SEMI-EDGES, REFLECTIONS AND COXETER GROUPS

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1. Introduction

There are numerous similarities between the covering theory of graphs and the covering theory of topological spaces. In algebraic topology, cf. e.g. Definition III.3.1 of [3], a map $p : X \to Y$ between arcwise connected, locally arcwise connected Hausdorff spaces is called a *covering map* if each element $y \in Y$ has an arcwise connected neighborhood U such that $p^{-1}(U)$ is a nonempty disjoint union of sets U_α on which $p_{|U_\alpha}$ is a homeomorphism $U_\alpha \xrightarrow{\approx} U$. In classical graph theory one replaces the topological spaces X and Y by connected graphs Γ_1 and Γ_2, the neighborhood U of a vertex $\mathbf{y} \in \Gamma_2$ by the induced subgraph $\Gamma_2(\mathbf{y})$ on the neighbors of $\mathbf{y}$ in Γ_2 and requires that for each preimage $\mathbf{x}$ in Γ_1 of $\mathbf{y}$ the restriction of p to $\Gamma_1(\mathbf{x})$ is a bijection resp. isomorphism onto $\Gamma_2(\mathbf{y})$.

More precisely, a surjection $\pi : \Gamma_1 \to \Gamma_2$ between connected graphs Γ_1, Γ_2 is called a 1-*covering map* if for each $\mathbf{x}$ in Γ_1 the map π induces a bijection between the set of vertices of $\Gamma_1(\mathbf{x})$ onto the set of vertices of $\Gamma_2(\mathbf{x}^\pi)$. On the other hand, a surjection $\pi : \Gamma_1 \to \Gamma_2$ between connected graphs Γ_1, Γ_2 is called a 2-*covering map* if for each $\mathbf{x}$ in Γ_1 the map π induces a graph isomorphism between the induced subgraph $\Gamma_1(\mathbf{x})$ onto the induced subgraph $\Gamma_2(\mathbf{x}^\pi)$.

The notion of 1-coverings (or *local isomorphisms*, as they are sometimes called) is important for the study of epimorphisms between point-line geometries under which the point row of each line is mapped bijectively onto the point row of its image and, dually, the line pencil of each point is mapped bijectively onto the line pencil of its image. In case of flag-transitive point-line geometries this corresponds to the study of injective completions of amalgams consisting of two groups (the point stabilizer and the line stabilizer). The geometry of the dihedral group $D_{2n} = \langle a, b \mid a^2 = b^2 = (ab)^n = 1\rangle$ probably is the easiest example. A $2n$-gon admits 1-covers by any $2kn$-gon, $k \geq 1$; this 1-covering corresponds to the group epimorphism $D_{2kn} = \langle a, b \mid a^2 = b^2 = (ab)^{kn} = 1\rangle \to D_{2n} = \langle a, b \mid a^2 = b^2 = (ab)^n = 1\rangle$ induced by factoring out the cyclic normal subgroup generated by all nth powers of the product ab. Of course, the universal 1-cover of a $2n$-gon is the tree of valency two, which corresponds to $D_{2\infty} = \langle a, b \mid a^2 = b^2 = 1\rangle$ being the universal completion of the amalgam $\langle a\rangle \cup \langle b\rangle$. For a thorough treatment of the relationship between covers of graphs and amalgams we refer the reader to [5], [6], [16], [17]. There also exist a number of articles on 1-coverings from a purely geometric point of view, especially for generalized polygons, see [7], [8], [14].

2-coverings, on the other hand, are important when studying locally homogeneous graphs. A graph Γ is called *locally homogeneous*, if for any pair $\mathbf{x}$, $\mathbf{y}$ of vertices of Γ the induced subgraphs $\Gamma(\mathbf{x})$ and $\Gamma(\mathbf{y})$ on the neighbors of $\mathbf{x}$, resp. $\mathbf{y}$ are isomorphic. It is clear from the definitions that any 2-cover of a locally homogeneous graph is again locally homogeneous with the same local structure. Therefore, when studying locally homogeneous graphs, it suffices to only consider the simply connected ones, i.e., those graphs that coincide with their universal 2-cover. For example, in [9] one can find a characterization of the line-hyperline graphs of a projective space of sufficiently large dimension by their local structure. The line-hyperline graph of some projective space $\mathbb{P}$ consists of the nonintersecting line-hyperline pairs of $\mathbb{P}$ (i.e., pairs of complemented subspaces of dimension two, resp. codimension two in the corresponding vector space) in which the pair (l, L) is adjacent to the pair (m, M) if and only if $l \subseteq M$ and $m \subseteq L$. It is easily seen that the local structure of the line-hyperline graph of $\mathbb{P}$ is given by the line-hyperline graph of an arbitrary hyperline of $\mathbb{P}$. Conversely, Theorem 1 of [9] says that this local property is characteristic for the line-hyperline graphs for sufficiently large dimension. The proof of that theorem heavily relies on the assumption that the locally homogeneous graphs under consideration are simply connected. Only after a successful classification of the simply connected graphs one realizes that they do not admit quotients with the same local structure (because their diameter is two), completing the proof.

Surowski's work [18] beautifully unifies the concepts of 1-coverings and 2-coverings by considering simplicial complexes. While topologists may not be surprised by the results presented in [18], the article describes graph theoretic covering theory in a language perfectly suited for geometers.

Malnic, Nedela and Skoviera [13] define a different notion of graphs. A *graph* $\Gamma = (V, D, \iota, -1)$ in their sense consists of a set V of *vertices*, a set D of *darts*, a map $\iota : D \to V$ and a permutation $-1 : D \to D : d \mapsto d^{-1}$ with $(-1)^2 = \mathrm{id}$. The map ι assigns to each dart its *initial vertex*, while the map -1 interchanges a dart and its *reverse*. The *terminal vertex* of a dart x is the initial vertex of x^{-1}. The orbits of -1 are called *edges*. Note that edges consist of one or two darts. An edge is called a *semi-edge* if its cardinality is one, a *loop* if its cardinality is two and both darts contained in this edge have the same initial vertex, and a *link* otherwise.

The authors of [13] justify this definition of a graph by applications in the theory of Cayley graphs. For example, Gross and Tucker [11] note that not all Cayley graphs are regular covers of bouquets of circles. Using the notion of a graph as in [13], however, one can prove that each Cayley graph is a regular cover of some *monopole*, i.e., a graph consisting of a unique vertex and an arbitrary number of darts, cf. 6.2 of [13], restated and reproved as Theorem 5.6 in the present paper. A *covering* ϕ in the sense of [13] is a graph surjection π such that for each vertex $\mathbf{x}$ in the domain the set of darts with initial vertex $\mathbf{x}$ is mapped bijectively under π onto the set of darts with initial vertex $\mathbf{x}^{\phi}$.

Neither topology nor Surowski's work [18] apply to the covering theory of graphs with semi-edges. Therefore it is one goal of the present paper to provide a suitable theory of coverings. To this end we define the notion of a *fundamental* 1*-covering* (see Definition 3.1) and prove that this covering is universal (cf. Definition 3.3).

Theorem 3.5 *Let Γ be a connected graph, let $\mathbf{x}$ be a vertex of Γ, and let $\widehat{\Gamma}$ be the fundamental* 1*-cover of Γ based at $\mathbf{x}$. Then the fundamental* 1*-covering $\pi : \widehat{\Gamma} \to \Gamma$ is universal.*

As a consequence of the preceding theorem we can classify all graphs that are 1-covers of a given graph. Results of this kind are typical in the context of topological spaces. In our context, however, we can achieve more: We give an explicit construction of each cover, comparable to the construction of covers of simplicial complexes in §55 of Seifert and Threlfall [15].

Later in this paper we turn our attention to reflections and local reflections of graphs, a *reflection* of a connected graph being an involutive automorphism of the graph that does not stabilize any vertex with the property that if one removes the darts of the graph that are normalized by the automorphism, then the graph becomes disconnected (see the beginning of Section 4). Applying the theory of reflections of graphs to the Cayley graph of a Coxeter group we obtain the following characterization.

Theorem 7.6 *The following statements are equivalent:*

(i) *(W, S) is a Coxeter system.*
(ii) *(W, S) satisfies the exchange condition.*
(iii) *The elements of S act as reflections on the Cayley graph* $\mathrm{Cay}(W, S)$.

In [12] this characterization is proven for the context of 'classical graphs' and used to identify as Coxeter groups reflection groups on certain topological spaces.

Finally, we combine the preceding theorem with our discussion of monopoles in Section 5 to obtain a characterization of the Cayley graphs of Coxeter groups.

Theorem 8.1 *Let $\pi : \Gamma \to (v, D, \iota, -1)$ be a* 1*-covering of a monopole. The graph Γ is the Cayley graph of a Coxeter group if and only if π is regular and any deck transformation in $\Delta(\pi)$ that interchanges two neighboring vertices of Γ acts as a reflection on Γ.*

2. Coverings

Following [13], a **graph** $\Gamma = (V, D, \iota, -1)$ consists of a set V of **vertices**, a set D of **darts**, a map $\iota : D \to V$ and a permutation $-1 : D \to D : d \mapsto d^{-1}$ with $(-1)^2 = \mathrm{id}$. The map ι assigns to each dart its **initial vertex**, while the map -1 interchanges a dart and its **reverse**. The **terminal vertex** of a dart x is the initial vertex of x^{-1}. The orbits of -1 are called **edges**. Note that edges consist of one or two darts. An edge is called a **semi-edge** if its cardinality is one, a **loop** if its cardinality is two and both darts contained in this edge have the same initial vertex, and a **link** otherwise.

A **monopole** is a graph consisting of one vertex and a number of darts.

A **morphism of graphs** $\phi : (V_1, D_1, \iota_1, -1_1) \to (V_2, D_2, \iota_2, -1_2)$ consists of maps $V_1 \to V_2$ and $D_1 \to D_2$ such that $\iota_1\phi = \phi\iota_2$ and $(-1_1)\phi = \phi(-1_2)$.

A **path of length** t is a sequence of t darts $\gamma = x_1 \cdots x_t$ such that the terminal vertex of x_k coincides with the initial vertex of x_{k+1} for all $1 \leq k \leq t-1$. A **path of length** 0 (or **trivial path**) is a vertex $\mathbf{x}$. The **initial vertex** of γ, denoted by γ^ι, is the initial vertex of x_1 (resp., $\mathbf{x}$ in case of a trivial path), the **terminal vertex** of γ is the terminal vertex of x_t (resp., $\mathbf{x}$ in case of a trivial path). If a path γ has initial vertex $\mathbf{x}$, then γ is **based** at $\mathbf{x}$. If its initial and terminal vertices coincide, then γ is called a **cycle** or a **closed path**.

If a path γ_1 terminates at some vertex $\mathbf{x}$ and a path γ_2 starts at the same vertex $\mathbf{x}$, then $\gamma_1\gamma_2$ is a path, called the **concatenation of** γ_1 **and** γ_2. Note that concatenation with the trivial path does not change the path. Furthermore, $(\gamma_1\gamma_2)^\iota = \gamma_1^\iota$ and $(\gamma_1\gamma_2)^{-1} = \gamma_2^{-1}\gamma_1^{-1}$.

A graph is **connected** if for each pair $\mathbf{x}$, $\mathbf{y}$ there exists a path from $\mathbf{x}$ to $\mathbf{y}$. We say that a vertex $\mathbf{y}$ is a **neighbor** of the vertex $\mathbf{x}$ if there exists a dart d with $d^\iota = \mathbf{x}$ and $(d^{-1})^\iota = \mathbf{y}$. Notice that the neighbor relation is symmetric. We denote the set of all neighbors of some vertex $\mathbf{x}$ by $B_1(\mathbf{x})$ ('B' stands for ball) and the set $B_1(\mathbf{x}) \cup \{\mathbf{x}\}$ by $D_1(\mathbf{x})$ ('D' stands for disk).

