the unitary semi-norm and the spectral radius are related by the inequality $\sigma_{\mathrm{U}}(h) \leq \rho(h)$.*

Proof. Choose a positive real number t such that $t > \rho(h)$ and therefore $\rho(t^{-1}h) < 1$. Proposition 5.1 yields a unitary element $u \in A^1$ such that $t^{-1}h = \frac{1}{2}(u + u^*)$. Hence $t^{-1}h$ belongs to the convex hull of $\mathrm{U}(A^1)$, so that $\sigma_{\mathrm{U}}(t^{-1}h) \leq 1$. We conclude that $\sigma_{\mathrm{U}}(h) \leq t$. □

5.5 Proposition. *The unitary semi-norm σ_{U} on a Mackey complete continuous quasi-inverse *-algebra A with continuous involution is continuous.*

Proof. Since σ_{U} is a semi-norm, it suffices to show that it is continuous at 0. Given $\varepsilon > 0$, let $\Omega \subseteq \mathbb{C}$ be the open disc with centre 0 and radius $\frac{\varepsilon}{2}$. Then A_Ω is a neighbourhood of 0 in A. By Lemma 5.4, every self-adjoint element $h \in A_\Omega$ satisfies $\sigma_{\mathrm{U}}(h) \leq \rho(h) < \frac{\varepsilon}{2}$. Let $V \subseteq A$ be a *-invariant balanced neighbourhood of 0 such that $\frac{1}{2}V + \frac{1}{2}V \subseteq A_\Omega$. Choose $a \in V$. Define self-adjoint elements of A by $h := \frac{1}{2}(a+a^*)$ and $k := \frac{1}{2i}(a-a^*)$. Then $h, k \in A_\Omega$, and $a = h + ik$. We conclude that $\sigma_{\mathrm{U}}(a) \leq \sigma_{\mathrm{U}}(h) + \sigma_{\mathrm{U}}(k) < \varepsilon$. □

5.6 Remark. Let A be a *-algebra such that the unitization $A^1 = A + \mathbb{C} \cdot 1$ is the linear span of $\mathrm{U}(A^1)$. Such an algebra is called a *U*-algebra* by Palmer, who develops the theory of these algebras in Section 10.4 of his monograph [21]. In particular, the *-representation theory of A is very similar to the *-representation theory of Banach *-algebras. This fact is based on the following observation. Let π be a *-representation of A on a pre-Hilbert space X, i.e. a homomorphism from A into the algebra of linear endomorphisms of X such that $\langle \pi(a).x, y \rangle = \langle x, \pi(a^*).y \rangle$ holds for all $a \in A$ and all $x, y \in X$. Then π is normed, which means that $\pi(a)$ is a bounded operator on X for every $a \in A$. Hence π extends to a representation of A on the Hilbert space completion of X. Moreover, the inequality $\|\pi(a)\| \leq \sigma_{\mathrm{U}}(a)$ holds for every $a \in A$, where the unitary semi-norm σ_{U} of A is constructed as in Definition 5.3. To prove this observation, it suffices to note that $\mathrm{id}_X - \pi(w)$ is a unitary operator whenever $1 - w$ is a unitary element of A^1, see [21, 10.3.8].

A similar argument shows that A has a greatest C*-semi-norm. Indeed, let σ be a non-zero C*-semi-norm on A. Then σ extends to a C*-semi-norm σ^1 on A^1 such that $\sigma^1(1) = 1$ (see [21, 9.5.3]). If $u \in A^1$ is unitary then $\sigma^1(u) = \sqrt{\sigma^1(u^*u)} = 1$. This entails that $\sigma^1(a) \leq \sigma_{\mathrm{U}}(a)$ holds for every element $a \in A^1$. Therefore, the Gelfand–Naĭmark semi-norm σ_{C^*} of A can be constructed as in Definition 3.7, and it satisfies the inequality $\sigma_{\mathrm{C}^*} \leq \sigma_{\mathrm{U}}$ (cf. [21, 10.3.9]).

5.7 Theorem. *Let A be a Mackey complete continuous quasi-inverse *-algebra with continuous involution, and let $A_{\sigma_{\mathrm{U}}}$ be the zero space of the unitary semi-norm. Then the norm-completion C of the quotient algebra $A/A_{\sigma_{\mathrm{U}}}$ is a C*-algebra.*

Proof. Assume first that A contains a unit element 1. Let $\|\cdot\|$ denote both the quotient norm induced by σ_{U} on $A/A_{\sigma_{\mathrm{U}}}$ and its extension to the completion C, and write $\pi\colon A \to C$ for the quotient projection followed by the inclusion of $A/A_{\sigma_{\mathrm{U}}}$ into C.

