Analyticity and naturality of the multi-variable functional calculus

Harald Biller*

20th April 2004

Abstract

Mackey-complete complex commutative continuous inverse algebras generalize complex commutative Banach algebras. After constructing the Gelfand transform for these algebras, we develop the functional calculus for holomorphic functions on neighbourhoods of the joint spectra of finitely many elements and for holomorphic functions on neighbourhoods of the Gelfand spectrum. To this end, we study the algebra of holomorphic germs in weak*-compact subsets of the dual. We emphasize the simultaneous analyticity of the functional calculus in both the function and its arguments and its naturality. Finally, we treat systems of analytic equations in these algebras.[1] [2]

Introduction

A continuous inverse algebra is a locally convex unital associative algebra in which the set of invertible elements is open and inversion is continuous. Such an algebra is called Mackey-complete if every smooth curve has a weak integral. This weak completeness property can also be defined in terms of the bounded structure, or in terms of the convergence of special Cauchy sequences.

Continuous inverse algebras were introduced by Waelbroeck [47]. They play a role in non-commutative geometry, in particular in K-theory [7, 10, 14, 36], and in the theory of pseudo-differential operators [23]. Currently, they are attracting attention in the theory of Lie groups and algebras of infinite dimension [18]. They appear as coordinate algebras in root-graded locally convex Lie algebras [34]. Linear Lie groups are most naturally defined as subgroups of continuous inverse algebras. These algebras have advantages

*Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstr. 7, 64289 Darmstadt, Germany; *e-mail address:* biller@mathematik.tu-darmstadt.de

[1] *Mathematics Subject Classification (2000):* 46H30; 32A38, 41A20, 46G20, 58B12

[2] *Keywords:* holomorphic functional calculus; commutative continuous inverse algebra; algebra of holomorphic germs

over Banach algebras (which are a special case), for instance because they lead to a rich supply of central extensions of Lie groups and algebras [32, 33]. This is due to the fact that unlike semi-simple commutative Banach algebras, commutative continuous inverse algebras can have non-zero derivations. A typical example is the algebra of smooth functions on a compact manifold.

In other ways, continuous inverse algebras are strikingly similar to Banach algebras. In particular, the theory of the Gelfand spectrum and the holomorphic functional calculus, which are probably the most important tools for commutative Banach algebras, can be worked out for complex commutative continuous inverse algebras. This is the purpose of the present paper. Section 1 establishes the basic properties of spectra and treats the Gelfand transform. Section 2 is a brief introduction to the differential calculus on locally convex vector spaces which we use. Section 3 develops the functional calculus for holomorphic functions on neighbourhoods of the joint spectrum of finitely many elements in a complex commutative continuous inverse algebra. Since it is based on Cauchy's integral formula, we have to assume that the algebra is Mackey-complete. Section 4 studies the algebra of holomorphic germs in a compact subset of the weak*-dual of a locally convex complex vector space E. The most obvious topology on this algebra, the locally convex direct limit topology, has to be modified in order to make the algebra multiplication continuous. Section 5 shows that the second topology differs from the first unless E has countable dimension. These two sections prepare Section 6, which is devoted to the functional calculus for holomorphic functions on neighbourhoods of the Gelfand spectrum in the weak*-dual of a complex commutative continuous inverse algebra. Section 7 treats systems of analytic equations in complex commutative continuous inverse algebras.

In the special case of commutative Banach algebras, the main results of Sections 1 to 3 and 6 are known. Here a new aspect is the analyticity of the functional calculus map $(f, a) \mapsto f[a]$. Naturality of the functional calculus with respect to algebra homomorphisms is a consequence which may not have received the attention it deserves. Sections 4 and 5 treat the algebra of holomorphic germs in a weak*-compactum K as the direct limit of the system of Banach algebras of bounded holomorphic functions on open neighbourhoods of K. The progress lies in the fact that this directed system is uncountable in general, so that we have to get by without the powerful theory of countable direct limits, which one uses in the framework of metrizable vector spaces (cf. [5] and Glöckner [20]). In Section 7, the solution of analytic equations is simplified by the use of implicitly defined holomorphic functions on the weak*-dual of the algebra and of the corresponding functional calculus. This approach allows us to treat systems of analytic equations with the same ease.

These results require a development of the theory which also applies to continuous inverse algebras. For this purpose, the original approach due to Šilov [42], Arens and Calderón [3], and Waelbroeck [45] seems more suitable than the approach by Bourbaki [12]. When Waelbroeck developed his theory in detail [46], even for complete commutative continuous inverse algebras, the joint spectrum which he used was larger than its modern version. Later, he sketched a modernized version of his n-variable holomorphic functional calculus [49, 50]. He also gave a detailed account for algebras with a certain bounded structure, where questions of continuity and analyticity do not arise [48]. Therefore, it seemed worthwhile to give a complete account of Waelbroeck's theory, which forms Section 3. Section 6 continues the short treatment of the case of commutative Banach algebras by Craw [15] and Taylor [43].

1 The Gelfand transform in commutative continuous inverse algebras

In our terminology, a locally convex algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is an associative algebra over $\mathbb{K}$ with a locally convex Hausdorff vector space topology such that the algebra multiplication is jointly continuous.

1.1 Definition. A *continuous inverse algebra over* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a locally convex algebra A over $\mathbb{K}$ with unit in which the set $A^\times$ of invertible elements is open and inversion is continuous.

If a continuous inverse algebra is commutative, Turpin [44] has proved that its topology can be described by a family of semi-norms which are submultiplicative. A locally convex algebra with this property is called locally multiplicatively convex, or (locally) m-convex for short. However, Żelazko [52] has constructed a non-commutative continuous inverse Fréchet algebra which is not locally multiplicatively convex.

1.2 Lemma. *Let A be a continuous inverse algebra over $\mathbb{C}$, and let $U \subseteq A$ be an open balanced neighbourhood of 0 such that $1 + U \subseteq A^\times$. For every element $a \in U$, the spectrum $\mathrm{Sp}(a) = \{\lambda \in \mathbb{C};\ \lambda \cdot 1 - a \in A^\times\}$ is contained in the open unit disc around 0.*

Proof. Choose $a \in U$. If $\lambda \in \mathbb{C}$ satisfies $|\lambda| \geq 1$ then $-\lambda^{-1} a \in U$, whence $\lambda \cdot 1 - a \in A^\times$, so that $\lambda \notin \mathrm{Sp}(a)$. The assertion follows. $\square$

1.3 Lemma. *Let A be a continuous inverse algebra over $\mathbb{C}$, and let $J \subseteq A$ be a proper closed two-sided ideal. Then the quotient algebra A/J is a continuous inverse algebra with respect to the quotient topology.*

Proof. The quotient algebra A/J is a locally convex algebra over $\mathbb{C}$. Let $\mathrm{pr}\colon A \to A/J$ denote the canonical projection. The image $\mathrm{pr}(A^\times)$ is a neighbourhood of the unit element of A/J which consists of invertible elements. The restriction and corestriction of the inversion map of A/J to $\mathrm{pr}(A^\times)$ is continuous. According to Glöckner [18, 2.8], this implies that A/J is a continuous inverse algebra. □