Let Γ be a connected graph. Two paths in Γ are 1**-homotopic** if one can be obtained from the other by a finite number of applications of the following operation, called **elementary** 1**-homotopy**: inserting or deleting a **return**, i.e., a cycle xx^{-1} of length two: replace xx^{-1} by x^ι or x^ι by xx^{-1}. A cycle that is homotopic to a cycle of length 0 is called 1**-homotopically trivial**. By $\amalg_1(\Gamma)$ denote the **1-fundamental groupoid** (also called **path groupoid**) of Γ, that is, the set of all 1-homotopy classes of paths in Γ endowed with the partial multiplication $([\gamma_1]_1, [\gamma_2]_1) \mapsto [\gamma_1\gamma_2]_1$ whenever the terminal vertex of γ_1 coincides with the initial vertex of γ_2; the subset $\amalg_1(\Gamma, \mathbf{x}) \subset \amalg_1(\Gamma)$ of 1-homotopy classes of paths in Γ with initial vertex $\mathbf{x}$ is called the **path groupoid of** Γ **based at** $\mathbf{x}$. The subset $\Pi_1(\Gamma, \mathbf{x}) \subset \amalg_1(\Gamma, \mathbf{x}) \subset \amalg_1(\Gamma)$ of all 1-homotopy classes of cycles in Γ based at some vertex $\mathbf{x}$ forms a group, the 1**-fundamental group** (or **path group**) **of** Γ **at** $\mathbf{x}$.

Lemma 2.1. *Let* Γ *be a connected graph and let* $\mathbf{x}$, $\mathbf{y}$ *be vertices of* Γ. *Then* $\Pi_1(\Gamma, \mathbf{x}) \cong \Pi_1(\Gamma, \mathbf{y})$ *via the isomorphism that, for a fixed path* γ *from* $\mathbf{y}$ *to* $\mathbf{x}$, *assigns to a homotopy class* $[\delta]_1$ *of cycles based at* $\mathbf{x}$ *the homotopy class* $\left[\gamma\delta\gamma^{-1}\right]_1$ *of cycles based at* $\mathbf{y}$.

Proof. As $\left[\gamma\delta_1\gamma^{-1}\right]_1\left[\gamma\delta_2\gamma^{-1}\right]_1 = \left[\gamma\delta_1\gamma^{-1}\gamma\delta_2\gamma^{-1}\right]_1 = \left[\gamma\delta_1\delta_2\gamma^{-1}\right]_1$ the given map is a group homomorphism. Certainly, it is bijective, so it is an isomorphism. $\square$

In a connected graph Γ, by the above lemma, two path groups with base points $\mathbf{x}$, $\mathbf{y}$ are isomorphic under the transition map that, for a fixed path γ from $\mathbf{y}$ to $\mathbf{x}$, assigns to a homotopy class $[\delta]_1$ of paths based at $\mathbf{x}$ the homotopy class $\left[\gamma\delta\gamma^{-1}\right]_1$ of paths based at $\mathbf{y}$. Any representative of this isomorphism class of path groups of Γ with base point is called the **path group of** Γ, denoted by $\Pi_1(\Gamma)$. The corresponding transition map is called a **base transformation**.

Definition 2.2. *Let $\Gamma = (V, D, \iota, -1)$ and $\widehat{\Gamma} = (\widehat{V}, \widehat{D}, \widehat{\iota}, \widehat{-1})$ be graphs. A graph epimorphism $\pi : \widehat{\Gamma} \to \Gamma$ is called a* 1**-covering** *if, for every vertex $\mathbf{x} \in \widehat{\Gamma}$ the map π sends the set of darts of $\widehat{\Gamma}$ with initial vertex $\mathbf{x}$ bijectively onto the set of darts of Γ with initial vertex $\mathbf{x}^\pi$. The graph $\widehat{\Gamma}$ is called a* 1**-cover** *of the graph Γ. The set $\pi^{-1}(x)$, $x \in V \cup D$ is called a* **fiber***; it is called a* **vertex fiber** *if $x \in V$ and a* **dart fiber** *if $x \in D$.*

A connected graph Γ is called 1**-simply connected** *if any* 1*-covering $\widehat{\Gamma} \to \Gamma$ with $\widehat{\Gamma}$ connected is an isomorphism.*

Lemma 2.3. *Let $\pi : \widehat{\Gamma} \to \Gamma$ be a* 1*-covering of graphs and let γ be an arbitrary path in Γ with initial vertex $\mathbf{x}$. Then for every vertex $\widehat{\mathbf{x}} \in \pi^{-1}(\mathbf{x})$ there exists a unique path $\widehat{\gamma}$ with $\widehat{\mathbf{x}}$ as initial vertex and $\widehat{\gamma}^\pi = \gamma$. Moreover,* 1*-homotopic paths of Γ lift to* 1*-homotopic paths of $\widehat{\Gamma}$, and* 1*-homotopic paths of $\widehat{\Gamma}$ are mapped onto* 1*-homotopic paths of Γ. In particular, if γ_1 and γ_2 are* 1*-homotopic in Γ, then their lifts $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ in $\widehat{\Gamma}$ with identical initial vertices have identical terminal vertices.*

Proof. See Proposition 4.2 of [13]. $\square$

Let $\pi : \widehat{\Gamma} \to \Gamma$ be a 1-covering of graphs and let ϕ be an automorphism of Γ. We say that ϕ **lifts to** $\widehat{\Gamma}$ if there exists an automorphism $\widehat{\phi}$ of $\widehat{\Gamma}$, a **lift of** ϕ, such that $\widehat{\phi}\pi = \pi\phi$. If ϕ lifts then so does ϕ^{-1}. If, more generally, all automorphisms in a group $A \leq \mathrm{Aut}\,\Gamma$ lift, then all those lifts form a group, the **lift** $\widehat{A} \leq \mathrm{Aut}\,\widehat{\Gamma}$ of A. (See [13] for conditions under which lifts exist.) The trivial group of automorphisms of Γ always lifts. The group of all of its lifts is called the **group of deck transformations** and is denoted by $\Delta(\pi)$.

Proposition 2.4. *Let $\pi : \widehat{\Gamma} \to \Gamma$ be a* 1*-covering of connected graphs. Then $\Delta(\pi)$ acts semi-regularly on $\widehat{\Gamma}$, that is, $\Delta(\pi)$ acts fixed point-freely on both the set of vertices and the set of darts of $\widehat{\Gamma}$.*

Proof. This follows directly from Lemma 2.3. $\square$

Definition 2.5. *Let $\pi : \widehat{\Gamma} \to \Gamma$ be a* 1*-covering of connected graphs. It is called* **regular***, if $\Delta(\pi)$ acts transitively on some, whence each, vertex fiber.*

The implication of transitivity on each vertex fiber by the transitivity on some vertex fiber follows from Lemma 2.3 as follows. First, one observes that by the connectedness of Γ, it is enough to show that if $\Delta(\pi)$ acts transitively on $\pi^{-1}(\mathbf{x})$ for some vertex $\mathbf{x}$ of Γ it acts transitively on $\pi^{-1}(\mathbf{y})$ for all neighbors $\mathbf{y}$ of $\mathbf{x}$. So, let d be a dart with $d^\iota = \mathbf{x}$ and $(d^{-1})^\iota = \mathbf{y}$. Then by Lemma 2.3, the path d lifts to a unique path $\widehat{d}$ for each $\widehat{\mathbf{x}} \in \pi^{-1}(\mathbf{x})$. Hence, if $\delta \in \Delta(\pi)$ maps $\widehat{\mathbf{x}}_1$ onto $\widehat{\mathbf{x}}_2$, it also maps the corresponding lift $\widehat{d_1}$ of d starting at $\widehat{\mathbf{x}}_1$ onto the lift $\widehat{d_2}$ of d starting at $\widehat{\mathbf{x}}_1$, whence $(\widehat{d_1^{-1}})^\iota$ gets mapped onto $(\widehat{d_2^{-1}})^\iota$. The transitivity of $\Delta(\pi)$ on the fiber of $\pi^{-1}(\mathbf{y})$ now follows from the fact that each element of the fiber of $\mathbf{y}$ is a neighbor of an element of the fiber of $\mathbf{x}$.

Remark 2.6. *It seems tempting to extend the theory of* 2*-coverings to graphs admitting loops and semi-edges as well. However, this does not lead to any new concept at all. For, when studying* 2*-coverings of graphs one requires that cycles of the form* $d_1d_2d_3$ *with* $d_1^\iota = (d_3^{-1})^\iota$ *are null-homotopic. For a semi-edge* d *this implies that* d *is null-homotopic, because both* dd *and* ddd *are null-homotopic. Similarly, if* l *is a loop based at the vertex* $\mathbf{x}$ *and* d *is any other dart based at* $\mathbf{x}$*, then* l *is null-homotopic because* ldd^{-1} *and* dd^{-1} *are null-homotopic. Therefore the only connected graph in which loops and semi-edges do not lift trivially is the graph consisting of one vertex and a loop, whose universal* 2*-cover is the complete graph on three vertices.*

3. Fundamental and universal covers

Definition 3.1. *Let* $\Gamma = (V, D, \iota, -1)$ *be a connected graph and let* $\mathbf{x}$ *be a vertex of* Γ*. Then the graph* $\widehat{\Gamma} = (\widehat{V}, \widehat{D}, \widehat{\iota}, \widehat{-1})$ *with*

$$\begin{aligned} \widehat{V} &= \Pi_1(\Gamma, \mathbf{x}), \\ \widehat{D} &= \left\{ ([\gamma]_1, d) \in \Pi_1(\Gamma, \mathbf{x}) \times D \mid d^\iota = (\gamma^{-1})^\iota \right\}, \\ \widehat{\iota} : \widehat{D} \to \widehat{V} &: ([\gamma]_1, d) \mapsto [\gamma]_1, \\ \widehat{-1} : \widehat{D} \to \widehat{D} &: ([\gamma]_1, d) \mapsto ([\gamma d]_1, d^{-1}) \end{aligned}$$

is called the **fundamental** 1**-cover of** Γ **based at** $\mathbf{x}$.

Proposition 3.2. *Let* Γ *be a connected graph, let* $\mathbf{x}$ *be a vertex of* Γ*, and let* $\widehat{\Gamma}$ *be the fundamental* 1*-cover of* Γ *based at* $\mathbf{x}$*. Then the canonical map* $\pi : \widehat{\Gamma} \to \Gamma$ *with*

$$\begin{aligned} \pi|_{\widehat{V}} : \widehat{V} \to V &: \gamma \mapsto (\gamma^{-1})^\iota \\ \pi|_{\widehat{D}} : \widehat{D} \to D &: ([\gamma]_1, d) \mapsto d \end{aligned}$$

is a 1*-covering map. Moreover,* $\pi : \widehat{\Gamma} \to \Gamma$ *is an isomorphism if and only if* $\Pi_1(\Gamma, \mathbf{x})$ *is trivial.*

Proof. Straightforward. $\square$

Definition 3.3. *Let* Γ, $\widehat{\Gamma}$ *be connected graphs and let* $\mathbf{x} \in \Gamma$, $\widehat{\mathbf{x}} \in \widehat{\Gamma}$ *be vertices. A* 1*-covering* $\pi : \widehat{\Gamma} \to \Gamma$ *mapping* $\widehat{\mathbf{x}}$ *onto* $\mathbf{x}$ *is called* **universal** *if, for any* 1*-covering* $\alpha : \Gamma_1 \to \Gamma$ *and any* $\mathbf{x}_1 \in \alpha^{-1}(\mathbf{x})$*, there exists a unique* 1*-covering map* $\beta : \widehat{\Gamma} \to \Gamma_1$ *with* $\pi = \beta\alpha$ *and* $\widehat{\mathbf{x}}^\beta = \mathbf{x}_1$.

