Bounded Imaginary Powers and H_∞-Calculus of the Stokes Operator in Two-Dimensional Exterior Domains

Helmut Abels*

September 1, 2004

Abstract

The present contribution deals with the Stokes operator A_q on $L^q_\sigma(\Omega)$, $1 < q < \infty$, where Ω is an exterior domain in $\mathbb{R}^2$ of class C^2. It is proved that A_q admits a bounded H_∞-calculus.This implies the existence of bounded imaginary powers of A_q, which has several important applications. – So far this property was only known for exterior domains in $\mathbb{R}^n$, $n \geq 3$. – In particular, this shows that A_q has maximal regularity on $L^q_\sigma(\Omega)$. For the proof the resolvent $(\lambda + A_q)^{-1}$ has to be analyzed for $|\lambda| \to \infty$ and $\lambda \to 0$. For large λ this is done using an approximate resolvent based on the results of [3], which were obtained by applying the calculus of pseudodifferential boundary value problems. For small λ we analyze the representation of the resolvent developed in [11] by a potential theoretical method.

Key words: Stokes operator, Stokes equations, exterior domains, bounded imaginary powers, H_∞-calculus
AMS-Classification: 35 Q 30, 76 D 07, 47 A 60, 47 F 05

1 Introduction and Main Result

Let $\Omega \subset \mathbb{R}^2$ be an exterior domain with C^2-boundary, i.e., $\mathbb{R}^2 \setminus \Omega$ is compact and $\partial\Omega$ is a C^2-manifold. Moreover, let $L^q_\sigma(\Omega) := \overline{\{f \in C_0^\infty(\Omega)^n : \operatorname{div} f = 0\}}^{L^q(\Omega)}$, $1 < q < \infty$, denote the space of solenoidal vector fields in $L_q(\Omega)^n$ with vanishing normal component on $\partial\Omega$.

In this article we consider the Stokes operator $A_q = -P_q\Delta$ on $L^q_\sigma(\Omega)$ with domain

$$\mathcal{D}(A_q) = \{f \in W^2_q(\Omega)^n : f|_{\partial\Omega} = 0\} \cap L^q_\sigma(\Omega)$$

*Department of Mathematics, Darmstadt University of Technology, Schloßgartenstr. 7, 64289 Darmstadt, Germany, e-mail: abels@mathematik.tu-darmstadt.de

where $P_q\colon L^q(\Omega)^n \to L^q_\sigma(\Omega)$ denotes the well-known Helmholtz projection, cf. Simader and Sohr [23]. Borchers and Varnhorn [11] proved that $-A_q$ generates a bounded and analytic semi-group. More precisely, they have shown that

$$\|(\lambda + A_q)^{-1}\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le \frac{C_{q,\delta}}{|\lambda|}, \qquad \lambda \in \Sigma_\delta, \tag{1.1}$$

where $\Sigma_\delta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \delta\}$ and $\delta \in (0, \pi)$. Earlier the same property was shown by Borchers and Sohr [10] for the case of exterior domains $\Omega \subset \mathbb{R}^n$ with $n \ge 3$. But the latter contribution could not settle the two-dimensional case since there are some additional difficulties in comparison to the case $n \ge 3$, cf. Remark 1.3 below and [11, Introduction] for further explanations.

Besides the fact that $-A_q$ generates a bounded analytic semi-group, an important property of the Stokes operator is that it possesses *bounded imaginary powers*, i.e.,

$$A_q^{iy} := \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{iy}(\lambda + A_q)^{-1}\, d\lambda$$

is a bounded operator satisfying

$$\|A_q^{iy}\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le C_{q,\delta} e^{(\pi-\delta)|y|}, \qquad y \in \mathbb{R}, \tag{1.2}$$

where $\delta \in (0, \pi)$, $1 < q < \infty$, and Γ is the negatively orientated boundary of Σ_δ. The latter property is not difficult to proved in the case $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, $n \ge 2$, cf. [18]. Besides these cases, the proof (1.2) is involved and most proofs use pseudodifferential operator techniques. This was done by Giga [16] for bounded domains in $\mathbb{R}^n$, $n \ge 2$, and by Giga and Sohr [17] for exterior domains in $\mathbb{R}^n$, $n \ge 3$, with smooth boundary. An alternative proof, which uses a perturbation theorem for the H_∞-calculus instead of pseudodifferential operators, was given by Noll and Saal [22] for bounded and exterior domains in $\mathbb{R}^n$, $n \ge 3$, with C^3-boundary. Moreover, (1.2) was proved for an infinite layer $\Omega = \mathbb{R}^{n-1} \times (-1, 1)$ in [4] and more generally for so-called asymptotically flat layers with $C^{1,1}$-boundary in [3].

The purpose of the present contribution is to prove:

Theorem 1.1 *Let $1 < q < \infty$ and let $\delta \in (0, \pi)$. Then A_q admits a* bounded H_∞-calculus *with respect to δ, i.e.,*

$$h(A_q) := \frac{1}{2\pi i}\int_\Gamma h(-\lambda)(\lambda + A_q)^{-1}\, d\lambda \tag{1.3}$$

is a bounded operator satisfying

$$\|h(A_q)\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le C_{q,\delta}\|h\|_\infty \tag{1.4}$$

for all $h \in H_\infty(\delta)$, where $H_\infty(\delta)$ denotes the Banach algebra of all bounded holomorphic functions $h\colon \Sigma_{\pi-\delta} \to \mathbb{C}$.

We note that in order to prove (1.4) for all $h \in H_\infty(\delta)$ it is sufficient to show the estimate for $h \in H(\delta)$, which consists of all $h \in H_\infty(\delta)$ such that

$$|h(z)| \le C \frac{|z|^s}{1+|z|^{2s}} \qquad \text{for all } z \in \Sigma_{\pi-\delta}$$

for some $s > 0$, cf. [7, Lemma 2.1]. For $h \in H(\delta)$ the integral (1.3) is well-defined as a Bochner integral on $\mathcal{L}(L^q_\sigma(\Omega))$ and for arbitrary $h \in H_\infty(\delta)$ the operator in (1.3) can be defined on a suitable dense subspace, cf. [7] for details.

The bounded H_∞-calculus was introduced by McIntosh [21] and generalizes the property of having bounded imaginary powers since choosing $h_y(z) := z^{iy}$, $y \in \mathbb{R}$, in (1.3)-(1.4) implies (1.2). Although in [4, 16, 17, 18] only (1.2) is proved, the proofs are easily modified to show (1.4).

The well-known result due to Dore and Venni [14, Theorem 3.2] and its extension by Giga and Sohr [18, Theorem 2.1] gives an important application of this abstract property:

Theorem 1.2 *Let $1 < p, q < \infty$ and let $0 < T \le \infty$. Then for every $f \in L^p(0,T; L^q_\sigma(\Omega))$ there is a unique solution $u \in W^1_p(0,T; L^q_\sigma(\Omega)) \cap L^p(0,T; \mathcal{D}(A_q))$ of*

$$\begin{aligned} u'(t) + A_q u(t) &= f(t), \qquad 0 < t < T, \\ u(0) &= 0 \end{aligned}$$

Moreover,

$$\|u'\|_{L^p(0,T;L^q_\sigma)} + \|A_q u\|_{L^p(0,T;L^q_\sigma)} \le C \|f\|_{L^p(0,T;L^q_\sigma)},$$

where C does not depend on T.

Therefore the Stokes operator A_q has *maximal regularity* on $L^q_\sigma(\Omega)$, $1 < q < \infty$.

Finally, we mention that the boundedness of A^{iy} and (1.2) can be used to characterize the domain of the fractional powers A^α_q, $0 < \alpha < 1$, as

$$\mathcal{D}(A^\alpha_q) = (L^q_\sigma(\Omega), \mathcal{D}(A_q))_{[\alpha]},$$

where $(.,.)_{[\alpha]}$ denotes the complex interpolation functor, cf. [17, Proposition 6.1].

