Holomorphic generation of continuous inverse algebras

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Abstract

We study complex commutative Banach algebras, and more generally continuous inverse algebras, in which the holomorphic functions of a fixed n-tuple of elements are dense. In particular, we characterize the compact subsets of $\mathbb{C}^n$ which appear as joint spectra of such n-tuples. The characterization is compared to several established notions of holomorphic convexity by means of approximation conditions.[1] [2]

Introduction

By a classic result, the joint spectra of topologically generating n-tuples in complex commutative Banach algebras are exactly the polynomially convex compact subsets of $\mathbb{C}^n$. The principal result of this paper is a similar characterization of the joint spectra of holomorphically generating n-tuples in complex commutative Banach algebras. Here holomorphic generation refers to the holomorphic functional calculus, which associates with every n-tuple $a \in A^n$ in a complex commutative Banach algebra A a continuous algebra homomorphism $\theta_a \colon \mathcal{O}(\mathrm{Sp}(a)) \to A$, where $\mathcal{O}(\mathrm{Sp}(a))$ denotes the algebra of germs of holomorphic functions near the joint spectrum $\mathrm{Sp}(a)$ in its natural inductive limit topology. The tuple a is said to generate A holomorphically if the image of θ_a is dense in A. We find (7.2 and 7.3) that a compact subset $K \subseteq \mathbb{C}^n$ is the joint spectrum of a holomorphically generating n-tuple in a complex commutative Banach algebra if and only if every homomorphism from $\mathcal{O}(K)$ into $\mathbb{C}$ is evaluation in a point of K. A compact subset of a Stein manifold with this property is called auto-spectral.

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Given a holomorphically generating n-tuple $a \in A^n$, one may strengthen the hypotheses by assuming that certain subalgebras $B \subseteq \mathcal{O}(\mathrm{Sp}(a))$ already have dense image under θ_a. This situation is interesting in its own right. Moreover, it helps to relate auto-spectrality to other holomorphic convexity conditions (7.4). If B consists of the germs of holomorphic functions defined in a fixed open neighbourhood $U \subseteq \mathbb{C}^n$ of $\mathrm{Sp}(a)$ then $\theta_a(B)$ is dense in A if and only if $\mathrm{Sp}(a)$ is holomorphically convex in U. Similarly, if B consists of the germs of holomorphic functions defined in holomorphically convex open neighbourhoods of $\mathrm{Sp}(a)$ then $\theta_a(B)$ is dense in A if and only if $\mathrm{Sp}(a)$ is a Stein compactum, i.e. it has a neighbourhood basis consisting of holomorphically convex open sets. Finally, let $B \subseteq \mathcal{O}(\mathrm{Sp}(a))$ be the algebra of germs of quotients of holomorphic functions defined in a fixed open neighbourhood $U \subseteq \mathbb{C}^n$ of $\mathrm{Sp}(a)$ such that the denominator does not vanish anywhere in $\mathrm{Sp}(a)$. Then $\theta_a(B)$ is dense in A if and only if $\mathrm{Sp}(a)$ is meromorphically convex in U. In fact, compact subsets of a Stein manifold X which are holomorphically convex with respect to some open neighbourhood can be characterized among the auto-spectral subsets of X by a certain approximation property (4.5). A similar characterization holds for Stein compacta (5.4) and for meromorphically convex compacta (6.12).

Section 1 provides several important tools, and Section 2 introduces auto-spectral compacta. Section 3 contains the direct proof that rationally convex compact subsets of $\mathbb{C}^n$ are auto-spectral. The three middle sections 4 to 6 treat holomorphic convexity, Stein compacta, and meromorphic convexity, respectively. The final Section 7 applies all this material to the theory of Banach algebras.

As the polynomials are contained in $\mathcal{O}(\mathrm{Sp}(a))$, every n-tuple which generates A in the usual sense generates A holomorphically. Therefore, we are considering a wider class of algebras, and polynomially convex compact subsets of $\mathbb{C}^n$ are examples of auto-spectral sets. The main benefit of the concept of holomorphic generation, however, lies in the following advantage of $\mathcal{O}(\mathrm{Sp}(a))$ over the algebra of polynomials. Even if $\mathcal{O}(\mathrm{Sp}(a))$ is not a Banach algebra, it is a complete locally convex algebra with open unit group and continuous inversion. Locally convex algebras with these properties are called complete continuous inverse algebras. Large parts of the theory of Banach algebras can be generalized to these algebras, and in fact they form a more natural class than Banach algebras for many questions, including those considered here. Continuous inverse algebras were introduced by Waelbroeck [45]. They play a role in non-commutative geometry, in particular in K-theory [8, 10, 12, 35], and in the theory of pseudo-differential operators [21]. Currently, they are attracting attention as the natural framework for Lie groups and algebras of infinite dimension [20]. They appear as coordinate algebras in root-graded locally convex Lie algebras [33]. Their role in the theory of Banach algebras is related to the fact that every complex commutative Banach algebra A is "sandwiched", for every choice of an n-tuple $a \in A^n$, between an algebra of holomorphic germs and an algebra of continuous functions by the functional calculus homomorphism $\theta_a\colon \mathcal{O}(\mathrm{Sp}(a)) \to A$ and the Gelfand homomorphism $\gamma_A\colon A \to C(\Gamma_A)$. If $a \in A^n$ holomorphically generates A then the Gelfand spectrum Γ_A is naturally homeomorphic to $\mathrm{Sp}(a) \subseteq \mathbb{C}^n$. Under this homeomorphism, the composition $\gamma_A \circ \theta_a\colon \mathcal{O}(\mathrm{Sp}(a)) \to A \to C(\Gamma_A)$ corresponds to the restriction map $\mathcal{O}(\mathrm{Sp}(a)) \to C(\mathrm{Sp}(a))$.

This observation could be applied in the theory of central extensions of infinite-dimensional Lie groups. Every complete commutative continuous inverse algebra A over $\mathbb{C}$ gives rise to a universal differential module $d\colon A \to \Omega^1(A)$ and a natural universal period homomorphism $\mathrm{per}\colon A^\times \to \mathrm{HC}_1(A)$, $a \mapsto [a^{-1}da]$, where $\mathrm{HC}_1(A) := \Omega^1(A)/\overline{\mathrm{im}(d)}$ is the first cyclic homology space of A. Note that the period homomorphism factors through $\pi_0(A) \cong A^\times/\exp(A)$, which is naturally isomorphic to the first Čech cohomology group of Γ_A because the analogue of the Arens–Royden Theorem can be proved for continuous inverse algebras [4]. If im(per) is discrete then the identity component of $\mathrm{SL}_m(A)$ has a universal central extension for every $m \in \mathbb{N}$. This condition is satisfied in all examples for which it has been checked, which is difficult because it depends on detailed understanding of $\mathrm{HC}_1(A)$. The examples include commutative C^*-algebras, for which the universal differential module vanishes (Maier [29]), the algebra of smooth functions on a compact manifold (Maier and Neeb [30]), and the algebra of compactly supported smooth functions on a non-compact manifold (Neeb [32]). In the light of the present paper, it would be interesting to decide whether the image of the universal period homomorphism of $\mathcal{O}(K)$ is discrete for a compact subset $K \subseteq \mathbb{C}^n$, at least if K satisfies one of the additional conditions studied here. As a first step, Neeb and Wagemann [34] have recently proved that the differential module of germs of holomorphic 1-forms in K is universal for $\mathcal{O}(K)$.

1 The algebras $\mathcal{O}(K)$ and $\mathcal{A}(K)$

A *continuous inverse algebra* is a locally convex unital algebra A over $\mathbb{C}$ such that the group $A^\times$ of invertible elements is open in A and inversion is continuous. We will usually assume that A is commutative. Then the *Gelfand spectrum* of A is the set Γ_A of (unital) algebra homomorphisms from A onto $\mathbb{C}$, which are automatically continuous. Under the topology of pointwise convergence on A, the Gelfand spectrum is a compact Hausdorff space, and a Gelfand Theory can be developed as in the case of Banach algebras [6].

We associate several algebras with each compact subset K of a second countable complex analytic manifold X. (We will always tacitly assume that all connected components of a manifold have the same dimension.) The algebra $\mathcal{O}(K)$ is the algebra of germs in K of holomorphic functions defined in open neighbourhoods of K in X. We topologize $\mathcal{O}(K)$ as the locally convex direct limit of the Fréchet algebras $\mathcal{O}(U)$ of holomorphic functions in U with the compact-open topology (or, equivalently, of the Banach algebras $\mathcal{O}^\infty(U)$ of bounded holomorphic functions with the supremum norm), where U varies over the open neighbourhoods of K in X. In this topology, $\mathcal{O}(K)$ is a complete continuous inverse Hausdorff algebra. Indeed, we may choose a metric d on X compatible with the topology and consider $\mathcal{O}(K)$ as the locally convex direct limit of the Banach algebras $\mathcal{O}^\infty(U_n)$, where U_n is the union of those connected components of $\left\{x \in X;\ d(x,K) < \frac{1}{n}\right\}$ which meet K. In this directed system, the connecting restriction maps are injective by the Identity Theorem. According to Dierolf and Wengenroth [13], a locally convex direct limit of a sequence of normed algebras with injective connecting maps is a locally m-convex algebra. In particular, inversion in $\mathcal{O}(K)$ is continuous on its domain (Michael [31, 2.8]). Moreover, the Arzela–Ascoli Theorem (see, for instance, Dugundji [15, XII.6.4]) entails that almost all connecting maps in the above directed system are compact. A locally convex direct limit of a sequence of Banach spaces with compact injective connecting maps is called a Silva space, and these spaces are complete Hausdorff spaces (see Floret [17, § 7]). The spectrum of an element $f \in \mathcal{O}(K)$ is the image of K under any representative of f, for which we just write $f(K)$. In particular, the spectral radius r in $\mathcal{O}(K)$ is given by $r(f) = \|f|_K\|_\infty$. Since the compositions of r with the limit maps are continuous, we find that r is a continuous semi-norm on $\mathcal{O}(K)$. We conclude that the unit group $\mathcal{O}(K)^\times$ is open in $\mathcal{O}(K)$. Further details of these arguments as well as a generalization to algebras of germs with infinite-dimensional domain and range can be found in [5].

Three more algebras associated to $K \subseteq X$ are constructed as follows. The restriction of an element of $\mathcal{O}(K)$ to K is a continuous complex-valued function on K. We obtain a Banach algebra $\mathcal{A}(K) \subseteq C(K)$ as the closure of the image of the restriction homomorphism $\mathcal{O}(K) \to C(K)$. We define a complete continuous inverse algebra $\mathcal{O}_X(K) \subseteq \mathcal{O}(K)$ as the closure of the image of the germ map $\mathcal{O}(X) \to \mathcal{O}(K)$, and a Banach algebra $\mathcal{A}_X(K) \subseteq \mathcal{A}(K)$ as the closure of the image of the restriction map $\mathcal{O}(X) \to \mathcal{A}(K)$.

Between the subalgebras of $C(K)$ obtained from these algebras by restriction, we have the inclusions

$$\mathcal{O}(X)|_K \subseteq \mathcal{O}_X(K)|_K \subseteq \left\{ \begin{array}{c} \mathcal{O}(K)|_K \\ \mathcal{A}_X(K) \end{array} \right\} \subseteq \mathcal{A}(K) \subseteq C(K).$$

All these algebras are different if $X = \mathbb{C}$ and K is the annulus

$$K = \left\{\zeta \in \mathbb{C};\ \tfrac{1}{2} \leq |\zeta| \leq 1\right\}.$$

Indeed, an element $f \in \mathcal{O}_X(K)|_K \setminus \mathcal{O}(X)|_K$ is defined by $f(\zeta) = (\zeta - 2)^{-1}$. All elements $g \in \mathcal{A}_X(K)$ satisfy $\oint_{|\zeta|=1} g(\zeta)\, d\zeta = 0$, so that we find a function

$g \in \mathcal{O}(K)|_K \setminus \mathcal{A}_X(K)$ by setting $g(\zeta) = \zeta^{-1}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos kt = \frac{(t-\pi)^2}{4} - \frac{\pi^2}{12}$ for all $t \in [0, 2\pi]$, the function

$$h\colon K \longrightarrow \mathbb{C},\ \zeta \longmapsto \sum_{k=1}^{\infty} \tfrac{1}{k^2}\, \zeta^k$$

is an element of $\mathcal{A}_X(K)$ which is not real-differentiable on the unit circle and hence does not belong to $\mathcal{O}(K)|_K$. The function $g+h$ belongs to $\mathcal{A}(K)$, but not to $\mathcal{O}(K)|_K \cup \mathcal{A}_X(K)$, and $\mathcal{A}(K) \neq C(K)$ because the elements of $\mathcal{A}(K)$ are holomorphic in the interior of K.

1.1 Remark. For a compact subset K of a complex analytic manifold, the algebra of continuous complex-valued functions on K which are holomorphic in the interior of K is another interesting closed subalgebra of $C(K)$. It is also sometimes denoted by $\mathcal{A}(K)$. This algebra and $\mathcal{A}(K)$ in our sense coincide for simple K, for instance if K is convex, but they are different in general. A (topologically complicated) compact subset of $\mathbb{C}$ for which this occurs is described by Gamelin [19, Section II.1]. We illustrate this phenomenon by three compact subsets of $\mathbb{C}^2$ which are increasingly complex and convincing. The first (and rather trivial) example is provided by $K := \{\zeta \in \mathbb{C}^2;\ |\zeta_1| < 1,\ \zeta_2 = 0\}$. Secondly, if $K \subseteq \mathbb{C}^2$ is the unit sphere then every element of $\mathcal{O}(K)$ extends to a holomorphic function on a neighbourhood of the unit ball (see, for instance, Range [36, II.1.6]). Therefore, every $f \in \mathcal{A}(K)$ satisfies $\oint_{|\zeta_1|=1} f(\zeta_1, 0)\, d\zeta_1 = 0$. While these two examples are "thin", the third is a compact subset $K \subseteq \mathbb{C}^2$ which is the closure of its interior. Define

$$\begin{aligned}
K_1 &:= \{\zeta \in \mathbb{C}^2;\ \|\zeta\|_2 \leq 3,\ |\zeta_1| \geq 1\}, \\
K_2 &:= \{\zeta \in \mathbb{C}^2;\ \|\zeta - (4,0)\|_2 \leq 1\}, \\
K_3 &:= \{\zeta \in \mathbb{C}^2;\ 5 \leq \|\zeta\|_2 \leq 6\}, \quad \text{and} \\
K &:= K_1 \cup K_2 \cup K_3.
\end{aligned}$$

Then $K_1 \cap K_2 = \{(3,0)\}$ and $K_2 \cap K_3 = \{(5,0)\}$, while K_1 and K_3 are disjoint. The interior K° is the disjoint union of K_1°, K_2°, and K_3°. Every element of $\mathcal{O}(K)$ is the germ of a holomorphic function defined in a connected open neighbourhood of K, and hence of a holomorphic function defined in an open neighbourhood of the compact ball with centre 0 and radius 6 by the extension phenomenon quoted above. Therefore, every $f \in \mathcal{A}(K)$ satisfies $\oint_{|\zeta_1|=2} f(\zeta_1, 0)\, d\zeta_1 = 0$. Thus an element of $\{f \in C(K);\ f|_{K^\circ} \in \mathcal{O}(K^\circ)\}$ which does not belong to $\mathcal{A}(K)$ is defined by $\zeta \mapsto \frac{1}{\zeta_1}$ on K_1 and $\zeta \mapsto \frac{1}{3}$ on $K_2 \cup K_3$. A slightly more complicated example for which the interior of K is even a Stein domain is described by Range [36, VII.2.2].