Let $h \in C$ be a self-adjoint element. Choose a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that $\lim_n \pi(a_n) = h$. Define self-adjoint elements of A by $h_n := \frac{1}{2}(a_n + a_n^*)$. Then $\lim_n \pi(h_n) = h$ because the involution on C is isometric. Continuity of the functional calculus implies that $\exp[ih] = \lim_n \exp[\pi(ih_n)]$. Proposition 5.5 shows that the projection π is continuous. Hence $\exp[\pi(ih_n)] = \pi(\exp[ih_n])$. Since the involution on A is continuous, the element $\exp[ih_n] \in A$ is unitary. Hence

$$\|\exp[\pi(ih_n)]\| = \|\pi(\exp[ih_n])\| = \sigma_{\mathrm{U}}(\exp[ih_n]) \leq 1.$$

This proves the inequality $\|\exp[ih]\| \leq 1$ for every self-adjoint element $h \in C$. We infer that $\|\exp[ih]\| = 1$ holds for every self-adjoint element $h \in C$. This implies that every self-adjoint element of C has real numerical range (see Bonsall and Duncan [7, 10.13]). The Vidav–Palmer Theorem (see [7, 38.14] or Palmer [21, 11.2.5]) shows that C is a C*-algebra.

Now assume that A does not have a unit element. Let $(a,\lambda) \in A^1 = A \oplus \mathbb{C}$ be a unitary element. Then $|\lambda| = 1$. Therefore, the image of the convex hull of $\mathrm{U}(A^1)$ under the product projection of A^1 onto $\mathbb{C}$ is the closed unit disc. Hence $(A^1)_{\sigma_{\mathrm{U}}} = A_{\sigma_{\mathrm{U}}}$. The first part of the proof shows that the completion of $A^1/A_{\sigma_{\mathrm{U}}}$ with respect to the norm induced by σ_{U} is a C*-algebra. Since $A/A_{\sigma_{\mathrm{U}}}$ is isometrically embedded in $A^1/A_{\sigma_{\mathrm{U}}}$, the theorem follows. $\square$

5.8 Corollary. *Let A be a Mackey complete continuous quasi-inverse *-algebra with continuous involution. Then the unitary semi-norm of A equals the greatest C*-semi-norm, i.e. $\sigma_{\mathrm{U}} = \sigma_{\mathrm{C}^*}$. If A has a unit element then $\{a \in A;\ \sigma_{\mathrm{C}^*}(a) \leq 1\}$ is the closed convex hull of* $\mathrm{U}(A)$.

For Banach *-algebras, this corollary is due to Palmer [19].

Proof. The theorem implies that σ_{U} is a C*-semi-norm, whence $\sigma_{\mathrm{U}} \leq \sigma_{\mathrm{C}^*}$. The opposite inequality is contained in Remark 5.6.

Assume that A is a unital algebra. Since σ_{U} is continuous on A by Proposition 5.5, the closed unit ball $\{a \in A;\ \sigma_{\mathrm{U}}(a) \leq 1\}$ is a closed convex subset of A, and this subset contains the unitary group $\mathrm{U}(A)$ and hence its closed convex hull. Conversely, choose $a \in A$ with $\sigma_{\mathrm{U}}(a) \leq 1$. Then every neighbourhood of a contains an element of the form λa with $0 < \lambda < 1$, and λa belongs to the convex hull of $\mathrm{U}(A)$ because $\sigma_{\mathrm{U}}(\lambda a) \leq \lambda < 1$. We conclude that a belongs to the closed convex hull of $\mathrm{U}(A)$. $\square$

6 Positive linear functionals

6.1 Definition. Let A be a complex *-algebra. A linear functional $\omega\colon A \to \mathbb{C}$ is called *positive* if $\omega(a^*a) \in \mathbb{R}_0^+$ holds for every $a \in A$. The set of all positive linear functionals on A is denoted by $\mathrm{Pos}(A, \mathbb{C})$.

6.2 Proposition. *Let A be a Mackey complete continuous quasi-inverse *-algebra. Then every positive linear functional $\omega\colon A \to \mathbb{C}$ has the following properties.*

(a) $\omega(a^*b) = \overline{\omega(b^*a)}$ *for all $a, b \in A$;*

(b) $|\omega(a^*b)|^2 \leq \omega(a^*a)\ \omega(b^*b)$ *for all $a, b \in A$;*

(c) $|\omega(x^*ax)| \leq \omega(x^*x)\sqrt{\rho(a^*a)}$ *for all $a, x \in A$;*

(d) $|\omega(x^*cx)| \leq \omega(x^*x)\ \rho(c)$ *for all $c, x \in A$ such that c is normal.*

Proof. Let $a, b \in A$. If $\lambda \in \mathbb{C}$ then

$$0 \leq \omega\big((a + \lambda b)^*(a + \lambda b)\big) = \omega(a^*a) + \lambda\omega(a^*b) + \overline{\lambda}\omega(b^*a) + |\lambda|^2\omega(b^*b).$$

In particular, all $\lambda \in \mathbb{C}$ satisfy $\lambda\omega(a^*b) + \overline{\lambda}\omega(b^*a) \in \mathbb{R}$. Setting $\lambda := 1$ and $\lambda := i$, we find that $\omega(a^*b) = \overline{\omega(b^*a)}$. If $\omega(b^*b) = 0$ then $0 \leq \omega(a^*a) + 2\,\mathrm{Re}(\lambda\omega(a^*b))$ holds for all $\lambda \in \mathbb{C}$. Hence $\omega(a^*b) = 0$, and property (b) follows. If $\omega(b^*b) \neq 0$ then (b) is proved by setting $\lambda := -\omega(b^*a)/\omega(b^*b)$.