The outline of the proof of Theorem 1.1 is as follows: Roughly speaking, one has to deal with two singularities of the Cauchy-integral (1.4). The first occurs since Γ is unbounded and $(\lambda + A_q)^{-1}$ is only of order $O(|\lambda|^{-1})$ as $|\lambda| \to \infty$. The second is due to the singularity of $(\lambda + A_q)^{-1}$ as $\lambda \to 0$. Hence we split $\Gamma = \Gamma_R \dot\cup \Gamma'_R$, where $\Gamma'_R := \Gamma \cap B_R(0)$ with suitable $R > 0$ and analyze each part separately. In order to analyze (1.3) with Γ replaced by Γ_R, an approximate resolvent R_λ is constructed that coincides with $(\lambda + A_q)^{-1}$ modulo terms of order $O((1 + |\lambda|)^{-1-\varepsilon})$ for some $\varepsilon > 0$, as $|\lambda| \to \infty$. The construction is based on results of [3], which were obtained by pseudodifferential operator techniques and were used in order to prove (1.4) for the Stokes operator on asymptotically flat layers. The latter analysis is done in Section 4. Finally, it remains to estimate (1.3) with Γ replaced by Γ'_R. For this we use the representation of $(\lambda + A_q)^{-1}$ developed in [10] in terms of the resolvent of the Stokes operator on $\mathbb{R}^2$ and some single and double layer potentials, cf. Section 5. But first of all we start with some preliminaries in Section 2 and introduce the so-called *reduced* Stokes operator in Section 3, which is needed in order to apply the results of [3].

Remark 1.3 The analysis of the resolvent as $|\lambda| \to 0$ in the present two-dimensional case is more difficult than in dimension $n \geq 3$ since the estimate

$$\|\nabla^2 u\|_q \leq C_{q,\delta} \|A_q u\|_q, \qquad \lambda \in \Sigma_\delta, u \in \mathcal{D}(A_q), \tag{1.5}$$

holds if and only if $1 < q < \frac{n}{2}$, cf. [11, Introduction] and [9]. Hence the method of [17, Section 4] for the case of an exterior domain in $\mathbb{R}^n$, $n \geq 3$, is not applicable since it is based on (1.5).

Acknowledgements: The author is grateful to Hermann Sohr for pointing out that the present problem was unsolved before and for motivating him to solve it.

2 Preliminaries

First of all, $\mathbb{N}$ will denote the set of natural numbers (without 0) and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a domain. Then $C^\infty(\Omega)$ denotes the set of all smooth $f\colon \Omega \to \mathbb{C}$, $C_0^\infty(\Omega)$ is the set of all $f \in C^\infty(\Omega)$ with compact support, and

$$C^\infty_{(0)}(\overline{\Omega}) := \{u = v|_{\overline{\Omega}} : v \in C_0^\infty(\mathbb{R}^n)\}.$$

The usual Lebesgue-space with respect to the Lebesgue measure on Ω and the $(n-1)$-dimensional surface measure on $\partial\Omega$ will be denoted by $L^q(\Omega)$, $L^q(\partial\Omega)$, resp., $1 \leq q \leq \infty$. Moreover, we use the abbreviations $\|.\|_q \equiv \|.\|_{L^q(\Omega)}$ and $\|.\|_{q,\partial\Omega} \equiv \|.\|_{L^q(\partial\Omega)}$. Furthermore, $L^q_{\mathrm{loc}}(\overline{\Omega})$, $1 \leq q \leq \infty$, is defined as the space of $f\colon \Omega \to \mathbb{C}$ such that $f \in L^q(B \cap \Omega)$ for all balls B with $B \cap \Omega \neq \emptyset$. The usual scalar product on $L^2(M)$ is denoted by $(.,.)_M$ for $M = \Omega, \partial\Omega$.

In the following the usual Sobolev-Slobodeckij spaces based on L^q, $1 < q < \infty$, are denoted by $W^s_q(\Omega)$ and $W^s_q(\partial\Omega)$, $s \geq 0$, with norms $\|.\|_{s,q}$ and $\|.\|_{s,q,\partial\Omega}$, respectively, cf. e.g. [6]. Moreover, $W^m_{q,0}(\Omega)$, $m \in \mathbb{N}$, denotes the closure of $C_0^\infty(\Omega)$ in $W^m_q(\Omega)$ and

$$W^{-m}_q(\Omega) := (W^m_{q',0}(\Omega))', \quad W^{-m}_{q,0}(\Omega) := (W^m_{q'}(\Omega))', \quad W^{-s}_q(\partial\Omega) := (W^s_{q'}(\partial\Omega))'$$

for $m \in \mathbb{N}$ and $s > 0$, where $\frac{1}{q} + \frac{1}{q'} = 1$.

Finally, the homogeneous Sobolev space of order 1 is defined as

$$\dot{W}^1_q(\Omega) := \left\{p \in L^q_{\mathrm{loc}}(\overline{\Omega}) : \nabla p \in L^q(\Omega)\right\}$$

normed by $\|\nabla \cdot\|_q$. If $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an exterior domain with C^1-boundary, for every $p \in \dot{W}^1_q(\Omega)$ there is a $\tilde{p} \in \dot{W}^1_q(\mathbb{R}^n)$ such that $\tilde{p}|_\Omega = p$ and $\|\nabla\tilde{p}\|_q \leq C\|\nabla p\|_q$, cf. e.g. [12, Theorem 1.2]. As a consequence we obtain the following useful lemma.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain with C^1-boundary and let $1 < q < \infty$. Then for every $p \in \dot{W}_q^1(\Omega)$ there is a decomposition $p = p_1 + p_2$ such that $p_1 \in W_q^1(\Omega)$ and $p_2 \in L^q_{\mathrm{loc}}(\overline{\Omega})$ with $\nabla p_2 \in W_q^1(\Omega)$ satisfying $\|(p_1, \nabla p_2)\|_{1,q} \leq C\|\nabla p\|_q$.*

Proof: Define for instance $p_1 \in W_q^1(\Omega)$ as

$$p_1 = \mathcal{F}^{-1}[(1-\varphi)(\xi)\mathcal{F}[\tilde{p}](\xi)]\big|_\Omega ,$$

where $\tilde{p}$ is as above, $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(\xi) = 1$ in a neighborhood of 0, and $\mathcal{F}$ denotes the Fourier transformation. ■

In the following let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain with C^m-boundary, $m \in \mathbb{N}$. Concerning traces recall that, if $m \geq s > j + \frac{1}{q}$, $j \in \mathbb{N}_0$, with $s - \frac{1}{q} \notin \mathbb{N}$, there is a bounded operator

$$\gamma_j \colon W_q^s(\Omega) \to W_q^{s-\frac{1}{q}}(\partial\Omega) \tag{2.1}$$

such that $\gamma_j u = \partial_\nu^j u|_{\partial\Omega}$ for all $f \in C_{(0)}^\infty(\overline{\Omega})$, cf. e.g. [6].