Remark 3.4. *By definition a universal* 1*-covering* $\pi : \widehat{\Gamma} \to \Gamma$ *is regular. Indeed, if* $\mathbf{x} \in \Gamma$, $\widehat{\mathbf{x}} \in \widehat{\Gamma}$ *with* $\widehat{\mathbf{x}}^\pi = \mathbf{x}$*, then, by the universality of* π*, for each* $\mathbf{x}_1 \in \pi^{-1}(\mathbf{x})$ *there exists an automorphism of* $\widehat{\Gamma}$ *mapping* $\widehat{\mathbf{x}}$ *onto* $\mathbf{x}_1$.

Theorem 3.5. *Let* Γ *be a connected graph, let* $\mathbf{x}$ *be a vertex of* Γ*, and let* $\widehat{\Gamma}$ *be the fundamental* 1*-cover of* Γ *based at* $\mathbf{x}$*. Then the fundamental* 1*-covering* $\pi : \widehat{\Gamma} \to \Gamma$ *is universal.*

Proof. Let $\alpha : \Gamma_1 \to \Gamma$ be an arbitrary 1-covering and let $\mathbf{x}_1 \in \alpha^{-1}(\mathbf{x})$. By Lemma 2.3 any class of paths $[\gamma]_1 \in \Pi_1(\Gamma, \mathbf{x})$ lifts to a unique class of paths $[\widehat{\gamma}]_1 \in \Pi_1(\Gamma_1, \mathbf{x}_1)$. Define a map $\beta : \widehat{\Gamma} \to \Gamma_1$ by $[\gamma]_1^\beta = (\widehat{\gamma}^{-1})^\iota$ and $([\gamma]_1, d)^\beta = \widehat{d}$, where $\widehat{d}$ is the unique lift of d based at $[\gamma]_1^\beta = (\widehat{\gamma}^{-1})^\iota$ by Lemma 2.3. It is clear that $\beta : \widehat{\Gamma} \to \Gamma_1$ is a 1-covering with $\pi = \beta\alpha$ and $\mathbf{x}^\beta = \mathbf{x}_1$. Uniqueness follows from connectedness. □

Corollary 3.6. *Let* Γ *be a connected graph and let* $\mathbf{x}$ *be a vertex of* Γ*. Then its fundamental* 1*-cover* $\widehat{\Gamma}$ *based at* $\mathbf{x}$ *is* 1*-simply connected.*

Proof. Let $\pi : \widehat{\Gamma} \to \Gamma$ be the fundamental 1-covering and let $\phi : \Gamma_1 \to \widehat{\Gamma}$ be some 1-covering. Then $\alpha := \phi\pi : \Gamma_1 \to \Gamma$ is a 1-covering. Let $\mathbf{x}_1 \in \phi^{-1}(\mathbf{x}) \subseteq \alpha^{-1}(\mathbf{x})$. By the universal property of π, there exists a 1-covering $\beta : \widehat{\Gamma} \to \Gamma_1$ mapping $(\mathbf{x})$ onto $\mathbf{x}_1$ with $\pi = \beta\alpha = \beta\phi\pi$. Then $\beta\phi$ is a 1-covering from $\widehat{\Gamma}$ onto itself with $(\mathbf{x})^{\beta\phi} = (\mathbf{x})$, whence it is the identity by the universal property of π. Therefore β is injective, whence bijective. Thus the graph morphism ϕ is the inverse of the graph morphism β and both β and ϕ are graph automorphisms. Hence $\widehat{\Gamma}$ is 1-simply connected. □

Let Γ be a connected graph. If $\widehat{\Gamma}_1$ is the fundamental 1-cover of Γ based at $\mathbf{x}$ and $\widehat{\Gamma}_2$ is the fundamental 1-cover of Γ based at $\mathbf{y}$, and, moreover, if γ is an arbitrary path in Γ from $\mathbf{y}$ to $\mathbf{x}$, then the map from $\widehat{\Gamma}_1$ to $\widehat{\Gamma}_2$ induced by $[\delta]_1 \mapsto [\gamma\delta]_1$ is an isomorphism of graphs. This proves the following.

Proposition 3.7. *Let* Γ *be a connected graph and let* $\mathbf{x}$, $\mathbf{y}$ *be vertices of* Γ*. Then the fundamental* 1*-cover of* Γ *based at* $\mathbf{x}$ *is isomorphic to the fundamental* 1*-cover of* Γ *at* $\mathbf{y}$ *via base transformation.* □

Hence we can suppress the base point of a fundamental 1-cover and speak of the fundamental 1-cover of Γ. If we speak of a fundamental 1-cover and a path group in the same context, then we always assume that the particular cover and the particular group have the same base vertex.

Corollary 3.8 (of Theorem 3.5). *Let* Γ *be a connected graph. The graph* Γ *is* 1*-simply connected if and only if* $\Pi_1(\Gamma)$ *is trivial.*

Proof. If $\Pi_1(\Gamma)$ is non-trivial, then the fundamental cover of Γ is not isomorphic to Γ by Proposition 3.2. Hence Γ is not simply connected.

Conversely, assume that Γ is not simply connected. Then it admits a 1-covering $\alpha : \Gamma_1 \to \Gamma$ for some Γ_1 that is not an isomorphism. If $\pi : \widehat{\Gamma} \to \Gamma$ is the canonical 1-covering of Γ by its fundamental 1-cover $\widehat{\Gamma}$, then, by Theorem 3.5, there exists a 1-covering $\beta : \widehat{\Gamma} \to \Gamma_1$ with $\pi = \beta\alpha$. If π were to be an isomorphism, then α would have to be injective (as β is surjective) which is not the case. So π is not an isomorphism and Proposition 3.2 implies that $\Pi_1(\Gamma)$ is non-trivial. □

Corollary 3.9 (of Corollary 3.8)**.** *Let Γ be a connected graph and let $\alpha : \Gamma_1 \to \Gamma$ be a 1-covering with 1-simply connected Γ_1. Then Γ_1 is isomorphic to the fundamental 1-cover $\widehat{\Gamma}$ of Γ.* □

Theorem 3.10. *Let $\Gamma = (V, D, \iota, -1)$ be a connected graph. Then, for any vertex $\mathbf{x}$ of Γ, there is a one-to-one correspondence between 1-covers of Γ based at $\mathbf{x}$ and subgroups of $\Pi_1(\Gamma, \mathbf{x})$. More precisely, if U is a subgroup of $\Pi_1(\Gamma, \mathbf{x})$, then the corresponding 1-cover of Γ is the graph $\Gamma_0 = (V_0, D_0, \iota_0, -1_0)$ with*

$$\begin{aligned} V_0 &= U \backslash \Pi_1(\Gamma, \mathbf{x}), \\ D_0 &= \left\{ (U[\gamma]_1, d) \in U \backslash \Pi_1(\Gamma, \mathbf{x}) \times D \mid d^\iota = (\gamma^{-1})^\iota \right\}, \\ \widehat{\iota} : D_0 \to V_0 &: (U[\gamma]_1, d) \mapsto U[\gamma]_1, \\ -1_0 : D_0 \to D_0 &: (U[\gamma]_1, d) \mapsto (U[\gamma d]_1, d^{-1}). \end{aligned}$$

Note that we use the symbol $\backslash$ in the theorem to denote right cosets, respectively, U-orbits for the left multiplication action on $\Pi_1(\Gamma, \mathbf{x})$.

Proof. Given a subgroup U of $\Pi_1(\Gamma, \mathbf{x})$ it is straightforward to check that the graph Γ_0 given above is a 1-cover of Γ.

Conversely, let $\alpha : \Gamma_0 \to \Gamma$ be a 1-covering. By Theorem 3.5 there exists a 1-covering $\beta : \widehat{\Gamma} \to \Gamma_0$, where $\widehat{\Gamma}$ is the fundamental cover of Γ. We can consider $\widehat{\Gamma}$ as the fundamental 1-cover of Γ_0 based at $\mathbf{x}^\beta$, since $\widehat{\Gamma}$ is 1-simply connected, so by Corollary 3.9 isomorphic to the fundamental 1-cover of Γ_0. By Lemma 2.3 the 1-covering α induces an embedding of $\Pi_1(\Gamma_0, \mathbf{x}^\beta)$ in $\Pi_1(\Gamma, \mathbf{x})$. (It induces a well-defined map as images of homotopic paths are homotopic. This induced map is injective as lifts of homotopic paths are homotopic.) We have identified $\Pi_1(\Gamma_0, \mathbf{x}^\beta)$ with a subgroup of $\Pi_1(\Gamma, \mathbf{x})$ and hence $\Gamma_0 = \Pi_1(\Gamma_0, \mathbf{x}^\beta) \backslash \widehat{\Gamma}$, finishing the proof. □

Corollary 3.11 (of Lemma 2.1 and Theorem 3.10)**.** *Let Γ be a connected graph. Then there is a one-to-one correspondence between equivalence classes of 1-covers of Γ without base vertex and conjugacy classes of subgroups of $\Pi_1(\Gamma)$.* □

Corollary 3.12 (of Theorem 3.10)**.** *Let $\pi : \Gamma_1 \to \Gamma_2$ be a 1-covering and let $\mathbf{y} = \mathbf{x}^\pi$ for some vertex $\mathbf{x}$ of Γ_1. Then π induces a group monomorphism $\pi_\# : \Pi(\Gamma_1, \mathbf{x}) \to \Pi(\Gamma_2, \mathbf{y})$.* □

Corollary 3.13 (of Theorem 3.10). *Let $\pi : \Gamma_1 \to \Gamma_2$ be a 1-covering with $\mathbf{y} = \mathbf{x}^\pi$ for some vertex $\mathbf{x} \in \Gamma_1$. Assume that $\alpha : \Gamma_0 \to \Gamma_2$ is another 1-covering, and let $\mathbf{w} \in \alpha^{-1}(\mathbf{y})$. Then a 1-covering $\beta : \Gamma_0 \to \Gamma_1$ with $\beta\pi = \alpha$ exists if and only if $\Pi_1(\Gamma_0, \mathbf{w})^{\alpha\#} \subseteq \Pi_1(\Gamma_1, \mathbf{x})^{\pi\#}$. Moreover, if such a β exists it is unique.* □

Corollary 3.14 (of Corollary 3.13). *Let $\pi : \Gamma_1 \to \Gamma_2$ be a 1-covering with $\mathbf{y} = \mathbf{x}^\pi$ for some vertex $\mathbf{x} \in \Gamma_1$ and let $\mathbf{x}_0 \in \pi^{-1}(\mathbf{x})$. Then there exists a deck transformation in $\Delta(\pi)$ taking $\mathbf{x}_0$ to $\mathbf{x}$ if and only if $\Pi_1(\Gamma_1, \mathbf{x}_0)^{\pi\#} = \Pi_1(\Gamma_1, \mathbf{x})^{\pi\#}$.*

Proof. Let $\phi \in \Delta(\pi)$ be the deck transformation taking $\mathbf{x}_0$ to $\mathbf{x}$. Then Corollary 3.13 applies with $\Gamma_0 = \Gamma_1$, $\mathbf{w} = \mathbf{x}_0$ and $\beta = \phi$ and $\alpha = \phi\pi = \pi$, yielding $\Pi_1(\Gamma_1, \mathbf{x}_0)^{(\phi\pi)\#} = \Pi_1(\Gamma_1, \mathbf{x}_0)^{\pi\#} \subseteq \Pi_1(\Gamma_1, \mathbf{x})^{\pi\#}$. The reverse inclusion follows by symmetry. □

Corollary 3.15 (of Lemma 2.1 and Corollary 3.14; cf. Corollary 5.5 of [13]). *Let $\pi : \Gamma_1 \to \Gamma_2$ be a 1-covering with $\mathbf{y} = \mathbf{x}^\pi$ for some vertex $\mathbf{x} \in \Gamma_1$. The subgroup $\Pi_1(\Gamma_1, \mathbf{x})^{\pi\#}$ of $\Pi_1(\Gamma_2, \mathbf{y})$ is normal if and only if π is a regular 1-covering.*