Let $a, x \in A$. Choose $\lambda \in \mathbb{R}$ with $\lambda > \rho(a^*a)$. Corollary 4.7 yields a self-adjoint element $k \in A^1 = A + \mathbb{C} \cdot 1$ such that $k^2 = \lambda - a^*a$. The inequality $\omega(x^*k^2x) \geq 0$ implies $\omega(x^*a^*ax) \leq \omega(x^*x) \cdot \lambda$. We conclude that $\omega(x^*a^*ax) \leq \omega(x^*x)\ \rho(a^*a)$. Together with property (b), this shows that

$$|\omega(x^*ax)|^2 \leq \omega(x^*x)\ \omega(x^*a^*ax) \leq \omega(x^*x)^2\rho(a^*a).$$

Thus we have proved (c). If $c \in A$ is normal then $\rho(c^*c) \leq \rho(c^*)\ \rho(c) = \rho(c)^2$ by Proposition 2.5. This implies (d). $\square$

6.3 Corollary. *Let A be a Mackey complete continuous inverse *-algebra. Then every positive linear functional $\omega\colon A \to \mathbb{C}$ has the following properties.*

(a) $\omega(a^*) = \overline{\omega(a)}$ *for all* $a \in A$*;*

(b) $|\omega(a)|^2 \leq \omega(1)\, \omega(a^*a)$ *for all* $a \in A$*;*

(c) $|\omega(a)| \leq \omega(1) \sqrt{\rho(a^*a)}$ *for all* $a \in A$*;*

(d) $|\omega(c)| \leq \omega(1)\, \rho(c)$ *for all normal elements* $c \in A$*.*

In particular, every positive linear functional on A *is continuous on the real subspace of normal elements of* A*.*

Proof. Properties (a)–(d) follow from the corresponding statements of Proposition 6.2 by setting suitable algebra elements equal to 1. The continuity assertion follows from Lemma 1.5. ☐

6.4 Proposition (Gelfand–Naĭmark–Segal Construction). *Let* A *be a Mackey complete continuous inverse algebra, and let* $\omega\colon A \to \mathbb{C}$ *be a positive linear functional. Then there exist a Hilbert space* $\mathcal{H}$*, a* **-representation* $\pi\colon A \to B(\mathcal{H})$*, and a vector* $v \in \mathcal{H}$ *such that* $\pi(A).v$ *is a dense subspace of* $\mathcal{H}$*, and all* $a \in A$ *satisfy* $\omega(a) = \langle \pi(a).v, v \rangle$*.*

Proof. Define a positive semidefinite sesquilinear form on A by $\langle a, b \rangle := \omega(b^*a)$. Let $R \subseteq A$ be the radical of this form. In other words,

$$R := \{a \in A;\ \forall\, b \in A\colon \langle a, b \rangle = 0\} = \{a \in A;\, \omega(a^*a) = 0\}\,,$$

where the second equality follows from Proposition 6.2. The form $\langle \cdot, \cdot \rangle$ induces a complex scalar product on A/R, which we denote by the same symbol. The Hilbert space $\mathcal{H}$ is defined as the completion of A/R.

Since R is a left ideal of A, a representation π of A on the pre-Hilbert space A/R is defined by $\pi(a).(x+R) := ax + R$ for $a, x \in A$. If $a \in A$ then

$$\begin{aligned} \|\pi(a)\|^2 &= \sup\left\{\|\pi(a).v\|^2;\ v \in A/R,\ \|v\| \leq 1\right\} \\ &= \sup\{\omega(x^*a^*ax);\ x \in A,\ \omega(x^*x) \leq 1\} \leq \rho(a^*a) \end{aligned}$$

by Proposition 6.2. Hence $\pi(a)$ is bounded, and it extends to a bounded operator on $\mathcal{H}$, which we also denote by $\pi(a)$. This yields a *-representation of A on $\mathcal{H}$ because

$$\begin{aligned} \big\langle \pi(a^*).(x+R),\ y+R \big\rangle &= \big\langle a^*x + R,\ y + R \big\rangle = \omega(y^*a^*x) \\ &= \omega\big((ay)^*x\big) = \big\langle x + R,\ ay + R \big\rangle = \big\langle x + R,\ \pi(a).(y+R) \big\rangle \end{aligned}$$

holds for all $a, x, y \in A$.

Set $v := 1 + R \in A/R \subseteq \mathcal{H}$. Then $\pi(A).v = A/R$ is dense in $\mathcal{H}$, and all $a \in A$ satisfy $\langle \pi(a).v, v \rangle = \omega(a)$. ☐

We have given the most direct approach to the Gelfand–Naĭmark–Segal construction. For some purposes, it is an advantage to consider $\mathcal{H}$ as a reproducing kernel Hilbert subspace of the linear dual of A (see Neeb [17, III.1]). In this picture, the representation π of A on $\mathcal{H}$ is the action by right multiplication in the argument of the function.