Furthermore, we recall that for $f \in L^q(\Omega)^n$ such that $\operatorname{div} f \in L^q(\Omega)$ it is possible to define a weak trace of the normal component $\gamma_\nu f \in W_q^{-\frac{1}{q}}(\partial\Omega)$ by

$$\langle \gamma_\nu f, v\rangle_{\partial\Omega} := (f, \nabla v)_\Omega + (\operatorname{div} f, v)_\Omega \qquad \text{for all } v \in W_{q'}^1(\Omega). \tag{2.2}$$

Moreover, we note that, if $f = f_0 + \nabla p$, $f_0 \in L^q_\sigma(\Omega)$, $p \in \dot{W}_q^1(\Omega)$, is the Helmholtz decomposition of $f \in L^q(\Omega)^n$, then p is uniquely determined as solution of the weak Neumann problem

$$\Delta p = \operatorname{div} f \qquad \text{in } \Omega \tag{2.3}$$

$$\partial_\nu p|_{\partial\Omega} = \nu \cdot f|_{\partial\Omega} \quad \text{on } \partial\Omega, \tag{2.4}$$

where ν denotes the exterior normal, (2.3) is understood in the sense of distributions, and (2.4) is understood as $\gamma_\nu(f - \nabla p) = 0$, cf. [23]. Because of the definition of γ_ν, the pressure $p \in \dot{W}_q^1(\Omega)$ solves (2.3)-(2.4) if and only if

$$(\nabla p, \nabla v)_\Omega = (f, \nabla v)_\Omega \qquad \text{for all } v \in \dot{W}_{q'}^1(\Omega).$$

Finally, the resolvent of the Laplace operator on $\mathbb{R}^n$, $n \geq 2$, is given by

$$(\lambda - \Delta_{\mathbb{R}^n})^{-1} f = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[f](\xi)}{\lambda + |\xi|^2}\right], \tag{2.5}$$

where $\mathcal{F}$ denotes the Fourier transformation. As a consequence of the Mikhlin multiplier theorem, cf. e.g. [8], one obtains for $u = (\lambda - \Delta_{\mathbb{R}^n})^{-1} f$

$$|\lambda|\|u\|_q + \|\nabla^2 u\|_q \leq C_{q,\delta}\|f\|_q, \qquad \lambda \in \Sigma_\delta, f \in L^q(\mathbb{R}^n), \tag{2.6}$$

where $1 < q < \infty$ and $0 < \delta < \pi$.

3 The Reduced Stokes Operator

In the following let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain with C^2-boundary.

In order to apply the results from [3] for the construction of an approximate resolvent for large λ, we need the *reduced* Stokes operator $A_{0,q}$ defined as

$$A_{0,q}u \;:=\; (-\Delta + \nabla K_N T)u, \qquad Tu := \gamma_\nu(\Delta - \nabla\operatorname{div})u|_{\partial\Omega},$$

for $u \in \mathcal{D}(A_{0,q}) := W^2_q(\Omega)^n \cap W^1_{q,0}(\Omega)^n$, where K_N denotes the Poisson operator of the Laplace equation with Neumann boundary conditions, i.e., $\Delta K_N = 0$ and $\partial_\nu K_N|_{\partial\Omega} = I$. Because of [23, Theorem 4.4], K_N exists and is a bounded operator

$$K_N : W^{-\frac{1}{q}}_{q,(0)}(\partial\Omega) := \left\{ a \in W^{-\frac{1}{q}}_q(\partial\Omega) : \langle a, 1\rangle_{\partial\Omega} = 0 \right\} \to \dot{W}^1_q(\Omega).$$

Note that $A_{q,0}$ is a densely defined unbounded operator on $L^q(\Omega)^n$ in contrast to the Stokes operator, which acts on the subspace $L^q_\sigma(\Omega)$.

It remains to justify that $\nabla K_N T : \mathcal{D}(A_{0,q}) \to L^q(\Omega)^n$. Since $\operatorname{div}(\Delta - \nabla\operatorname{div})u = 0$, $T : W^2_q(\Omega)^n \to W^{-\frac{1}{q}}_q(\Omega)$ is a bounded operator and by (2.2)

$$\langle Tu, v\rangle_{\partial\Omega} := ((\Delta - \nabla\operatorname{div})u, \nabla v)_\Omega \qquad \text{for all } v \in W^1_{q'}(\Omega). \tag{3.1}$$

The latter identity implies that even $Tu \in W^{-\frac{1}{q}}_{q,(0)}(\partial\Omega)$ for $u \in W^2_q(\Omega)$, which can be seen as follows: Since $\partial\Omega$ is compact, we may assume that also $\operatorname{supp} u$ is compact. Then choosing $v \in C^\infty_{(0)}(\overline{\Omega})$ such that $v \equiv 1$ on $\operatorname{supp} u \cup \partial\Omega$ yields $\langle Tu, 1\rangle_{\partial\Omega} = 1$.

There is the following alternative description of T: Introducing local coordinates $Tu = \operatorname{div}_\tau \partial_\nu u_\tau|_{\partial\Omega}$ for every $u \in C^\infty_{(0)}(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$, cf. [20, Lemma A.1]. Here u_τ denotes the tangential components of u and $\operatorname{div}_\tau a_\tau := \partial_{\kappa_1} a_{\kappa_1} + \ldots + \partial_{\kappa_{n-1}} a_{\kappa_{n-1}}$ for $a \in W^s_q(\partial\Omega)$, $s - \frac{1}{q} \notin \mathbb{Z}$, where $\kappa_1(y), \ldots, \kappa_{n-1}(y)$ denotes a basis of the tangential space $T_y\partial\Omega$ for each $y \in \partial\Omega$ such that $\kappa_1(y), \ldots, \kappa_{n-1}(y), \nu(y)$ is a positively orientated orthonormal basis of $\mathbb{R}^n$. Hence by (2.1)

$$T : W^{2+s}_q(\Omega)^n \to W^{s-\frac{1}{q}}_q(\partial\Omega) \quad \text{for all } 0 \geq s > -1 + \frac{1}{q}. \tag{3.2}$$

Remark 3.1 As the usual Stokes operator is associated to the Stokes resolvent equation, i.e.,

$$(\lambda - \Delta)u + \nabla p = f \quad \text{in } \Omega, \tag{3.3}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{3.4}$$

$$u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, \tag{3.5}$$

the reduced Stokes operator is associated to the *reduced Stokes resolvent equations*

$$(\lambda - \Delta + \nabla K_N T)u = f \quad \text{in } \Omega, \tag{3.6}$$

$$u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \tag{3.7}$$

The reduced system (3.6)-(3.7) is obtained from (3.3)-(3.5) by expressing the pressure p in terms of the data f and the velocity field u, which goes back to the work of Grubb and Solonnikov [20, Section 4 and 5], cf. [2, Section 3] for details.

The construction in Section 4 is based on the following lemma.

Lemma 3.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain with $C^{1,1}$-boundary, let $1 < q < \infty$, and assume that $(\lambda + A_{0,q})^{-1}$ exists for some $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. Then $(\lambda + A_q)^{-1}$ exists and*

$$A_{0,q}|_{L^q_\sigma(\Omega)} = A_q, \qquad (\lambda + A_{0,q})^{-1}|_{L^q_\sigma(\Omega)} = (\lambda + A_q)^{-1}. \tag{3.8}$$

Proof: The first statement can be seen as follows: If $u \in \mathcal{D}(A_{0,q}) \cap L^q_\sigma(\Omega)$, then $\operatorname{div}(-\Delta u + \nabla K_N T u) = 0$ in the sense of distributions and

$$\gamma_\nu(-\Delta u + \nabla K_N T u) = -\gamma_\nu \Delta u + \partial_\nu K_N T u|_{\partial\Omega} = 0$$

in the sense of (2.2). Hence $-\Delta u = (-\Delta + \nabla K_N T)u - \nabla K_N T u$ is the Helmholtz decomposition of $-\Delta u$, i.e., $(-\Delta + \nabla K_N T)u = P_q(-\Delta)u = A_q u$.