Proof. The base transformation of Lemma 2.1 between bases $\mathbf{x}_0, \mathbf{x}$ inside the fiber $\pi^{-1}(\mathbf{y})$ proves that $\Pi_1(\Gamma_1, \mathbf{x})^{\pi\#}$ and $\Pi_1(\Gamma_1, \mathbf{x}_0)^{\pi\#}$ are conjugate inside $\Pi_1(\Gamma_2, \mathbf{y})$. On the other hand, by Corollary 3.14, we have $\Pi_1(\Gamma_1, \mathbf{x}_0)^{\pi\#} = \Pi_1(\Gamma_1, \mathbf{x})^{\pi\#}$ for all $\mathbf{x}_0 \in \pi^{-1}(\mathbf{y})$ if and only if $\Delta(\pi)$ acts transitively on $\pi^{-1}(\mathbf{y})$, i.e., if and only if π is a regular 1-covering. □

Corollary 3.16 (of Corollary 3.15). *Let $\pi : \Gamma_1 \to \Gamma_2$ be a regular 1-covering with $\mathbf{y} = \mathbf{x}^\pi$ for some vertex $\mathbf{x} \in \Gamma_1$. Then*

$$\Delta(\pi) \cong \Pi_1(\Gamma_2, \mathbf{y})/\Pi_1(\Gamma_1, \mathbf{x})^{\pi\#}.$$

□

Corollary 3.17 (of Corollaries 3.8 and 3.16). *Let $\pi : \Gamma_1 \to \Gamma_2$ be a regular 1-covering with $\mathbf{y} = \mathbf{x}^\pi$ for some vertex $\mathbf{x} \in \Gamma_1$. If, moreover, Γ_1 is 1-simply connected, then*

$$\Delta(\pi) \cong \Pi_1(\Gamma_2, \mathbf{y}).$$

□

Remark 3.18. *The concept of a* fundamental cover *for simplicial complexes already exists in the literature, e.g. §55 of Seifert and Threlfall* [15]. *The purpose of Section 3 is to provide an extension of the simplicial covering theory to graphs with semi-edges.*

4. Reflections and local reflections

For an automorphism σ of a connected graph $\Gamma = (V, D, \iota, -1)$ set

$$\begin{aligned} \mathrm{Fix}_\sigma(V) &:= \{v \in V \mid v^\sigma = v\}, \text{ the set of fixed vertices, and} \\ \mathrm{Norm}_\sigma(D) &:= \left\{d \in D \mid d \neq d^\sigma = d^{-1}\right\}, \text{ the set of properly normalized darts.} \end{aligned}$$

An involution σ of a connected graph $\Gamma = (V, D, \iota, -1)$ is called a **reflection**, if $\mathrm{Fix}_\sigma(V) = \emptyset$ and if $\Gamma_\sigma = (V, D_\sigma, \iota_\sigma, -1_\sigma)$ with $D_\sigma = D \backslash \mathrm{Norm}_\sigma(D)$ and $\iota_\sigma = \iota|_{D_\sigma}$, $-1_\sigma = -1|_{D_\sigma}$ is disconnected. We write $x \sim_\sigma x'$ if x and x' are vertices of the same connected component in Γ_σ and we say that σ **separates** x and x' if we have $x \not\sim_\sigma x'$. For $x \in V$ set

$$V_+^\sigma(x) = \{y \in V \mid x \sim_\sigma y\} \quad \text{and} \quad V_-^\sigma(x) = \{y \in V \mid x \not\sim_\sigma y\}.$$

An involution σ of a connected graph Γ is called a **local reflection** if there exists a vertex v with $D_1(v) \cap D_1(v^\sigma) \neq \emptyset$ and if, for any vertex v with $D_1(v) \cap D_1(v^\sigma) \neq \emptyset$, the restriction of σ to $D_1(v) \cap D_1(v^\sigma)$ is a reflection.

Proposition 4.1. *Let $\Gamma = (V, D, \iota, -1)$ be a connected graph. Any local reflection $\sigma : \Gamma \to \Gamma$ has the following properties. The set $\mathrm{Fix}_\sigma(V)$ is empty, the graph Γ_σ consists of one or two connected components, and, for every dart d with $d \neq d^\sigma = d^{-1}$, the graph $(V, D_\sigma \cup \{d, d^{-1}\}, \iota|_{D_\sigma \cup \{d, d^{-1}\}}, -1|_{D_\sigma \cup \{d, d^{-1}\}})$ is connected.*

Proof. Suppose $v \in \mathrm{Fix}_\sigma(V)$. Then $D_1(v) \cap D_1(v^\sigma)$ contains $v = v^\sigma$, so the restriction of σ to $D_1(v) \cap D_1(v^\sigma)$ is not a reflection.

Suppose Γ_σ consists of more than two connected components. Since darts of Γ with $d \neq d^\sigma = d^{-1}$ can only connect connected components of Γ_σ that are interchanged by σ, the original graph Γ cannot be connected, a contradiction. Hence Γ_σ consists of one or two connected components.

The last statement is trivially true if Γ_σ is connected. If it is disconnected, then, by the above, it consists of two connected components, and the connected components have to be interchanged by σ. Adding any dart of Γ with $d \neq d^\sigma = d^{-1}$ obviously connects the two components. $\square$

Corollary 4.2. *Let $\Gamma = (V, D, \iota, -1)$ be a connected graph. A local reflection of Γ is a reflection if and only if Γ_σ consists of two connected components.* $\square$

Theorem 4.3. *Let $\Gamma = (V, D, \iota, -1)$ be a connected graph. Any local reflection σ of Γ can be lifted to a reflection $\widehat{\sigma}$ of $\widehat{\Gamma} = (\widehat{V}, \widehat{D}, \widehat{\iota}, \widehat{-1})$ with*

$$\begin{aligned} \widehat{V} &= V \times \{-1, 1\}, \\ \widehat{D} &= D \times \{-1, 1\}, \\ \widehat{\iota} : \widehat{D} \to \widehat{V} &: (d, \pm 1) \mapsto (d^\iota, \pm 1) \\ \widehat{-1} : \widehat{D} \to \widehat{D} &: (d, \pm 1) \mapsto \begin{cases} (d^{-1}, \pm 1), & \text{if } d \notin \mathrm{Norm}_\sigma(D), \\ (d^{-1}, \mp 1), & \text{if } d \in \mathrm{Norm}_\sigma(D), \end{cases} \end{aligned}$$

mapping $(v, \pm 1)$ onto $(v^\sigma, \mp 1)$ and $(d, \pm 1)$ onto $(d^\sigma, \mp 1)$. The canonical map $\pi : \widehat{\Gamma} \to \Gamma$ is a 1-covering and $\sigma\pi = \pi\widehat{\sigma}$.

Proof. The only thing to prove is the fact that $\widehat{\sigma}$ is a reflection. It is a graph morphism, since

$$\begin{aligned}
((d,\pm 1)^{\iota})^{\widehat{\sigma}} &= (d^{\iota},\pm 1)^{\widehat{\sigma}} \\
&= (d^{\iota\sigma},\mp 1) \\
&= (d^{\sigma\iota},\mp 1) \\
&= (d^{\sigma},\mp 1)^{\iota} \\
&= ((d,\pm 1)^{\widehat{\sigma}})^{\iota} \quad \text{and} \\
((d,\pm 1)^{-1})^{\widehat{\sigma}} &= (d^{-1},\epsilon(\pm 1))^{\widehat{\sigma}} \\
&= (d^{(-1)\sigma},\epsilon(\mp 1)) \\
&= (d^{\sigma(-1)},\epsilon(\mp 1)) \\
&= (d^{\sigma},\mp 1)^{-1} \\
&= ((d,\pm 1)^{\widehat{\sigma}})^{-1}
\end{aligned}$$

where $\epsilon = 1$ if $d \notin \mathrm{Norm}_{\sigma}(D)$ and $\epsilon = -1$ if $d \in \mathrm{Norm}_{\sigma}(D)$. Because $\widehat{\sigma}^2 = \mathrm{id}$ the morphism $\widehat{\sigma}$ is an automorphism. Certainly, the graph $\widehat{\Gamma}$ is connected and $\mathrm{Fix}_{\widehat{\sigma}}(\widehat{V}) = \emptyset$. The set $\mathrm{Norm}_{\widehat{\sigma}}(\widehat{D})$ equals $\{(d,\pm 1) \mid d \in \mathrm{Norm}_{\sigma}(D)\}$ and the graph $\widehat{\Gamma}_{\widehat{\sigma}}$ consists of two connected components. Therefore $\widehat{\sigma}$ is a reflection. $\square$

Lemma 4.4. *If σ is a (local) reflection on a graph $\Gamma = (V, D, \iota, -1)$ and ϕ is a graph automorphism of Γ, then $\phi^{-1}\sigma\phi$ is a (local) reflection and we have $\mathrm{Norm}_{\phi^{-1}\sigma\phi}(D) = (\mathrm{Norm}_{\sigma}(D))^{\phi}$.*

Proof. We have $d \in (\mathrm{Norm}_{\sigma}(D))^{\phi}$ if and only if $d^{\phi^{-1}} \in \mathrm{Norm}_{\sigma}(D)$, which is equivalent to $d^{\phi^{-1}\sigma} = (d^{\phi^{-1}})^{-1} = (d^{-1})^{\phi^{-1}}$, whence $d^{\phi^{-1}\sigma\phi} = d^{-1}$, which in turn is equivalent to $d \in \mathrm{Norm}_{\phi^{-1}\sigma\phi}(D)$. In the same way, we can prove $\mathrm{Fix}_{\phi^{-1}\sigma\phi}(V) = \mathrm{Fix}_{\sigma}(V)^{\phi}$, and in our case that set is empty. Since ϕ is a graph automorphism, the graph $\Gamma_{\phi^{-1}\sigma\phi} = (V, D_{\phi^{-1}\sigma\phi}, \iota_{\phi^{-1}\sigma\phi}, -1_{\phi^{-1}\sigma\phi})$ is not connected, so $\phi^{-1}\sigma\phi$ is a (local) reflection. $\square$

Let Γ be a graph and let $\alpha : W \to \mathrm{Aut}(\Gamma)$ be a group action Γ. The action α is **free** if no vertex of Γ is fixed by a group element other than the identity. This implies that α is injective, so we may think of W as a subgroup of $\mathrm{Aut}(\Gamma)$.

Lemma 4.5. *Let $\Gamma = (V, D, \iota, -1)$ be a graph and let W act freely on Γ. We have*

$$\mathrm{Norm}_{w}(D) \cap \mathrm{Norm}_{v}(D) = \emptyset$$

for each pair v, w of distinct involutions in W. In particular, any involution $r \in W$ acting as a local reflection of Γ is uniquely determined by the set $\mathrm{Norm}_{r}(D)$.