Conditions on compact subsets $K \subseteq \mathbb{C}^n$ under which $\mathcal{A}(K)$ equals the algebra of continuous complex-valued functions on K which are holomorphic in the interior of K have been studied extensively; see Gamelin [19, Section VIII.8] for $n = 1$ and Range [36, VII.2.1] for $n > 1$.

For the following lemma, recall the notion of the *joint spectrum* of an n-tuple $a = (a_1, \dots, a_n)$ in a commutative continuous inverse algebra A over $\mathbb{C}$. This is the compact subset of $\mathbb{C}^n$ defined as

$$\mathrm{Sp}_A(a_1, \dots, a_n) := \left\{ \big(\chi(a_1), \dots, \chi(a_n)\big);\ \chi \in \Gamma_A \right\}.$$

As in the case of Banach algebras, the joint spectrum of $a \in A^n$ is the set of $\lambda \in \mathbb{C}^n$ such that the ideal of A generated by $\lambda_1 - a_1, \dots, \lambda_n - a_n$ is proper.

The compact sets in which we are most interested are joint spectra of n-tuples in continuous inverse algebras, so they are subsets of $\mathbb{C}^n$. In Section 4, however, we will also be led to consider more general ambient manifolds, namely, envelopes of holomorphy of open subsets of $\mathbb{C}^n$. The natural class of manifolds for our theory is the class of *Stein manifolds*. These can be defined as those complex analytic manifolds X which admit a biholomorphic embedding $\iota\colon X \hookrightarrow \mathbb{C}^n$ onto a closed submanifold of some space $\mathbb{C}^n$. Their intrinsic characterization will be recalled in Section 4. Their most important property for us is the existence of an open neighbourhood $U \subseteq \mathbb{C}^n$ of $\iota(X)$ and of a holomorphic map $\rho\colon U \to X$ which is a retraction for ι, i.e. which satisfies $\rho \circ \iota = \mathrm{id}_X$. This fact is due to Docquier and Grauert [14]. A proof can also be found in the monograph by Gunning and Rossi [23, VIII.C.8]. In fact, a Stein manifold is a holomorphic neighbourhood retract in any complex manifold in which it is embedded as a closed submanifold, see Siu [42, Corollary 1].

Here, Stein manifolds give rise to another pair of continuous inverse algebras, which will be used in Section 5. Let K be a compact subset of a Stein manifold. The closure in $\mathcal{O}(K)$ of the algebra of germs of holomorphic functions defined in Stein open neighbourhoods of K will be called $\mathcal{O}_{\mathrm{St}}(K)$. The closure of its image under the restriction map $\mathcal{O}(K) \to \mathcal{A}(K)$ will be called $\mathcal{A}_{\mathrm{St}}(K)$.

1.2 Lemma (Spectra of $\mathcal{O}(K)$ and of $\mathcal{A}(K)$). *Let X be a second countable complex analytic manifold, and let $K \subseteq X$ be a compact subset. Let $A \subseteq \mathcal{O}(K)$ be a closed unital subalgebra, and set $B := \overline{A|_K} \subseteq C(K)$, the closure of the image of A under the restriction homomorphism $\mathcal{O}(K) \to C(K)$. Then the spectral radii in A and in B are given by the supremum norm on K. The restriction map $f \mapsto f|_K\colon A \to B$ induces a homeomorphism from Γ_B onto Γ_A. In particular, if $f \in A^n$ then $\mathrm{Sp}_A(f) = \mathrm{Sp}_B(f|_K)$.*

In the important special case that $A = \mathcal{O}(K)$, this result is due to Harvey and Wells [24, 2.4]. It often allows us to switch between A and B. The algebra A is useful because it consists of germs of holomorphic functions. The algebra B is only defined in terms of A, but the description of its topology is more concrete, and it has the advantage of being a Banach algebra.

Proof. The spectrum of an element $f \in C(K)$ is $f(K)$. Similarly, if f is a holomorphic function defined in an open neighbourhood of K in X then the germ $\tilde{f}$ of f in K satisfies $\mathrm{Sp}_{\mathcal{O}(K)}(\tilde{f}) = f(K)$. Hence the spectral radii in $C(K)$ and in $\mathcal{O}(K)$ are the supremum norm on K. In a Banach algebra, the spectral radius of an element of a closed subalgebra with respect to that subalgebra equals the spectral radius with respect to the whole algebra (see, for instance, Rudin [40, 10.18]). In a continuous inverse algebra, the corresponding fact can be proved in a similar way [3, 2.3]. This proves the assertion about spectral radii in A and in B.

The map $\rho^*\colon \Gamma_B \to \Gamma_A$ induced by the restriction map $\rho\colon A \to B$ is continuous, and it is injective because $\rho(A)$ is dense in B. Since characters of continuous inverse algebras are majorized by the spectral radius, every element $\chi \in \Gamma_A$ factors through ρ and induces a character of B. This proves that ρ^* is surjective, and it is a homeomorphism because the spectra are compact.

Finally, we choose $f \in A^n$ and calculate

$$\mathrm{Sp}_A(f) = \left\{\chi^{\times n}(f);\ \chi \in \Gamma_A\right\} = \left\{\chi^{\times n}(f|_K);\ \chi \in \Gamma_B\right\} = \mathrm{Sp}_B(f|_K).$$

(Here $\chi^{\times n}(f) := (\chi(f_1), \ldots, \chi(f_n))$.) □

1.3 Remark. In the situation of Lemma 1.2, note that $B \subseteq C(K)$ is a realization of the completed quotient of $A \subseteq \mathcal{O}(K)$ with respect to the spectral radius seminorm, or, equivalently, a realization of the closure of the image of A under the Gelfand homomorphism into $C(\Gamma_A)$. In particular, the set $\{\mathrm{ev}_\zeta;\ \zeta \in \partial K\} \subseteq \Gamma_B$ of evaluations in boundary points of K contains the Šilov boundary of B.

Under mild completeness assumptions, a commutative continuous inverse algebra A over $\mathbb{C}$ admits an n-variable holomorphic functional calculus, of which we recall the main statements. The appropriate completeness hypothesis is *Mackey-completeness*, which means that the Riemann integral $\int_0^1 \gamma(t)\, dt$ exists for every smooth curve $\gamma\colon [0,1] \to A$. This is equivalent to the convergence of all members of a certain class of Cauchy sequences. A convenient and comprehensive reference for this concept is Section 2 of the monograph by Kriegl and Michor [28].

In the situation of the preceding paragraph, choose an n-tuple $a \in A^n$. The holomorphic functional calculus provides a continuous homomorphism of unital algebras $f \mapsto f[a]\colon \mathcal{O}(\mathrm{Sp}(a)) \to A$ which maps the germ of the j-th coordinate function $\zeta \mapsto \zeta_j\colon \mathbb{C}^n \to \mathbb{C}$ to a_j. For Banach algebras, the construction is due to Šilov [41] and Arens and Calderón [1]. Bourbaki [11, I § 4] presents an alternative approach. For complete continuous inverse algebras, the holomorphic functional calculus is due to Waelbroeck, who developed an early variant in [44] and sketched the modern version in [46] and in [47]. A detailed account can be found in [6].

A property of the holomorphic functional calculus which is stressed by Waelbroeck [47] and which we will use several times is its naturality with respect to homomorphisms $\varphi\colon A \to B$ between Mackey-complete commutative continuous inverse algebras over $\mathbb{C}$. For an n-tuple $a \in A^n$ and a holomorphic germ $f \in \mathcal{O}(\mathrm{Sp}_A(a))$, this means that $\varphi(f[a]) = f[\varphi^{\times n}(a)]$. (Note that the right-hand side is defined because $\mathrm{Sp}_B\big(\varphi^{\times n}(a)\big) \subseteq \mathrm{Sp}_A(a)$.) Since the holomorphic functional calculus in the algebra $\mathbb{C}$ is given by application of the function, a special case of naturality is the observation that $\chi(f[a]) = f\big(\chi^{\times n}(a)\big)$ holds for each $\chi \in \Gamma_A$.

1.4 Lemma (Functional calculus in $\mathcal{O}(K)$ and in $\mathcal{A}(K)$). *Assume that A is either a closed subalgebra of $C(K)$ for some compact Hausdorff space K, or a closed subalgebra of $\mathcal{O}(K)$ for some compact subset K of a second countable complex analytic manifold X. Let $a \in A^n$, let $U \subseteq \mathbb{C}^n$ be an open neighbourhood of $\mathrm{Sp}_A(a)$, and let $f \in \mathcal{O}(U)$. Then $f[a] = f \circ a$.*

Proof. In both cases, $a|_K$ is a continuous map from K into $\mathbb{C}^n$. For each $x \in K$, the evaluation homomorphism $\mathrm{ev}_x\colon A \to \mathbb{C}$, $g \mapsto g(x)$ belongs to Γ_A. Hence $a(K) = \{\mathrm{ev}_x^{\times n}(a);\ x \in K\} \subseteq \mathrm{Sp}_A(a)$, so that we can form $f \circ a$.

Assume that A is a closed subalgebra of $C(K)$ for some compact Hausdorff space K. For any $x \in K$, naturality of the holomorphic functional calculus yields

$$f[a](x) = \mathrm{ev}_x(f[a]) = f\big(\mathrm{ev}_x^{\times n}(a)\big) = (f \circ a)(x).$$

(In particular, the composition $f \circ a$ is an element of A.)

Assume that A is a closed subalgebra of $\mathcal{O}(K)$ for some compact subset K of a complex analytic manifold X. Naturality of the holomorphic functional calculus with respect to the inclusion map $\iota\colon A \hookrightarrow \mathcal{O}(K)$ means that $\iota(f[a]) = f[\iota^{\times n}(a)]$, and it implies that we may assume that $A = \mathcal{O}(K)$. First consider the case that $K = \{\zeta\}$, a single point. Lemma 1.2 implies that $\Gamma_{\mathcal{O}(K)} = \{\mathrm{ev}_\zeta\}$. Hence $\mathrm{Sp}_{\mathcal{O}(K)}(a) = \{a(\zeta)\}$, and we may assume that U is an open polydisc in $\mathbb{C}^n$. If f is a coordinate function, the result is a fundamental property of the holomorphic functional calculus. Since every element of $\mathcal{O}(U)$ has a power series expansion around the centre of U which converges on U, the coordinate functions generate a dense subalgebra of $\mathcal{O}(U)$, and the result extends to all $f \in \mathcal{O}(U)$. In the case that K consists of more than one point, define a continuous homomorphism $\varphi_\zeta\colon \mathcal{O}(K) \to \mathcal{O}(\{\zeta\})$ for each $\zeta \in K$ by assigning to $f \in \mathcal{O}(K)$ its germ in ζ. By naturality,

$$\varphi_\zeta(f[a]) = f\big[\varphi_\zeta^{\times n}(a)\big] = f \circ \big(\varphi_\zeta^{\times n}(a)\big) = \varphi_\zeta(f \circ a).$$

Since $f[a]$ and $f \circ a$ have the same germ at every $\zeta \in K$, we conclude that they are equal. (This argument was adapted from Waelbroeck [47, 5.2].) $\square$

2 Auto-spectral compacta

This section introduces the important concept of an auto-spectral compact subset of a Stein manifold. The properties of these manifolds which are most relevant for our purposes were briefly reviewed before Lemma 1.2.

2.1 Lemma. *Let X be a Stein manifold, let $K \subseteq X$ be compact, and let A be a closed unital subalgebra of $\mathcal{O}(K)$ with $\mathcal{O}_X(K) \subseteq A$. Choose a closed (biholomorphic) embedding $\iota\colon X \hookrightarrow \mathbb{C}^n$, and let $\tilde{\iota} \in A^n$ be the germ of ι. Then $\mathrm{Sp}_A(\tilde{\iota}) = \iota(K)$ if and only if every character of A is evaluation in a point of K.*

Proof. If Γ_A consists of evaluations in points of K then $\mathrm{Sp}_A(\tilde{\iota}) = \iota(K)$. Conversely, assume that this equation holds, and choose $\chi \in \Gamma_A$. Choose an open neighbourhood $U \subseteq \mathbb{C}^n$ of $\iota(X)$ and a holomorphic map $\rho\colon U \to X$ such that $\rho \circ \iota = \mathrm{id}_X$. Let $\zeta \in K$ be defined by $\iota(\zeta) = \chi^{\times n}(\tilde{\iota})$. Let $f \in A$. Then $\tilde{f} := f \circ \rho \in \mathcal{O}(\iota(K)) = \mathcal{O}(\mathrm{Sp}_A(\tilde{\iota}))$, and $\tilde{f}[\tilde{\iota}] = \tilde{f} \circ \iota = f$ by Lemma 1.4. Hence

$$\chi(f) = \chi\big(\tilde{f}[\tilde{\iota}]\big) = \tilde{f}\big(\chi^{\times n}(\tilde{\iota})\big) = \tilde{f}(\iota(\zeta)) = f(\zeta)$$

by naturality of the holomorphic functional calculus. We conclude that χ is evaluation in $\zeta \in K$. $\square$

Note that we do not need the theory of Stein manifolds if we content ourselves with the case that $X = \mathbb{C}^n$ and $\iota = \mathrm{id}_{\mathbb{C}^n}$. The latter remark applies to large parts of the present paper. Also note that in this case, the hypothesis $\mathcal{O}_X(K) \subseteq A$ just means that A contains the germs of the coordinate functions.