In order to prove the second relation let $u = (\lambda + A_{0,q})^{-1} f$ with $f \in L^q_\sigma(\Omega)$. Then applying div and γ_ν to (3.6) we conclude for $g = \operatorname{div} u$

$$(\lambda - \Delta) g = 0 \quad \text{in } \Omega, \tag{3.9}$$

$$\partial_\nu g = 0 \quad \text{on } \partial\Omega, \tag{3.10}$$

where $\partial_\nu g|_{\partial\Omega} = \gamma_\nu \nabla g$. Because of Proposition 3.3 below, $g = \operatorname{div} u = 0$. Therefore $u \in L^q_\sigma(\Omega)$ and $(\lambda + A_q)u = (\lambda + A_{0,q})u = f$. Since by the first statement $\lambda + A_q = (\lambda + A_{q,0})|_{L^q_\sigma(\Omega)}$ is injective, we finally conclude that $(\lambda + A_q)^{-1} f = u = (\lambda + A_{q,0})^{-1} f$ for every $f \in L^q_\sigma(\Omega)$. ■

It remains to prove:

Proposition 3.3 *Let $1 < q < \infty$ and let $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. Then there is only one solution $g \in W^1_q(\Omega)$ with $g = \operatorname{div} u$ for some $u \in \mathcal{D}(A_{0,q})$ satisfying (3.9)-(3.10) namely $g = 0$.*

Proof: If $\lambda = 0$, this is a consequence of the (unique) Helmholtz decomposition, cf. [23]. Hence it remains to consider $\lambda \neq 0$. Then by definition of γ_ν, the system (3.9)-(3.10) is equivalent to

$$\lambda(g, v) + (\nabla g, \nabla v) = 0 \qquad \text{for all } v \in W^1_{q'}(\Omega). \tag{3.11}$$

If $q = 2$, this implies $g = 0$ by choosing $v = g$. Hence in the case $q \neq 2$ it is sufficient to prove $g \in W^1_2(\Omega)$. If $q > 2$, let $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ for $|x| \geq R + 1$ and $\psi(x) = 0$ for $|x| \leq R$, where $R > 0$ is chosen such that $\partial\Omega \subset B_R(0)$. Then $\tilde{g} := \psi g \in W^1_q(\mathbb{R}^n)$ solves

$$(\lambda - \Delta)\tilde{g} = -2(\nabla\psi) \cdot \nabla\tilde{g} - (\Delta\psi)\tilde{g} =: f \qquad \text{in } \mathbb{R}^n,$$

where $f \in L^q(\mathbb{R}^n)$ has compact support. Therefore $\tilde{g} \in W_r^2(\mathbb{R}^n)$ for every $1 < r \le q$, which implies $g \in W_2^1(\Omega)$ if $q > 2$. Finally, (3.9)-(3.10) imply that $\Delta g = \lambda g$ and $\partial_\nu g|_{\partial\Omega} = 0$. Hence the same procedure of the elliptic regularity theory as in the proof of [15, Lemma 5.4] yields $g \in W_q^2(\Omega) \hookrightarrow W_{q_1}^1(\Omega)$, where $\frac{1}{q_1} = \frac{1}{q} - \frac{1}{n}$. Thus in the case $q_1 \ge 2$ we conclude $g \in W_2^1(\Omega)$. If $q_1 < 2$, we repeat this argument finitely many times until $g \in W_{q_m}^1(\Omega)$ with $q_m > 2$. ■

4 Analysis of the Resolvent for Large λ

In order to estimate (1.3) with Γ replaced by $\Gamma_R := \Gamma \setminus \overline{B_R(0)}$, $R > 0$, it is sufficient to construct an approximate resolvent R_λ satisfying

$$(\lambda + A_{0,q})^{-1} = R_\lambda + S_\lambda,$$

where $\|S_\lambda\|_{\mathcal{L}(L^q(\Omega))} \le C_{q,\delta}(1+|\lambda|)^{-1-\varepsilon}$ for some $\varepsilon > 0$ and

$$\left\| \int_{\Gamma_R} h(-\lambda) R_\lambda \, d\lambda \right\|_{\mathcal{L}(L^q(\Omega))} \le C_{q,\delta} \|h\|_\infty$$

for all $h \in H(\delta)$. The operator R_λ can be constructed using the calculus of pseudodifferential boundary value problems developed by Grubb [19] in a version with non-smooth coefficients, cf. [3, 1]. This approach was already used in [3] to prove the existence of a bounded H_∞-calculus in asymptotically flat layers. Since this construction is mainly based on localization and a similar approximation of $(\lambda + A_{0,q})^{-1}$ in a curved half-space $\mathbb{R}^n_\gamma = \{(x', x_n) \in \mathbb{R}^n : x_n > \gamma(x')\}$, $\gamma \in C^{1,1}(\mathbb{R}^{n-1})$, it can easily be modified to the case of an exterior domain – as well as many other classes of domains.

For the present contribution it is not necessary to recall the precise construction of the approximate resolvent R_λ in the curved half-space and all the operators belonging to calculus of pseudodifferential boundary value problems with Hölder-continuous coefficients. We refer to [3] for the details. For the following analysis, it is sufficient to recall the following theorem, which summarizes results obtained in [3].

Theorem 4.1 *Let $\mathbb{R}^n_\gamma$, $n \ge 2$, $\gamma \in C^{1,1}(\mathbb{R}^{n-1})$ be a curved half-space, $1 < q < \infty$, and let $\delta \in (0, \pi)$. Then there is a bounded operator $R_{\gamma,\lambda} \colon L^q(\mathbb{R}^n_\gamma)^n \to W_q^2(\mathbb{R}^n_\gamma)^n$, which is independent of q, such that*

$$(\lambda - \Delta + \nabla \widetilde{K}_{\gamma,N} T) R_{\gamma,\lambda} f = f + S_{\gamma,\lambda} f \quad \text{in } \mathbb{R}^n_\gamma, \tag{4.1}$$

$$R_{\gamma,\lambda} f = 0 \qquad \text{on } \partial\mathbb{R}^n_\gamma \tag{4.2}$$

for every $f \in L^q(\mathbb{R}^n_\gamma)^n$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, where $\|S_{\gamma,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} \le C_{q,\delta}(1+|\lambda|)^{-\varepsilon}$ uniformly in $\lambda \in \Sigma_\delta$ for some $\varepsilon > 0$. Here $\widetilde{K}_{\gamma,N} \colon W_q^{-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma) \to W_q^1(\mathbb{R}^n_\gamma)$ is a bounded *operator, which is independent of q, satisfying*

$$\Delta \widetilde{K}_{\gamma,N} a = R'_\gamma a \qquad in\ \mathbb{R}^n_\gamma, \tag{4.3}$$
$$\partial_\nu \widetilde{K}_{\gamma,N} a|_{\partial\Omega} = a + S'_\gamma a \quad on\ \partial\mathbb{R}^n_\gamma, \tag{4.4}$$

where $R'_\gamma \colon W_q^{-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_\gamma) \to W^{-1}_{q,0}(\mathbb{R}^n_\gamma)$ *and* $S'_\gamma \colon W_q^{-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_\gamma) \to W_q^{-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma)$ *are bounded operators. Moreover, for every* $R > 0$

$$(1+|\lambda|)\|R_{\gamma,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} + \|\nabla^2 R_{\gamma,\lambda}\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} \leq C_{q,\delta}, \qquad \lambda \in \Sigma_\delta, \tag{4.5}$$
$$\left\| \int_{\Gamma_R} h(-\lambda) R_{\gamma,\lambda}\, d\lambda \right\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} \leq C_{q,\delta}\|h\|_\infty, \qquad h \in H(\delta). \tag{4.6}$$

Proof: First of all $\widetilde{K}_{\gamma,N} \equiv \widetilde{K}_1$ is defined in [3, Section 5.5] as

$$\widetilde{K}_{\gamma,N} \;=\; F^{*,-1}\underline{k}_1(D_x, x')F_0^*,$$

where $\underline{k}_1(D_x, x')$ is a Poisson operator of order -1 in R-form with $C^{0,1}$-coefficients in the sense of [3, Section 4], $F\colon \mathbb{R}^n_+ \to \mathbb{R}^n_\gamma$ is defined by $F(x) = (x', x_n + \gamma(x'))$, $(F^{*,-1}f)(x) := f(F^{-1}(x))$ is the push-forward of a function $f\colon \mathbb{R}^n_+ \to \mathbb{C}$ by F, and $(F_0^* a)(y) := a(F_0(y))$ is the pull-back of a function $a\colon \partial\mathbb{R}^n_\gamma \to \mathbb{C}$ by $F_0 := F|_{\partial\mathbb{R}^n_+}$. The statements on $\widetilde{K}_{\gamma,N}$ are a consequence of [3, Lemma 5.15] using

$$L^q(\mathbb{R}^{n-1}; L^1(0,b)) \hookrightarrow W^{-1}_{q,0}(\mathbb{R}^n_+), \qquad H^{-1}_q(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)) \hookrightarrow W^{-1}_{q,0}(\mathbb{R}^n_+),$$

and the fact that $F^*\colon W^{-1}_{q,0}(\mathbb{R}^n_\gamma) \to W^{-1}_{q,0}(\mathbb{R}^n_+)$ is an isomorphism.