Proof. Notice $\mathrm{Norm}_{v}(D) = \mathrm{Norm}_{v^{-1}}(D)$. Suppose $d \in \mathrm{Norm}_{w}(D) \cap \mathrm{Norm}_{v}(D)$ and, thus, $d \in \mathrm{Norm}_{w}(D) \cap \mathrm{Norm}_{v^{-1}}(D)$. Since the operation of W is free, the equality $(d^\iota)^{wv^{-1}} = ((d^{wv^{-1}})^\iota) = ((d^{-1})^{v^{-1}})^\iota = d^\iota$ implies $wv^{-1} = 1$, whence $w = v$. The second statement is clear since $\mathrm{Norm}_r(D)$ is not empty for a local reflection $r \in W$. □

Definition 4.6. *Let W be a group generated by a symmetric subset S, i.e., $S^{-1} = S$, with $1 \notin S$. A* **word** *(on S) is a sequence $(s_1, s_2, \ldots, s_n)$ of elements of S. Let $w \in W$. The* **length** *$\ell_S(w)$ or simply $\ell(w)$ of w (with respect to S) is the smallest integer $n \geq 0$ such that $w = s_1 s_2 \cdots s_n$ for a word $(s_1, s_2, \ldots, s_n)$. A* **reduced decomposition** *of w (with respect to S) is a word $(s_1, s_2, \ldots, s_n)$ such that we have $w = s_1 s_2 \cdots s_n$ and $n = \ell(w)$.*

Lemma 4.7. *Let $\Gamma = (V, D, \iota, -1)$ be a graph and let W act on Γ. Let S be a generating subset of W such that every element of S acts as a reflection of Γ. Suppose there is a vertex $\mathbf{x}$ such that for every $s \in S$ there exists a $d_{\mathbf{x},\mathbf{x}^s} \in D$ with $d^\iota_{\mathbf{x},\mathbf{x}^s} = \mathbf{x}$ and $(d^{-1}_{\mathbf{x},\mathbf{x}^s})^\iota = (d_{\mathbf{x}^s,\mathbf{x}})^\iota = \mathbf{x}^s$. Furthermore suppose for all $s, t \in S$, $w \in W$, the relation $\mathbf{x}^s = \mathbf{x}^{wtw^{-1}}$ implies $s = wtw^{-1}$. Then the stabilizer W_x is trivial.*

Proof. Suppose, we have $\mathbf{x}^w = \mathbf{x}$ for a $w \in W$ of minimal length $n \geq 1$. Take a reduced decomposition $(s_1, s_2, \ldots, s_n)$ of w with respect to S. Consider the path $d_{\mathbf{x},\mathbf{x}^{s_n}} d_{\mathbf{x}^{s_n},\mathbf{x}^{s_{n-1}s_n}} \cdots d_{\mathbf{x}^{s_2\cdots s_n},\mathbf{x}^{s_1 s_2 \cdots s_n}}$ with initial and terminal vertex $\mathbf{x}$ (the dart $d_{\mathbf{x}^{s_j\cdots s_n},\mathbf{x}^{s_{j-1}s_j\cdots s_n}}$ exists as $d_{\mathbf{x},\mathbf{x}^{s_{j-1}}}$ exists and $s_j \cdots s_n$ is an automorphism of Γ). The vertices $\mathbf{x}$ and $\mathbf{x}^{s_n}$ are separated by s_n. So there must be an index $1 < j \leq n$, such that s_n separates $\mathbf{x}^{s_j s_{j+1} \cdots s_n}$ and $\mathbf{x}^{s_{j-1}s_j \cdots s_n}$. In other words, we have $\mathbf{x}^{(s_j s_{j+1} \cdots s_n)s_n} = \mathbf{x}^{s_{j-1}s_j\cdots s_n}$ or equivalently $\mathbf{x}^{(s_j s_{j+1}\cdots s_n)s_n(s_j s_{j+1}\cdots s_n)^{-1}} = \mathbf{x}^{s_{j-1}}$. By hypothesis we get

$$(s_j s_{j+1} \cdots s_n) s_n (s_j s_{j+1} \cdots s_n)^{-1} = s_{j-1}$$

or equivalently $s_j s_{j+1} \cdots s_n = s_{j-1} s_j \cdots s_{n-1}$ whence

$$s_1 s_2 \cdots s_n = s_1 s_2 \cdots s_{j-2} s_j s_{j+1} \cdots s_{n-1},$$

which is a contradiction to the minimality of n. □

Proposition 4.8. *Let Γ be a connected graph, let W act on Γ and let S be a generating subset of W. Suppose there is a vertex $\mathbf{x} \in V$ such that for each dart d with initial vertex $\mathbf{x}$ there is an $s \in S$ with $\mathbf{x}^s = (d^{-1})^\iota$. Then the action of W on the set of vertices of Γ is transitive.*

Proof. By a straightforward induction argument. □

5. Covers of monopoles

Definition 5.1. *An* **automaton** *is a triple $A = (S, X, \tau)$ where S is a set, the set of* **states**, *X is a set, the set of* **inputs**, *and $\tau : S \times X \to S$ is a map, the* **transition map**. *(By iteration we can and sometimes will consider τ as a map from $S \times X^*$ into S, where X^* is the monoid of of all words over the alphabet X.) The* **transition semi-group** *G_A of the automaton A consists of the transformations $g : S \to S$ such that there exists an $x \in X^*$ such that*

$s^g = (s,x)^\tau$ *for all* $s \in S$. *The automaton* A *is called a* **group automaton** *if its transition semi-group* G_A *is a group.*

Obviously, in a group automaton the map $\tau_x : S \to S : s \mapsto (s,x)^\tau$ is a permutation of S for arbitrary $x \in X$. Therefore we can add the symbol x^{-1} to X and define the map $\tau_{x^{-1}}$ as the inverse map of τ_x. Then X admits an involution -1 with the property $(\mathbf{x}^{-1})^{-1} = \mathbf{x}$. If a group automaton $A = (S, X, \tau)$ satisfies $X^{-1} = X$, then it is called **symmetric**. A group automaton is called **homogeneous** if $s^{\tau_x} = s^{\tau_y}$ for some $s \in S$ implies $s^{\tau_x} = s^{\tau_y}$ for all $s \in S$.

The **transition graph** $\Gamma = (V, D, \iota, -1)$ of a symmetric group automaton $A = (S, X, \tau)$ consists of

$$\begin{array}{rcl} \widehat{V} & = & S, \\ \widehat{D} & = & S \times X, \\ \iota : D \to V & : & (s,x) \mapsto s, \\ -1 : D \to D & : & (s,x) \mapsto ((s,x)^\tau, x^{-1}). \end{array}$$

Theorem 5.2. *Any transition graph of a symmetric group automaton is a* 1*-cover of a monopole and vice versa.*

Proof. Certainly, the canonical projection $S \times X \to X$ extends to a 1-covering of the monopole $(\{S\}, X, \bar{\iota}, \overline{-1})$ with $\bar{\iota} : X \to \{S\}$ the constant map and $\overline{-1} : X \to X$ the inversion map.

Conversely, let $\widehat{\Gamma} = (\widehat{V}, \widehat{D}, \widehat{\iota}, \widehat{-1})$ be a 1-cover of the monopole $\Gamma = (v, D, \iota, -1)$. Then $(\widehat{V}, D, \tau)$ with $(x,d)^\tau$ defined as the terminal vertex in $\widehat{V}$ of the unique lift of the path d of length one to a path of $\widehat{\Gamma}$ with initial vertex x by Lemma 2.3, is an automaton. It even is a group automaton, because $\tau_{d^{-1}}$ is the inverse of τ_d. Moreover, $D^{-1} = D$ implies that the group automaton is symmetric. Obviously, $\widehat{\Gamma}$ is the transition graph of $(\widehat{V}, D, \tau)$. □

An automaton $A = (S, X, \tau)$ is called **strongly connected** if for each pair s, t of states there exists an $x \in X^*$ with $(s,x)^\tau = t$. The transition graph of a symmetric group automaton is connected if and only if the corresponding automaton is strongly connected.

Lemma 5.3. *Let* $\widehat{\Gamma}$ *be the transition graph of a strongly connected symmetric group automaton* (S, X, τ) *and let* $\Gamma = (v, D, \iota, -1)$ *be a monopole* 1*-covered by* $\widehat{\Gamma}$. *The* 1*-covering* $\widehat{\Gamma} \to \Gamma$ *is regular if and only if* (S, X, τ) *is homogeneous and for each* $x \in X$ *the map* τ_x *can be extended to an automorphism of* $\widehat{\Gamma}$.

Proof. Suppose for each $x \in X$ the map τ_x can be extended to an automorphism of $\widehat{\Gamma}$ and (S, X, τ) is homogeneous. There exists an equivalence relation on X defined by $x \equiv y$ if and only if $s^{\tau_x} = s^{\tau_y}$ for some, whence by homogeneity of (S, X, τ), all $s \in S$. Certainly $(S, X/{\equiv}, \bar{\tau})$ is a homogeneous strongly connected symmetric group automaton and for each $\bar{x} \in X/\equiv$

the map $\bar{\tau}_{\bar{x}}$ can be extended to an automorphism of the transition graph of $(S, X/\equiv, \bar{\tau})$. All edges except at most one of the transition graph of $(S, X/\equiv, \bar{\tau})$ are links. Moreover, each link is uniquely determined by its initial and terminal vertices. It is clear that each map $\tau_{\bar{x}}$ induces a (uniquely determined) deck transformation of the 1-covering from the transition graph of $(S, X/\equiv, \bar{\tau})$ onto some monopole with dart set $X/\equiv$. But now it is trivial to extend τ_x to a deck transformation of the 1-covering $\widehat{\Gamma} \to \Gamma$. The strong connectedness of (S, X, τ) yields transitivity of the group of deck transformations on the vertex fibers, whence $\widehat{\Gamma} \to \Gamma$ is regular.

Conversely, assume that $\widehat{\Gamma} \to \Gamma$ is regular. Then for each pair $\mathbf{x}$, $\mathbf{y}$ of neighbors in $\widehat{\Gamma}$ there exists a deck transformation mapping $\mathbf{x}$ to $\mathbf{y}$. This implies the homogeneity of (S, X, τ). Moreover, the restriction of this automorphism to the set of vertices coincides with the map τ_d where d is the dart in Γ that lifts to some dart $\widehat{d}$ in $\widehat{\Gamma}$ with initial vertex $\mathbf{x}$ and terminal vertex $\mathbf{y}$. □

Example 5.4 (An inhomogeneous group automaton). *Let $S = \{1, 2, 3, 4\}$, let $X = \left\{a, a^{-1}, b, b^{-1}, c, c^{-1}\right\}$ and define τ as follows.*

	a	a^{-1}	b	b^{-1}	c	c^{-1}
1	2	4	2	2	4	4
2	3	1	1	1	3	3
3	4	2	4	4	2	2
4	1	3	3	3	1	1

Notice that the map τ_a can be extended to an automorphism of the transition graph Γ of the group automaton (S, X, τ) but not to a deck transformation of the canonical 1-covering. Indeed, if an automorphism of Γ induced by τ_a is a deck transformation, then it has to preserve the fiber $\pi^{-1}(a)$ of the 1-covering $\pi : \Gamma \to (\{S\}, X, \iota, -1)$. But then this automorphism has to interchange the fibers $\pi^{-1}(b)$ and $\pi^{-1}(c)$ and, thus, is not a deck transformation.

In [13] one can find a characterization of graphs that are regular 1-covers of monopoles. Here is a brief reminder of their result.

Definition 5.5. *A* **voltage space** *on a connected graph Γ is a triple (F, G, ξ) where G is a group acting on a non-empty set F and $\xi : \Pi_1(\Gamma) \to G$ is a homomorphism of groupoids. The group G is called the* **voltage group**, *the set F is the* **abstract fiber** *and $[\gamma]_1^{\xi}$ is the voltage of the 1-homotopy class of the path γ.*

In case G acts on $F = G$ by right translation, then (F, G, ξ) is called a **Cayley voltage space**.

To any voltage space (F, G, ξ) over some graph Γ there exists the associated graph $\widehat{\Gamma} = (\widehat{V}, \widehat{D}, \widehat{\iota}, \widehat{-1})$ with

$$\begin{aligned} \widehat{V} &= V \times F, \\ \widehat{D} &= D \times F, \\ \widehat{\iota} : \widehat{D} \to \widehat{V} &: (d, f) \mapsto (d^{\iota}, f), \\ \widehat{-1} : \widehat{D} \to \widehat{D} &: (d, f) \mapsto (d^{-1}, f^{(d^{\xi})}). \end{aligned}$$

One can check that the canonical projection $\pi : \widehat{\Gamma} \to \Gamma$ induced by $(d, f)^{\pi} = d$ is a 1-covering.