6.5 Corollary. *Every positive linear functional ω on a Mackey complete continuous inverse *-algebra A is continuous with respect to the greatest C^*-semi-norm σ_{C^*} of A.*

Proof. In the notation of Proposition 6.4, the map $a \mapsto \|\pi(a)\| \colon A \to \mathbb{R}$ is a C*-semi-norm, so that all $a \in A$ satisfy

$$|\omega(a)| = |\langle \pi(a).v, v\rangle| \leq \|\pi(a).v\| \cdot \|v\| \leq \|\pi(a)\| \cdot \|v\|^2 \leq \|v\|^2\, \sigma_{C^*}(a). \qquad \square$$

6.6 Proposition (Automatic continuity). *Let A be a Mackey complete continuous inverse *-algebra A. Assume that the involution of A is continuous or that A is a Fréchet space. Then every *-representation of A on a Hilbert space is continuous. In particular, every positive linear functional on A is continuous.*

Proof. If $\pi \colon A \to B(\mathcal{H})$ is a *-representation of A on a Hilbert space $\mathcal{H}$ then $a \mapsto \|\pi(a)\| \colon A \to \mathbb{R}$ is a C*-semi-norm on A. Hence the assertions follow immediately from Proposition 3.8, Theorem 3.9, and Proposition 6.4. $\square$

6.7 Definition. In any complex *-algebra A, the *positive cone* is defined as

$$\mathrm{Pos}(A) := \{h \in \mathrm{Sym}(A);\ \forall\, \omega \in \mathrm{Pos}(A, \mathbb{C}) \colon \omega(h) \geq 0\}\,.$$

6.8 Proposition. *In a Mackey complete continuous inverse *-algebra A, the positive cone* $\mathrm{Pos}(A)$ *equals the closed convex cone in* $\mathrm{Sym}(A)$ *generated by* $\{a^*a;\ a \in A\}$.

Proof. Let $P \subseteq \mathrm{Sym}(A)$ be the closed convex cone generated by the set $\{a^*a;\ a \in A\}$. Since $\mathrm{Pos}(A)$ is a convex cone and a closed subset of $\mathrm{Sym}(A)$, we have $P \subseteq \mathrm{Pos}(A)$. Conversely, assume that $h \in \mathrm{Sym}(A) \setminus P$. The Hahn–Banach Theorem (see, for instance, Rudin [28, 3.4]) yields a continuous linear functional $\varphi \colon \mathrm{Sym}(A) \to \mathbb{R}$ such that $\varphi(h) \notin \varphi(P)$. We may assume that $\varphi(h) < 0$ and $\varphi(P) \subseteq \mathbb{R}_0^+$. Define

$$\omega \colon A \longrightarrow \mathbb{C}, \quad a \longmapsto \varphi\left(\frac{a + a^*}{2}\right) + i\varphi\left(\frac{a - a^*}{2i}\right).$$

If $a \in A$ then $\omega(ia) = \varphi\big(i\frac{a-a^*}{2}\big) + i\varphi\big(\frac{a+a^*}{2}\big) = i\omega(a)$. Hence ω is a $\mathbb{C}$-linear functional on A. The functionals ω and φ coincide on $\mathrm{Sym}(A)$, so that $\omega(P) \subseteq \mathbb{R}_0^+$. We conclude that $\omega \in \mathrm{Pos}(A, \mathbb{C})$ and $\omega(h) < 0$, whence $h \notin \mathrm{Pos}(A)$. $\square$

7 Hermitian continuous inverse *-algebras

In this section, we study a condition which establishes a connection between the involution and the properties of spectra in a complex *-algebra. As the following example shows, this connection can be quite loose in general.

7.1 Example. Let A be the Banach algebra $\mathbb{C}\times\mathbb{C}$, which is unital and semi-simple. Define an involution on A by $(\zeta_1, \zeta_2)^* := (\overline{\zeta_2}, \overline{\zeta_1})$. The spectrum of (ζ_1, ζ_2) is $\{\zeta_1, \zeta_2\}$. The self-adjoint elements of A are exactly those of the form $(\zeta, \overline{\zeta})$ for some $\zeta \in \mathbb{C}$. Every positive linear functional on A vanishes (cf. Bonsall and Duncan [7, 37.16]).

The unitary elements of A are the elements $(\zeta, 1/\overline{\zeta})$ with $\zeta \in \mathbb{C}^\times$. It follows easily that the convex hull of the unitary group $\mathrm{U}(A)$ equals A. In particular, the unitary semi-norm σ_{U} of A is trivial. The same holds for the Gelfand–Naĭmark semi-norm σ_{C^*} because $\sigma_{\mathrm{C}^*} \leq \sigma_{\mathrm{U}}$.