The operator $R_{\gamma,\lambda} \equiv R_{0,\lambda}$ is defined in [3, Section 5.6]. Then (4.1)-(4.2) and the estimate of $S_{\gamma,\lambda}$ is the statement of [3, Lemma 5.17] if $\nabla\widetilde{K}_{\gamma,N} T R_{\gamma,\lambda}$ is replaced by the operator $\widetilde{G}_\lambda$ defined in [3, Lemma 5.17]. However the estimate

$$\left\| \nabla\widetilde{K}_{\gamma,N} T R_\lambda - \widetilde{G}_\lambda \right\|_{\mathcal{L}(L^q(\mathbb{R}^n_\gamma))} \leq C_{q,\delta}(1+|\lambda|)^{-\varepsilon}, \qquad \lambda \in \Sigma_\delta,$$

for some $\varepsilon > 0$ is shown in the last part of the proof of [3, Lemma 5.18].

Finally, (4.5) follows from [3, Estimate (5.35)] and (4.6) is a consequence of [3, Theorem 5.13], [3, Lemma 5.14], and [3, Theorem 3.2]. ■

Remark 4.2 The operator $\widetilde{K}_{\gamma,N}$ of the latter theorem is an approximate Poisson operator to the Laplace equation with Neumann boundary condition on $\mathbb{R}^n_\gamma$. As stated above, it is constructed explicitly in [3] and is an operator of the calculus of pseudodifferential boundary value problems necessary to construct the approximate resolvent of the reduced Stokes operator.

We will use the latter theorem to construct an approximate resolvent in an exterior domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with $C^{1,1}$-boundary. In order to localize Ω let $U_j \subset \overline{\Omega}$, $j = 0, \ldots, m$ be relatively open sets such that

1. each $U_j \cap \overline{\Omega}$, $j = 1, \ldots, m$, is bounded and coincides (after rotation) with a relatively open subset of $\overline{\mathbb{R}^n_{\gamma_j}}$, where $\mathbb{R}^n_{\gamma_j}$, $\gamma_j \in C^{1,1}(\mathbb{R}^{n-1})$, is a curved half-space.
2. $\partial\Omega \subset \bigcup_{j=1}^m U_j$, $U_0 \cap \partial\Omega = \emptyset$, and $\overline{\Omega} \subseteq \bigcup_{j=0}^m U_j$.

Moreover, let $\varphi_j \in C^\infty_{(0)}(\overline{\Omega})$, $j = 0, \ldots, m$, be a partition of unity on $\overline{\Omega}$ such that $\operatorname{supp}\varphi_j \subset U_j$, $j = 0, \ldots, m$. Finally, let $\psi_j \in C^\infty_{(0)}(\overline{\Omega})$, $j = 0, \ldots, m$, such that $\psi_j = 1$ on $\operatorname{supp}\varphi_j$ and again $\operatorname{supp}\psi_j \subset U_j$, $j = 0, \ldots, m$. Now we define the approximate resolvent R_λ as

$$R_\lambda f = \sum_{j=0}^m \psi_j R_{j,\lambda}\varphi_j f, \qquad f \in L^q(\Omega)^n,$$

where $R_{j,\lambda} = R_{\gamma_j,\lambda}$, $j = 1, \ldots, m$, is the approximate resolvent on $\mathbb{R}^n_{\gamma_j}$ due to Theorem 4.1 and $R_{0,\lambda} = (\lambda - \Delta_{\mathbb{R}^n})^{-1}$ is the resolvent of the Laplace operator on $\mathbb{R}^n$. Moreover, we define the approximate Poisson operator

$$\widetilde{K}_N a = \sum_{j=1}^m \psi_j \widetilde{K}_{\gamma_j,N}\varphi_j a, \qquad a \in W_q^{-\frac{1}{q}}(\partial\Omega),$$

where $\widetilde{K}_{\gamma_j,N}$ is the operator due to Theorem 4.1 for $\mathbb{R}^n_{\gamma_j}$. Now we have

Lemma 4.3 *Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain with $C^{1,1}$-boundary, let $\widetilde{K}_N$ be as above, and let K_N be the Poisson operator of the Neumann problem as defined in Section 3. Then there is some $\varepsilon > 0$ such that*

$$\|\nabla(K_N - \widetilde{K}_N)Tu\|_q \leq C_q\|u\|_{2-\varepsilon,q},$$

for all $u \in W_q^{2-\varepsilon}(\Omega)^n$.

Proof: Let $f \in L^{q'}(\Omega)^n$ be arbitrary and let $f = f_0 + \nabla p$, $f_0 \in L^{q'}_\sigma(\Omega)$, $p \in \dot{W}^1_{q'}(\Omega)$, be its Helmholtz decomposition. By Lemma 2.1 $p = p_1 + p_2$, where $p_1 \in W^1_{q'}(\Omega)$ and $p_2 \in L^{q'}_{\mathrm{loc}}(\overline{\Omega})$ with $\nabla p_2 \in W^1_{q'}(\Omega)$ and $\|(p_1, \nabla p_2)\|_{1,q'} \leq C_{q'}\|\nabla p\|_{q'}$.

Then by (4.3)-(4.4)

$$\begin{aligned}
&(\nabla(K_N - \widetilde{K}_N)Tu, f)_\Omega = (\nabla(K_N - \widetilde{K}_N)Tu, \nabla p)_\Omega \\
&= (\nabla(K_N - \widetilde{K}_N)Tu, \nabla p_2)_\Omega + \langle (I - \partial_\nu \widetilde{K}_N)Tu, p_1\rangle_{\partial\Omega} - (\Delta\widetilde{K}_N Tu, p_1)_\Omega,
\end{aligned}$$

where

$$\begin{aligned}
|\langle (I - \partial_\nu\widetilde{K}_N)Tu, p_1\rangle_{\partial\Omega}| &\leq \sum_{j=1}^m |\langle \psi_j S'_{\gamma_j}\varphi_j Tu, p_1\rangle_{\partial\Omega}| \\
&\leq C\|Tu\|_{W_q^{-\frac{1}{q}-\varepsilon}(\partial\Omega)}\|p_1\|_{1,q'} \leq C\|u\|_{2-\varepsilon,q}\|f\|_{q'}
\end{aligned}$$

by (3.2) for $\varepsilon > 0$ suitably small. Moreover,

$$
\begin{aligned}
&|(\Delta \widetilde{K}_N Tu, p_1)_\Omega| \\
&\quad \leq \sum_{j=1}^m |(\psi_j R'_{\gamma_j} \varphi_j Tu, p_1)_\Omega| + \sum_{j=1}^m |(2(\nabla\psi_j)\cdot \nabla \widetilde{K}_{\gamma_j,N}\varphi_j Tu + (\Delta\psi_j)\widetilde{K}_{\gamma_j,N}\varphi_j Tu, p_1)_\Omega|,
\end{aligned}
$$

where

$$
|(\psi_j R'_{\gamma_j}\varphi_j Tu, p_1)_\Omega| \leq C\|R'_{\gamma_j}\varphi_j Tu\|_{W^{-1}_{q,0}(\mathbb{R}^n_{\gamma_j})}\|p_1\|_{1,q'} \leq C\|u\|_{2-\varepsilon,q}\|f\|_{q'}
$$

for some $\varepsilon > 0$. Since $p_1 \in W^1_{q'}(\Omega) \hookrightarrow L^{s'}(\Omega)$ for some $s' > q'$ and $\widetilde{K}_{\gamma_j,N}\colon W_s^{-\frac{1}{s}}(\partial\mathbb{R}^n_{\gamma_j}) \to W^1_s(\mathbb{R}^n_\gamma)$, $\frac{1}{s}+\frac{1}{s'} = 1$,