2.2 Definition. Let X be a Stein manifold, and choose a closed embedding $\iota\colon X \hookrightarrow \mathbb{C}^n$. A compact subset $K \subseteq X$ is called *auto-spectral* if the following conditions are satisfied, all of which are equivalent by Lemmas 1.2 and 2.1.

(i) Every character of $\mathcal{O}(K)$ is evaluation in a point of K.

(ii) $\mathrm{Sp}_{\mathcal{O}(K)}(\tilde{\iota}) = \iota(K)$.

(iii) Every character of $\mathcal{A}(K)$ is evaluation in a point of K.

(iv) $\mathrm{Sp}_{\mathcal{A}(K)}(\iota|_K) = \iota(K)$.

Auto-spectral compact sets seem to have been introduced by Wells [48] under the name of "holomorphically convex compact sets". Some of their basic properties had already been obtained by Rossi [37]. In view of a result due to Harvey and Wells [24, 3.4], auto-spectral sets are what Grauert and Remmert [22, IV.1.1] call compact Stein subsets.

It is easy to describe the effect of replacing $\mathcal{O}(K)$ by a closed subalgebra in the preceding definition.

2.3 Lemma. *Let X be a Stein manifold, let $K \subseteq X$ be compact, and let $A \subseteq \mathcal{O}(K)$ be a closed unital subalgebra with $\mathcal{O}_X(K) \subseteq A$. Then the following conditions are equivalent:*

(i) *Every character of A is evaluation in a point of K.*

(ii) *K is auto-spectral, and $A = \mathcal{O}(K)$.*

(iii) *K is auto-spectral, and $\overline{A|_K} = \mathcal{A}(K)$.*

Proof. Assume that condition (i) holds. Choose a closed embedding $\iota\colon X \hookrightarrow \mathbb{C}^n$, an open neighbourhood $U \subseteq \mathbb{C}^n$ of $\iota(X)$, and a holomorphic retraction $\rho\colon U \to X$ for ι. Then K is auto-spectral because

$$\iota(K) \subseteq \mathrm{Sp}_{\mathcal{O}(K)}(\tilde{\iota}) \subseteq \mathrm{Sp}_A(\tilde{\iota}) = \iota(K).$$

For any $f \in \mathcal{O}(K)$, we can form the element $(f \circ \rho)[\tilde{\iota}] \in A$, and Lemma 1.4 shows that this element is equal to f. Thus we have proved condition (ii), which in turn trivially implies condition (iii). Finally, condition (iii) implies (i) by Lemma 1.2. $\square$

Let X be a Stein manifold. For $K \subseteq U \subseteq X$ with K compact and U open, Corollary 4.5 will show that K is holomorphically convex in U if and only if K is auto-spectral and $\mathcal{O}_U(K) = \mathcal{O}(K)$. In Proposition 5.4, we will see that a compact subset $K \subseteq X$ is a Stein compactum if and only if it is auto-spectral and the restrictions of functions defined in Stein open neighbourhoods of K form a dense subset of $\mathcal{A}(K)$.

Auto-spectrality is a convexity condition in the sense of the following proposition and its corollary.

2.4 Proposition. *The intersection of any family of auto-spectral subsets of a Stein manifold is again auto-spectral.*

Proof. Let $(K_j)_{j\in J}$ be a family of auto-spectral subsets of a Stein manifold X, and set $K := \bigcap_{j\in J} K_j$. Choose a closed embedding $\iota\colon X \hookrightarrow \mathbb{C}^n$. For each $j \in J$, consider the natural map from $\mathcal{O}(K_j)$ into $\mathcal{O}(K)$. This yields the middle inclusion in

$$\iota(K) \subseteq \mathrm{Sp}_{\mathcal{O}(K)}(\tilde{\iota}) \subseteq \mathrm{Sp}_{\mathcal{O}(K_j)}(\tilde{\iota}) = \iota(K_j).$$

We conclude that $\mathrm{Sp}_{\mathcal{O}(K)}(\tilde{\iota}) = \iota(K)$. $\square$

2.5 Corollary. *Every compact subset K of a Stein manifold X is contained in a smallest auto-spectral subset of X, the* auto-spectral hull of K in X.

Proof. This follows from the preceding proposition, provided that K is contained in an auto-spectral subset of X. Now Corollary 4.5 will yield that the holomorphically convex hull of K in X is auto-spectral. (The argument is easier for the case that $X = \mathbb{C}^n$. Indeed, Remark 3.3 and Proposition 3.5 below imply that all convex compact subsets of $\mathbb{C}^n$ are auto-spectral.) $\square$

Waelbroeck [47, 5.2] defined auto-spectral subsets of $\mathbb{C}^n$ under the name of "analytic compact sets", and he essentially proved the following proposition about them. As was recalled after Remark 1.3, Mackey-completeness is the weak completeness assumption used in the construction of the holomorphic functional calculus.

2.6 Proposition. *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$, let $a \in A^n$, and let $K \subseteq \mathbb{C}^n$ be an auto-spectral compact set. Then a continuous homomorphism $\varphi\colon \mathcal{O}(K) \to A$ with $\varphi^{\times n}(\widetilde{\mathrm{id}}_{\mathbb{C}^n}) = a$ exists if and only if $\mathrm{Sp}(a) \subseteq K$. If this is the case then φ is uniquely determined by the equation $\varphi(f) = f[a]$ for all $f \in \mathcal{O}(K)$.*

Proof. Waelbroeck's proof for Banach algebras [47, 5.2] essentially applies to the present situation. If such a homomorphism φ exists then $\mathrm{Sp}_A(a) = \mathrm{Sp}_A(\varphi^{\times n}(\widetilde{\mathrm{id}}_{\mathbb{C}^n})) \subseteq \mathrm{Sp}_{\mathcal{O}(K)}(\widetilde{\mathrm{id}}_{\mathbb{C}^n}) = K$. Moreover, all $f \in \mathcal{O}(K)$ satisfy

$$\varphi(f) = \varphi\Big(f\left[\widetilde{\mathrm{id}}_{\mathbb{C}^n}\right]\Big) = f\left[\varphi^{\times n}\Big(\widetilde{\mathrm{id}}_{\mathbb{C}^n}\Big)\right] = f[a]$$

by Lemma 1.4 and naturality of the holomorphic functional calculus. Conversely, if $\mathrm{Sp}_A(a) \subseteq K$ then $\varphi\colon \underset{\sim}{f} \mapsto f[a]$ is a continuous homomorphism from $\mathcal{O}(K)$ into A which maps $\widetilde{\mathrm{id}}_{\mathbb{C}^n}$ to a. $\square$

2.7 Corollary (Uniqueness of the holomorphic functional calculus). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$, let $a \in A^n$, and let $K \subseteq \mathbb{C}^n$ be the auto-spectral hull of $\mathrm{Sp}(a)$ in $\mathbb{C}^n$. Then $f \mapsto \underset{\sim}{f}[a]$ is the unique continuous homomorphism from $\mathcal{O}(K)$ into A which maps $\mathrm{id}_{\mathbb{C}^n}$ to a.* $\square$

2.8 Remark. The corollary is the uniqueness statement for the restriction of the functional calculus to functions which are holomorphic on a neighbourhood of the auto-spectral hull of the joint spectrum. By contrast, the full holomorphic calculus for holomorphic functions defined in a neighbourhood of the joint spectrum is unique only under certain additional conditions. (One such condition can be found in Bourbaki [11, I § 4], another condition is due to Zame [50].)

In his original definition of the holomorphic functional calculus [44], Waelbroeck only developed it for functions which are holomorphic on a neighbourhood of the rationally convex hull (see the following section) of what is now called the joint spectrum of an n-tuple. Since rationally convex compact subsets of $\mathbb{C}^n$ are auto-spectral, Waelbroeck thus achieved uniqueness of his functional calculus.

2.9 Proposition (Gelfand spectrum and connected components). *Let X be a second countable complex analytic manifold, and let $K \subseteq X$ be a compact subset. For each closed subset $L \subseteq K$, let $\rho_L \colon \mathcal{O}(K) \to \mathcal{O}(L)$ be the restriction homomorphism. Then*

$$\Gamma_{\mathcal{O}(K)} = \bigcup_{L \in \operatorname{comp}(K)} \left\{ \gamma \circ \rho_L ;\ \gamma \in \Gamma_{\mathcal{O}(L)} \right\},$$

where $\operatorname{comp}(K)$ denotes the set of connected components of K, and the union is disjoint.

Proof. The key tool for the proof is the set of idempotent elements of $\mathcal{O}(K)$. At each point of K, the germ of an idempotent is either 1 or 0. Since an idempotent element induces a continuous function on K, its support is an open and closed subset of K. Conversely, for each open and closed subset $L \subseteq K$, there is a unique idempotent $e_L \in \mathcal{O}(K)$ with support L, which is constructed in the following way. Choose disjoint open neighbourhoods U of L and V of $K \setminus L$ in the ambient manifold X, and let $e_L \in \mathcal{O}(K)$ be the germ of the function which is 1 on U and 0 on V. Note that e_L only depends on L and not on the choice of the neighbourhoods U and V.

We first prove that the union in the proposition is disjoint. Let $L_1, L_2 \subseteq K$ be different connected components, and choose $\gamma_j \in \Gamma_{\mathcal{O}(L_j)}$ for $j \in \{1, 2\}$. In a compact Hausdorff space, the connected component of a point p is the intersection of the open and closed neighbourhoods of p (see Engelking [16, 6.1.23]). By compactness, there is an open and closed subset $L \subseteq K$ such that $L_1 \subseteq L$ and $L_2 \cap L = \emptyset$. Now $\gamma_1(\rho_{L_1}(e_L)) = 1$ and $\gamma_2(\rho_{L_2}(e_L)) = 0$. We conclude that $\gamma_1 \circ \rho_{L_1} \neq \gamma_2 \circ \rho_{L_2}$.

Let $\gamma \in \Gamma_{\mathcal{O}(K)}$. We have to find a connected component $L \subseteq K$ and a character $\gamma' \in \Gamma_{\mathcal{O}(L)}$ such that $\gamma = \gamma' \circ \rho_L$. Define

$$\mathcal{S} := \left\{ L' \subseteq K ;\ L' \text{ is open and closed in } K, \text{ and } \gamma(e_{L'}) = 1 \right\}.$$

If $L_1, L_2 \in \mathcal{S}$ then $\gamma(e_{L_1 \cap L_2}) = \gamma(e_{L_1} \cdot e_{L_2}) = 1$, so that $L_1 \cap L_2 \in \mathcal{S}$. Hence $\mathcal{S}$ is closed under finite intersections. Since $\emptyset \notin \mathcal{S}$, compactness of K implies that the intersection $L := \bigcap \mathcal{S}$ is not empty. If $L' \subseteq K$ is open and closed then either $L' \in \mathcal{S}$ or $K \setminus L' \in \mathcal{S}$. This entails that L is connected. The restriction homomorphism ρ_L maps $\mathcal{O}(K)$ onto $\mathcal{O}(L)$ because every neighbourhood of L contains an open and closed subset of K.

We claim that the kernel of ρ_L is contained in the kernel of γ. Indeed, let $f \in \mathcal{O}(U)$ for an open neighbourhood $U \subseteq X$ of K such that the germ $\tilde{f}$ of f in K satisfies $\rho_L(\tilde{f}) = 0$. Then f vanishes on a neighbourhood V of L. There is an open and closed subset $L' \subseteq K$ such that $L \subseteq L' \subseteq V$. Since $\gamma(e_{L'}) = 1$ and $\tilde{f} = (1 - e_{L'})\tilde{f}$, we find that $\gamma(\tilde{f}) = 0$. This proves the claim.

We conclude that there is an algebra homomorphism $\gamma' \colon \mathcal{O}(L) \to \mathbb{C}$ which satisfies $\gamma = \gamma' \circ \rho_L$. □

2.10 Corollary. *A compact subset of a Stein manifold is auto-spectral if and only if each of its connected components is auto-spectral.*

Proof. If a compact subset K of a Stein manifold X has only auto-spectral connected components then Proposition 2.9 shows that K is auto-spectral. Conversely, let $L \subseteq K$ be a connected component which is not auto-spectral. Choose a closed embedding $\iota \colon X \hookrightarrow \mathbb{C}^n$. Then $\iota(L)$ is a proper subset of $L' := \mathrm{Sp}_{\mathcal{A}(L)}(\iota|_L)$. Since $\mathcal{A}(L)$ does not contain any non-trivial idempotent, the Šilov Idempotent Theorem (see Bonsall and Duncan [9, 21.5]) implies that $\Gamma_{\mathcal{A}(L)}$ and hence L' are connected. Hence L' is not contained in $\iota(K)$. Since $\mathrm{Sp}_{\mathcal{A}(K)}(\iota|_K)$ contains L', it properly contains $\iota(K)$. We conclude that K is not auto-spectral. □

Zame [49, 3.4] gives a completely different proof of this corollary in terms of the cohomology of coherent analytic sheaves.

3 Rational convexity

We introduce the concept of a rationally convex compact subset of $\mathbb{C}^n$. It will be easy to prove that such a set is auto-spectral.

3.1 Definition. Let $\mathcal{P}(\mathbb{C}^n)$ denote the algebra of complex-valued polynomial functions on $\mathbb{C}^n$. Define the *rationally convex hull* of a compact subset $K \subseteq \mathbb{C}^n$ as

$$\widehat{K}_{\mathcal{R}(\mathbb{C}^n)} := \bigcap_{p \in \mathcal{P}(\mathbb{C}^n)} p^{-1}\big(p(K)\big).$$

A compact subset $K \subseteq \mathbb{C}^n$ is called *rationally convex* if $K = \widehat{K}_{\mathcal{R}(\mathbb{C}^n)}$.

Note that $K \mapsto \widehat{K}_{\mathcal{R}(\mathbb{C}^n)}$ is a hull operation in the sense that it preserves inclusion, that $K \subset \widehat{K}_{\mathcal{R}(\mathbb{C}^n)}$, and that $\widehat{K}_{\mathcal{R}(\mathbb{C}^n)}$ is its own rationally convex hull.

3.2 Example. Every compact subset of $\mathbb{C}$ is rationally convex (use the identity function).

3.3 Remark. Using linear polynomials, we find that an affine complex hyperplane which does not meet K does not meet $\widehat{K}_{\mathcal{R}(\mathbb{C}^n)}$. Since every affine real hyperplane is the union of affine complex hyperplanes, this entails that $\widehat{K}_{\mathcal{R}(\mathbb{C}^n)}$ is contained in the convex hull of K. In particular, $\widehat{K}_{\mathcal{R}(\mathbb{C}^n)}$ is compact, and every convex compact subset of $\mathbb{C}^n$ is rationally convex.