7.2 Definition. A continuous quasi-inverse *-algebra A is called *hermitian* if every self-adjoint element has real spectrum.

For the following theorem, recall the definition and basic properties of the Raĭkov–Pták functional τ (Definition 3.4 and Lemma 3.5). The use of τ as the principal tool in the theory of hermitian Banach *-algebras is due to Pták [23].

7.3 Theorem (Pták's Theorem for continuous inverse *-algebras). *The following are equivalent for a Mackey complete continuous inverse *-algebra A:*

(i) *A is hermitian;*

(ii) *$\rho \leq \tau$ (Pták's criterion);*

(iii) *$\sigma_{\mathrm{C}^*} = \tau$ (Raĭkov's criterion);*

(iv) *$\rho(c) = \tau(c)$ for all normal elements $c \in A$;*

(v) *$\tau(a + a^*) \leq 2\tau(a)$ for all $a \in A$.*

Since Lemma 3.5 yields the inequality $\sigma_{\mathrm{C}^*} \leq \tau$ in every continuous inverse *-algebra, these conditions are also equivalent to the inequality $\rho \leq \sigma_{\mathrm{C}^*}$, which plays a central role in Palmer's approach [21, Section 10.4].

The following proof of Theorem 7.3 is self-contained. It uses some specific material for continuous inverse algebras, but most arguments can be found, for the case of Banach *-algebras, in Pták's exposition [24, Section 5]; see also Bonsall and Duncan [7, § 41].

Proof. We will first prove the equivalence of conditions (i), (ii), and (iv). Lemma 3.5 shows that (ii) implies (iv). Assume that (iv) holds. If A is not hermitian, there is a self-adjoint element $h \in A$ such that $i \in \mathrm{Sp}(h)$. Let $\alpha \in \mathbb{R}$. Then $i + i\alpha$ belongs to the spectrum of the normal element $h + i\alpha$, so that

$$\begin{aligned}(1+\alpha)^2 &\leq \rho(h+i\alpha)^2 = \tau(h+i\alpha)^2 \\ &= \rho\big((h+i\alpha)^*(h+i\alpha)\big) = \rho(h^2+\alpha^2) \leq \rho(h^2) + \alpha^2.\end{aligned}$$

Hence $1 + 2\alpha \leq \rho(h^2)$ holds for all $\alpha \in \mathbb{R}$. This contradiction proves that A is hermitian.

Assume that A is hermitian. We claim that $1 - a$ is invertible for each $a \in A$ which satisfies $\tau(a) < 1$. Indeed, Corollary 4.7 yields a self-adjoint element $h \in A$ such that $h^2 = 1 - a^*a$, and h is invertible because $1 - a^*a$ is invertible. We calculate

$$\begin{aligned}(1+a^*)(1-a) &= 1 + a^* - a - a^*a \\ &= h^2 + a^* - a = h\big(1 + h^{-1}(a^* - a)h^{-1}\big)h.\end{aligned}$$

Since $ih^{-1}(a^* - a)h^{-1}$ is a self-adjoint element, it has real spectrum, so that the displayed expressions are invertible. Hence $1-a$ is left invertible. By the same argument, the adjoint $(1-a)^* = 1-a^*$ is left invertible because $\tau(a^*) = \tau(a) < 1$. Thus $1 - a$ is invertible as we claimed. Let $a \in A$ be arbitrary, and choose $\lambda \in \mathbb{C}$ with $|\lambda| > \tau(a)$, so that $\tau(\lambda^{-1}a) = |\lambda|^{-1}\tau(a) < 1$. Then the elements $1 - \lambda^{-1}a$ and $\lambda - a$ of A are invertible, so that $\lambda \notin \mathrm{Sp}(a)$. We conclude that $\rho(a) \leq \tau(a)$. Thus (i) implies (ii), and we have proved the equivalence of (i), (ii), and (iv).

Now we will assume conditions (i) and (ii) and derive (iii). If $a \in A$ then

$$\tau(a^*a) = \sqrt{\rho(a^*a\,a^*a)} = \sqrt{\rho(a^*a)^2} = \tau(a)^2.$$

Let $h, k \in A$ be self-adjoint elements. Then

$$\rho(hk) \leq \tau(hk) = \sqrt{\rho(khhk)} = \sqrt{\rho(h^2k^2)}\ .$$

It follows by induction that $\rho(hk) \leq \rho(h^{2^n}k^{2^n})^{2^{-n}}$ holds for all $n \in \mathbb{N}$. If $\rho(h) < 1$ and $\rho(k) < 1$ then Lemma 1.6 yields that $\lim_m h^m = 0 = \lim_m k^m$. Proposition 1.2 implies that $\lim_m h^m k^{2m} h^m = 0$, whence Lemma 1.5 shows that $\lim_m \rho(h^m k^{2m} h^m) = 0$. Since $\rho(h^{2m}k^{2m}) = \rho(h^m k^{2m} h^m)$, we infer that $\rho(hk) < 1$. For general self-adjoint $h, k \in A$, choose $\alpha, \beta \in \mathbb{R}$ with $\rho(h) < \alpha$ and $\rho(k) < \beta$. We have proved that $\rho\big((\alpha^{-1}h)(\beta^{-1}k)\big) < 1$ and hence $\rho(hk) < \alpha\beta$. We conclude that

$$\rho(hk) \leq \rho(h)\ \rho(k) \qquad (h = h^*,\ k = k^*).$$

This implies that all $a, b \in A$ satisfy

$$\tau(ab) = \sqrt{\rho(b^*a^*ab)} = \sqrt{\rho(a^*abb^*)} \leq \sqrt{\rho(a^*a)\ \rho(bb^*)} = \tau(a)\ \tau(b).$$