$$
\begin{aligned}
&|(2(\nabla\psi_j)\cdot\nabla\widetilde{K}_{\gamma_j,N}\varphi_j Tu + (\Delta\psi_j)\widetilde{K}_{\gamma_j,N}\varphi_j Tu, p_1)_\Omega| \\
&\quad \leq C\|\varphi_j Tu\|_{W_s^{-\frac{1}{s}}(\partial\mathbb{R}^n_{\gamma_j})}\|p_1\|_{1,q'} \leq C\|Tu\|_{W_q^{-\frac{1}{q}-\varepsilon}(\partial\mathbb{R}^n_{\gamma_j})}\|f\|_{q'} \leq C\|u\|_{2-\varepsilon,q}\|f\|_{q'},
\end{aligned}
$$

where we have used that $\operatorname{supp}\varphi_j$ is compact and $-\frac{1}{s} \leq -\frac{1}{q} - \varepsilon$ for suitably small $\varepsilon > 0$. The term $(\nabla\widetilde{K}_N Tu, \nabla p_2)_\Omega$ is estimated in the same way using $\nabla p_2 \in W^1_{q'}(\Omega)$. Finally, by (3.1)

$$
\begin{aligned}
&|(\nabla K_N Tu, \nabla p_2)_\Omega| = |((\Delta - \nabla\operatorname{div})u, \nabla p_2)_\Omega| \\
&\quad \leq |(\nabla u, \nabla^2 p_2)_\Omega| + |(\partial_\nu u, \nabla p_2)_{\partial\Omega}| + |(\operatorname{div} u, \Delta p_2)_\Omega| + |(\operatorname{div} u, \partial_\nu p_2)_{\partial\Omega}| \\
&\quad \leq C\left(\|\nabla u\|_q\|\nabla^2 p_2\|_{q'} + \|\nabla u\|_{L^q(\partial\Omega)}\|\nabla p_2\|_{L^{q'}(\partial\Omega)}\right) \\
&\quad \leq C\|u\|_{2-\varepsilon,q}\|\nabla p_2\|_{1,q'} \leq C\|u\|_{2-\varepsilon,q}\|f\|_{q'}
\end{aligned}
$$

for some $\varepsilon > 0$. ■

Using the latter lemma and Theorem 4.1 we obtain:

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain with $C^{1,1}$-boundary, $\delta \in (0,\pi)$, and let $1 < q < \infty$. Then for every $R > 0$*

$$
\left\|\int_{\Gamma_R} h(-\lambda)(\lambda + A_q)^{-1}\,d\lambda\right\|_{\mathcal{L}(L^q(\Omega))} \leq C_{q,\delta,R}\|h\|_\infty
$$

for every $h \in H(\delta)$.

Proof: First of all, by (2.6), (4.5), and interpolation

$$
\|R_{j,\lambda}\varphi_j f\|_{s,q} \leq C_{q,\delta,R}(1+|\lambda|)^{-1+\frac{s}{2}}\|f\|_q, \qquad \lambda \in \Sigma_\delta, |\lambda| \geq R, \tag{4.7}
$$

for all $s \in [0,2]$, $f \in L^q(\Omega)^n$, and $j = 0, \ldots, m$. Moreover, by (4.1)

$$\begin{aligned}
&(\lambda - \Delta + \nabla K_N T) R_\lambda f \\
&\quad = f + \sum_{j=1}^m \psi_j S_{\gamma_j,\lambda} \varphi_j f - \sum_{j=0}^m (2(\nabla \psi_j) \cdot \nabla R_{j,\lambda} \varphi_j f + (\Delta \psi_j) R_{j,\lambda} \varphi_j f) \\
&\qquad + (\nabla K_N T - \nabla \widetilde{K}_N T) R_\lambda f.
\end{aligned}$$

Hence (4.7), Theorem 4.1, and Lemma 4.3 imply

$$(\lambda - \Delta + \nabla K_N T) R_\lambda = I + S'_\lambda,$$

where $\|S'_\lambda\|_{\mathcal{L}(L^q(\Omega))} \leq C_{q,\delta}(1+|\lambda|)^{-1-\varepsilon}$ uniformly in $\lambda \in \Sigma_\delta$ for some $\varepsilon > 0$. Therefore $(\lambda + A_{0,q})^{-1}$ exists for all $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R'$ for some $R' > 0$ and

$$(\lambda + A_{0,q})^{-1} = R_\lambda + S_\lambda,$$

where $\|S_\lambda\|_{\mathcal{L}(L^q(\Omega))} \leq C_{q,\delta}(1+|\lambda|)^{-\varepsilon}$ uniformly in $\lambda \in \Sigma_\delta$, $|\lambda| \geq R$. Since $(\lambda + A_{0,q})^{-1}|_{L^q_\sigma(\Omega)} = (\lambda + A_q)^{-1}$, we conclude that

$$(\lambda + A_q)^{-1} = R_\lambda|_{L^q_\sigma(\Omega)} + S_\lambda|_{L^q_\sigma(\Omega)},$$

for $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R'$. Moreover,

$$\left\| \int_{\Gamma_R} h(-\lambda) B_\lambda \, d\lambda \right\|_{\mathcal{L}(L^q(\Omega))} \leq C_{q,\delta} \|h\|_\infty \qquad \text{for } B_\lambda = R_\lambda, S_\lambda,$$

for $R \geq R'$ because of (4.6) and $\|S_\lambda\| \leq C_{q,\delta}(1+|\lambda|)^{-1-\varepsilon}$. Finally, since $(\lambda + A_q)^{-1}$ is uniformly bounded on each compact subset of $\overline{\Sigma_\delta} \setminus \{0\}$, $R > 0$ can be chosen arbitrarily. ■

5 Analysis of the Resolvent for Small λ

First of all, recall that the resolvent of the Stokes operator $A_{q,\mathbb{R}^2}$ on $L^q(\mathbb{R}^2)$, $1 < q < \infty$, can be written as

$$\begin{aligned}
(\lambda + A_{q,\mathbb{R}^2})^{-1} &= P(\lambda - \Delta_{\mathbb{R}^2})^{-1}, \qquad \text{where} \\
Pf &= \mathcal{F}^{-1}\left[\left(I - \frac{\xi\xi^T}{|\xi|^2}\right)\mathcal{F}[f](\xi)\right]
\end{aligned} \tag{5.1}$$

for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Moreover, $(\lambda - \Delta_{\mathbb{R}^2})^{-1}$ can be represented as

$$(\lambda - \Delta_{\mathbb{R}^2})^{-1} f = e_\lambda * f, \qquad \text{where} \quad e_\lambda(x) = K_0(\sqrt{\lambda}|x|).$$

Here $\sqrt{\lambda}$ denotes the unique square root of λ with $\operatorname{Re}\sqrt{\lambda} > 0$ and $K_n(z)$, $n \in \mathbb{N}_0$, is the modified Bessel function of order n, cf. [5, page 375]. We note that $K_0'(z) = -K_1(z)$. From the definition it follows that

$$K_0(z) = O(\ln z), \qquad K_1(z) = O(z^{-1}) \qquad \text{as } z \to 0. \tag{5.2}$$

Furthermore, we will use that

$$|K_n(z)| \le C_{n,\delta} e^{-\alpha|z|} \qquad \text{for } z \in \Sigma_{\delta/2}, |z| \ge 1 \tag{5.3}$$

with $0 < \delta < \pi$ and for some $\alpha > 0$ depending on δ, cf. [5, page 378].