It was observed in [13] (see also [10]) that with each regular 1-covering $\pi : \widehat{\Gamma} \to \Gamma$ one can associate a Cayley voltage space as follows. Choose G to be the group of deck transformations $\Delta(\pi)$ and label the elements of each vertex fiber by G so that the left action of $\Delta(\pi)$ on $\widehat{\Gamma}$ induces the action of $\Delta(\pi)$ on itself by left translation on each labelled vertex fiber. The homomorphism $\xi : \Pi_1(\Gamma) \to \Delta(\pi)$ is given by $d^{\xi} = g_1^{-1} g_2$ for each dart d if there exists a dart in $\pi^{-1}(d)$ passing from the element labelled g_1 in the vertex fiber of d^{ι} to the element labelled g_2 in the vertex fiber of $(d^{-1})^{\iota}$. Note that ξ is well-defined. Indeed, if there exists another dart in $\pi^{-1}(d)$ with initial vertex labelled g_3 and terminal vertex labelled g_4, then, by the left action of $G = \Delta(\pi)$, we have $g_3 g_1^{-1} = g_4 g_2^{-1}$ and hence $g_1^{-1} g_2 = g_3^{-1} g_4$.

Given a group G and a generating multiset $S = S^{-1}$ of G (i.e., we have a map $\alpha : S \to G$ with $\langle S^{\alpha} \rangle = G$ and $(s^{-1})^{\alpha} = (s^{\alpha})^{-1}$ for all $s \in S$), the **Cayley graph** $\mathrm{Cay}(G, S)$ is the graph $(G, G \times S, \iota, -1)$ where $(g, s)^{\iota} = g$ and $(g, s)^{-1} = (g s^{\alpha}, s^{-1})$.

Theorem 5.6 (Malnic, Nedela, Skoviera [13]). *Any Cayley graph is a regular 1-cover of a monopole and vice versa.*

Proof. Consider the Cayley graph $\mathrm{Cay}(G, S)$ of the group G with respect to the generating multiset $S = S^{-1}$. Then the map induced by the canonical projection $G \times S \mapsto S$ extends to a regular 1-covering of the monopole $(\{G\}, S, \bar{\iota}, \overline{-1})$ with $\bar{\iota} : S \to \{G\}$ the constant map and $\overline{-1} : S \to S$ the inversion map.

To prove the converse, recall that by the above any regular 1-covering $\pi : \widehat{\Gamma} \to \Gamma$ gives rise to some Cayley voltage space $(\Delta(\pi), \Delta(\pi), \xi)$. Suppose we have a 1-covering of some monopole $\Gamma = (v, D, \iota, -1)$. Then the graph $\widehat{\Gamma} = (\widehat{V}, \widehat{D}, \widehat{\iota}, \widehat{-1})$ associated to the voltage space $(\Delta(\pi), \Delta(\pi), \xi)$ consists of

$$\begin{aligned} \widehat{V} &= \{v\} \times \Delta(\pi) = \Delta(\pi), \\ \widehat{D} &= D \times \Delta(\pi), \\ \widehat{\iota} : \widehat{D} \to \widehat{V} &: (d, f) \mapsto (d^{\iota}, f) = (v, f) = f, \\ \widehat{-1} : \widehat{D} \to \widehat{D} &: (d, f) \mapsto (d^{-1}, f^{(d^{\xi})}) = (d^{-1}, f d^{\xi}), \end{aligned}$$

which is the Cayley graph of $\Delta(\pi)$ with respect to the generating multiset D. $\square$

Corollary 5.7. *Let $\Gamma = (V, D, \iota, -1)$ be a connected graph, suppose the group W acts regularly on Γ and let $S \subseteq W$ be a symmetric, generating multiset. If for a fixed vertex $\mathbf{x}$ of Γ the set of darts with initial vertex $\mathbf{x}$ equals $\{d_s \in D \mid s \in S\}$ with $(d_s^{-1})^\iota = s.\mathbf{x}$ and $d_s = d_t$ if and only if $s = t$, then the map $\phi_{\mathbf{x}} : W \times S \to D : (w, s) \mapsto w.d_s$ induces an equivariant graph isomorphism from the Cayley graph* $\mathrm{Cay}(W, S)$ *of W with respect to S to the graph Γ.*

Proof. Let $(\mathbf{x}, \{d_s \in D \mid s \in S\}, \iota', -1')$ be a monopole where $\iota' : \{d_s \in D \mid s \in S\} \to \{\mathbf{x}\}$ is the constant map and $-1' : \{d_s \in D \mid s \in S\} \to \{d_s \in D \mid s \in S\} : d_s \mapsto d_{s^{-1}}$. The map $\pi : (V, D, \iota, -1) \to (\mathbf{x}, \{d_s \in D \mid s \in S\}, \iota', -1')$ with $(w.d_s)^\pi = d_s$ is a regular 1-covering. Indeed, it is a morphism as $d_s^{-1} = s.d_{s^{-1}}$ and $w.d_s^{-1} = ws.d_{s^{-1}}$ whence $(-1)\pi = \pi(-1')$ (the condition $\iota\pi = \pi\iota'$ is satisfied trivially), it is a 1-covering since the $w.d_s$, $s \in S$, are the darts with initial vertex $w.\mathbf{x}$, and it is regular as the group W is a lift of the identity acting regularly on Γ. In particular, by Proposition 2.4, we have $W \cong \Delta(\pi)$. Hence, by the theorem, $(V, D, \iota, -1)$ can be considered as the Cayley graph of $\Delta(\pi) \cong W$ with respect to S. The equivariance follows by the action $W \to \mathrm{Aut}(\mathrm{Cay}(W, S)) : g \mapsto \{(w, s) \mapsto (gw, s)\}$ on the Cayley graph and the action $W \to \mathrm{Aut}(\Gamma) : g \mapsto \{w.d_s \mapsto gw.d_s\}$ on Γ. $\square$

Remark 5.8. *Consider the action of W on its Cayley graph with respect to S, and suppose the elements of S act as reflections. Due to Lemma 4.4, the set $R := \{wsw^{-1} : s \in S, w \in W\}$ consists of reflections.*

Corollary 5.9 (of Theorem 5.6)**.** *Any Cayley graph $(G, G \times S, \iota, -1)$ of a group G generated by a multiset S consisting of involutions is a regular* 1-*cover of the monopole $(\{G\}, S, \iota', -1')$ (with ι' the constant map and $-1'$ the identity map) admitting only semi-edges. Conversely, if $\pi : \Gamma \to (v, D, \iota, -1)$ is a regular* 1-*covering onto a monopole admitting only semi-edges, then Γ is the Cayley graph of a group $G \cong \Delta(\pi)$ generated by involutions.* $\square$

Corollary 5.10 (of Lemma 5.3 and Theorem 5.6)**.** *The transition graph Γ of some strongly connected symmetric group automaton (S, X, τ) is a Cayley graph if and only if (S, X, τ) is homogeneous and for each $x \in X$ the map τ_x can be extended to an automorphism of Γ.* $\square$

6. Chambers

For the next definition recall that $R = \{wsw^{-1} : s \in S, w \in W\}$.

Definition 6.1. *Let Γ be a connected graph and let W act freely on Γ. Moreover, assume W is generated by a subset S of elements that act as reflections on Γ. Since the action of W is free the elements of S are involutions, so S is symmetric, i.e., $S^{-1} = S$. Define the equivalence relation $\sim := \bigcap_{s \in R} \sim_s$ on X. The equivalence classes of this relation are called* **chambers**. *Define* *the* **chamber graph** Γ_C *on the chambers of* Γ *with darts* $d_{C,C'}$ *with initial vertex* C *and terminal vertex* C' *for distinct chambers* C, C' *if there exists a dart* d *of* Γ *with* $d^\iota = \mathbf{x} \in C$ *and* $(d^{-1})^\iota = \mathbf{x}' \in C'$. *Moreover, define* $(d_{C,C'})^{-1} = d_{C',C}$. *The group* W *also acts on* Γ_C *and the elements of* S *act as reflections on* Γ_C. *Indeed, for each* $s \in S$, *the graph* Γ_s *(cf. the definition of a reflection at the beginning of Section 4) consists of two connected components, and any chamber is completely contained in one component.*

For $s \in R$, *we shall call* $\mathrm{Norm}_s(D)$ *a* **wall** *of a chamber* C, *if* $\mathrm{Norm}_s(D)$ *contains a dart whose initial vertex is contained in* C. *By* S_C *we denote the set of all* $s \in R$ *for which* $\mathrm{Norm}_s(D)$ *is a wall of the chamber* C. *With* S *also* R *consists of involutions, and so does* $S_C \subseteq \tilde{S}$.

Let $s \in R$. *We say that a path* $d_{C_0,C_1} d_{C_1,C_2} \cdots d_{C_{n-1},C_n}$ **crosses** s, *if the set of indices* $0 \leq j < n$ *with* $C_j \not\sim_s C_{j+1}$ *is not empty. The wall* s *is* **crossed** n **times** *by a path if the set of indices* $0 \leq j < n$ *with* $C_j \not\sim_s C_{j+1}$ *has* n *elements.*

Lemma 6.2. *If the dart* $d_{C,C'}$ *exists, then there is a unique element* $s \in R$ *separating* C *and* C'. *For this element we have* $C^s = C'$.

Proof. Since $d_{C,C'}$ is a dart of the chamber graph, there exists a dart d of Γ with $d^\iota = \mathbf{x} \in C$ and $(d^{-1})^\iota = \mathbf{x}' \in C'$. Since C and C' are distinct, there is an $s \in S$ separating $\mathbf{x}$ and $\mathbf{x}'$. This means $d \in \mathrm{Norm}_s(D)$, so we have $\mathbf{x}^s = \mathbf{x}'$ and hence $C^s = C'$. For any reflection r separating $\mathbf{x}$ and $\mathbf{x}'$ we get $d \in \mathrm{Norm}_r(D)$ and can conclude $s = r$ due to Lemma 4.5, which proves the uniqueness of s. □

Recall that W acts freely on Γ.

Proposition 6.3. *The distance between two chambers* C *and* C' *is equal to the number of elements* $s \in R$ *separating* C *and* C'.

Proof. Let C and C' be chambers and let $d_{C,C_1} d_{C_1,C_2} \cdots d_{C_{n-1},C'}$ be a minimal path from C to C' of length n. Let m be the number of elements $s \in R$ separating C and C'. For every $s \in R$ separating C and C' there is an index $0 \leq j < n$ such that s separates C_j and C_{j+1} and s is uniquely determined by this property due to Lemma 6.2. This entails $m \leq n$.

Now we assume $m < n$. So there is an $s \in R$ separating C and C' such that there are two different indices j and j' with $C_j \not\sim_s C_{j+1}$ and $C_{j'} \not\sim_s C_{j'+1}$. Without loss of generality, we can assume $j < j'$ and $C_{j+1} \sim_s C_{j'}$. But then the path $d_{C,C_1} \cdots d_{C_{j-1},C_j = C^s_{j+1}} \cdots d_{C^s_{j'} = C_{j'+1},C_{j'+2}} \cdots d_{C_{n-1},C'}$ has the length $n-2$, which is a contradiction to the fact that the path is minimal. □

Corollary 6.4. *Minimal galleries cross every* $s \in R$ *at most once.* □

Proposition 6.5. *The group* W *is generated by* S_C *and acts transitively on the set of chambers and* R *is the set of all reflections in* W.