1.4 Lemma (Arens [2], Waelbroeck [46]). *In a continuous inverse algebra A over $\mathbb{C}$, the following statements hold.*

(a) *Every element has non-empty compact spectrum.*

(b) *If A is a skew field then A is topologically isomorphic to $\mathbb{C}$.*

Proof. The spectrum of an arbitrary element is closed by definition and bounded by Lemma 1.2. Waelbroeck [46, II.1.2] has proved that it is non-empty. Statement (b) is due to Arens [2]. Concise modern proofs are given by Glöckner [18, 4.3 and 4.15]. As Glöckner himself observes [18, 4.15], the standing completeness hypothesis of [18, Section 4] is not used in the proofs of these results. □

1.5 Lemma. *Let M be a maximal proper ideal in a commutative continuous inverse algebra A over $\mathbb{C}$. Then M is the kernel of some unital algebra homomorphism from A onto $\mathbb{C}$.*

Proof. Since M is disjoint from the open set $A^\times$, the same holds for its closure $\overline{M}$, which is therefore a proper ideal. Maximality of M implies that M is closed. By the Lemma 1.3, the quotient A/M is a continuous inverse algebra over $\mathbb{C}$. Since A is commutative, the quotient A/M is a field, whence it is isomorphic to $\mathbb{C}$ by Lemma 1.4. Hence M is the kernel of a complex homomorphism. □

1.6 Definition. Let A be a complex algebra with unit.

(a) Define the *Gelfand spectrum* of A as $\Gamma_A := \mathrm{Hom}(A, \mathbb{C})$ with the topology of pointwise convergence on A. Note that $0 \notin \Gamma_A$ because we require homomorphisms to respect the unit elements.

(b) Each element $a \in A$ gives rise to a function $\hat{a}$ from the linear dual of A into $\mathbb{C}$ by $\hat{a}(\varphi) := \varphi(a)$, which is continuous with respect to the topology of pointwise convergence. The restriction $\hat{a}|_{\Gamma_A}\colon \Gamma_A \to \mathbb{C}$ is called the *Gelfand transform* of a. The map $a \mapsto \hat{a}|_{\Gamma_A}\colon A \to C(\Gamma_A)$, which is a homomorphism of unital algebras, is called the *Gelfand homomorphism* of the algebra A.

The restriction $\hat{a}|_{\Gamma_A}$ is often denoted by $\hat{a}$. We use the more complicated notation because $\hat{a}$ in our sense will play an important role in Section 6.

1.7 Theorem (The Gelfand transform). *In a commutative continuous inverse algebra A over $\mathbb{C}$, the following statements hold.*

(a) *Every element* $a \in A$ *satisfies*

$$\mathrm{Sp}(a) = \{\chi(a);\ \chi \in \Gamma_A\} = \hat{a}(\Gamma_A).$$

(b) *The Gelfand spectrum* Γ_A *is a compact Hausdorff space.*

(c) *The Gelfand homomorphism is continuous with respect to the topology of uniform convergence on* $C(\Gamma_A)$*. Its kernel is the Jacobson radical of* A*.*

(d) *The spectral radius* $a \mapsto \rho(a) = \left\|\hat{a}|_{\Gamma_A}\right\|_\infty : A \to \mathbb{R}_0^+$ *is a continuous algebra semi-norm on* A *with the Jacobson radical as its zero space.*

Proof. (a) Choose $\lambda \in \mathbb{C} \setminus \mathrm{Sp}(a)$ and $\chi \in \Gamma_A$. Since $\lambda - a$ is invertible, the same holds for $\lambda - \chi(a)$, whence $\lambda \neq \chi(a)$. This proves that $\hat{a}(\Gamma_A) \subseteq \mathrm{Sp}(a)$. To prove the reverse inclusion, choose $\lambda \in \mathrm{Sp}(a)$. Then $(\lambda - a)A$ is a proper ideal of A. Therefore, we may choose a maximal proper ideal $M \subseteq A$ such that $\lambda - a \in M$. By Lemma 1.5, the maximal ideal M is the kernel of some $\chi \in \Gamma_A$. Hence $\lambda = \chi(a)$.

(b) The Gelfand spectrum is compact because it is a closed subspace of the product

$$\prod_{a \in A} \mathrm{Sp}(a) \subseteq \mathbb{C}^A.$$

(c) The kernel of the Gelfand homomorphism is the intersection of all maximal ideals. This is one possible definition of the Jacobson radical.

To prove continuity, it suffices to prove that the Gelfand homomorphism is continuous at 0. Choose $\varepsilon > 0$, and let $U \subseteq A$ be an open balanced neighbourhood of 0 such that $1 + U \subseteq A^\times$. By Lemma 1.2 and part (a), every element $a \in \varepsilon U$ satisfies $\left\|\hat{a}|_{\Gamma_A}\right\|_\infty < \varepsilon$.

(d) This follows immediately from (a) and (c). □

1.8 Proposition (The joint spectrum). *Let* A *be a commutative continuous inverse algebra over* $\mathbb{C}$*. Recall that the joint spectrum of an* n*-tuple* $a \in A^n$ *is the compact subset of* $\mathbb{C}^n$ *defined as*

$$\mathrm{Sp}(a) := \{(\chi(a_1), \ldots, \chi(a_n));\ \chi \in \Gamma_A\}.$$

If $U \subseteq \mathbb{C}^n$ *is an open subset then*

$$A_U := \{a \in A^n;\ \mathrm{Sp}(a) \subseteq U\}$$

is an open subset of A^n*.*

Proof. Choose $a \in A_U$, and choose $\varepsilon > 0$ such that the open polydisc $V \subseteq \mathbb{C}^n$ around 0 of polyradius $(\varepsilon, \ldots, \varepsilon)$ satisfies $\mathrm{Sp}(a) + V \subseteq U$. Let $W \subseteq A$ be an open balanced neighbourhood of 0 such that $1 + W \subseteq A^\times$. Lemma 1.2 and Theorem 1.7 (a) imply that the open neighbourhood $a + (\varepsilon W)^n$ of a in A^n is contained in A_U. □

2 Differentiable functions between locally convex vector spaces

The concepts of differentiability and smoothness which we use go back to Michal and Bastiani [4], see also Hamilton [25] and Milnor [31]. The required generality and the connections to analyticity have been worked out by Glöckner [19].

2.1 Definition. Let E and F be locally convex real vector spaces, and let $U \subseteq E$ be open. A map $f\colon U \to F$ is called *continuously differentiable*, and we write $f \in C^1(U,F)$, if the directional derivative

$$df(x;v) := \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}$$

exists for all $x \in U$ and all $v \in E$ and if the map $df\colon U \times E \to F$ which is thus defined is continuous.