Moreover, it is well-known that

$$\|(\lambda - \Delta_{\mathbb{R}^2})^{-1} f\|_r \;\le\; C_{s,q,\delta}|\lambda|^{-1+\frac{1}{q}-\frac{1}{r}}\|f\|_q, \qquad \lambda \in \Sigma_\delta, \tag{5.4}$$

for $1 < q < \infty$, $q \le r \le \infty$, and $\delta \in (0,\pi)$, cf. e.g. proof of [11, Proposition 4.1]. Because of the identity $(\lambda + A_{q,\mathbb{R}^2})^{-1} = (\lambda - \Delta_{\mathbb{R}^2})^{-1}P$ and the continuity $P\colon L^q(\mathbb{R}^2)^2 \to L^q(\mathbb{R}^2)^2$, the same is true for the Stokes resolvent.

Finally, $A_{q,\mathbb{R}^2}$ possesses a bounded H_∞-calculus for all $\delta \in (0,\pi)$. In particular,

$$\left\| \frac{1}{2\pi i}\int_\Gamma h(-\lambda)(\lambda + A_{q,\mathbb{R}^2})^{-1}\, d\lambda \right\|_{\mathcal{L}(L^q(\mathbb{R}^2))} \le C_\delta \|h\|_\infty, \qquad h \in H(\delta), \tag{5.5}$$

cf. [13, Theorem 7.2].

Remark 5.1 The estimate (5.5) also holds with Γ replaced by $\Gamma_R = \Gamma \setminus B_R(0)$, $R > 0$, cf. [3, Lemma 5.14].

Next we recall the representation of the resolvent of the Stokes operator developed in [11], which is

$$(\lambda + A_q)^{-1} f = \big((\lambda + A_{q,\mathbb{R}^2})^{-1} f_0\big)\big|_\Omega - B^{\cdot}_\lambda K_\lambda^{-1} f_\lambda, \tag{5.6}$$

where $f_\lambda = (\lambda + A_{q,\mathbb{R}^2})^{-1} f_0|_{\partial\Omega}$ and f_0 denotes the extension of f by zero to $\mathbb{R}^2$. Moreover,

$$B^{\cdot}_\lambda \phi = D^{\cdot}_\lambda \phi - \eta E^{\cdot}_\lambda M\phi + \frac{4\pi\alpha}{\ln\sqrt{\lambda}} E^{\cdot}_\lambda \phi, \qquad \alpha, \eta \in \mathbb{C} \setminus \{0\}, \tag{5.7}$$

and $K_\lambda = B^{\cdot}_\lambda|_{\partial\Omega}$. Here $D^{\cdot}_\lambda$ is a double layer potential, which satisfies

$$\|D^{\cdot}_\lambda \phi\|_q \;\le\; \begin{cases} C_{q,\delta}|\lambda|^{\frac{1}{2}-\frac{1}{q}}\|\phi\|_{\infty,\partial\Omega} & \text{if } 1 < q < 2, \\ C_{q,\delta}|\ln\lambda|\|\phi\|_{\infty,\partial\Omega} & \text{if } q = 2, \\ C_{q,\delta}\|\phi\|_{\infty,\partial\Omega} & \text{if } q > 2 \end{cases} \tag{5.8}$$

uniformly in $\lambda \in \Sigma_\delta$, $\delta \in (0,\pi)$, with $|\lambda| \leq \frac{1}{2}$ and $\phi \in C^0(\partial\Omega)^n$ with $\int_{\partial\Omega} \nu \cdot \phi \, d\sigma = 0$, cf. [11, Estimate (4.15)]. Moreover, $E^{\cdot}_\lambda$ is the single layer potential defined by

$$E^{\cdot}_\lambda \phi = \int_{\partial\Omega} E^{(r,c)}_\lambda (x-y)\phi(y)\, d\sigma(y), \qquad \phi \in C^0(\partial\Omega)^n,$$

where the matrix $E^{(r,c)}_\lambda(x) = (E^\lambda_{jk}(x))_{j,k=1,2}$ is defined by

$$E^\lambda_{jk}(x) \;=\; \frac{1}{2\pi}\left(\delta_{jk} e_1(\sqrt{\lambda}|x|) + \frac{x_j x_k}{|x|^2} e_2(\sqrt{\lambda}|x|)\right)$$

and

$$e_1(\kappa) = K_0(\kappa) + \kappa^{-1}K_1(\kappa) - \kappa^{-2}, \qquad e_2(\kappa) = -K_0(\kappa) - 2\kappa^{-1}K_1(\kappa) + 2\kappa^{-2}.$$

Finally, $M\phi = \phi - \phi_M$, where ϕ_M denotes the mean-value of ϕ on $\partial\Omega$.

Because of [11, Proposition 3.8] K_λ is invertible for all $\lambda \in \Sigma_\delta$ with $|\lambda| \leq R$ for some $R > 0$ and

$$\|K_\lambda^{-1}\|_{\mathcal{L}(C^0(\partial\Omega))} \leq C_\delta, \qquad \lambda \in \Sigma_\delta \cap B_R(0). \tag{5.9}$$

In particular, this implies

$$\|D^{\cdot}_\lambda K_\lambda^{-1} f_\lambda\|_q \leq C|\lambda|^{-1+\varepsilon}\|f\|_q \tag{5.10}$$

by (5.4) and (5.8) for some $\varepsilon > 0$, where $\int_{\partial\Omega} \nu \cdot f_\lambda \, d\sigma = 0$ since $\operatorname{div} f_\lambda = 0$ and K_λ^{-1} preserves this property, cf. [11, Lemma 3.7]. Hence the latter term corresponds to an absolutely integrable part in (1.3).

As shown in [11, Section 3]

$$\hat{E}^{(r,c)}_\lambda(\xi) := \mathcal{F}[E^{(r,c)}_\lambda](\xi) = \frac{1}{2\pi(\lambda + |\xi|^2)}\left(I - \frac{\xi\xi^T}{|\xi|^2}\right).$$

Therefore the single layer potential $E^{\cdot}_\lambda$ can be represented as $E^{\cdot}_\lambda = PE'_\lambda$, where

$$(E'_\lambda \phi)(x) = \int_{\partial\Omega} K_0(\sqrt{\lambda}|x-y|)\phi(y)\, d\sigma(y)$$

and P defined as above is bounded on $L^q(\mathbb{R}^2)$, $1 < q < \infty$, and is independent of λ. Hence using (5.6) and (5.7) in (1.3) and estimating each term separately we can replace $E^{\cdot}_\lambda$ by E'_λ.

Summarizing it suffices to estimate (1.3) with $(\lambda + A_q)^{-1}$ replaced by $E'_\lambda M K_\lambda^{-1} f_\lambda$ and $\frac{1}{\ln\sqrt{\lambda}} E'_\lambda K_\lambda^{-1} f_\lambda$, which is done in the following two lemmata.

Lemma 5.2 *Let $1 < q < \infty$, $\delta \in (0,\pi)$, $\varepsilon > 0$, and let $E'_\lambda M$ be defined as above. Then*

$$\|E'_\lambda M\phi\|_q \leq \begin{cases} C_{q,\delta}|\lambda|^{\frac{1}{2}-\frac{1}{q}}\|\phi\|_\infty & \text{if } 1 < q < 2, \\ C_{q,\delta,\varepsilon}|\lambda|^{-\varepsilon}\|\phi\|_\infty & \text{if } q \geq 2 \end{cases}$$

for all $\phi \in C^0(\partial\Omega)$ uniformly in $\lambda \in \Sigma_\delta \cap B_1(0)$. In particular, there is an $a < 1$ and $R > 0$ such that for all $f \in L^q(\mathbb{R}^2)^2$

$$\|E'_\lambda M K_\lambda^{-1} f_\lambda\|_q \le C_{q,\delta}|\lambda|^{-a}\|f\|_q, \qquad \lambda \in \Sigma_\delta, |\lambda| \le R.$$