Our next aim is to prove that the set

$$P := \{h \in \mathrm{Sym}(A);\ \mathrm{Sp}(h) \subseteq \mathbb{R}_0^+\}$$

is a convex cone. (In fact, Corollary 7.7 will show that $P = \mathrm{Pos}(A)$.) If $h \in P$ then $\alpha h \in P$ holds for all $\alpha \geq 0$. We have to prove that $h + k \in P$ if $h, k \in P$. Since A is hermitian, it suffices to show that $1 + h + k \in A^\times$. Since $1 + h,\ 1 + k \in A^\times$, we may define self-adjoint elements $u, v \in A^\times$ by $u := h(1+h)^{-1}$ and $v := k(1+k)^{-1}$. The Spectral Mapping Theorem gives $\rho(u) < 1$ and $\rho(v) < 1$. We have seen that this implies $\rho(uv) < 1$, and so $1 - uv \in A^\times$. Since

$$1 + h + k = (1+h)(1-uv)(1+k),$$

it follows that $1 + h + k \in A^\times$. We conclude that $h + k \in P$.

If $h, k \in A$ are arbitrary self-adjoint elements then $\rho(h) \pm h,\ \rho(k) \pm k \in P$, whence $\rho(h) + \rho(k) \pm (h + k) \in P$. This implies that

$$\rho(h+k) \leq \rho(h) + \rho(k) \qquad (h = h^*,\ k = k^*).$$

Let $a \in A$, and set $h := a + a^*$ and $k := i(a - a^*)$. Then

$$\rho(h^2 + k^2) - h^2 = \big(\rho(h^2 + k^2) - (h^2 + k^2)\big) + k^2 \in P,$$

and so

$$\begin{aligned} \rho(a + a^*) &= \sqrt{\rho(h^2)} \leq \sqrt{\rho(h^2 + k^2)} = \sqrt{2\rho(a^*a + aa^*)} \\ &\leq \sqrt{2(\rho(a^*a) + \rho(aa^*))} = \sqrt{2(\tau(a)^2 + \tau(a)^2)} = 2\tau(a). \end{aligned}$$

Let $a, b \in A$. Then

$$\begin{aligned} \tau(a+b)^2 &= \rho\big((a+b)^*(a+b)\big) = \rho(a^*a + b^*b + a^*b + b^*a) \\ &\leq \rho(a^*a) + \rho(b^*b) + \rho(a^*b + b^*a) \\ &\leq \tau(a)^2 + \tau(b)^2 + 2\tau(a^*b) \\ &\leq \tau(a)^2 + \tau(b)^2 + 2\tau(a^*)\,\tau(b) = \big(\tau(a) + \tau(b)\big)^2. \end{aligned}$$

Thus we have proved that τ is a C*-semi-norm. Hence $\tau \leq \sigma_{\mathrm{C}^*}$, and we infer from Lemma 3.5 that $\sigma_{\mathrm{C}^*} = \tau$. This completes the proof of condition (iii).

If condition (iii) holds then τ is sub-additive, which implies (v). Assume that (v) holds, and let $c \in A$ be a normal element. Write $c = h + ik$ with self-adjoint elements $h, k \in A$. Then $hk = kh$, so that $\rho(c) \leq \rho(h) + \rho(k)$ by Proposition 2.5. Moreover,

$$\rho(h) = \tau(h) = \tau\left(\tfrac{c}{2} + \tfrac{c^*}{2}\right) \leq 2\tau\left(\tfrac{c}{2}\right) = \tau(c)$$

and, similarly, $\rho(k) \leq \tau(c)$. We conclude that $\rho(c) \leq 2\tau(c)$. If $n \in \mathbb{N}$ then

$$\begin{aligned}\rho(c)^{2n} &= \rho(c^n)^2 \\ &\leq 4\tau(c^n)^2 = 4\rho\big((c^*)^n c^n\big) = 4\rho\big((c^*c)^n\big) = 4\rho(c^*c)^n = 4\tau(c)^{2n}.\end{aligned}$$

Hence $\rho(c) \leq \sqrt[n]{2}\,\tau(c)$ for all $n \in \mathbb{N}$ and so $\rho(c) \leq \tau(c)$. Since $\rho(c) \geq \tau(c)$ holds without further assumptions, we have proved condition (iv). □

7.4 Remark (Further characterizations of hermitian algebras). (a) Let A be a Mackey complete continuous inverse *-algebra. Then A is hermitian if and only if there is a constant $C > 0$ such that all $a \in A$ satisfy $\rho(a^*a) \leq C\sigma_{\mathrm{C}^*}(a^*a)$. This condition was observed by Palmer [21, 10.4.8].