One of the basic properties of a continuously differentiable map $f\colon U \to F$ is the fundamental theorem of calculus which states that whenever $x, y \in U$ and U contains the line segment from x to y then

$$f(y) - f(x) = \int_0^1 df(x + t(y-x); y-x)\, dt.$$

Integrals are always understood in the weak sense (see Rudin [40, 3.26]). The present integral certainly exists in the completion of F, but it actually belongs to F just because it equals the left-hand side. This formula implies that a continuously differentiable map f is continuous, and that its derivative df is real-linear in its second argument (Glöckner [19, 1.9]).

Let $E_1, E_2, \ldots, E_n$ and F be locally convex real vector spaces, and let $U_1 \subseteq E_1, \ldots, U_n \subseteq E_n$ be open. Let $f\colon U_1 \times \cdots \times U_n \to F$ be a map, and let $x_1 \in U_1, \ldots, x_n \in U_n$. If the partial directional derivative of f at $(x_1, \ldots, x_n)$ with respect to the j-th argument in direction $v \in E_j$ exists, it is called the j-th partial derivative $\partial_j f(x_1, \ldots, x_n; v)$. The map f is continuously differentiable if and only if $\partial_j f\colon U_1 \times \cdots \times U_n \times E_j \to F$ exists and is continuous for each $j \in \{1, \ldots, n\}$ (cf. Glöckner [19, 1.10]). In this case, its derivative satisfies

$$df(x_1, \ldots, x_n; v_1, \ldots, v_n) = \sum_{j=1}^{n} \partial_j f(x_1, \ldots, x_n; v_j) \qquad (x_j \in U_j, v_j \in E_j).$$

2.2 Definition. (a) Let E and F be locally convex real vector spaces, and let $U \subseteq E$ be open. Inductively, a map $f\colon U \to F$ is called *k times continuously differentiable*, and we write $f \in C^k(U,F)$, if $f \in C^1(U,F)$ and $df \in C^{k-1}(U \times E, F)$.

(b) A map $f\colon U \to F$ is called *smooth* if it belongs to

$$C^\infty(U,F) := \bigcap_{k\in\mathbb{N}} C^k(U,F).$$

2.3 Definition. Let E and F be locally convex complex vector spaces, and let $U \subseteq E$ be open. A smooth map $f\colon U \to F$ is called *complex analytic* if the continuous real-linear map $df(x;\cdot)\colon E \to F$ is complex-linear for all $x \in U$.

Glöckner [19, 2.5] has proved that this holds if and only if f is continuous, and for every $x \in U$, there exists a zero-neighbourhood V in E such that $x + V \subseteq U$ and $f(x+h) = \sum_{n=0}^{\infty} \beta_n(h)$ for all $h \in V$ as a pointwise limit, where each $\beta_n\colon E \to F$ is a continuous homogeneous polynomial over $\mathbb{C}$ of degree n. This is the definition of complex analyticity given by Bochnak and Siciak [8, 5.6].

2.4 Definition. (a) A sequence $(x_n)_{n\in\mathbb{N}}$ in a locally convex real vector space E is called a *Mackey–Cauchy sequence* if there is a net $(t_{m,n})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of positive real numbers which converges to 0 such that the set

$$\left\{\frac{x_m - x_n}{t_{m,n}};\ m,n \in \mathbb{N}\right\}$$

is a bounded subset of E. Every Mackey–Cauchy sequence is a Cauchy sequence.

(b) The locally convex real vector space E is called *Mackey complete* if every Mackey–Cauchy sequence in E converges. This holds if and only if every smooth curve $\alpha\colon [a,b] \to E$ (where $a, b \in \mathbb{R}$) has a Riemann integral $\int_a^b \alpha(t)\,dt$ in E (see Kriegl and Michor [29, 2.14]). Another equivalent condition is that every bounded subset of E is contained in a convex balanced bounded subset $B \subseteq E$ such that the Minkowski functional of B in the linear span of B is a complete norm [29, 2.2].

Let E and F be locally convex complex vector spaces, and let $U \subseteq E$ be open. Assume that F is Mackey complete. Glöckner [18, 1.4] has proved that a continuously differentiable map $f\colon U \to F$ with df complex-linear in the second argument is complex analytic.

2.5 Proposition (Parameter-dependent integrals). *Let $a, b \in \mathbb{R}$ with $a \le b$, let X be a topological space, and let F be a locally convex real vector space. Let $f\colon [a,b] \times X \to F$ be a continuous map such that*

$$g(x) := \int_a^b f(t,x)\,dt$$

exists in F for every $x \in X$. Then $g\colon X \to F$ is continuous.

Suppose, in addition, that X is an open subset of a locally convex real vector space E, that $\partial_2 f\colon [a,b]\times X\times E\to F$ exists and is continuous, and that

$$g_1(x,v) := \int_a^b \partial_2 f(t,x;v)\ dt$$

exists in F for every $x\in X$ and every $v\in E$. Then $g\in C^1(X,F)$ with $dg=g_1$.

Note that integrals of continuous curves automatically exist in F if F is sequentially complete.

Proof. Choose a point $x_0\in X$ and a closed convex zero-neighbourhood $U\subseteq F$. Since

$$\varphi\colon [a,b]\times X\longrightarrow F,\ (t,x)\longmapsto f(t,x)-f(t,x_0)$$

is continuous and maps the compact set $[a,b]\times\{x_0\}$ to 0, there is a neighbourhood $V\subseteq X$ of x_0 such that $\varphi([a,b]\times V)\subseteq U$. For any $x\in V$, the Hahn–Banach Theorem entails that

$$g(x)-g(x_0)=\int_a^b \big(f(t,x)-f(t,x_0)\big)\ dt=\int_a^b \varphi(t,x)\ dt\in(b-a)U.$$

This shows that g is continuous at x_0.

For the second part, choose $x\in X$, $v\in E$, and $h\in\mathbb{R}^\times$ such that X contains the line segment from x to $x+hv$. Then

$$\begin{aligned}\frac{g(x+hv)-g(x)}{h} &= \frac{1}{h}\int_a^b\big(f(t,x+hv)-f(t,x)\big)\ dt\\ &= \int_a^b\int_0^1 \partial_2 f(t,x+shv;v)\ ds\ dt.\end{aligned}$$