Proof: First of all, since

$$\|E'_\lambda \phi\|_r \le C_{r,\delta}|\lambda|^{-\frac{1}{r}}\|\phi\|_\infty, \qquad \lambda \in \Sigma_\delta, \tag{5.11}$$

for $1 < r < \infty$, cf. [11, Equation (4.14)], and $L^r(\Omega \cap B_R) \hookrightarrow L^q(\Omega \cap B_R)$ if $r \ge q$, it is sufficient to estimate the $L^q(\Omega_R)$-norm of $E'_\lambda M\phi$ with $\Omega_R := \Omega \setminus B_R(0)$, where $R > 1$ is chosen such that $\partial\Omega \subset B_{R-1}(0)$. Since $\int_{\partial\Omega} M\phi(y)\, d\sigma(y) = 0$,

$$E'_\lambda M\phi = \int_{\partial\Omega} \Big(K_0(\sqrt{\lambda}|x-y|) - K_0(\sqrt{\lambda}|x|)\Big)\, M\phi(y)\, d\sigma(y).$$

Moreover,

$$\Big|K_0(\sqrt{\lambda}|x-y|) - K_0(\sqrt{\lambda}|x|)\Big| \le C \sup_{s\in[|x-y|,|x|]} |K_0'(\sqrt{\lambda}s)||\lambda|^{\frac{1}{2}}|y|.$$

Using $K_0'(z) = -K_1(z)$ and (5.2), we conclude

$$\begin{aligned} &\int_{R\le|x|\le|\lambda|^{-\frac{1}{2}}} \Big|K_0(\sqrt{\lambda}|x-y|) - K_0(\sqrt{\lambda}|x|)\Big|^q\, dx \\ \le\ & C\int_{R\le|x|\le|\lambda|^{-\frac{1}{2}}} |x|^{-q}\, dx \le \begin{cases} C_\delta|\lambda|^{\frac{q}{2}-1} & \text{if } 1 < q < 2, \\ C_\delta|\ln\lambda| & \text{if } q = 2, \\ C_\delta & \text{if } q > 2, \end{cases} \end{aligned}$$

where we have used $|x-y| \ge \frac{|x|}{R}$. Similarly, by (5.3)

$$\begin{aligned} &\int_{|x|>|\lambda|^{-\frac{1}{2}}} \Big|K_0(\sqrt{\lambda}|x-y|) - K_0(\sqrt{\lambda}|x|)\Big|^q\, dx \\ \le\ & C|\lambda|^{\frac{q}{2}} \int_{|x|>|\lambda|^{-\frac{1}{2}}} e^{-c_q|\lambda|^{\frac{1}{2}}|x|}\, dx = C'|\lambda|^{\frac{q}{2}-1} \end{aligned}$$

with some $c_q > 0$. This implies the statement of the theorem. ∎

Lemma 5.3 *Let $1 < q < \infty$, $\delta \in (0,\pi)$, and let $E'_\lambda K_\lambda^{-1} f_\lambda$ be defined as above. Then*

$$\left\|\int_{\Gamma'_R} h(-\lambda)\frac{1}{\ln\sqrt{\lambda}} E'_\lambda K_\lambda^{-1} f_\lambda\, d\lambda\right\|_q \le C_{q,\delta}\|h\|_\infty\|f\|_q \tag{5.12}$$

for all $h \in H(\delta)$ and $f \in L^q(\mathbb{R}^2)^2$, where $R > 0$ is as in (5.9).

Proof: First of all,

$$\|E'_\lambda K_\lambda^{-1} f_\lambda\|_r \le C|\lambda|^{-1+\frac{1}{q}-\frac{1}{r}}\|f\|_q$$

for all $q \le r < \infty$ due to (5.11), (5.4), and (5.9). Hence we can replace the $L^q(\Omega)$ on the left-hand side of (5.12) by $L^q(\Omega_{R'})$ with $\Omega_{R'} := \Omega \setminus B_{R'}(0)$, where $R' > 1$ is chosen such that $\partial\Omega \subset B_{R'-1}(0)$. Moreover, because of the Marcinkiewisz interpolation theorem, cf. [8, Theorem 1.3.1], and since $1 < q < \infty$ is arbitrary, it is sufficient to prove (5.12) with the $L^q(\Omega_{R'})$-norm replaced by the weak L^q-norm on $\Omega_{R'}$, i.e.,

$$\|f\|_{L^q_*(\Omega_{R'})} := \sup_{\sigma>0}\{\sigma m(f,\sigma)^{\frac{1}{q}}\}, \quad \text{where } m(f,\sigma) = |\{x \in \Omega_{R'} : |f(x)| > \sigma\}|\,.$$

Moreover,

$$\left|\int_{\Gamma'_R} h(-\lambda)\frac{1}{\ln\sqrt{\lambda}}(E'_\lambda K_\lambda^{-1} f_\lambda)(x)\,d\lambda\right| \le \|h\|_\infty \int_{\Gamma'_R}\left|\frac{1}{\ln\sqrt{\lambda}}(E'_\lambda K_\lambda^{-1} f_\lambda)(x)\right|\,d|\lambda|.$$

Now by the definition of E'_λ and (5.9)

$$\left|\frac{1}{\ln\sqrt{\lambda}}(E'_\lambda K_\lambda^{-1} f_\lambda)(x)\right| \le C_q \int_{\partial\Omega} |K_0(\sqrt{\lambda}|x-y|)|\,d\sigma(y)\||\ln\lambda|^{-1} f_\lambda\|_{\infty,\partial\Omega}.$$

Setting $g_\lambda = \int_{\partial\Omega}|K_0(\sqrt{\lambda}|x-y|)|\,d\sigma(y)$, we conclude

$$\begin{aligned}
&\left|\int_{\Gamma'_R} h(-\lambda)\frac{1}{\ln\sqrt{\lambda}}(E_\lambda K_\lambda^{-1} f_\lambda|_{\partial\Omega})(x)\,d\lambda\right| \\
&\le\ C\|h\|_\infty \left(\int_{\Gamma'_R} g_\lambda(x)^q\,d|\lambda|\right)^{\frac{1}{q}} \left(\int_{\Gamma'_R} |\ln\lambda|^{-q'}\|f_\lambda\|^{q'}_{\infty,\partial\Omega}\,d|\lambda|\right)^{\frac{1}{q'}}.
\end{aligned}$$

Because of (5.4)

$$\int_{\Gamma'_R} |\ln\lambda|^{-q'}\|f_\lambda\|^{q'}_{\infty,\partial\Omega}\,d|\lambda| \le C_q \int_{\Gamma'_R} |\ln\lambda|^{-q'}|\lambda|^{-1}\,d|\lambda|\|f\|_q^{q'} \le C'_q\|f\|_q^{q'}.$$

Furthermore,

$$\begin{aligned}
\int_{\Gamma'_R} g_\lambda(x)^q\,d|\lambda| &\le C\sum_{\pm}\sup_{y\in\partial\Omega}\int_0^R \left|K_0\left(s^{\frac{1}{2}}e^{\pm i\frac{\delta}{2}}|x-y|\right)\right|^q ds \\
&\le C\sup_{y\in\partial\Omega}\sum_{\pm}\int_0^\infty \left|K_0\left(t^{\frac{1}{2}}e^{\pm i\frac{\delta}{2}}\right)\right|^q dt|x-y|^{-2} \\
&\le C'_{q'}(1+|x|)^{-2}
\end{aligned}$$

because $\operatorname{dist}(x,\partial\Omega) \ge 1$ and by (5.2)-(5.3). Hence $\left(\int_{\Gamma'_R} g_\lambda(x)^q d|\lambda|\right)^{\frac{1}{q}} \in L^q_*(\Omega_R)$, which finishes the proof. ■

Proof of Theorem 1.1: Let $R > 0$ be small enough such that K_λ^{-1} exists for $\lambda \in \Sigma_\delta$, $|\lambda| \leq R$. By Theorem 4.4 it is sufficient to prove (1.4) with Γ replaced by $\Gamma'_R = \Gamma \cap B_R(0)$ and arbitrary $h \in H(\delta)$. But this is a consequence of the identity (5.6), Remark 5.1, (5.10), Lemma 5.2, and Lemma 5.3. ■

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