Indeed, Palmer's condition follows immediately from Raĭkov's condition $\sigma_{\mathrm{C}^*} = \tau$. Conversely, assume that Palmer's condition holds. Let $h \in A$ be self-adjoint. If $n \in \mathbb{N}$ then $\rho(h)^{2n} = \rho((h^n)^2) \leq C\sigma_{\mathrm{C}^*}((h^n)^2) \leq C\sigma_{\mathrm{C}^*}(h)^{2n}$. Hence $\rho(h) \leq \sigma_{\mathrm{C}^*}(h)$. This implies $\tau \leq \sigma_{\mathrm{C}^*}$, and we conclude that $\sigma_{\mathrm{C}^*} = \tau$, so that A is hermitian.

Pták's condition $\rho \leq \tau$ can also be proved directly from the inequality $\rho(h) \leq \sigma_{\mathrm{C}^*}(h)$ for all $h \in \mathrm{Sym}(A)$ (cf. Palmer [21, 10.2.11]). Namely, choose $a \in A$ with $\sigma_{\mathrm{C}^*}(a) \leq \frac{1}{3}$. Then

$$\begin{aligned}\rho(a + a^* - a^*a) &\leq \sigma_{\mathrm{C}^*}(a + a^* - a^*a) \\ &\leq \sigma_{\mathrm{C}^*}(a) + \sigma_{\mathrm{C}^*}(a^*) + \sigma_{\mathrm{C}^*}(a^*)\,\sigma_{\mathrm{C}^*}(a) \\ &\leq \big(1 + \sigma_{\mathrm{C}^*}(a)\big)^2 - 1 \\ &< 1.\end{aligned}$$

Hence $(1-a^*)(1-a) = 1-(a+a^*-a^*a)$ is invertible. Similarly, the element $(1-a)(1-a^*)$ is invertible. We conclude that $1 - a \in A^\times$. Let $a \in A$ be arbitrary, and choose $\lambda \in \mathbb{C}$ with $|\lambda| > 3\sigma_{\mathrm{C}^*}(a)$, so that $\sigma_{\mathrm{C}^*}(\lambda^{-1}a) = |\lambda|^{-1}\sigma_{\mathrm{C}^*}(a) < \frac{1}{3}$. Then the elements $1 - \lambda^{-1}a$ and $\lambda - a$ of A are invertible, so that $\lambda \notin \mathrm{Sp}(a)$. This implies that $\rho(a) \leq 3\sigma_{\mathrm{C}^*}(a)$. If $n \in \mathbb{N}$ then $\rho(a)^n = \rho(a^n) \leq 3\sigma_{\mathrm{C}^*}(a^n) \leq 3\sigma_{\mathrm{C}^*}(a)^n$. We conclude that $\rho \leq \sigma_{\mathrm{C}^*}$. Since $\sigma_{\mathrm{C}^*} \leq \tau$, we have proved that $\rho \leq \tau$.

(b) Pták [24, 5.10] proved that the following are equivalent for a unital Banach *-algebra A:

(i) A is hermitian;

(ii) $\mathrm{Sp}(u) \subseteq \{\zeta \in \mathbb{C};\ |\zeta| = 1\}$ for all $u \in \mathrm{U}(A)$;

(iii) ρ is bounded on $\mathrm{U}(A)$.

Pták's proof also applies to Mackey complete continuous inverse *-algebras.

7.5 Proposition ($C^*(A)$ for hermitian A). *Let A be a Mackey complete continuous inverse *-algebra, and let $\pi\colon A \to C^*(A)$ be the natural homomorphism from A into its enveloping C*-algebra. Then A is hermitian if and only if π is equispectral, which means that $\pi^{-1}\big(C^*(A)^\times\big) = A^\times$.*

Note that $\pi^{-1}\big(C^*(A)^\times\big) = A^\times$ holds if and only if every $a \in A$ satisfies $\mathrm{Sp}(\pi(a)) = \mathrm{Sp}(a)$. This explains the word "equispectral".

Proof. If π is equispectral then A is hermitian because C^*-algebras are hermitian (see, for instance, Rudin [28, 11.28]).