3.4 Example. For $n \geq 2$, the rationally convex hull of the unit sphere $S \subseteq \mathbb{C}^n$ is the unit ball $B \subseteq \mathbb{C}^n$. To prove this, recall that every element of $\mathcal{O}(S)$ extends to an element of $\mathcal{O}(B)$ (see, for instance, Range [36, II.1.6]). Now suppose that $\zeta \in B \setminus \widehat{S}_{\mathcal{R}(\mathbb{C}^n)}$. Then there is a polynomial $p \in \mathcal{P}(\mathbb{C}^n)$ such that $p(\zeta) \notin p(S)$, and we may assume that $p(\zeta) = 0$. The germ of $\frac{1}{p}$ in S is an element of $\mathcal{O}(S)$ which does not extend to an element of $\mathcal{O}(B)$, which is a contradiction.

Note that the same extension phenomenon entails that S is not autospectral.

3.5 Proposition. *Every rationally convex compact subset of $\mathbb{C}^n$ is autospectral.*

Proof. Let $K \subseteq \mathbb{C}^n$ be a rationally convex compact subset. Choose $\lambda \in \mathbb{C}^n \setminus K$. We have to show that $\lambda \notin \operatorname{Sp}_{\mathcal{O}(K)}\big(\widetilde{\mathrm{id}}_{\mathbb{C}^n}\big)$. There is a polynomial $p \in \mathcal{P}(\mathbb{C}^n)$ such that $p(\lambda) = 0 \notin p(K)$. Expanding p at λ, we find a representation

$$p(\zeta) = \sum_{k \in \mathbb{N}_0{}^n} c_k (\zeta_1 - \lambda_1)^{k_1} \cdot \ldots \cdot (\zeta_n - \lambda_n)^{k_n} \qquad (\zeta \in \mathbb{C}^n)$$

with coefficients $c_k \in \mathbb{C}^n$, where $c_0 = 0$. We rewrite this as

$$p(\zeta) = \sum_{j=1}^{n} (\zeta_j - \lambda_j) q_j(\zeta) \qquad (\zeta \in \mathbb{C}^n)$$

with suitable polynomials $q_j \in \mathcal{P}(\mathbb{C}^n)$. Set $U := \mathbb{C}^n \setminus p^{-1}(\{0\})$, and define $f_1, \ldots, f_n \in \mathcal{O}(U)$ by $f_j := -\frac{q_j}{p}$. Then all $\zeta \in U$ satisfy

$$1 = \sum_{j=1}^{n} (\lambda_j - \zeta_j) f_j(\zeta).$$

This proves that the ideal of $\mathcal{O}(K)$ generated by the elements $\lambda_j - \zeta_j$ is all of $\mathcal{O}(K)$, so that $\lambda \notin \operatorname{Sp}_{\mathcal{O}(K)}\big(\widetilde{\mathrm{id}}_{\mathbb{C}^n}\big)$. $\square$

4 Holomorphic convexity

This section uses the envelope of holomorphy of an open subset U of a Stein manifold in order to study the Gelfand spectrum of $\mathcal{A}_U(K)$, the closure in $C(K)$ of $\{f|_K;\ f \in \mathcal{O}(U)\}$, for a compact subset $K \subseteq U$. In particular, we show that K is auto-spectral if it is holomorphically convex in U, which is a fundamental concept in complex analysis. More precisely, the holomorphically convex compact subsets of U are characterized among the auto-spectral compact subsets of U by an approximation property.

4.1 Lemma (Evaluation homomorphisms). *Let X be a second countable complex analytic manifold, and let $K \subseteq X$ be a compact subset. Let $\rho\colon \mathcal{O}(X) \to \mathcal{A}_X(K)$ be the restriction map, and choose a point $\zeta \in X$. Then the evaluation homomorphism $\mathrm{ev}_\zeta\colon \mathcal{O}(X) \to \mathbb{C}$, $f \mapsto f(\zeta)$ has the form $\chi \circ \rho$ for some character $\chi \in \Gamma_{\mathcal{A}_X(K)}$ if and only if $|f(\zeta)| \leq \|f|_K\|_\infty$ holds for all $f \in \mathcal{O}(X)$.*

The set of all these points,

$$\widehat{K}_{\mathcal{O}(X)} := \left\{\zeta \in X;\ \forall\, f \in \mathcal{O}(X)\colon |f(\zeta)| \leq \left\|f|_K\right\|_\infty\right\},$$

is called the holomorphically convex hull *of K in X. For each point $\zeta \in \widehat{K}_{\mathcal{O}(X)}$, there is a unique character $\widetilde{\mathrm{ev}}_\zeta \in \Gamma_{\mathcal{A}_X(K)}$ such that $\mathrm{ev}_\zeta = \widetilde{\mathrm{ev}}_\zeta \circ \rho$. Moreover, the map*

$$\zeta \longmapsto \widetilde{\mathrm{ev}}_\zeta\colon \widehat{K}_{\mathcal{O}(X)} \longrightarrow \Gamma_{\mathcal{A}_X(K)}$$

is continuous. For $\zeta \in K$ and $f \in \mathcal{A}_X(K)$, we have $\widetilde{\mathrm{ev}}_\zeta(f) = f(\zeta)$.

Proof. If $\mathrm{ev}_\zeta = \chi \circ \rho$ for some $\chi \in \Gamma_{\mathcal{A}_X(K)}$ then all $f \in \mathcal{O}(X)$ satisfy

$$|f(\zeta)| = |\,\mathrm{ev}_\zeta(f)| = \left|\chi\big(f|_K\big)\right| \leq \left\|f|_K\right\|_\infty.$$

Conversely, if $|f(\zeta)| \leq \|f|_K\|_\infty$ holds for all $f \in \mathcal{O}(X)$ then ev_ζ factors through ρ, and the induced complex homomorphism of $\mathrm{im}(\rho) \subseteq \mathcal{A}_X(K)$ is continuous and hence extends to a character of $\mathcal{A}_X(K)$. This character is uniquely determined by ev_ζ because $\mathrm{im}(\rho)$ is a dense subalgebra of $\mathcal{A}_X(K)$.

In order to prove that the map $\zeta \mapsto \widetilde{\mathrm{ev}}_\zeta\colon \widehat{K}_{\mathcal{O}(X)} \to \Gamma_{\mathcal{A}_X(K)}$ is continuous, we have to show that the map $\zeta \mapsto \widetilde{\mathrm{ev}}_\zeta(f)$ is continuous for every $f \in \mathcal{A}_X(K)$. It suffices to take f from the dense subalgebra $\mathrm{im}(\rho)$. But if $f \in \mathcal{O}(X)$ then $\widetilde{\mathrm{ev}}_\zeta(f|_K) = f(\zeta)$ depends continuously on ζ.

Choose $\zeta \in K$. Then $\widetilde{\mathrm{ev}}_\zeta(f) = f(\zeta)$ holds if $f \in \mathrm{im}(\rho)$. By continuity, this equation extends to all $f \in \mathcal{A}_X(K)$. □

The holomorphically convex hull $\widehat{K}_{\mathcal{O}(X)}$ is an important concept in complex analysis. Note that it is a closed subset of X. If $U \subseteq X$ is an open subset with $K \subseteq U$ then $\widehat{K}_{\mathcal{O}(U)} \subseteq \widehat{K}_{\mathcal{O}(X)}$. Moreover, if X is an open subset of $\mathbb{C}^n$ then $\widehat{K}_{\mathcal{O}(X)}$ is contained in the convex hull of K, as one sees by using the functions $\zeta \mapsto e^{\langle\zeta,\alpha\rangle}$, where $\alpha \in \mathbb{C}^n$.

4.2 Example. (a) Let $K \subseteq U \subseteq \mathbb{C}$ with K compact and U open. Then $\widehat{K}_{\mathcal{O}(U)}$ is the union of K with those bounded connected components of $\mathbb{C} \setminus K$ which are contained in U. Indeed, let K' be this union. The Maximum Modulus Theorem (see Rudin [39, 10.24]) implies that $K' \subseteq \widehat{K}_{\mathcal{O}(U)}$. Conversely, let $\zeta \in U \setminus K'$, choose disjoint open neighbourhoods V_1 of K' and V_2 of ζ in $\mathbb{C}$, and let $f \in \mathcal{O}(V_1 \cup V_2)$ be the characteristic function of V_2. Since $\mathbb{C} \setminus U$ meets every bounded connected component of $\mathbb{C} \setminus (K' \cup \{\zeta\})$, Runge's Theorem [39, 13.6] yields a complex rational function g with poles only in $\mathbb{C} \setminus U$ such that $|f(\eta) - g(\eta)| < \frac{1}{2}$ for every $\eta \in K' \cup \{\zeta\}$. Then $g|_U \in \mathcal{O}(U)$ satisfies $|g(\zeta)| > \frac{1}{2} > \|g|_{K'}\|_\infty \geq \|g|_K\|_\infty$, and we conclude that $\zeta \notin \widehat{K}_{\mathcal{O}(U)}$.

(b) In higher dimensions, holomorphically convex hulls need not be compact. For the classic example, consider the compact unit polydisc $D := \big\{\zeta \in \mathbb{C}^2;\ |\zeta_1|, |\zeta_2| \leq 1\big\}$ and let $K := \{\zeta \in D;\ \zeta_1 = 0 \text{ or } |\zeta_2| = 1\}$. Then every holomorphic function defined in an open neighbourhood of K extends to a holomorphic function on an open neighbourhood of D (see, for example, Range [36, II.1.1]). In particular, the set K is not auto-spectral.

For a connected open neighbourhood $U \subseteq \mathbb{C}^2$ of K, we claim that $\widehat{K}_{\mathcal{O}(U)} = U \cap D$. Indeed, the left-hand side is contained in the right-hand side because D is convex. The reverse inclusion follows from the Maximum Modulus Theorem (in its one-variable version, actually).

(c) Recall that the polynomially convex hull of a compact subset $K \subseteq \mathbb{C}^n$ is the compact set $\{\zeta \in \mathbb{C}^n;\ \forall p \in \mathcal{P}(\mathbb{C}^n)\colon |p(\zeta)| \leq \|p|_K\|_\infty\}$. An open subset $U \subseteq \mathbb{C}^n$ is called polynomially convex if it contains the polynomially convex hull of each of its compact subsets. For such an open subset U, the polynomials are dense in $\mathcal{O}(U)$ (see Gunning and Rossi [23, I.F.9]).

If $K \subseteq U \subseteq \mathbb{C}^n$ with K compact and U open and polynomially convex then $\widehat{K}_{\mathcal{O}(U)}$ is the polynomially convex hull of K. Indeed, assume that $\zeta \in U$ belongs to the polynomially convex hull of K, choose $f \in \mathcal{O}(U)$, and let $\varepsilon > 0$. Then there is a polynomial $p \in \mathcal{P}(\mathbb{C}^n)$ such that $|f(\zeta) - p(\zeta)| < \varepsilon$ and $\|(f-p)|_K\|_\infty < \varepsilon$, whence

$$|f(\zeta)| < |p(\zeta)| + \varepsilon \leq \big\|p|_K\big\|_\infty + \varepsilon < \big\|f|_K\big\|_\infty + 2\varepsilon.$$

Thus $|f(\zeta)| \leq \|f|_K\|_\infty$, and we conclude that $\zeta \in \widehat{K}_{\mathcal{O}(U)}$.

(d) Every compact subset $K \subseteq \mathbb{R}^n$ is a polynomially convex subset of $\mathbb{C}^n$. Indeed, by (c), it suffices to show that K is holomorphically convex in $\mathbb{C}^n$. Let $\zeta \in \mathbb{C}^n \setminus K$. If $\zeta \in \mathbb{R}^n$ then $\zeta \notin \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ because the polynomials are dense in $C(K \cup \{\zeta\})$ by the Stone–Weierstrass Theorem (see Hewitt and Stromberg [25, 7.34]). If $\operatorname{Im}\zeta_j < 0$ for some $j \in \{1, \ldots, n\}$ then the entire function $\xi \mapsto e^{i\xi_j} : \mathbb{C}^n \to \mathbb{C}$ separates ζ from K because $|e^{i\zeta_j}| = e^{\operatorname{Re} i\zeta_j} > 1$. Similarly, if $\operatorname{Im}\zeta_j > 0$ then one uses the entire function $\xi \mapsto e^{-i\xi_j}$.

A complex analytic manifold is called *holomorphically convex* if for every compact subset, the holomorphically convex hull is compact. For instance, the preceding example shows that all open subsets of $\mathbb{C}$ and all polynomially convex open subsets of $\mathbb{C}^n$ are holomorphically convex manifolds. Stein manifolds can be characterized in terms of holomorphic convexity. Indeed, a second countable complex analytic manifold X of complex dimension n is a Stein manifold if and only if it is holomorphically convex, the holomorphic functions separate the points of X, and for every $\zeta \in X$, one can find n holomorphic functions on X which form a coordinate system at ζ. In fact, the last two conditions are equivalent if X is holomorphically convex (Hörmander [27, 5.2.12] and Taylor [43, Exercise 11.13]). Moreover, in the presence of the other conditions, holomorphic convexity of X is equivalent to the property that every continuous homomorphism from $\mathcal{O}(X)$ into $\mathbb{C}$ is evaluation in a point of X. These facts are proved in many monographs on complex analysis; see, for instance, Hörmander [27, 5.1.3, 5.1.5 and 5.3.9] and Gunning and Rossi [23, VII.C.5 and VII.C.13]. Note that an open subset of $\mathbb{C}^n$ is a Stein manifold if and only if it is holomorphically convex.

Let X be a Stein manifold. A *Riemann domain over* X is a pair (Y, π) consisting of a second countable complex analytic manifold Y and an analytic local diffeomorphism $\pi\colon Y \to X$. Following Hörmander [27, 5.4.4], we also require that the holomorphic functions on Y separate points. For example, any open subset of X will be considered as a Riemann domain together with the inclusion map. A *holomorphic extension* of a Riemann domain (Y, π) over X is a Riemann domain (Y', π') over X such that Y' contains Y as an open submanifold, we have $\pi'|_Y = \pi$, and every $f \in \mathcal{O}(Y)$ has a unique holomorphic extension $\hat{f} \in \mathcal{O}(Y')$. By the Open Mapping Theorem (see Rudin [40, 2.12]), the restriction map $\mathcal{O}(Y') \to \mathcal{O}(Y)$ is an isomorphism of Fréchet spaces.