In particular, the integral on the right-hand side exists in F if $h\neq 0$, and by hypothesis also if $h=0$. Applying the first part of the proposition twice, we find that the right-hand side depends continuously on (x,v,h). In particular, its limit for $h\to 0$ exists and satisfies

$$dg(x;v)=\int_a^b \partial_2 f(t,x;v)\ dt,$$

and this depends continuously on $(x,v)\in X\times E$. □

2.6 Corollary. *Let $U\subseteq\mathbb{R}$ be an open interval, choose $a,b\in U$, let E and F be locally convex real vector spaces, let $V\subseteq E$ be open, and let $f\in C^\infty(U\times V,F)$. If F is Mackey complete then*

$$g\colon V\longrightarrow F,\ x\longmapsto\int_a^b f(t,x)\ dt$$

is smooth. □

2.7 Corollary. *Let $a, b \in \mathbb{R}$ with $a \leq b$, let $U \subseteq \mathbb{C}$ be open, and let $\gamma\colon [a,b] \to U$ be a smooth curve. Let E and F be locally convex complex vector spaces, let $V \subseteq E$ be open, and let $f\colon U \times V \to F$ be a complex analytic map. If F is Mackey complete then*

$$g\colon V \longrightarrow F,\ x \longmapsto \int_\gamma f(\zeta, x)\, d\zeta$$

is complex analytic. □

3 Holomorphic functions on subsets of $\mathbb{C}^n$

Like Waelbroeck [46], we base the n-variable functional calculus on the following result from complex analysis in several variables. Recall that an open polynomial polyhedron $U \subseteq \mathbb{C}^n$ is a subset defined by a finite set of polynomials in n variables, $\mathcal{P}_0 \subseteq \mathcal{P}(\mathbb{C}^n)$, as $U := \{\zeta \in \mathbb{C}^n;\ \forall p \in \mathcal{P}_0\colon |p(\zeta)| < 1\}$.

3.1 Theorem (Oka's Extension Theorem [35], cf. Allan [1]). *Let $\mathcal{P}(\mathbb{C}^n)$ denote the polynomial functions on $\mathbb{C}^n$, let $p_1, \ldots, p_k \in \mathcal{P}(\mathbb{C}^n)$, and define the "Oka map"*

$$\mu\colon \mathbb{C}^n \longrightarrow \mathbb{C}^{n+k},\ \zeta \longmapsto (\zeta, p_1(\zeta), \ldots, p_k(\zeta)).$$

Let $U \subseteq \mathbb{C}^n$ be an open polydisc around 0, let $D \subseteq \mathbb{C}$ be the open unit disc, set $V := U \times D^k$, and define an open polynomial polyhedron $W := \mu^{-1}(V) \subseteq U$. Then the algebra homomorphism

$$\mu^*\colon \mathcal{O}(V) \longrightarrow \mathcal{O}(W),\ f \longmapsto f \circ \mu|_W$$

is surjective, continuous, and open, and its kernel is generated by the functions

$$(\zeta_1, \ldots, \zeta_{n+k}) \longmapsto \zeta_{n+j} - p_j(\zeta_1, \ldots, \zeta_n)\colon V \longrightarrow \mathbb{C} \qquad (j \in \{1, \ldots, k\}).\ \square$$

Recall that the *polynomially convex hull* of a compact subset $K \subseteq \mathbb{C}^n$ is the compact set

$$\big\{\zeta \in \mathbb{C}^n;\ \forall p \in \mathcal{P}(\mathbb{C}^n)\colon |p(\zeta)| \leq \big\|p|_K\big\|_\infty\big\},$$

which contains K. A subset of $\mathbb{C}^n$ is called *polynomially convex* if it contains the polynomially convex hull of each of its compact subsets. Oka's Extension Theorem implies the following density theorem (see Gunning and Rossi [24, I.F.9]).

3.2 Corollary. *For any polynomially convex open subset $U \subseteq \mathbb{C}^n$, the polynomials are dense in $\mathcal{O}(U)$.* □

3.3 Proposition (The functional calculus for holomorphic functions on polynomially convex neighbourhoods of the joint spectrum). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$, and let $U \subseteq \mathbb{C}^n$ be a polynomially convex open subset. Then there is a unique map*

$$\Theta_{A,U} \colon \mathcal{O}(U) \times A_U \longrightarrow A,$$

where $A_U = \{a \in A^n;\ \mathrm{Sp}(a) \subseteq U\}$ is an open subset of A^n by Proposition 1.8, such that for each $a \in A_U$, the map

$$\Theta_{A,U}(\cdot, a) \colon \mathcal{O}(U) \longrightarrow A,\ f \longmapsto \Theta_{A,U}(f,a)$$

is a continuous unital algebra homomorphism which maps the j-th coordinate function $\zeta \mapsto \zeta_j$ to a_j. Moreover, the map $\Theta_{A,U}$ is complex analytic.

Proof. By Corollary 3.2, a continuous unital algebra homomorphism on $\mathcal{O}(U)$ is uniquely determined by the images of the coordinate functions. Therefore, there is at most one map $\Theta_{A,U}$ with the required properties.

Assume first that U is an open polydisc with centre 0 and polyradius $r \in (\mathbb{R}^+)^n$, i.e. assume that

$$U = \{\zeta \in \mathbb{C}^n;\ |\zeta_1| < r_1, \ldots, |\zeta_n| < r_n\}.$$

Fix $\varepsilon > 0$ with $\varepsilon < \min\{r_1, \ldots, r_n\}$. Let $V \subseteq \mathbb{C}^n$ be the open polydisc around 0 with polyradius $(r_1 - \varepsilon, \ldots, r_n - \varepsilon)$. For $f \in \mathcal{O}(U)$ and $a \in A_V \subseteq A^n$, set

$$\begin{aligned}\Theta_{A,U}(f,a) := \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = r_1 - \frac{\varepsilon}{2}} \cdots \int_{|\zeta_n| = r_n - \frac{\varepsilon}{2}} & f(\zeta_1, \ldots, \zeta_n) \\ & \cdot (\zeta_1 - a_1)^{-1} \cdots (\zeta_n - a_n)^{-1}\, d\zeta_n \cdots d\zeta_1.\end{aligned}$$

Since inversion in A is complex analytic (Glöckner [18, 3.2]) and evaluation $\mathcal{O}(U) \times U \to \mathbb{C}$ is continuous, the integrand is a complex analytic function of $(\zeta, f, a) \in (U \setminus \overline{V}) \times \mathcal{O}(U) \times A_V$. By Corollary 2.7, the n-fold integral exists and defines a complex analytic map $\mathcal{O}(U) \times A_V \to A$. Moreover, this map is linear in its first argument.

If $f(\zeta_1, \ldots, \zeta_n) = {\zeta_1}^{k_1} \cdots {\zeta_n}^{k_n}$ for certain $k_j \in \mathbb{N}_0$ then

$$\begin{aligned}\Theta_{A,U}(f,a) &= \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = r_1 - \frac{\varepsilon}{2}} {\zeta_1}^{k_1} (\zeta_1 - a_1)^{-1}\, d\zeta_1 \\ &\qquad \cdot \ldots \cdot \int_{|\zeta_n| = r_n - \frac{\varepsilon}{2}} {\zeta_n}^{k_n} (\zeta_n - a_n)^{-1}\, d\zeta_n \\ &= {a_1}^{k_1} \cdot \ldots \cdot {a_n}^{k_n}.\end{aligned}$$

This follows from the one-variable case,

$$\begin{aligned}
\int_{|\zeta_j|=r_j-\frac{\varepsilon}{2}} {\zeta_j}^{k_j}(\zeta_j-a_j)^{-1}\,d\zeta_j &= \int_{|\zeta_j|=r_j-\frac{\varepsilon}{2}} {\zeta_j}^{k_j-1}(1-\tfrac{1}{\zeta_j}a_j)^{-1}\,d\zeta_j \\
&= \int_{|\zeta_j|=r_j-\frac{\varepsilon}{2}} \sum_{m=0}^{\infty} {\zeta_j}^{k_j-1-m}{a_j}^m\,d\zeta_j \\
&= \sum_{m=0}^{\infty} {a_j}^m \int_{|\zeta_j|=r_j-\frac{\varepsilon}{2}} {\zeta_j}^{k_j-1-m}\,d\zeta_j \\
&= 2\pi i\cdot {a_j}^{k_j}
\end{aligned}$$

(see [18, 4.9] for details). By linearity and continuity, the map $\Theta_{A,U}(\cdot,a)$ is a continuous unital algebra homomorphism which satisfies $\Theta_{A,U}(p,a)=p(a)$ for every polynomial $p\in\mathcal{P}(\mathbb{C}^n)$.