Proof. By Proposition 4.8 the subgroup $\tilde{W} := \langle S_C \rangle$ of W generated by the elements of S_C acts transitively on the set of chambers. Since R consists of reflections and generates W, it suffices to show that every reflection lies in $\tilde{W}$. Let r be a reflection and let $d \in \mathrm{Norm}_r(D)$. Then $d^\iota = \mathbf{x}_1$ lies in a chamber $\tilde{C}$. There is a $w \in \tilde{W}$, such that $C^w = \tilde{C}$. Since $\mathrm{Norm}_r(D)$ is a wall of $\tilde{C}$, the set $\mathrm{Norm}_r(D)^{w^{-1}} = \mathrm{Norm}_{wrw^{-1}}(D)$ (cf. Lemma 4.4) is a wall of C. We get $wrw^{-1} \in S$. This means $r \in \tilde{W}$ and $r \in R$. Since R consists of reflections, it is precisely the set of all reflections. Since it generates W and its elements lie in $\tilde{W}$, the set S_C generates W. □

Proposition 6.6. *The group W acts regularly on the set of chambers.*

Proof. Let C be a chamber. Then d_{C,C^s} is a dart of the chamber graph for every $s \in S_C$. If we have $C^{\tilde{s}} = C^s$ for $s \in S_C$ and $\tilde{s} \in R$ then we get $\tilde{s} = s$ by Lemma 6.2. So by Lemma 4.7 the stabilizer W_C is trivial. Since W acts transitively on the set of chambers, this implies regularity. □

Corollary 6.7. *The chamber graph Γ_C is isomorphic to the Cayley graph $\mathrm{Cay}(W, S_C)$. Moreover, this isomorphism is W-equivariant.*

Proof. This follows by the proposition and Corollary 5.7. □

Corollary 6.8. *There exists a 1-1 correspondence between words on S and paths in Γ_C with initial vertex C. Two words on S represent the same element of W if and only if the terminal vertices of the corresponding paths based at C coincide.* □

The following result states for a very special case that S_C is a minimal generating set for W. We will need this result for the proof of Theorem 7.4.

Lemma 6.9. *Let C be a chamber and let $s_1 \neq s_2 \in S_C$. Suppose that W is generated by $S = \{s_1, s_2\}$. Then $S = S_C$.*

Proof. According to [2] Ch. IV § 1, the pair (W, S) is a Coxeter system, as S consists of two involutions. Let Γ_1 be the Cayley graph of W with respect to S, and let Γ_2 be the Cayley graph of W with respect to S_C. Both graphs have the same set of vertices and every dart in Γ_1 is a dart in Γ_2. By Corollary 6.7, the chamber graph can be identified with Γ_2. It is easily verified by studying the Cayley graph of dihedral groups (i.e. groups generated by two involutions) that, if we add a non-empty set of edges to the graph G_1, then the resulting graph is not G-invariant or the group elements $\{w^{-1}sw : w \in W,\ s \in S\}$ do not all act as reflections on it. We conclude that both graphs Γ_1 and Γ_2 have the same set of darts. Since the number of darts emanating from 1 in the Cayley graph is just the number of generators, we obtain $|S| = |S_C|$, and thus $S_C = S$. □

The following result shows an interesting way of characterizing the finiteness of the group W.

Definition 6.10. *Let C be a chamber. Set $-C := \bigcap_{s \in R} X_-^s(C)$, so $-C$ is the (possibly empty) set of all vertices, which are separated by every $s \in R$ from the vertices in C.*

Theorem 6.11. *Let C be a chamber. The group W is finite if and only if the set $-C$ is non-empty; in this case $-C$ is a chamber. If W is finite then the following hold:*

(i) *The map $C \mapsto -C$ is an equivariant involution of the chamber graph.*
(ii) $S_C = S_{-C}$.
(iii) $-C = \bigcap_{s \in S_C} X^s_-(C)$.
(iv) *The chamber $-C$ is the unique chamber with maximal distance from the chamber C.*

Proof. Assume that $-C$ is non-empty and let $\mathbf{x}, \mathbf{y} \in -C$. The vertices $\mathbf{x}$, $\mathbf{y}$ cannot be separated by any $s \in R$, as $\mathbf{x}$, $\mathbf{y}$ both are separated by all $s \in R$ from any $\mathbf{z} \in C$. Therefore $-C$ is a chamber. All elements of R separate C and $-C$. The chamber graph is connected, because the chambers partition the vertex set of Γ and Γ is connected, so by Proposition 6.3 the set R is finite. As a consequence, the set of all chambers is finite. Since W acts regularly on this set, it is a finite group.

Now let W be finite. So the set of chambers is finite. Let $\tilde{C}$ be a chamber with maximal distance n from C. Suppose there is an $s \in R$ that does not separate $\tilde{C}$ and C. Then the chamber $\tilde{C}^s$ has distance $m \leq n$ from C. Let $d_{C,C_1} \cdots d_{C_{m-1},C_m=\tilde{C}^s}$ be a minimal path from C to $\tilde{C}^s$. There is an index $0 \leq j < m$ such that $C_j^s = C_{j+1}$. The path $d_{C,C_1} \cdots d_{C_j=C^s_{j+1},C^s_{j+2}} \cdots d_{C^s_{m-1},C^s_m=\tilde{C}}$ has the length $m-1 < n$, which is a contradiction. So we have $\tilde{C} = -C$. We have proved the first assertion and part (iv).

To prove (i) let m be the distance between C and $-C$. Since W acts transitively on the chamber graph, for any chamber $\tilde{C}$ the distance of any chamber $\tilde{C}$ and $-\tilde{C}$ is equal to m. So the given map is an involution. Now let $w \in W$ and C a chamber. Then the distance between C^w and $(-C)^w$ equals the distance m between C and $-C$, which entails $-(C^w) = (-C)^w$. So the map is equivariant. If two chambers C and $\tilde{C}$ are adjacent, then the distance between $-C$ and $\tilde{C}$ is $m-1$. Thus $-C$ and $-\tilde{C}$ are separated by one and only one element of R. So they are adjacent.

For part (ii) let C be a chamber. Then $\mathrm{Norm}_s(D)$ is a wall of C if and only if C and C^s are adjacent. This is equivalent to $-C$ and $-C^s$ adjacent.

Finally, we prove (iii). Suppose C and C' are separated by every element of S_C. Now let $\tilde{s} \in R$ and $s \in S_C$. Suppose $\tilde{s}$ separates C and C'. Since S_C generates W, it suffices to show that $s\tilde{s}s^{-1}$ separates C and C'. Certainly $s\tilde{s}s^{-1}$ separates C^s and C'^s. If $s\tilde{s}s^{-1} = s$, then $\tilde{s} = s$ we are done. Otherwise, recall that s is the only element of R which separates C and C^s. The same is true for C' and C'^s. So neither C and C^s nor C' and C'^s are separated by $\tilde{s}$. This means $\tilde{s}$ separates C^s and C'^s and, thus, $s\tilde{s}s^{-1}$ separates C and C'. $\square$

Corollary 6.12 (of Corollary 6.8 and Theorem 6.11). *If W is finite, there exists a unique longest element w_0 in W with respect to S and it has length $|R|$.* □

For the case of dihedral groups we will need statement Theorem 6.11 (iii) without requiring W to be finite:

Lemma 6.13. *If $S_C = \{s_1, s_2\}$ then*

$$-C = \bigcap_{s \in S_C} X^s_-(C).$$

Proof. If the set $D := \bigcap_{s \in S_C} X^s_-(C)$ is empty then so is $-C$ by definition. So we suppose that D is not empty and deduce that W is finite. Then we are done by Theorem 6.11 (iii). So let $\tilde{C}$ be a chamber in D with minimal distance m to C. Recall the statement of Proposition 6.3. The distance between C^{s_1} and $\tilde{C}$ is $m-1$. So there is a path of length $m-1$ starting with the dart $d_{\tilde{C},C'}$ and ending in C^{s_1}. The chamber C' must lie outside of D due to the minimality of the distance between C and $\tilde{C}$. The path does not cross s_1, so the path $d_{\tilde{C},C'}$ must cross s_2. Thus $\mathrm{Norm}_{s_2}(D)$ is a wall of $\tilde{C}$. By swapping the roles of s_1 and s_2 we see that E^{s_1} is also a wall of $\tilde{C}$. Due to transitive action of W on chambers, every chamber has two walls. We obtain $D = \tilde{C}$. Now we can proceed as in the proof of Theorem 6.11 (iii). □

7. Coxeter groups

Definition 7.1. *Let W be a group generated by a subset S of proper involutions. For s and $t \in S$ let $m(s,t) \in \mathbb{N} \cup \infty$ be the order of st in W. Set $I := \{(s,t) \in S \times S \mid m(s,t) < \infty\}$. The pair (W,S) is called a* **Coxeter system**, *if W has a presentation of the form $W = \langle S \mid (st)^{m(s,t)} = 1 \text{ for all } (s,t) \in I \rangle$. Then the group W is called a* **Coxeter group**.

In Section 8 we want to relate Coxeter groups to the Cayley graphs and covers of monopoles from Section 5. Therefore we have to allow for repetition of generators and trivial generators in Coxeter groups. More precisely, if W is a Coxeter group with respect to a generating system $S \subset W$ of involutions and $\alpha : D \to S \cup \{1_W\}$ is a surjection, then we call W a **Coxeter group with respect to the generating multiset** *D. In view of the map -1 in a graph we set $d^{-1} := d$ for each $d \in D$. The relation $\equiv$ from the proof of Lemma 5.3 shows that one can pass between the two notions without problem.*

Remark 7.2. *The existence of a presentation of the form in the above definition is equivalent to either of the following conditions:*

(i) *The system (W,S) satisfies the following universal property: Let G be a group and let $f : S \to G$ be a map with $(f(s)f(t))^{m(s,t)} = 1$ for all $(s,t) \in I$. Then there is a unique homomorphism $\tilde{f} : W \to G$ extending f.*

(ii) *Let $\overline{W}$ be a group and let $f : \overline{W} \to W$ be a homomorphism. Assume that there is a map $h : S \to \overline{W}$ with $f\big(h(s)\big) = s$ and $\big(h(s)h(t)\big)^{m(s,t)} = 1$ for $(s,t) \in I$ and such that the elements $h(s)$ with $s \in S$ generate $\overline{W}$. Then f is injective.*

Definition 7.3. *We say that the pair (W,S) satisfies the* **exchange condition** *if the following holds. Let $w \in W$ and $s \in S$, such that $\ell(sw) \leq \ell(w)$. Then for every reduced decomposition $(s_1, s_2, \dots, s_n)$ of w, there is an integer $1 \leq j \leq n$, such that $ss_1s_2 \cdots s_{j-1} = s_1s_2 \cdots s_j$.*

Theorem 7.4. *For any chamber C the pair (W, S_C) is a Coxeter system.*

Proof. Let $\overline{W}$ be a group and let $f : \overline{W} \to W$ be a homomorphism. Assume there exists a map $h : S_C \to \overline{W}$ with $f(h(s)) = s$ and $(h(s)h(s'))^{m(s,s')} = 1$ for $(s,s') \in I$ and such that the elements $h(s)$ with $s \in S_C$ generate $\overline{W}$. If we show that f is injective, we are done by Remark 7.2.

Since each $s \in S_C$ is an involution, so is each $h(s) \in \overline{W}$, $s \in S_C$, whence every element $g \in \overline{W}$ can be represented as $g = h(s_n)h(s_{n-1}) \cdots h(s_1)$ with $n \in \mathbb{N}$ and $s_1, s_2, \dots, s_n \in S_C$. Suppose $g \in \ker f$ and let C be a chamber. This means $d_{C,C^{s_1}} d_{C^{s_1},C^{s_2s_1}} \cdots d_{C^{s_{n-1}\cdots s_1},C^{s_n\cdots s_1}}$ is a cycle.