An *envelope of holomorphy* of a Riemann domain (Y, π) over X is a holomorphic extension (E, ε) of (Y, π) which is as large as possible, in the sense of the following universal property: if (Y', π') is a holomorphic extension of (Y, π) then there is a unique analytic map $\varphi\colon Y' \to E$ such that $\varphi|_Y = \mathrm{id}_Y$. Note that $\varepsilon \circ \varphi = \pi'$ because both restrict to π and that $\varphi^*\colon \mathcal{O}(E) \to \mathcal{O}(Y')$ is an isomorphism. Since the holomorphic functions on Y' separate points, the map φ is injective, and hence an open embedding by the Open Mapping Theorem (see Range [36, I.1.21]). If (E', ε') is another envelope of holomorphy of (Y, π), the universal property yields a unique analytic diffeomorphism $\varphi\colon E \to E'$ such that $\varphi|_Y = \mathrm{id}_Y$. According to a classic result (see Rossi [38], and Hörmander [27, 5.4.3 and 5.4.5] or Gunning and Rossi [23, I.G.11] for the case $X = \mathbb{C}^n$), every Riemann domain (Y, π) over a Stein manifold X has an envelope of holomorphy (E, ε). Since (E, ε) is unique up to a natural analytic diffeomorphism, one usually speaks of *the* envelope of holomorphy of (Y, π). The envelope of holomorphy can also be characterized as the unique holomorphic extension which is a Stein manifold (see Hörmander [27, 5.4.2 and 5.4.3]).

4.3 Proposition. *Let X be a Stein manifold, let (Y,π) be a Riemann domain over X with envelope of holomorphy (E,ε), and let $K \subseteq Y$ be a compact subset. Then $\mathcal{A}_Y(K) = \mathcal{A}_E(K)$, and the map*

$$\varphi\colon \widehat{K}_{\mathcal{O}(E)} \longrightarrow \Gamma_{\mathcal{A}_Y(K)},\ \zeta \longmapsto \widetilde{\mathrm{ev}}_\zeta$$

is a homeomorphism. The equation $\widehat{K}_{\mathcal{O}(Y)} = \widehat{K}_{\mathcal{O}(E)}$ holds if and only if $\widehat{K}_{\mathcal{O}(Y)}$ is compact.

The fact that φ is a homeomorphism was first observed by Rossi [37, 2.3], cf. Gunning and Rossi [23, VII.A.7].

Proof. The definition of a holomorphic extension implies that $\mathcal{A}_Y(K) = \mathcal{A}_E(K)$.

The assertion that φ is a homeomorphism follows from the fact that E is a Stein manifold. Indeed, $\widehat{K}_{\mathcal{O}(E)}$ is compact because E is holomorphically convex, and φ is bijective because every continuous homomorphism from $\mathcal{O}(E)$ into $\mathbb{C}$ is evaluation in a unique point of E. Hence φ is a continuous bijection between compact Hausdorff spaces and therefore a homeomorphism.

If $\widehat{K}_{\mathcal{O}(Y)} = \widehat{K}_{\mathcal{O}(E)}$ then $\widehat{K}_{\mathcal{O}(Y)}$ is compact. Conversely, assume compactness of $\widehat{K}_{\mathcal{O}(Y)}$. Since $\widehat{K}_{\mathcal{O}(Y)} = Y \cap \widehat{K}_{\mathcal{O}(E)}$, this implies that $\varphi\big(\widehat{K}_{\mathcal{O}(Y)}\big)$ is an open and closed subset of $\Gamma_{\mathcal{A}_Y(K)}$. By the Šilov Idempotent Theorem (see, for instance, Bonsall and Duncan [9, 21.5]), the characteristic function of $\varphi\big(\widehat{K}_{\mathcal{O}(Y)}\big)$ in $\Gamma_{\mathcal{A}_Y(K)}$ is the Gelfand transform of an idempotent $e \in \mathcal{A}_Y(K)$. If $\zeta \in K$ then $e(\zeta) = \widetilde{\mathrm{ev}}_\zeta(e) = 1$. Hence $e = 1$, and $\varphi\big(\widehat{K}_{\mathcal{O}(Y)}\big) = \Gamma_{\mathcal{A}_Y(K)}$. We conclude that $\widehat{K}_{\mathcal{O}(Y)} = \widehat{K}_{\mathcal{O}(E)}$. □

4.4 Corollary (Spectrum of $\mathcal{A}_U(K)$). *Let X be a Stein manifold, and let $K \subseteq U \subseteq X$ with K compact and U open. Choose a closed embedding $\iota\colon X \hookrightarrow \mathbb{C}^n$. Then the following conditions are equivalent:*

(i) $\mathrm{Sp}_{\mathcal{A}_U(K)}(\iota|_K) \subseteq \iota(U)$;

(ii) $\Gamma_{\mathcal{A}_U(K)} = \Big\{\widetilde{\mathrm{ev}}_\zeta;\ \zeta \in \widehat{K}_{\mathcal{O}(U)}\Big\}$;

(iii) $\widehat{K}_{\mathcal{O}(U)}$ *is compact;*

(iv) $\mathrm{Sp}_{\mathcal{A}_U(K)}(\iota|_K) = \iota\big(\widehat{K}_{\mathcal{O}(U)}\big)$.

Proof. Let (E,ε) be the envelope of holomorphy of (U, id_U). Let $\zeta \in \widehat{K}_{\mathcal{O}(E)}$. According to Lemma 4.1, the evaluation homomorphism $\mathrm{ev}_\zeta\colon \mathcal{O}(E) \to \mathbb{C}$ induces a character $\widetilde{\mathrm{ev}}_\zeta$ of $\mathcal{A}_E(K) = \mathcal{A}_U(K)$. Since $\varepsilon|_U = \mathrm{id}_U$, we find that

$$\iota(\varepsilon(\zeta)) = \mathrm{ev}_\zeta^{\times n}(\iota \circ \varepsilon) = \widetilde{\mathrm{ev}}_\zeta^{\times n}(\iota \circ \varepsilon|_K) = \widetilde{\mathrm{ev}}_\zeta^{\times n}(\iota|_K).$$

Proposition 4.3 implies that $\mathrm{Sp}_{\mathcal{A}_U(K)}(\iota|_K) = \iota\big(\varepsilon\big(\widehat{K}_{\mathcal{O}(E)}\big)\big)$.

By the same proposition, condition (ii) is equivalent to the equation $\widehat{K}_{\mathcal{O}(U)} = \widehat{K}_{\mathcal{O}(E)}$, which is equivalent to (iii). If $\widehat{K}_{\mathcal{O}(U)} = \widehat{K}_{\mathcal{O}(E)}$ then $\mathrm{Sp}_{\mathcal{A}_U(K)}(\iota|_K) = \iota\big(\varepsilon\big(\widehat{K}_{\mathcal{O}(E)}\big)\big) = \iota\big(\widehat{K}_{\mathcal{O}(U)}\big)$, which is (iv). Condition (iv) implies (i). Since

$$\iota\big(\widehat{K}_{\mathcal{O}(U)}\big) = \iota\big(U \cap \widehat{K}_{\mathcal{O}(E)}\big) \subseteq \iota\big(\varepsilon\big(\widehat{K}_{\mathcal{O}(E)}\big)\big) = \mathrm{Sp}_{\mathcal{A}_U(K)}(\iota|_K)$$

and $\widehat{K}_{\mathcal{O}(U)}$ is closed in U, condition (i) implies (iii). □

4.5 Corollary (Auto-spectrality and holomorphic convexity). *Let U be a Riemann domain over a Stein manifold, and let $K \subseteq U$ be compact. Then the following conditions are equivalent:*

(i) *K is holomorphically convex in U (i.e. $\widehat{K}_{\mathcal{O}(U)} = K$).*

(ii) *K is auto-spectral, and $\mathcal{O}_U(K) = \mathcal{O}(K)$.*

(iii) *K is auto-spectral, and $\mathcal{A}_U(K) = \mathcal{A}(K)$.*

Proof. Since U is an open subset of its envelope of holomorphy, which is a Stein manifold, Lemma 1.2 and Corollary 4.4 show that condition (i) holds if and only if $\Gamma_{\mathcal{O}_U(K)}$ consists of evaluations in points of K. By Lemma 2.3, this is equivalent to both (ii) and (iii). □

4.6 Corollary (Auto-spectrality and polynomial convexity). *A compact subset $K \subseteq \mathbb{C}^n$ is polynomially convex if and only if it is auto-spectral and the polynomials are dense in $\mathcal{A}(K)$ or, equivalently, in $\mathcal{O}(K)$.*

Proof. The polynomially convex hull of a compact subset $K \subseteq \mathbb{C}^n$ equals $\widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ by Example 4.2, and the polynomials are dense in $\mathcal{A}_{\mathbb{C}^n}(K)$ and in $\mathcal{O}_{\mathbb{C}^n}(K)$. Therefore, the assertions follow from Corollary 4.5. □

5 Stein compacta

Every compact subset of a Stein manifold which is holomorphically convex with respect to some open neighbourhood is a Stein compactum, and every Stein compactum is auto-spectral. In addition to the proofs of these facts, this section contains several examples. One of them, which is due to Björk [7], is an auto-spectral subset of $\mathbb{C}^2$ which is not a Stein compactum.

Let X be a second countable complex analytic manifold such that $\mathcal{O}(X)$ separates points. For a compact subset $K \subseteq X$, let $\mathcal{U}^{\mathrm{St}}(K)$ be the set of Stein open neighbourhoods of K. (Note that an open subset of X is a Stein manifold if and only if it is holomorphically convex.) A *Stein compactum* in X is a compact subset $K \subseteq X$ such that $\mathcal{U}^{\mathrm{St}}(K)$ is a neighbourhood basis of K.

5.1 Example. (a) Let X be a second countable complex analytic manifold such that $\mathcal{O}(X)$ separates points. Then any compact subset $K \subseteq X$ such that $\widehat{K}_{\mathcal{O}(X)} = K$ is a Stein compactum. Indeed, K has a neighbourhood basis consisting of open analytic polyhedra, and every such neighbourhood is holomorphically convex and hence a Stein manifold. The proofs given by Range [36, II.3.9 and II.3.10] for subsets of $\mathbb{C}^n$ carry over to the present situation.

(b) Every compact subset of $\mathbb{C}$ is a Stein compactum by Example 4.2. However, not every compact subset of $\mathbb{C}$ has an open neighbourhood in which it is holomorphically convex. As an example, consider the compact set

$$([0,1] + \{0,i\}) \cup i[0,1] \cup \bigcup_{n\in\mathbb{N}} \left(\frac{1}{n} + i[0,1]\right).$$

Example 5.1 implies that every compact subset of a Stein manifold X has a relatively compact holomorphically convex open neighbourhood. Since the intersection of two holomorphically convex open subsets of X is holomorphically convex, a compact subset of X is a Stein compactum if (and only if) it is an intersection of holomorphically convex open sets. In particular, the intersection of any family of Stein compacta in X is again a Stein compactum. Therefore, every compact subset $K \subseteq X$ is contained in a smallest Stein compactum, which we denote by $\widehat{K}_{\mathrm{St}}$. This compactum can also be described as $\widehat{K}_{\mathrm{St}} = \bigcap \mathcal{U}^{\mathrm{St}}(K)$.

For a compact subset K of a Stein manifold, recall that $\mathcal{O}_{\mathrm{St}}(K)$ was defined as the closure in $\mathcal{O}(K)$ of the algebra of germs of holomorphic functions defined in some member of $\mathcal{U}^{\mathrm{St}}(K)$. The closure of the image of $\mathcal{O}_{\mathrm{St}}(K)$ under the restriction map $\mathcal{O}(K) \to \mathcal{A}(K)$ is the Banach algebra $\mathcal{A}_{\mathrm{St}}(K)$.

5.2 Proposition. *For a compact subset K of a Stein manifold X, the restriction homomorphism $\rho\colon \mathcal{A}(\widehat{K}_{\mathrm{St}}) \to \mathcal{A}_{\mathrm{St}}(K)$, $f \mapsto f|_K$ is an isomorphism.*

Proof. Set $A := \left\{ f|_{\widehat{K}_{\mathrm{St}}};\ f \in \mathcal{O}(\widehat{K}_{\mathrm{St}}) \right\} \subseteq C(\widehat{K}_{\mathrm{St}})$. We claim that all $f \in A$ satisfy $\|f\|_\infty = \|f|_K\|_\infty$. It is clear that the left-hand side is greater than or equal to the right-hand side. Conversely, choose $U \in \mathcal{U}^{\mathrm{St}}(K)$, and note that $\widehat{K}_{\mathrm{St}} \subseteq \widehat{K}_{\mathcal{O}(U)}$ because the right-hand side is a Stein compactum which contains K. If $f \in \mathcal{O}(U)$ then

$$\left\|f|_{\widehat{K}_{\mathrm{St}}}\right\|_\infty \leq \left\|f|_{\widehat{K}_{\mathcal{O}(U)}}\right\|_\infty \leq \left\|f|_K\right\|_\infty$$

by the definition of $\widehat{K}_{\mathcal{O}(U)}$. This proves the claim. Hence the restriction $\rho|_A\colon A \to \mathcal{A}_{\mathrm{St}}(K)$ is a dense isometric embedding, and ρ is the completion of $\rho|_A$. □

5.3 Corollary. *Let $K \subseteq \mathbb{C}^n$ be a compact subset. Then for every $\zeta \in \widehat{K}_{\mathrm{St}}$, evaluation in ζ induces a unique character $\widetilde{\mathrm{ev}}_\zeta \colon \mathcal{A}_{\mathrm{St}}(K) \to \mathbb{C}$.* $\square$

5.4 Proposition (Auto-spectrality and Stein compacta). *Let X be a Stein manifold, and let $K \subseteq X$ be compact. Choose a closed embedding $\iota \colon X \hookrightarrow \mathbb{C}^n$.*

(a) *We have $\Gamma_{\mathcal{A}_{\mathrm{St}}(K)} = \left\{\widetilde{\mathrm{ev}}_\zeta;\ \zeta \in \widehat{K}_{\mathrm{St}}\right\}$ and $\mathrm{Sp}_{\mathcal{A}_{\mathrm{St}}(K)}(\iota|_K) = \iota\big(\widehat{K}_{\mathrm{St}}\big)$.*

(b) *The set K is a Stein compactum if and only if it is auto-spectral and satisfies $\mathcal{A}_{\mathrm{St}}(K) = \mathcal{A}(K)$ or, equivalently, $\mathcal{O}_{\mathrm{St}}(K) = \mathcal{O}(K)$.*

The fact that Stein compacta are auto-spectral is due to Rossi [37, 2.12].