We have defined the restriction of $\Theta_{A,U}$ to $\mathcal{O}(U)\times A_V$. By uniqueness of $\Theta_{A,U}(\cdot,a)$, different choices of V lead to compatible definitions. Since the open sets A_V with V as above cover A_U, the proposition holds if U is an open polydisc.

Now let $U\subseteq\mathbb{C}^n$ be a general polynomially convex open subset. Choose $a\in A_U$. Let $U'\subseteq\mathbb{C}^n$ be a (bounded) open polydisc with centre 0 such that $\mathrm{Sp}(a)\subseteq U'$. The polynomially convex hull of $\mathrm{Sp}(a)$ is contained in U. Hence for each ζ in the compact set $L:=\overline{U'}\setminus U$, there is a $p_\zeta\in\mathcal{P}(\mathbb{C}^n)$ such that $|p_\zeta(\zeta)|>1>\|p_\zeta|_{\mathrm{Sp}(a)}\|_\infty$. Choose a finite subset $\{p_1,\dots,p_k\}\subseteq\{p_\zeta;\zeta\in L\}$ such that the open sets $\{\zeta\in\mathbb{C}^n;|p_j(\zeta)|>1\}$ cover L. Define

$$\mu\colon\mathbb{C}^n\longrightarrow\mathbb{C}^{n+k},\ \zeta\longmapsto(\zeta,p_1(\zeta),\dots,p_k(\zeta)).$$

Let $D\subseteq\mathbb{C}$ be the open unit disc, set $V':=U'\times D^k\subseteq\mathbb{C}^{n+k}$, and define an open polynomial polyhedron $V:=\mu^{-1}(V')\subseteq U'$. By the choice of the polynomials p_j, we have $\mathrm{Sp}(a)\subseteq V\subseteq U$. By Oka's Theorem 3.1, the algebra homomorphism

$$\mu^*\colon\mathcal{O}(V')\longrightarrow\mathcal{O}(V),\ f\longmapsto f\circ\mu|_V$$

is surjective, continuous, and open, and its kernel is generated by the polynomial functions

$$q_j\colon V'\longrightarrow\mathbb{C},\ (\zeta_1,\dots,\zeta_{n+k})\longmapsto\zeta_{n+j}-p_j(\zeta_1,\dots,\zeta_n)\qquad(j\in\{1,\dots,k\}).$$

By the spectral mapping theorem for polynomials, we have $\mathrm{Sp}(\mu(a))=\mu(\mathrm{Sp}(a))\subseteq V'$ whenever $a\in A_V$. The map $\Theta_{A,V'}$ which we have constructed in the first part of the proof leads to a complex analytic map

$$\Phi\colon\mathcal{O}(V')\times A_V\longrightarrow A,\ (f,a)\longmapsto\Theta_{A,V'}(f,\mu(a)).$$

For each $a\in A_V$, we have $\Phi(q_j,a)=p_j(a)-p_j(a)=0$, so that the homomorphism $\Phi(\cdot,a)\colon\mathcal{O}(V')\to A$ factors through μ^*. Hence there is a map

$\Psi\colon \mathcal{O}(V) \times A_V \to A$ such that $\Phi = \Psi \circ (\mu^* \times \mathrm{id}_{A_V})$. For each $a \in A_V$, the map $\Psi(\cdot, a)$ is a unital algebra homomorphism, and since the j-th coordinate function on V is the μ^*-image of the j-th coordinate function on V', its image under $\Psi(\cdot, a)$ is a_j. According to Glöckner [19, 2.10], the map Ψ is complex analytic because $\mu^* \times \mathrm{id}_{A^n}$ is a linear quotient map. The restriction of $\Theta_{A,U}$ to $\mathcal{O}(U) \times A_V$ can thus be defined by $\Theta_{A,U}(f, a) := \Psi(f|V, a)$ for $f \in \mathcal{O}(U)$ and $a \in A_V$. As in the case of polydiscs, different choices of V lead to compatible definitions, and the open sets A_V with $V \subseteq U$ an open polynomial polyhedron cover A_U. This completes the proof. □

The construction of V in the second part of the proof will be used again, so that we formulate it as a lemma.

3.4 Lemma. *Let $K \subseteq U \subseteq \mathbb{C}^n$ with K compact and polynomially convex and U open. Then there is an open polynomial polyhedron $V \subseteq U$ such that $K \subseteq V$.* □

3.5 Lemma. *Let A be a commutative continuous inverse algebra over $\mathbb{C}$ which is topologically generated by finitely many elements $a_1, \ldots, a_n$. Then $\mathrm{Sp}(a_1, \ldots, a_n)$ is polynomially convex.*

Proof (cf. Bonsall and Duncan [9, II.19.11]). Write $\Gamma_A \subseteq A'$ for the Gelfand spectrum of A and $\rho\colon A \to \mathbb{R}_0^+$ for the spectral radius, which is a continuous algebra seminorm by Theorem 1.7. Set $a := (a_1, \ldots, a_n)$. If $p \in \mathcal{P}(\mathbb{C}^n)$ then

$$\begin{aligned}\left\|p|_{\mathrm{Sp}(a)}\right\|_\infty &= \sup\{\left|p\big(\chi(a_1), \ldots, \chi(a_n)\big)\right|; \chi \in \Gamma_A\} \\ &= \sup\{|\chi(p(a))|; \chi \in \Gamma_A\} = \rho(p(a)).\end{aligned}$$

Choose $\lambda \in \mathbb{C}^n \setminus \mathrm{Sp}(a)$. Then there are elements $b_1, \ldots, b_n \in A$ such that $1 = (\lambda_1 - a_1)b_1 + \cdots + (\lambda_n - a_n)b_n$. Since we can approximate the elements b_j by polynomials in a, we find $p_1, \ldots, p_n \in \mathcal{P}(\mathbb{C}^n)$ such that

$$\rho\left(1 - \sum_{j=1}^{n} (\lambda_j - a_j) p_j(a)\right) < 1.$$

Define $p \in \mathcal{P}(\mathbb{C}^n)$ by $p(\zeta) = 1 - \sum_{j=1}^n (\lambda_j - \zeta_j) p_j(\zeta)$. Then $p(\lambda) = 1 > \rho(p(a)) = \|p|_{\mathrm{Sp}(a)}\|_\infty$, which shows that λ does not belong to the polynomially convex hull of $\mathrm{Sp}(a)$. □