Let k be the maximal value occuring as a distance between C and $C^{s_is_{i-1}\cdots s_1}$ for some $0 < i < n$, and let i be the maximal value for which $C^{s_is_{i-1}\cdots s_1}$ has distance k from C. Set $w := s_is_{i-1} \cdots s_1$. The chamber C^{s_iw} is closer to C than C^w. (See Figure 1.) Since $w^{-1}s_iw$ separates C^{s_iw} and C^w, it also separates C and C^w by Corollary 6.4. So s_i separates $C^{w^{-1}}$ and C. In the same way we show that s_{i+1} separates $C^{w^{-1}}$ and C.

We now consider the subgroup W' of W generated by the subset $S' = \{s_i, s_{i+1}\}$. Set

$$R' := \{w^{-1}sw : w \in W',\ s \in S'\}.$$

Now W' is generated by the W'-invariant set of reflections R'. We shall call the chambers with respect to this set of reflections *large* chambers in order to distinguish them from the *small* chambers which come from the set of reflections R. Let C' be the large chamber that contains C. From Lemma 6.9 we know $S' = S_{C'}$. The chamber graph with respect to R' can be identified with the Cayley graph of the dihedral group W' with respect to S'. Since

$$C^{w^{-1}} \subseteq \bigcap_{s \in S_{C'}} X^s_-(C'),$$

the set $-C'$ is a large chamber due to Lemma 6.13. So the group W' is finite due to Theorem 6.11. Thus the Cayley graph of W' with respect to S' is a cycle.

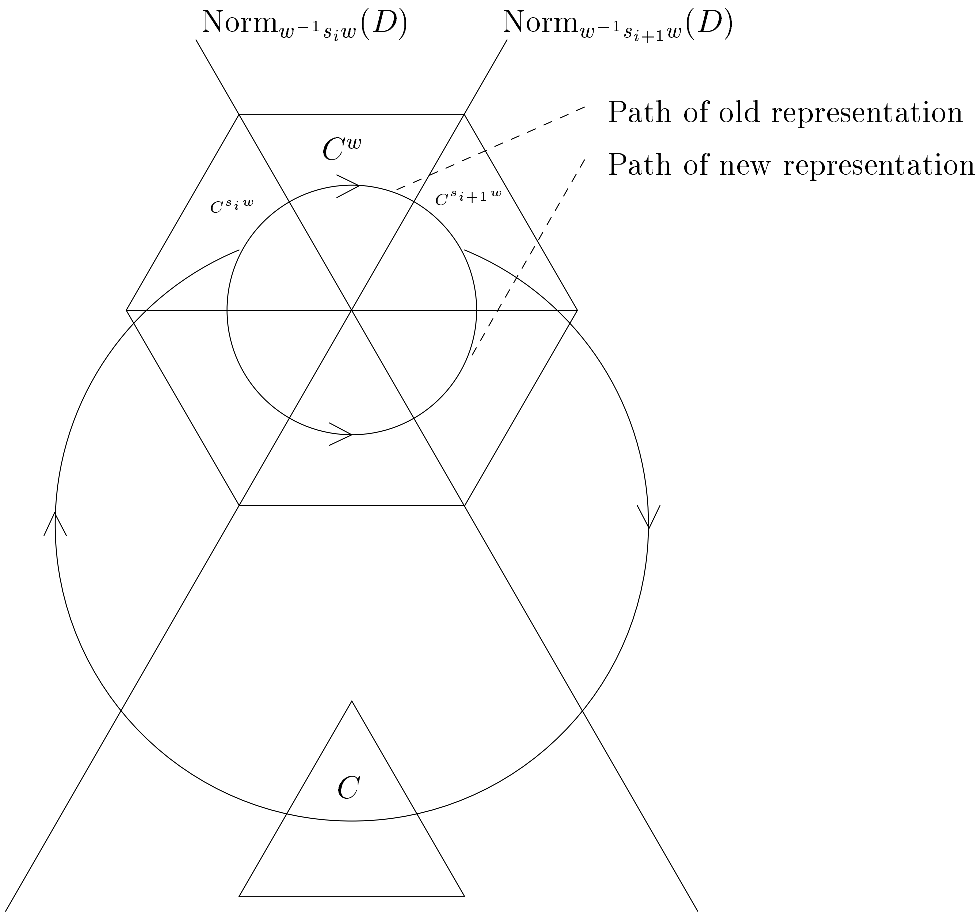

FIGURE 1. Paths corresponding to the old and the new representations of g

The large chambers C' and $-C'$ have distance $|R'| = m(s_i, s_{i+1})$ by the definition of $-C'$. Set

$$\widehat{w} \;:=\; \begin{cases} (s_i s_{i+1})^{\frac{m(s_i,s_{i+1})}{2}} & \text{if } m(s_i,s_{i+1}) \text{ is even.} \\ (s_i s_{i+1})^{\frac{m(s_i,s_{i+1})-1}{2}} s_i & \text{if } m(s_i,s_{i+1}) \text{ is odd.} \end{cases}$$

Since $\widehat{w}.C' = -C'$, the large chamber $-C'$ contains the small chamber $\widehat{C} := \widehat{w}.C$. So the small chambers C and $\widehat{C}$ have distance at least $|R'|$.

This implies that the paths

$$d_{C,C^{s_i}}\ d_{C^{s_i},C^{s_{i+1}s_i}}\ d_{C^{s_{i+1}s_i},C^{s_is_{i+1}s_i}} \quad \ldots,\ \ d_{\widehat{C}^{s_i},\widehat{C}} \text{ and}$$
$$d_{C,C^{s_{i+1}}}\ d_{C^{s_{i+1}},C^{s_is_{i+1}}}\ d_{C^{s_is_{i+1}},C^{s_{i+1}s_is_{i+1}}} \quad \ldots,\ \ d_{\widehat{C}^{s_{i+1}},\widehat{C}}$$

are minimal (with respect to R) and have length $|R'|$.

Due to Theorem 6.11 (iii), the small chambers C and $\widehat{C}$ are separated by each of the elements of R', and the small chambers $C^{w^{-1}}$ and $\widehat{C}$ are separated by none of the elements of R. Since the distance between C and $C^{w^{-1}}$ is k, the distance $w^{-1}.C$ and $\widehat{C}$ is $k - |R'|$. So each of the small chambers

$$C^{s_i},\ C^{s_{i+1}s_i},\ C^{s_is_{i+1}s_i},\ C^{s_{i+1}s_is_{i+1}s_i},\ \ldots,\ C^{s_{i+1}}$$

has distance less than k from $C^{w^{-1}}$. So the w-translates of these elements have distance less than k from C.

We obtain the new representation

$$g = h(s_n)h(s_{n-1})\cdots h(s_{i+2})\underbrace{\big(h(s_i)h(s_{i+1})\big)^{m(s_i,s_{i+1})-1}}_{=h(s_{i+1})h(s_i)}h(s_{i-1}),\ldots,h(s_1).$$

The corresponding cyclic path has a smaller k or a smaller index i for which the maximal distance k occurs. (See Figure 1.) By induction we obtain $g = 1$.

□

If we specialize our action of W to the action on the Cayley graph, then chambers consist of only one vertex, since the action on the points and on the chambers is simply transitive. In this case we have the following corollary of Theorem 6.11.

Theorem 7.5. *Let (W, S) be a Coxeter system and let $R = \{wsw^{-1} : s \in S, w \in W\}$. Then the following conditions are equivalent:*

(i) *W is finite.*
(ii) *R is finite.*
(iii) *There is a $w \in W$ such that 1 is separated from w by every $s \in R$.* □

If condition (iii) is satisfied, then w is the unique group element with maximal length and it has length $|R|$, cf. Corollary 6.12.

Theorem 7.6. *The following statements are equivalent:*

(i) *(W, S) is a Coxeter system.*
(ii) *(W, S) satisfies the exchange condition.*
(iii) *The elements of S act as reflections on* $\mathrm{Cay}(W, S)$.

Proof. The implication (i) $\Longrightarrow$ (ii) is well-known, see e.g. [4]. The implication (iii) $\Longrightarrow$ (i) is a consequence of Theorem 7.4 and Corollary 6.7.

So now we assume (ii). Our proof is inspired by the ideas in [2] Ch. IV, § 1.7. Let $s \in S$ and set

$$P_s := \{w \in W : \ell(w) < \ell(sw)\}.$$

Due to [2] Ch. IV, § 1.5 Proposition 4 the group W is the disjoint union of P_s and $s.P_s = \{w \in W : \ell(w) > \ell(sw)\}$. Let (w, s') be a dart with initial vertex $w \in W$ and terminal vertex ws' for $s' \in S$. Assume $w \in P_s$ and $ws' \in P_s^s$. By showing $(w, s') \in \mathrm{Norm}_s(D)$ in the Cayley graph $\mathrm{Cay}(W, S)$, we prove that s is a reflection. So we are done if we show $sw = ws'$.

Set $n := \ell(w)$. Due to $w \in P_s$, we have $\ell(sw) = n+1$. Using this and $ws' \in P_s^s$ we get $n \leq \ell(sws') < \ell(ws') \leq n+1$. We conclude $\ell(sws') = n$ and $\ell(ws') = n+1$. Now let $(s_1, s_2, \dots, s_n)$ be a reduced decomposition of w. By setting $s_{n+1} := s'$, we obtain the reduced decomposition $(s_1, s_2, \dots, s_{n+1})$ of ws'. The exchange condition tells us that there is an index $1 \leq j \leq n+1$, such that we have $ss_1s_2\cdots s_{j-1} = s_1s_2\cdots s_j$. Suppose $j < n+1$ for a moment. Then we get $sw = ss_1s_2\cdots s_n = s_1s_2\cdots s_js_j\cdots s_n = s_1s_2\cdots s_{j-1}s_{j+1}\cdots s_n$, which is a contradiction to $\ell(sw) = n+1$. So we have $j = n+1$ and the above equation yields $sw = ws'$. □

Corollary 7.7. *If (W,S) is a Coxeter system, then the set $R = \{wsw^{-1} : s \in S, w \in W\}$ is exactly the subset of W consisting of those elements which operate as reflections on the Cayley graph of W with respect to S. Any subgroup generated by a subset of R is a Coxeter group.*

Proof. Let (W,S) be a Coxeter system and consider the action of W on its Cayley graph with respect to S. This action is free. By Proposition 6.5, the set R is the set of reflections in W. Any subgroup of W operates freely on the Cayley graph of W with respect to S. If this subgroup is generated by reflections, then it is a Coxeter group by Theorem 7.4. □

8. Covers of monopoles, revisited

Theorem 8.1. *Let $\pi : \Gamma \to (v, D, \iota, -1)$ be a 1-covering of a monopole admitting semi-edges only. The graph Γ is the Cayley graph of a Coxeter group if and only if π is regular and any deck transformation in $\Delta(\pi)$ that interchanges two neighboring vertices of Γ acts as a reflection on Γ.*

Proof. Theorem 5.6 implies that Γ is the Cayley graph of $\Delta(\pi)$ with respect to the generating multiset D (with embedding map $\xi : D \to \Delta(\pi)$ as defined before Theorem 5.6) if and only if π is regular. The set of vertices of Γ equals $\Delta(\pi)$. By regularity of π and by Lemma 4.4 it suffices to study deck transformations that interchange the vertex 1 of Γ and some neighboring vertex d^ξ of Γ. But any such deck transformation is given by the maps $\Delta(\pi) \to \Delta(\pi) : w \mapsto d^\xi w$ on the vertices of Γ and $D \times \Delta(\pi) \to D \times \Delta(\pi) : (d, w) \mapsto (d, d^\xi w)$ on the darts. (For, the given map is a deck transformation interchanging 1 and d^ξ, so it has to be the one we were studying as $\Delta(\pi)$ acts regularly on Γ.) By Theorem 7.6 the group $\Delta(\pi)$ is a Coxeter group with respect to the generating multiset D if and only if the elements of D that do not act as the identity on Γ act as reflections on Γ. The theorem is proved. □

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