Proof. If $U \in \mathcal{U}^{\mathrm{St}}(K)$ then

$$\iota(K) \subseteq \mathrm{Sp}_{\mathcal{A}(K)}(\iota|_K) \subseteq \mathrm{Sp}_{\mathcal{A}_U(K)}(\iota|_K) = \iota(\widehat{K}_{\mathcal{O}(U)}),$$

where the last equation follows from Corollary 4.4. Now assume that K is a Stein compactum. Then $\mathcal{O}_{\mathrm{St}}(K) = \mathcal{O}(K)$ and $\mathcal{A}_{\mathrm{St}}(K) = \mathcal{A}(K)$ hold by definition. Moreover, $K = \bigcap_{U \in \mathcal{U}^{\mathrm{St}}(K)} \widehat{K}_{\mathcal{O}(U)}$, which implies that $\iota(K) = \mathrm{Sp}_{\mathcal{A}(K)}(\iota|_K)$. Thus K is auto-spectral.

Since $\widehat{K}_{\mathrm{St}}$ is auto-spectral, statement (a) now follows immediately from Proposition 5.2.

Conversely, assume that K is auto-spectral. If $\mathcal{O}_{\mathrm{St}}(K) = \mathcal{O}(K)$ then $\mathcal{A}_{\mathrm{St}}(K) = \mathcal{A}(K)$. Assume that the latter equation holds. Then

$$\iota(K) = \mathrm{Sp}_{\mathcal{A}(K)}(\iota|_K) = \mathrm{Sp}_{\mathcal{A}_{\mathrm{St}}(K)}(\iota|_K) = \iota(\widehat{K}_{\mathrm{St}})$$

by statement (a). We conclude that $K = \widehat{K}_{\mathrm{St}}$, which means that K is a Stein compactum. $\square$

5.5 Example. Define D and K as in Example 4.2 (b). The arguments in that example show that $D \subseteq \mathrm{Sp}_{\mathcal{A}(K)}(\mathrm{id}_K)$ and that $\widehat{K}_{\mathrm{St}} = D$. Hence $\mathrm{Sp}_{\mathcal{A}(K)}(\mathrm{id}_K) = D$.

5.6 Remark. A subset $S \subseteq \mathbb{C}^n$ is called a *Reinhardt subset* (*with centre* 0) if for all $\zeta \in S$ and all $\eta \in \mathbb{C}^n$ with $|\eta_1| = \cdots = |\eta_n| = 1$, the point $(\eta_1\zeta_1, \ldots, \eta_n\zeta_n)$ also belongs to S. If this even holds for all $\eta \in \mathbb{C}^n$ with $|\eta_1|, \ldots, |\eta_n| \leq 1$ then S is called a *complete* Reinhardt subset (with centre 0).

For such subsets, there is a particularly easy characterization of holomorphic convexity. Let $U \subseteq \mathbb{C}^n$ be an open connected Reinhardt subset with $0 \in U$. Then U is holomorphically convex if and only if it is complete and *logarithmically convex,* which means that the subset

$$\big\{(t_1, \ldots, t_n) \in \mathbb{R}^n;\ (e^{t_1}, \ldots, e^{t_n}) \in U\big\}$$

of $\mathbb{R}^n$ is convex (see, for instance, Hörmander [27, 2.5.5 and 2.5.8]). If U is not holomorphically convex, this shows that there is a unique smallest holomorphically convex open Reinhardt subset $V \subseteq \mathbb{C}^n$ such that $U \subseteq V$. Every holomorphic function on U has a unique holomorphic extension to V (see Hörmander [27, 2.4.6]). In other words, the domain V is a realization of the envelope of holomorphy of U.

Let $K \subseteq \mathbb{C}^n$ be a compact connected Reinhardt subset with $0 \in K$. By the above discussion, $\widehat{K}_{\mathrm{St}}$ is the smallest complete logarithmically convex Reinhardt subset of $\mathbb{C}^n$ which contains K, and every $f \in \mathcal{O}(K)$ extends uniquely to an element of $\mathcal{O}(\widehat{K}_{\mathrm{St}})$. In particular, K is a Stein compactum if and only if it is a complete Reinhardt set and logarithmically convex. Moreover, this holds if and only if K is auto-spectral (cf. Björk [7, 4.4]). Indeed, if $\zeta \in \widehat{K}_{\mathrm{St}} \setminus K$ then evaluation in ζ is a character of $\mathcal{O}(\widehat{K}_{\mathrm{St}})$ and hence of $\mathcal{O}(K)$.

Similar facts can be shown for Reinhardt sets which do not contain their centre. In particular, a compact connected Reinhardt subset of $\mathbb{C}^n$ is a Stein compactum if and only if it is auto-spectral.

For $0 \leq r_1 \leq R_1$ and $0 \leq r_2 \leq R_2$, we define a compact Reinhardt subset of $\mathbb{C}^2$ by

$$K(r_1, r_2; R_1, R_2) := \left\{\zeta \in \mathbb{C}^2;\ r_1 \leq |\zeta_1| \leq R_1,\ r_2 \leq |\zeta_2| \leq R_2\right\}.$$

For instance, the set K from Example 4.2 (b) can concisely be written as $K(0,0;0,1) \cup K(0,1;1,1)$. Using the functions $\zeta \mapsto \zeta_j$ and $\zeta \mapsto \zeta_j^{-1}$ for $j \in \{1,2\}$, we find that each $K(r_1, r_2; R_1, R_2)$ is holomorphically convex in any sufficiently small open neighbourhood. In particular, $K(r_1, r_2; R_1, R_2)$ is a Stein compactum. We will now use these sets in order to illustrate two important phenomena. The first of the following two examples is essentially due to Björk [7].

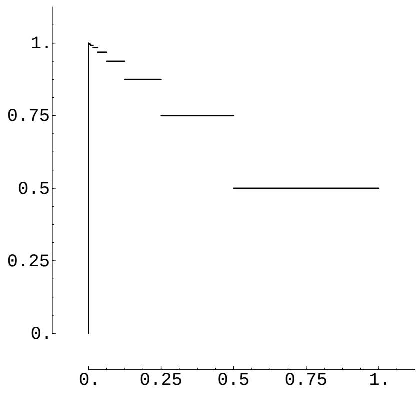
5.7 Example. The compact subset

$$K := \underbrace{K(0,0;0,1)}_{=:K_0} \cup \bigcup_{n \in \mathbb{N}} \underbrace{K(2^{-n}, 1-2^{-n}; 2^{-n+1}, 1-2^{-n})}_{=:K_n} \subseteq \mathbb{C}^2$$

is auto-spectral by Corollary 2.10, but it is not a Stein compactum. Indeed, let $U \subseteq \mathbb{C}^2$ be a holomorphically convex open neighbourhood of K. Then the connected component U_0 of 0 in U contains $K_0 \cup \bigcup_{n>N} K_n$ for some $N \in \mathbb{N}$. Hence U_0 contains $\bigcup_{n>N} K(0,0;2^{-n+1}, 1-2^{-n})$. But then U_0 must also contain K_N. Descending inductively, we find that U must contain

$$K(0,0;0,1) \cup \bigcup_{n \in \mathbb{N}} K(0,0;2^{-n+1}, 1-2^{-n}),$$

and $\widehat{K}_{\mathrm{St}}$ is the logarithmically convex hull of this compact Reinhardt set.

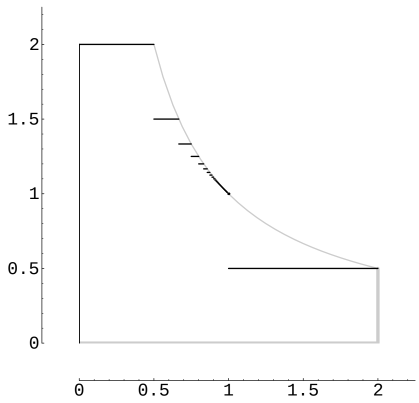


Example 5.7

From a similar example, Björk [7] deduces a compact connected auto-spectral subset of $\mathbb{C}^3$ which is not a Stein compactum.

5.8 Example. This example will show that the auto-spectral hull of a compact subset of $\mathbb{C}^m$ cannot be computed by repeatedly assigning $K \mapsto \mathrm{Sp}_{\mathcal{O}(K)}(\widetilde{\mathrm{id}}_{\mathbb{C}^m})$. (Since this assignment preserves inclusion, the spectrum $\mathrm{Sp}_{\mathcal{O}(K)}(\widetilde{\mathrm{id}}_{\mathbb{C}^m})$ is contained in the auto-spectral hull of K.) Define

$$K_0 := K(0,0;0,2) \cup K(1,1;1,1) \cup K\big(1,\tfrac{1}{2};\ 2,\tfrac{1}{2}\big) \\ \cup \bigcup_{n\in\mathbb{N}} K\big(\tfrac{n-1}{n},\ \tfrac{n+1}{n};\ \tfrac{n}{n+1},\ \tfrac{n+1}{n}\big).$$



Example 5.8

For $n \in \mathbb{N}$, set $K_n := \mathrm{Sp}_{\mathcal{O}(K_{n-1})}(\widetilde{\mathrm{id}}_{\mathbb{C}^2})$. Using Proposition 2.9 and Remark 5.6, one inductively computes that

$$K_n = K_0 \cup \Big\{\zeta \in \mathbb{C}^2;\ |\zeta_1| \le \tfrac{n}{n+1},\ |\zeta_2| \le 2,\ |\zeta_1 \cdot \zeta_2| \le 1\Big\}.$$

Hence the sets K_n form a strictly increasing sequence. In particular, assigning $\mathrm{Sp}_{\mathcal{O}(K)}\big(\widetilde{\mathrm{id}}_{\mathbb{C}^m}\big)$ to a compact set $K \subseteq \mathbb{C}^m$ is not a hull operation. The union $\bigcup_{n\in\mathbb{N}} K_n$ is not closed. Its closure is the set

$$K_\infty := K_0 \cup \big\{\zeta \in \mathbb{C}^2;\ |\zeta_1| \le 1,\ |\zeta_2| \le 2,\ |\zeta_1 \cdot \zeta_2| \le 1\big\}.$$

It is still not auto-spectral. The auto-spectral hull of K_0 is

$$\big\{\zeta \in \mathbb{C}^2;\ |\zeta_1| \le 2,\ |\zeta_2| \le 2,\ |\zeta_1 \cdot \zeta_2| \le 1\big\}.$$

It coincides with $\mathrm{Sp}_{\mathcal{O}(K_\infty)}\big(\widetilde{\mathrm{id}}_{\mathbb{C}^2}\big)$.

Note that not every holomorphic function defined in an open neighbourhood of K_0 extends to a holomorphically convex open neighbourhood. An example is provided by any non-constant locally constant function defined in a neighbourhood of K_0. A compact connected set for which this phenomenon occurs can be derived from Exercise II.3.13 in Range [36].

6 Meromorphic convexity

In this section, we relate holomorphic convexity and Stein compacta to the concept of rational convexity, which has been introduced in Section 3.

6.1 Definition. Let X be a second countable complex analytic manifold, and let $K \subseteq X$ be a compact subset. The *meromorphically convex hull* of K in X is defined as

$$\widehat{K}_{\mathcal{M}(X)} := \bigcap_{f\in\mathcal{O}(X)} f^{-1}\big(f(K)\big).$$

Note that every open subset $U \subseteq X$ with $K \subseteq U$ satisfies $\widehat{K}_{\mathcal{M}(U)} \subseteq \widehat{K}_{\mathcal{M}(X)}$.

6.2 Remark. Let $D(f(K))$ denote the smallest closed disc around 0 in $\mathbb{C}$ which contains $f(K)$. Then the holomorphically convex hull of K in X can be expressed as $\widehat{K}_{\mathcal{O}(X)} = \bigcap_{f\in\mathcal{O}(X)} f^{-1}\big(D(f(K))\big)$. This observation proves that $\widehat{K}_{\mathcal{M}(X)} \subseteq \widehat{K}_{\mathcal{O}(X)}$.

6.3 Lemma. *Let X be a second countable complex analytic manifold, let $K \subseteq X$ be a compact subset, and let $\zeta_0 \in \widehat{K}_{\mathcal{M}(X)}$. Then all $f, g \in \mathcal{O}(X)$ with $0 \notin g(K)$ satisfy*

$$\left|\frac{f(\zeta_0)}{g(\zeta_0)}\right| \le \left\| \frac{f}{g}\Big|_K \right\|_\infty .$$

Proof. The holomorphic function

$$h\colon X \longrightarrow \mathbb{C}, \quad \zeta \longmapsto f(\zeta_0)\cdot g(\zeta) - f(\zeta)\cdot g(\zeta_0)$$

vanishes in ζ_0. Hence there is an element $\zeta \in K$ such that $h(\zeta) = 0$. This equation is equivalent to $\frac{f(\zeta_0)}{g(\zeta_0)} = \frac{f(\zeta)}{g(\zeta)}$. $\square$

The lemma shows that to some extent, the definition of meromorphic convexity fits into the general concept of convexity with respect to a fixed set of functions. However, a meromorphic function need not be the quotient of two global holomorphic functions. In order to understand the situation, we briefly recall the definition of a meromorphic function on a complex analytic manifold X. For each $\zeta \in X$, let $\mathcal{M}_\zeta$ be the field of fractions of the domain $\mathcal{O}_\zeta := \mathcal{O}(\{\zeta\})$. In the disjoint union $\mathcal{M} := \bigcup_{\zeta\in X}\mathcal{M}_\zeta$, consider the subsets $\{f_\zeta/g_\zeta;\ \zeta \in U\}$, where $U \subseteq X$ is a connected open subset, $f, g \in \mathcal{O}(U)$, f_ζ and g_ζ are the germs at $\zeta \in U$, and g is not the zero function on U. These subsets are the basis of a topology which turns $\mathcal{M}$ with the natural projection onto X into a sheaf, the *sheaf of germs of meromorphic functions*. A meromorphic function on an open subset $U \subseteq X$ is a section of $\mathcal{M}$ over U. Note that a meromorphic function on X need not give rise to a continuous function from X into the Riemann sphere $\mathbb{C} \cup \{\infty\}$. This problem already occurs for the meromorphic function on $\mathbb{C}^2$ given by $\zeta \mapsto \zeta_1/\zeta_2$.