3.6 Lemma (cf. Arens and Calderón [3, 2.3]). *Let A be a commutative continuous inverse algebra over $\mathbb{C}$, let $a = (a_1, \ldots, a_n) \in A^n$, and let $U \subseteq \mathbb{C}^n$ be an open neighbourhood of $\mathrm{Sp}_A(a)$. Then there exists a topologically finitely generated closed unital subalgebra B of A containing $a_1, \ldots, a_n$ such that $\mathrm{Sp}_B(a) \subseteq U$.*

Proof. The proof given by Bonsall and Duncan [9, II.20.3] for Banach algebras applies to continuous inverse algebras if the norm is replaced by the spectral radius. □

3.7 Corollary. *Let A be a commutative continuous inverse algebra over $\mathbb{C}$, let $U \subseteq \mathbb{C}^n$ be open, and choose $a \in A_U$. Then there is a natural number $k \in \mathbb{N}$, a k-tuple $b \in A^k$, and an open polynomial polyhedron $V \subseteq \mathbb{C}^{n+k}$ such that*

$$\mathrm{Sp}(a,b) := \mathrm{Sp}(a_1,\ldots,a_n,b_1,\ldots,b_k) \subseteq V \subseteq \mathrm{pr}_{n,n+k}^{-1}(U),$$

where $\mathrm{pr}_{n,n+k}\colon \mathbb{C}^{n+k} \to \mathbb{C}^n$ denotes the projection onto the first n coordinates.

Proof. The Arens–Calderón trick (Lemma 3.6) yields a closed unital subalgebra B of A which is topologically generated by $a_1,\ldots,a_n$ and finitely many additional elements $b_1,\ldots,b_k \in A$ and satisfies $\mathrm{Sp}_B(a) \subseteq U$. Since $\mathrm{Sp}_B(a) = \mathrm{pr}_{n,n+k}(\mathrm{Sp}_B(a,b))$, we have

$$\mathrm{Sp}_A(a,b) \subseteq \mathrm{Sp}_B(a,b) \subseteq {\mathrm{pr}_{n,n+k}}^{-1}(U).$$

Lemmas 3.4 and 3.5 yield an open polynomial polyhedron $V \subseteq {\mathrm{pr}_{n,n+k}}^{-1}(U)$ such that $\mathrm{Sp}_A(a,b) \subseteq \mathrm{Sp}_B(a,b) \subseteq V$. □

3.8 Theorem (The n-variable holomorphic functional calculus). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. Then there is a unique family of maps*

$$\Theta_{A,U}\colon \mathcal{O}(U) \times A_U \longrightarrow A,$$

where U varies over the open subsets of all spaces $\mathbb{C}^n$ ($n \in \mathbb{N}$) and $A_U = \{a \in A^n; \mathrm{Sp}(a) \subseteq U\}$, such that the following two conditions hold.

(i) *For each $a \in A_U \subseteq A^n$, the map*

$$\Theta_{A,U}(\cdot,a)\colon \mathcal{O}(U) \longrightarrow A,\ f \longmapsto \Theta_{A,U}(f,a)$$

is a continuous unital algebra homomorphism which maps the j-th coordinate function $\zeta \mapsto \zeta_j$ to a_j.

(ii) *Let $m \leq n$, let $\mathrm{pr}_{m,n}\colon \mathbb{C}^n \to \mathbb{C}^m$ be the projection onto the first m coordinates, and let $V \subseteq \mathbb{C}^n$ and $U \subseteq \mathbb{C}^m$ be open sets with $\mathrm{pr}_{m,n}(V) \subseteq U$. If $f \in \mathcal{O}(U)$ and $a \in A_V \subseteq A^n$ then*

$$\Theta_{A,V}\big(f \circ \mathrm{pr}_{m,n}|_V, (a_1,\ldots,a_n)\big) = \Theta_{A,U}\big(f,(a_1,\ldots,a_m)\big).$$

Moreover, the unique family $(\Theta_{A,U})_U$ satisfying (i) *and* (ii) *consists of complex analytic maps.*

Note that condition (ii) is a special case of compatibility of the holomorphic functional calculus with composition of holomorphic maps (see Theorem 3.11 below). Also note that if $V, U \subseteq \mathbb{C}^n$ are open subsets with $V \subseteq U$ then condition (ii) says that $\Theta_{A,V}(f|_V, a) = \Theta_{A,U}(f, a)$ for all $f \in \mathcal{O}(U)$ and all $a \in A_V \subseteq A_U \subseteq A^n$. For fixed $a \in A^n$, we thus obtain a continuous algebra homomorphism from the locally convex direct limit $\mathcal{O}(\mathrm{Sp}(a)) := \varinjlim \mathcal{O}(U)$, where U varies over the open neighbourhoods of $\mathrm{Sp}(a)$ in $\mathbb{C}^n$, into A.

When no ambiguities arise, one writes $\Theta_{A,U}(f, a) =: f[a]$.

Proof. Let us first prove uniqueness of the family $(\Theta_{A,U})_U$. Choose $n \in \mathbb{N}$, an open subset $U \in \mathbb{C}^n$, and an n-tuple $a \in A_U$. Corollary 3.7 yields a natural number $k \in \mathbb{N}$, a k-tuple $b \in A^k$, and an open polynomial polyhedron $V \subseteq \mathbb{C}^{n+k}$ such that $\mathrm{pr}_{n,n+k}(V) \subseteq U$ and $\mathrm{Sp}(a, b) \subseteq V$. Then $\Theta_{A,V}$ must be the map defined in Proposition 3.3, by the uniqueness assertion of that proposition. Condition (ii) of the present theorem shows that

$$\Theta_{A,U}(f, a) = \Theta_{A,V}\big(f \circ \mathrm{pr}_{n,n+k}|_V, (a, b)\big)$$

for all $f \in \mathcal{O}(U)$, where we write (a, b) for $(a_1, \ldots, a_n, b_1, \ldots, b_k) \in A^{n+k}$.

We would like to use this equation in order to define $\Theta_{A,U}$. Before proving that this definition is independent of the choices we made, note that it will yield complex analytic maps. Indeed, if we fix b and V then the same definition can be used for all $f \in \mathcal{O}(U)$ and all $a \in \{x \in A^n;\ \mathrm{Sp}_A(x, b) \subseteq V\}$. The latter is an open subset of A_U, and the subsets of this form cover A_U if k, b, and V are allowed to vary.