Conversely, assume that A is hermitian, so that it satisfies $\rho \leq \sigma_{\mathrm{C}^*}$. Choose $a \in A$ such that $\pi(a)$ is invertible in $C^*(A)$. We have to prove that $a \in A^\times$. Since $\pi(A)$ is dense in $C^*(A)$, we may choose $b \in A$ such that $\|\pi(a)^{-1} - \pi(b)\| < \|\pi(a)\|^{-1}$. We calculate

$$\rho(1-ab) \leq \sigma_{\mathrm{C}^*}(1-ab) = \|1-\pi(ab)\| = \big\|\pi(a)\big(\pi(a)^{-1} - \pi(b)\big)\big\| < 1.$$

The analogous calculation yields the inequality $\rho(1-ba) < 1$. Hence $ab, ba \in A^\times$, and we conclude that $a \in A^\times$. □

7.6 Proposition (Spectrum and states). *Let A be a Mackey complete hermitian continuous inverse *-algebra. Then every normal element $c \in A$ satisfies*

$$\mathrm{conv}(\mathrm{Sp}(c)) = \{\omega(c);\ \omega \in \mathrm{Pos}(A, \mathbb{C}),\ \omega(1) = 1\}.$$

Proof. Let $\pi\colon A \to C^*(A)$ be the natural homomorphism from A into its enveloping C*-algebra. By Corollary 6.5, every positive linear functional on A factors through a positive linear functional on $\pi(A)$, which extends by continuity to a positive linear functional on $C^*(A)$. In other words,

$$\mathrm{Pos}(A, \mathbb{C}) = \{\omega \circ \pi;\ \omega \in \mathrm{Pos}(C^*(A), \mathbb{C})\}.$$

In view of Proposition 7.5, the assertion follows from the corresponding result for C*-algebras (see, for instance, Bonsall and Duncan [7, § 38]).

A different proof can be found in Palmer's monograph [21, 10.4.21]. □

The set $\{\omega \in \mathrm{Pos}(A, \mathbb{C});\ \omega(1) = 1\}$ is called the set of *states* of A. The extreme points of this convex set are called the *pure states* of A. A refinement of the preceding proposition (Palmer [21, 10.4.21]) asserts that the spectrum of a normal element $c \in A$ is contained in the image of c under the set of pure states.

7.7 Corollary (Shirali–Ford Theorem). *Every Mackey complete hermitian continuous inverse *-algebra satisfies*

$$\mathrm{Pos}(A) = \big\{h \in \mathrm{Sym}(A);\ \mathrm{Sp}(h) \subseteq \mathbb{R}_0^+\big\}.$$

In particular, $\mathrm{Sp}(a^*a) \subseteq \mathbb{R}_0^+$ holds for all $a \in A$. □

7.8 Remark. Kaplansky [14] conjectured in 1949 that every element a of a hermitian Banach *-algebra satisfies $\mathrm{Sp}(a^*a) \subseteq \mathbb{R}_0^+$. This conjecture was finally proved by Shirali and Ford [30] in 1970. The particularly conceptual proof based on Proposition 7.5 which we have given is due to Fragoulopoulou [10]. Several other proofs have been given; see Bonsall and Duncan [7, 41.5], Pták [24, 5.9], and Palmer [21, 10.4.2].

Section 10.4 of Palmer's monograph contains a number of additional properties of *-algebras which satisfy $\rho \leq \sigma_{\mathrm{C}^*}$. Palmer develops his theory for *-algebras in which some sub-multiplicative semi-norm dominates the spectral radius. The reader must be alert to the fact that this hypothesis is sometimes used without being explicitly stated (e.g. [21, 10.4.4 and 10.4.16]). Palmer's work is very comprehensive, which makes it sometimes difficult to read. For these reasons, we have decided to give proofs for results such as Propositions 7.5 and 7.6 which are also contained in Palmer's book [21, 10.4.18 and 10.4.21].

7.9 Proposition. *The Jacobson radical of a Mackey complete hermitian continuous inverse *-algebra A equals the *-ideal* $\{a \in A;\ \sigma_{\mathrm{C}^*}(a) = 0\}$.

Proof. Proposition 3.6 shows that σ_{C^*} vanishes on $\mathrm{rad}(A)$. Conversely, if A is hermitian then $\rho \leq \sigma_{\mathrm{C}^*}$, so that $\{a \in A;\ \sigma_{\mathrm{C}^*}(a) = 0\}$ consists of quasi-invertible elements and hence is contained in $\mathrm{rad}(A)$. $\square$

7.10 Proposition. *Let A be a Mackey complete hermitian continuous inverse *-algebra, and let $B \subseteq A$ be a closed unital *-subalgebra. Then B is hermitian, and $B^\times = A^\times \cap B$.*

The condition $B^\times = A^\times \cap B$ holds if and only if every $b \in B$ satisfies $\mathrm{Sp}_B(b) = \mathrm{Sp}_A(b)$. Hence a subalgebra $B \subseteq A$ which satisfies this condition might be called "equispectral".

Proof. Choose $b \in A^\times \cap B$. Then $b^* \in A^\times$ and $b^*b \in A^\times$, so that $\mathrm{Sp}_A(b^*b) \subseteq \mathbb{R}^+$. Proposition 1.7 shows that $\mathrm{Sp}_B(b^*b) = \mathrm{Sp}_A(b^*b)$, whence $b^*b \in B^\times$. We conclude that $b^{-1} = (b^*b)^{-1}b^* \in B$, so that $b \in B^\times$. Hence B is an equispectral subalgebra. In particular, it is hermitian. $\square$

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