Since $\mathcal{O}_\zeta \subseteq \mathcal{M}_\zeta$, we may define the *singular set* of a meromorphic function m on X as the subset $S(m) := \{\zeta \in X;\ m(\zeta) \notin \mathcal{O}_\zeta\}$. This is a closed subvariety of X (see Gunning and Rossi [23, VIII.B.4]), and the restriction of m to $R(m) := X \setminus S(m)$ is a holomorphic function. A singular point $\zeta_0 \in S(m)$ is called a *pole* of m if $\lim_{\zeta\to\zeta_0,\,\zeta\in R(m)} m(\zeta) = \infty$. The singular points which are not poles are called *points of indeterminacy* of m.

6.4 Lemma. *Let X be a second countable complex analytic manifold such that $\mathcal{O}(X)$ separates points, let $K \subseteq X$ be compact, and choose $\zeta \in X$. Then the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *hold between the following statements.*

(i) *There exists $f \in \mathcal{O}(X)$ such that $f(\zeta) \notin f(K)$, i.e. $\zeta \notin \widehat{K}_{\mathcal{M}(X)}$.*

(ii) *There exists a meromorphic function m on X such that $K \cup \{\zeta\} \subseteq R(m)$ and $|m(\zeta)| > \|m|_K\|_\infty$.*

(iii) *There exists a meromorphic function m on X such that $K \subseteq R(m)$ and ζ is a pole of m.*

(iv) *There exists a meromorphic function m on X such that $K \subseteq R(m)$ and $\zeta \in S(m)$.*

If X is a Stein manifold and $H^2(X;\mathbb{Z}) = 0$ then the four statements are equivalent.

Proof. Assume that $f \in \mathcal{O}(X)$ satisfies $f(\zeta) \notin f(K)$. After adding a suitable constant to f, we may assume that $0 < |f(\zeta)| < |f(\eta)|$ holds for every $\eta \in K$. Then $m := 1/f$ is a meromorphic function on X with $K \cup \{\zeta\} \subseteq R(m)$ and $|m(\zeta)| > \|m|_K\|_\infty$.

Assume that m is a meromorphic function on X with these properties. We may assume that every connected component of X meets $K \cup \{\zeta\}$. The meromorphic function $m - m(\zeta)$ neither has zeros nor singularities in a neighbourhood of K. If m is not locally constant at ζ then $(m - m(\zeta))^{-1}$ is a meromorphic function on X with the properties stipulated in statement (iii). If m is locally constant at ζ then the connected component of ζ in X does not meet K. Since $\mathcal{O}(X)$ separates points, it is easy to construct a meromorphic function on X which is regular in a neighbourhood of K and has a pole at ζ.

The implication (iii) $\Rightarrow$ (iv) is trivial.

Assume that X is a Stein manifold and that m is a meromorphic function on X such that $K \subseteq R(m)$ and $\zeta \in S(m)$. Then there are $f, g \in \mathcal{O}(X)$ such that $m = f/g$ (Gunning and Rossi [23, VIII.B.10]), and $g(\zeta) = 0$ because $\zeta \in S(m)$. Assume, moreover, that $H^2(X;\mathbb{Z}) = 0$. Then we may choose the holomorphic functions f and g such that the germs f_ζ and g_ζ are relatively prime for each $\zeta \in X$ (Gunning and Rossi [23, VIII.B.3 and 13]). Then $S(m)$ is exactly the set of zeros of g. In particular, $0 \notin g(K)$ because $K \subseteq R(m)$. Thus under these additional assumptions, statement (iv) implies (i). $\square$

The preceding lemma shows that there is no obvious choice of the definition of meromorphic convexity on a general complex analytic manifold. Our definition is the strongest and also the easiest.

Our distinction of poles and points of indeterminacy follows Range [36, VI, § 4]. Rossi [37] calls $S(m)$ the poleset of m. Rossi's paper is an important source for the present section, in particular for Lemmas 6.6, 6.7, and 6.11. However, some of Rossi's arguments seem to disregard the possible presence of points of indeterminacy, so that it seemed worthwhile to adapt his proofs. This also yields an extension of Rossi's results beyond the framework of Stein manifolds.

6.5 Remark. Let $U \subseteq \mathbb{C}^n$ be open and polynomially convex, and let $K \subseteq U$ be compact. Then $\widehat{K}_{\mathcal{M}(U)}$ coincides with the rationally convex hull of K, i.e.

$$\widehat{K}_{\mathcal{M}(U)} = \bigcap_{p \in \mathcal{P}(\mathbb{C}^n)} p^{-1}\big(p(K)\big).$$

Indeed, the forward inclusion is trivial, and the reverse inclusion follows easily from the fact that the polynomials are dense in $\mathcal{O}(U)$, which was mentioned in Example 4.2.

For a meromorphically convex compact subset of a complex analytic manifold, a simple compactness argument yields what might be called a neighbourhood basis of meromorphic polyhedra.

6.6 Lemma (Meromorphic polyhedra). *Let X be a second countable complex analytic manifold, let $K \subseteq X$ be a meromorphically convex compact subset, and let $U \subseteq X$ be a relatively compact open neighbourhood of K. Then there is a finite set $F \subseteq \mathcal{O}(X)$ of holomorphic functions on X such that*

$$K \subseteq \{\zeta \in U;\ \forall f \in F\colon |f(\zeta)| > 1\}$$

and $\{\zeta \in U;\ \forall f \in F\colon |f(\zeta)| \geq 1\}$ is compact.

Proof. For each boundary point $\zeta \in \partial U$, there is a holomorphic function $f_\zeta \in \mathcal{O}(X)$ such that $f_\zeta(\zeta) = 0$ and $f_\zeta(K) \subseteq \{\eta \in \mathbb{C};\ |\eta| \geq 2\}$. Set $U_\zeta := \{\eta \in X;\ |f(\eta)| < 1\}$. Since these open sets cover the compact boundary ∂U, there is a finite subset $F' \subseteq \partial U$ such that $\partial U \subseteq \bigcup_{\zeta \in F'} U_\zeta$. Set $F := \{f_\zeta;\ \zeta \in F'\}$. Then $K \subseteq \{\zeta \in U;\ \forall f \in F\colon |f(\zeta)| > 1\}$, and

$$\{\zeta \in U;\ \forall f \in F\colon |f(\zeta)| \geq 1\} = U \setminus \bigcup_{\zeta \in F'} U_\zeta$$

is compact. □

6.7 Lemma (Meromorphically convex compacta are Stein). *Let X be a second countable complex analytic manifold such that $\mathcal{O}(X)$ separates points, and let $K \subseteq X$ be compact subset which is meromorphically convex in X. Then K is a Stein compactum.*

Proof. Let $U \subseteq X$ be a relatively compact open neighbourhood of K. Choose a finite subset $F \subseteq \mathcal{O}(X)$ as in Lemma 6.6. Define an open neighbourhood of K by $V := U \setminus \bigcup_{f \in F} f^{-1}(\{0\})$. If $f \in F$ then $f^{-1}|_V \in \mathcal{O}(V)$. The holomorphically convex hull of K in V satisfies

$$\begin{aligned}
\widehat{K}_{\mathcal{O}(V)} &\subseteq \left\{\zeta \in V;\ \forall f \in F\colon |f(\zeta)^{-1}| \leq \left\|f^{-1}|_K\right\|_\infty\right\} \\
&\subseteq \left\{\zeta \in V;\ \forall f \in F\colon |f(\zeta)^{-1}| \leq 1\right\} \\
&= \{\zeta \in U;\ \forall f \in F\colon |f(\zeta)| \geq 1\}\,.
\end{aligned}$$

The right-hand side is a compact subset of V, whence $\widehat{K}_{\mathcal{O}(V)}$ is compact. Example 5.1 shows that $\widehat{K}_{\mathcal{O}(V)}$ is a Stein compactum.

Since the relatively compact open neighbourhood $U \subseteq X$ of K can be chosen arbitrarily small, the set K is an intersection of Stein compacta and hence a Stein compactum. □

6.8 Remark. Hörmander and Wermer [26] constructed a smoothly embedded disc in $\mathbb{C}^2$ which is a Stein compactum but not rationally convex (cf. Forstnerič [18]).

6.9 Proposition. *Let X be a second countable complex analytic manifold such that $\mathcal{O}(X)$ separates points. Then a compact subset $K \subseteq X$ is a Stein compactum if and only if K has arbitrarily small open neighbourhoods $U \subseteq X$ such that $\widehat{K}_{\mathcal{M}(U)}$ is compact.*

Proof. A Stein compactum $K \subseteq X$ has arbitrarily small holomorphically convex open neighbourhoods, and for each such neighbourhood $U \subseteq X$, the set $\widehat{K}_{\mathcal{M}(U)}$ is a closed subset of $\widehat{K}_{\mathcal{O}(U)}$ and hence compact. Conversely, if $K \subseteq X$ is a compact subset with arbitrarily small open neighbourhoods $U \subseteq X$ such that $\widehat{K}_{\mathcal{M}(U)}$ is compact then the preceding lemma shows that K is an intersection of Stein compacta and hence a Stein compactum. □

6.10 Proposition (Meromorphically convex manifolds are Stein). *Let U be an open subset of a Stein manifold X. Then U is holomorphically convex (i.e. a Stein manifold) if and only if $\widehat{K}_{\mathcal{M}(U)}$ is compact for every compact subset $K \subseteq U$.*

It is conceivable that this also holds for more general complex analytic manifolds U.

Proof. If U is holomorphically convex then $\widehat{K}_{\mathcal{M}(U)} \subseteq \widehat{K}_{\mathcal{O}(U)}$ is compact for each compact subset $K \subseteq U$.

Conversely, assume that every compact subset of U has compact meromorphically convex hull in U. We claim that U is Hartogs pseudoconvex. The meaning of this claim is as follows. Let d be the complex dimension of U. Extending the notation introduced before Example 5.7 to subsets of $\mathbb{C}^d$ in the obvious way, define $K \subseteq \mathbb{C}^d$ by $K := K(\mathbf{0}; 0, \ldots, 0, 0, 1) \cup K(0, \ldots, 0, 0, 1; 0, \ldots, 0, 1, 1)$, and $\hat{K} \subseteq \mathbb{C}^d$ by $\hat{K} := K(\mathbf{0}; 0, \ldots, 0, 1, 1)$. Let φ be a biholomorphic embedding of a neighbourhood of $\hat{K}$ in $\mathbb{C}^d$ into X such that $L := \varphi(K) \subseteq U$. What we claim is that $\hat{L} := \varphi(\hat{K})$ is also contained in U. The key to the proof of this claim is Hartogs' result that every holomorphic function defined in a neighbourhood of L extends holomorphically to a neighbourhood of $\hat{L}$ (see Range [36, II.2.2]). Pick $\zeta \in \hat{L} \cap U$. We claim that $\zeta \in \widehat{L}_{\mathcal{M}(U)}$. Otherwise, there exists $f \in \mathcal{O}(U)$ such that $f(\zeta) = 0 \notin f(L)$. Then $1/f$ is a holomorphic function near L which does not extend to $\hat{L}$. This contradiction shows that $\hat{L} \cap U \subseteq \widehat{L}_{\mathcal{M}(U)}$. Hence $\hat{L} \cap U = \hat{L} \cap \widehat{L}_{\mathcal{M}(U)}$ is open and closed in $\hat{L}$. Since this set is not empty and $\hat{L}$ is connected, this proves our claim that $\hat{L} \subseteq U$, i.e. that U is Hartogs pseudoconvex.

Each Hartogs pseudoconvex open subset of $\mathbb{C}^n$ is holomorphically convex (see Range [36, II.5.8 and VI.1.17]). In the case that $X = \mathbb{C}^n$, we have thus completed the proof of the proposition.

In the case that X is a general Stein manifold, choose a closed embedding $\iota \colon X \hookrightarrow \mathbb{C}^n$, an open neighbourhood $V \subseteq \mathbb{C}^n$ of $\iota(X)$, and a holomorphic retraction $\rho\colon V \to X$ for ι. By shrinking V, we may assume that V is holomorphically convex (see Siu [42]). Set $W := \rho^{-1}(U) \subseteq V$, and choose a compact subset $K \subseteq W$. We claim that $\widehat{K}_{\mathcal{M}(W)}$ is compact. We have $\widehat{K}_{\mathcal{M}(W)} \subseteq \widehat{K}_{\mathcal{O}(W)} \subseteq \widehat{K}_{\mathcal{O}(V)}$, and the latter set is compact. Thus it suffices to show that $\widehat{K}_{\mathcal{M}(W)}$ is closed in V. Set $L := \rho(K)$, and choose $\zeta \in W \setminus \rho^{-1}\big(\widehat{L}_{\mathcal{M}(U)}\big)$. Then there exists $f \in \mathcal{O}(U)$ such that $f(\rho(\zeta)) \notin f(L) = f(\rho(K))$. Using $f \circ \rho \in \mathcal{O}(W)$, we find that $\zeta \notin \widehat{K}_{\mathcal{M}(W)}$. Thus $\widehat{K}_{\mathcal{M}(W)} \subseteq \rho^{-1}(\widehat{L}_{\mathcal{M}(U)})$, and the latter set is closed in V. Since $\widehat{K}_{\mathcal{M}(W)}$ is closed in W, we conclude that it is closed in V and hence indeed compact.

The first part of the proof now shows that W is holomorphically convex, whence the same holds for $U = \iota^{-1}(W)$. □

In the case that $X = \mathbb{C}^n$, Proposition 6.10 also follows from Lemma 6.7 and the Behnke–Stein Exhaustion Theorem [2]. A similar result is contained in the same paper by Behnke and Stein.

To end this section, we characterize meromorphic convexity in terms of approximation by meromorphic functions. This result is analogous to our characterizations of holomorphically convex compacta in Corollary 4.5 and of Stein compacta in Proposition 5.4.