Return to the situation of the first paragraph. Choose a number $l \in \mathbb{N}$, an l-tuple $b' \in A^l$, and an open polynomial polyhedron $V' \subseteq \mathbb{C}^{n+l}$ such that $\mathrm{pr}_{n,n+l}(V') \subseteq U$ and $\mathrm{Sp}(a, b') \subseteq V'$. We have to show that

$$\Theta_{A,V'}\big(f \circ \mathrm{pr}_{n,n+l}|_{V'}, (a, b')\big) = \Theta_{A,V}\big(f \circ \mathrm{pr}_{n,n+k}|_V, (a, b)\big)$$

holds for all $f \in \mathcal{O}(U)$. Define $\mathrm{pr} := \mathrm{pr}_{n+k,n+k+l} \colon \mathbb{C}^{n+k+l} \to \mathbb{C}^{n+k}$ and

$$\mathrm{pr}' \colon \mathbb{C}^{n+k+l} \longrightarrow \mathbb{C}^{n+l},\ \zeta \longmapsto (\zeta_1, \zeta_2, \ldots, \zeta_n,\ \zeta_{n+k+1}, \zeta_{n+k+2}, \ldots, \zeta_{n+k+l}).$$

Then $\mathrm{Sp}(a, b, b')$ is contained in $W := \mathrm{pr}^{-1}(V) \cap \mathrm{pr}'^{-1}(V')$, and this is an open polynomial polyhedron in $\mathbb{C}^{n+k+l}$.

The map $g \mapsto \Theta_{A,W}(g \circ \mathrm{pr}|_W, (a, b, b')) \colon \mathcal{O}(V) \to A$ is a continuous unital algebra homomorphism which maps the j-th coordinate function to a_j if $j \in \{1, \ldots, n\}$, and to b_{j-n} if $j \in \{n+1, \ldots, n+k\}$. Since the polynomials are dense in $\mathcal{O}(V)$ by Corollary 3.2, we infer that

$$\Theta_{A,W}\big(g \circ \mathrm{pr}|_W, (a, b, b')\big) = \Theta_{A,V}\big(g, (a, b)\big) \qquad (g \in \mathcal{O}(V)).$$

Similarly, we find that

$$\Theta_{A,W}\big(h \circ \mathrm{pr}'|_W, (a, b, b')\big) = \Theta_{A,V'}\big(h, (a, b')\big) \qquad (h \in \mathcal{O}(V')).$$

Putting this together, we conclude that all $f \in \mathcal{O}(U)$ satisfy

$$\begin{aligned}\Theta_{A,V'}\big(f \circ \mathrm{pr}_{n,n+l}|_{V'}, (a,b')\big) &= \Theta_{A,W}\big(f \circ \mathrm{pr}_{n,n+l} \circ \mathrm{pr}'|_W, (a,b,b')\big) \\ &= \Theta_{A,W}\big(f \circ \mathrm{pr}_{n,n+k+l}|_W, (a,b,b')\big) \\ &= \Theta_{A,W}\big(f \circ \mathrm{pr}_{n,n+k} \circ \mathrm{pr}|_W, (a,b,b')\big) \\ &= \Theta_{A,V}\big(f \circ \mathrm{pr}_{n,n+k}|_V, (a,b)\big)\end{aligned}$$

as required. □

3.9 Theorem (Naturality of $\Theta_{A,U}$ in A). *Let $\varphi\colon A \to B$ be a unital homomorphism of Mackey-complete commutative continuous inverse algebras over $\mathbb{C}$, and let $U \subseteq \mathbb{C}^n$ be open. If $f \in \mathcal{O}(U)$ and $a \in A_U \subseteq A^n$ then*

$$\varphi\big(\Theta_{A,U}(f,a)\big) = \Theta_{B,U}\big(f, (\varphi(a_1), \ldots, \varphi(a_n))\big)$$

or, writing $\varphi^{\times n}(a) := (\varphi(a_1), \ldots, \varphi(a_n))$, just $\varphi(f[a]) = f[\varphi^{\times n}(a)]$.

Proof. Assume first that U is polynomially convex. Choose $a \in A_U$, and note that $\mathrm{Sp}_B(\varphi^{\times n}(a)) \subseteq \mathrm{Sp}_A(a)$, so that $\varphi^{\times n}(a) \in B_U$. We have two continuous unital algebra homomorphisms from $\mathcal{O}(U)$ into B,

$$f \longmapsto \varphi\big(\Theta_{A,U}(f,a)\big) \qquad \text{and} \qquad f \longmapsto \Theta_{B,U}\big(f, \varphi^{\times n}(a)\big).$$

Since both map the j-th coordinate function to $\varphi(a_j)$, these homomorphisms are equal by Corollary 3.2, which proves the theorem for polynomially convex open sets U.

For a general open subset $U \subseteq \mathbb{C}^n$, let $a \in A_U$, and choose $k \in \mathbb{N}$, $b \in A^k$ and $V \subseteq \mathbb{C}^{n+k}$ as in Corollary 3.7. Then

$$\begin{aligned}\varphi\big(\Theta_{A,U}(f,a)\big) &= \varphi\Big(\Theta_{A,V}\big(f \circ \mathrm{pr}_{n,n+k}|_V, (a,b)\big)\Big) \\ &= \Theta_{B,V}\big(f \circ \mathrm{pr}_{n,n+k}|_V, \varphi^{\times(n+k)}(a,b)\big) \\ &= \Theta_{B,U}\big(f, \varphi^{\times n}(a)\big).\end{aligned}$$

This completes the proof. □

3.10 Corollary (Spectral Mapping Theorem). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. Let $U \subseteq \mathbb{C}^n$ be open, and let $f\colon U \to \mathbb{C}^m$ be complex analytic. If $a \in A_U \subseteq A^n$ then*

$$\mathrm{Sp}\big(f_1[a], \ldots, f_m[a]\big) = f\big(\mathrm{Sp}(a)\big).$$

Proof. If $f \in \mathcal{O}(U)$ and $\zeta \in U$, note that $\Theta_{\mathbb{C},U}(f,\zeta) = f(\zeta)$. Let Γ_A denote the Gelfand spectrum of A. Each $a \in A_U$ satisfies

$$\begin{aligned}
&\mathrm{Sp}\big(f_1[a],\dots,f_m[a]\big)\\
&\quad= \Big\{\Big(\chi\big(\Theta_{A,U}(f_1,a)\big),\dots,\chi\big(\Theta_{A,U}(f_m,a)\big)\Big);\chi\in\Gamma_A\Big\}\\
&\quad= \Big\{\Big(\Theta_{\mathbb{C},U}\big(f_1,\chi^{\times n}(a)\big),\dots,\Theta_{\mathbb{C},U}\big(f_m,\chi^{\times n}(a)\big)\Big);\chi\in\Gamma_A\Big\}\\
&\quad= \Big\{\Big(f_1\big(\chi^{\times n}(a)\big),\dots,f_m\big(\chi^{\times n}(a)\big)\Big);\chi\in\Gamma_A\Big\}\\
&\quad= f\big(\mathrm{Sp}(a)\big)
\end{aligned}$$

as we claimed. □

3.11 Theorem (Compatibility of $\Theta_{A,U}$ with composition). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. Let $V \subseteq \mathbb{C}^n$ and $U \subseteq \mathbb{C}^m$ be open subsets, and let $f\colon V \to U$ be complex analytic. If $g \in \mathcal{O}(U)$ and $a \in A_V \subseteq A^n$ then*

$$\Theta_{A,V}(g\circ f,a) = \Theta_{A,U}\Big(g,\big(\Theta_{A,V}(f_1,a),\dots,\Theta_{A,V}(f_m,a)\big)\Big)$$

or, in short, $(g\circ f)[a] = g\big[f[a]\big]$.