If X is a complex analytic manifold and $K \subseteq X$ compact, define an algebra of meromorphic functions on X by

$$\mathcal{M}_K(X) = \{f/g;\ f, g \in \mathcal{O}(X),\ 0 \notin g(K)\}\,.$$

6.11 Lemma (Meromorphic approximation, cf. Rossi [37, 3.4]). *Let X be a second countable complex analytic manifold such that $\mathcal{O}(X)$ separates points, and let $K \subseteq X$ be a meromorphically convex compact subset. Then the subalgebra $\mathcal{M}_K(X)|_K = \{f|_K;\ f \in \mathcal{M}_K(X)\}$ is dense in $\mathcal{A}(K)$.*

Proof. Let $U \subseteq X$ be an open neighbourhood of K, let $f \in \mathcal{O}(U)$, and let $\varepsilon > 0$. We have to construct an element $g \in \mathcal{M}_K(X)$ such that all $\zeta \in K$ satisfy $|f(\zeta) - g(\zeta)| < \varepsilon$. We may assume that U has compact closure in X, and also that U is a Stein manifold, by Lemma 6.7. Choose $F = \{f_1, \dots, f_m\} \subseteq \mathcal{O}(X)$ as in Lemma 6.6, and choose a closed biholomorphic embedding $\iota\colon U \to \mathbb{C}^n$. The map

$$h\colon U \longrightarrow \mathbb{C}^{m+n}, \quad \zeta \longmapsto \big(f_1(\zeta), \dots, f_m(\zeta), \iota_1(\zeta), \dots, \iota_n(\zeta)\big)$$

is a closed biholomorphic embedding. Set

$$Y := \big\{\zeta \in \mathbb{C}^{m+n};\ |\zeta_1| > 1, \dots, |\zeta_m| > 1\big\}$$

and

$$V := h^{-1}(Y) = \{\zeta \in U;\ \forall j \in \{1, \dots, m\}\colon |f_j(\zeta)| > 1\}\,.$$

Then Y is a Stein manifold, V is an open neighbourhood of K, and $h(V) = Y \cap h(U)$ is a closed submanifold of Y. Every holomorphic function on $h(V)$ has a holomorphic extension to Y (Gunning and Rossi [23, VIII.A.18]). Hence there exists $k \in \mathcal{O}(Y)$ such that $k \circ h|_V = f|_V$. By Laurent extension (see Range [36, II.1.4]), there is a Laurent polynomial

$$p \in \mathbb{C}\big[\zeta_1, \zeta_1^{-1}, \dots, \zeta_m, \zeta_m^{-1}, \zeta_{m+1}, \dots, \zeta_{m+n}\big]$$

such that all $\zeta \in h(K)$ satisfy $|k(\zeta) - p(\zeta)| < \varepsilon$. In other words, all $\zeta \in K$ satisfy $|f(\zeta) - p(h(\zeta))| < \varepsilon$. Since none of the f_j has a zero in K, the composition $g := p \circ h$ is an element of $\mathcal{M}_K(X)$, and it has the desired approximation property. $\square$

6.12 Proposition (Auto-spectrality and meromorphic convexity). *Let X be a second countable complex analytic manifold such that $\mathcal{O}(X)$ separates points, and let $K \subseteq X$ be compact. Then the following conditions are equivalent:*

(i) *K is meromorphically convex in X (i.e. $\widehat{K}_{\mathcal{M}(X)} = K$).*

(ii) *K is auto-spectral, and the subalgebra of germs of elements of $\mathcal{M}_X(K)$ in K is dense in $\mathcal{O}(K)$.*

(iii) *K is auto-spectral, and $\mathcal{M}_X(K)|_K$ is dense in $\mathcal{A}(K)$.*

Proof. If condition (i) holds then K is auto-spectral by Lemma 6.7 and Proposition 5.4, so that statement (iii) follows from Lemma 6.11. By Lemmas 1.2 and 2.3, statements (ii) and (iii) are equivalent to each other and to the condition that the Gelfand spectrum of $A := \overline{\mathcal{M}_X(K)|_K}$ consists of the evaluations in points of K. It remains to show that this implies condition (i).

Let $\zeta \in \widehat{K}_{\mathcal{M}(X)}$. No element of $\mathcal{M}_K(X)$ has a singularity at ζ, so that evaluation in ζ is a homomorphism from $\mathcal{M}_K(X)$ onto $\mathbb{C}$. By Lemma 6.3, every $f \in \mathcal{M}_K(X)$ satisfies $|f(\zeta)| \leq \|f|_K\|_\infty$. Hence evaluation in ζ induces a character of A. Thus if Γ_A consists of the evaluations in points of K then K is meromorphically convex in X. $\square$

6.13 Corollary (Auto-spectrality and rational convexity). *A compact subset $K \subseteq \mathbb{C}^n$ is rationally convex if and only if it is auto-spectral and the algebra of rational functions on $\mathbb{C}^n$ without singularities in K is dense in $\mathcal{A}(K)$ or, equivalently, in $\mathcal{O}(K)$.*

Proof. The rationally convex hull of a compact subset $K \subseteq \mathbb{C}^n$ equals $\widehat{K}_{\mathcal{M}(\mathbb{C}^n)}$ by Remark 6.5. A rational function can be written as a quotient of relatively prime polynomials, and then the singular set equals the set of zeros of the denominator. Therefore, every rational function on $\mathbb{C}^n$ without singularities in K is an element of $\mathcal{M}_X(K)$. Since the polynomials are dense in $\mathcal{A}_{\mathbb{C}^n}(K)$ and in $\mathcal{O}_{\mathbb{C}^n}(K)$, every element of $\mathcal{M}_K(\mathbb{C}^n)$ can be approximated by rational functions without singularities in K, both in the topology of $\mathcal{A}(K)$ and in the topology of $\mathcal{O}(K)$. Therefore, the assertions follow from Proposition 6.12. $\square$

6.14 Remark. The strategy of proof for Lemma 6.6 and Proposition 6.12 can immediately be applied in the context of holomorphic convexity, yielding the following results.

Let K be a compact subset of a second countable complex analytic manifold X.

(a) Assume that $\widehat{K}_{\mathcal{O}(X)} = K$, and let $U \subseteq X$ be a relatively compact open neighbourhood of K. Then there is a finite subset $F \subseteq \mathcal{O}(X)$ such that $\{\zeta \in U;\ \forall f \in F \colon |f(\zeta)| \leq 1\}$ is a compact neighbourhood of K.

(b) The equation $\widehat{K}_{\mathcal{O}(X)} = K$ holds if and only if K is auto-spectral and satisfies $\mathcal{A}_X(K) = \mathcal{A}(K)$ or, equivalently, $\mathcal{O}_X(K) = \mathcal{O}(K)$.

Thus Corollary 4.5 is generalized from Riemann domains over Stein manifolds to complex analytic manifolds in which the holomorphic functions separate points. However, Proposition 4.3 contains additional insights, and its proof is more elementary, at least in the case of Riemann domains over $\mathbb{C}^n$.

7 Holomorphic generation

The final section relates our previous results to Mackey-complete complex commutative continuous inverse algebras A which are generated by n-tuples $a \in A^n$ in the sense of the holomorphic functional calculus. In this situation, the joint spectrum $\mathrm{Sp}(a)$ is an auto-spectral set, and the algebra A is "sandwiched" between $\mathcal{O}(\mathrm{Sp}(a))$ and $\mathcal{A}(\mathrm{Sp}(a))$. The joint spectrum of the n-tuple a also shows whether a generates the algebra A in a stronger sense. (Recall that Mackey-completeness is just the weak completeness assumption used in the construction of the holomorphic functional calculus.)

7.1 Lemma. *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$, let $a \in A^n$, and set $K := \mathrm{Sp}_A(a)$. Let $\theta \colon \mathcal{O}(K) \to A$, $f \mapsto f[a]$ be the functional calculus homomorphism, and let $\gamma \colon A \to C(\Gamma_A)$, $x \mapsto \hat{x}$ be the Gelfand homomorphism. Then the composition $\gamma \circ \theta \colon \mathcal{O}(K) \to C(\Gamma_A)$ equals the homomorphism $\hat{a}^*$ induced by the continuous surjection*

$$\hat{a} \colon \Gamma_A \longrightarrow K,\ \chi \longmapsto \big(\chi(a_1), \ldots, \chi(a_n)\big).$$

Proof. The statement follows from a short calculation by means of naturality of the holomorphic functional calculus. Indeed, if $f \in \mathcal{O}(K)$ and $\chi \in \Gamma_A$

then

$$\big((\gamma\circ\theta)(f)\big)(\chi) = \big(\gamma(f[a])\big)(\chi) = \chi(f[a])$$
$$= f\big(\chi^{\times n}(a)\big) = f\big(\hat{a}(\chi)\big) = (f\circ\hat{a})(\chi) = \big(\hat{a}^*(f)\big)(\chi).$$

$\square$

The preceding observation is particularly interesting in the case that $\hat{a}$ is a homeomorphism. In this case, we can use $\hat{a}$ to identify Γ_A and K and think of $\hat{a}^*$ as the restriction map from $\mathcal{O}(K)$ into $C(K)$.

7.2 Theorem (Holomorphically generated algebras). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$, let $a \in A^n$, and set $K := \mathrm{Sp}_A(a)$ and $\theta\colon \mathcal{O}(K) \to A$, $f \mapsto f[a]$. Assume that θ has dense image. (In this situation, we say that the n-tuple a* generates *the algebra A* holomorphically.*) Then*

$$\hat{a}\colon \Gamma_A \longrightarrow K,\ \chi \longmapsto \big(\chi(a_1),\dots,\chi(a_n)\big)$$

is a homeomorphism. Let $\gamma\colon A \to C(K)$, $x \mapsto \hat{x}\circ\hat{a}^{-1}$ be the homomorphism induced by $\hat{a}$ and the Gelfand homomorphism. Then $\overline{\mathrm{im}(\gamma)} = \mathcal{A}(K)$, and the composition

$$\mathcal{O}(K) \xrightarrow{\theta} A \xrightarrow{\gamma} \mathcal{A}(K) \xrightarrow{\iota} C(K),$$

where ι is the inclusion, equals the restriction homomorphism $f \mapsto f|_K$. The induced maps

$$\Gamma_{C(K)} \xrightarrow{\iota^*} \Gamma_{\mathcal{A}(K)} \xrightarrow{\gamma^*} \Gamma_A \xrightarrow{\theta^*} \Gamma_{\mathcal{O}(K)}$$

are homeomorphisms. In particular, the joint spectrum K is an auto-spectral compact subset of $\mathbb{C}^n$.

Proof. By definition, the map $\hat{a}$ is a continuous surjection. As both Γ_A and K are compact Hausdorff spaces, it suffices to show that $\hat{a}$ is injective. If $\chi \in \Gamma_A$ and $f \in \mathcal{O}(K)$ then $\chi(f[a]) = f(\hat{a}(\chi))$ by naturality of the holomorphic functional calculus. Therefore, $\hat{a}(\chi)$ uniquely determines $\chi|_{\mathrm{im}(\theta)}$ and hence χ.

Choose $f \in \mathcal{O}(K)$. If $\chi \in \Gamma_A$ then $\theta(f)\widehat{\ }(\chi) = \chi(f[a]) = f(\hat{a}(\chi))$. Thus if $\zeta \in K$ then $\gamma(\theta(f))(\zeta) = \theta(f)\widehat{\ }(\hat{a}^{-1}(\zeta)) = f(\zeta)$. This proves that $\gamma\circ\theta\colon \mathcal{O}(K) \to C(K)$ is the restriction homomorphism. Since θ has dense image, we conclude that $\mathcal{A}(K) = \overline{\mathrm{im}(\gamma\circ\theta)} = \overline{\mathrm{im}(\gamma)}$, so that we may consider γ as a map into $\mathcal{A}(K)$.

Lemma 1.2 shows that $(\gamma\circ\theta)^* = \theta^*\circ\gamma^*$ is a homeomorphism. Since θ^* is injective and γ^* is surjective, we find that both θ^* and γ^* are homeomorphisms. The map $(\iota\circ\gamma)^* = \gamma^*\circ\iota^*$ is a homeomorphism by construction.

Since every character of the algebra $C(K)$ is evaluation in a point of K, the same holds for the algebras $\mathcal{A}(K)$ and $\mathcal{O}(K)$. We conclude that K is auto-spectral. $\square$

7.3 Remark. Let $K \subseteq \mathbb{C}^n$ be an auto-spectral compact set. Then $\mathcal{O}(K)$ is holomorphically generated by the n-tuple $\widetilde{\mathrm{id}}_{\mathbb{C}^n}$, and $\mathrm{Sp}_{\mathcal{O}(K)}(\widetilde{\mathrm{id}}_{\mathbb{C}^n}) = K$. The analogous statement holds for the Banach algebra $\mathcal{A}(K)$. Thus the auto-spectral compact subsets of $\mathbb{C}^n$ are exactly the joint spectra of holomorphically generating n-tuples in Mackey-complete commutative continuous inverse algebras (or in commutative Banach algebras).

An n-tuple a in a commutative continuous inverse algebra A may generate the algebra not only holomorphically, but in a stronger sense. For instance, the algebra A may be the topological closure of the subalgebra generated by a. Such a situation yields certain additional necessary conditions on the spectrum of a, which are also sufficient if the n-tuple a is assumed to be holomorphically generating. Several situations of this kind are studied in the following corollary.

7.4 Corollary (Holomorphic generation by subalgebras). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$ which is generated holomorphically by $a \in A^n$. Set $K := \mathrm{Sp}_A(a)$.*

(a) *For a unital subalgebra $B \subseteq \mathcal{O}(K)$ which contains the germs of the coordinate functions, the following are equivalent:*

  (i) *$\{f[a];\ f \in B\}$ is dense in A;*

  (ii) *B is dense in $\mathcal{O}(K)$;*

  (iii) *$B|_K$ is dense in $\mathcal{A}(K)$.*

(b) *Let $U \subseteq \mathbb{C}^n$ be an open neighbourhood of K. Then the subalgebra $\{f[a];\ f \in \mathcal{O}(U)\}$ is dense in A if and only if K is holomorphically convex in U.*

(c) *The unital subalgebra generated by $\{a_1, \ldots, a_n\}$ is dense in A if and only if K is polynomially convex.*

(d) *The subalgebra $\{f[a];\ f \in \mathcal{O}_{\mathrm{St}}(K)\}$ is dense in A if and only if K is a Stein compactum.*

(e) *Let $U \subseteq \mathbb{C}^n$ be an open neighbourhood of K. Then the subalgebra of elements of the form $(f/g)[a]$, where $f, g \in \mathcal{O}(U)$ and $0 \notin g(K)$, is dense in A if and only if K is meromorphically convex in U.*

(f) *The subalgebra of elements of the form $f[a]$, where f is a rational function on $\mathbb{C}^n$ without singularities in K, is dense in A if and only if K is rationally convex.*

The forward implication in assertion (c), which is classic at least in the case of commutative Banach algebras (see Bonsall and Duncan [9, 19.11]), was one piece of motivation for the present paper.

Proof. First note that K is auto-spectral by Theorem 7.2. For the proof of assertion (a), Theorem 7.2 also yields that condition (i) implies (iii), which implies (ii) by Lemma 2.3. The definitions show that (ii) implies (i).

To prove assertions (b) to (f), choose the subalgebra $B \subseteq \mathcal{O}(K)$ in part (a) suitably and use, respectively, Corollary 4.5, Corollary 4.6, Proposition 5.4, Proposition 6.12, and Corollary 6.13. $\square$

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