Proof. Assume first that U is polynomially convex. For each $a \in A_V$, we write

$$\Theta_{A,V}(f,a) := \big(\Theta_{A,V}(f_1,a),\dots,\Theta_{A,V}(f_m,a)\big) \in A^m,$$

and we obtain two continuous unital algebra homomorphisms from $\mathcal{O}(U)$ into A,

$$g \longmapsto \Theta_{A,V}(g\circ f,a) \qquad \text{and} \qquad g \longmapsto \Theta_{A,U}\big(g,\Theta_{A,V}(f,a)\big).$$

Since both homomorphisms map the j-th coordinate function to $\Theta_{A,V}(f_j,a)$, they are equal by Corollary 3.2, which proves the theorem for the case that U is polynomially convex.

In the general case, let $a \in A_V$, and set $b := \Theta_{A,V}(f,a) \in A^m$, so that $\mathrm{Sp}(b) = f(\mathrm{Sp}(a)) \subseteq U$. Corollary 3.7 yields a number $k \in \mathbb{N}$, a k-tuple $c \in A^k$, and an open polynomial polyhedron $W \subseteq \mathbb{C}^{m+k}$ such that $\mathrm{Sp}(b,c) \subseteq W \subseteq {\mathrm{pr}_{m,m+k}}^{-1}(U)$. Set $h := f \times \mathrm{id}_{\mathbb{C}^k} : V \times \mathbb{C}^k \to U \times \mathbb{C}^k$. Then $\Theta_{A,V\times\mathbb{C}^k}(h,(a,c)) = (b,c)$ and

$$h\big(\mathrm{Sp}(a,c)\big) = \mathrm{Sp}\Big(\Theta_{A,V\times\mathbb{C}^k}\big(h,(a,c)\big)\Big) = \mathrm{Sp}(b,c) \subseteq W,$$

whence $\mathrm{Sp}(a,c) \subseteq h^{-1}(W) =: V' \subseteq V \times \mathbb{C}^k$. The case which we already have proved yields

$$\begin{aligned}
\Theta_{A,U}(g,b) &= \Theta_{A,W}\big(g\circ \mathrm{pr}_{m,m+k}|_W,(b,c)\big)\\
&= \Theta_{A,W}\Big(g\circ \mathrm{pr}_{m,m+k}|_W,\Theta_{A,V\times\mathbb{C}^k}\big(h,(a,c)\big)\Big)\\
&= \Theta_{A,W}\Big(g\circ \mathrm{pr}_{m,m+k}|_W,\Theta_{A,V'}\big(h|_{V'},(a,c)\big)\Big)\\
&= \Theta_{A,V'}\big(g\circ \mathrm{pr}_{m,m+k}\circ h|_{V'},(a,c)\big)\\
&= \Theta_{A,V'}\big(g\circ f\circ \mathrm{pr}_{m,m+k}|_{V'},(a,c)\big)\\
&= \Theta_{A,V}(g\circ f,a),
\end{aligned}$$

which was to be proved. □

3.12 Proposition (Compatibility of $\Theta_{A,U}$ with differentiation). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$, and let $U\subseteq\mathbb{C}^n$ be open. Let $j\in\{1,\dots,n\}$ and $b\in A$. Then the $(j{+}1)$-th partial derivative of the functional calculus map $\Theta_{A,U}$ at $(f,a)\in\mathcal{O}(U)\times A_U\subseteq\mathcal{O}(U)\times A^n$ satisfies*

$$\partial_{j+1}\Theta_{A,U}(f,a_1,\dots,a_n;b)=\Theta_{A,U}(\partial_j f,a_1,\dots,a_n)\cdot b.$$

Proof. Let $\rho(b)$ be the spectral radius of b. Choose $\varepsilon>0$ such that the euclidean $(2+\rho(b))\varepsilon$-neighbourhood of $\mathrm{Sp}(a)$ in $\mathbb{C}^n$ is contained in U. Let $V\subseteq\mathbb{C}^n$ be the open ε-neighbourhood of $\mathrm{Sp}(a)$, and let $W\subseteq\mathbb{C}$ be the open disc of radius $1+\rho(b)$ around 0. For every $\zeta\in\mathbb{C}$ with $0<|\zeta|<\varepsilon$, define $g_\zeta\in\mathcal{O}(V\times W)$ by

$$g_\zeta(\xi,\eta)=\frac{f(\xi_1,\dots,\xi_j+\zeta\eta,\dots,\xi_n)-f(\xi_1,\dots,\xi_n)}{\zeta}\,.$$

Define $g_0\in\mathcal{O}(V\times W)$ by $g_0(\xi,\eta)=\partial_j f(\xi)\cdot\eta$. In view of Theorem 3.11, we have to prove that

$$\lim_{\zeta\to0,\zeta\neq0}\Theta_{A,V\times W}\big(g_\zeta,(a,b)\big)=\Theta_{A,V\times W}\big(g_0,(a,b)\big).$$

Since $\Theta_{A,V\times W}$ is continuous, it suffices to prove that $\lim_{\zeta\to0}g_\zeta=g_0$ holds in $\mathcal{O}(V\times W)$. This equation follows from uniform continuity of $\partial_j f$ on compact sets by means of the integral representation

$$g_\zeta(\xi,\eta)=\int_0^1\partial_j f(\xi_1,\dots,\xi_j+t\zeta\eta,\dots,\xi_n)\cdot\eta\;dt,$$

which holds for all $(\xi,\eta)\in V\times W$ and all $\zeta\in\mathbb{C}$ with $|\zeta|<\varepsilon$. □

4 Holomorphic germs in weak*-duals

In Section 6, we will develop the functional calculus for holomorphic functions which are defined on open neighbourhoods of the Gelfand spectrum of a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. In the present section and in Section 5, we prepare this by studying the algebra of germs of holomorphic functions in a general compact subset of a weak*-dual vector space.

Let E be a locally convex complex vector space, and consider the topological dual E' with the weak*-topology. For $x \in E^n$, define $\hat{x}\colon E' \to \mathbb{C}^n$, $\varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_n))$. Note that every continuous linear map from E' into $\mathbb{C}^n$ has this form (see Rudin [40, 3.10]). The Hahn–Banach Separation Theorem entails that the map $\hat{x}$ is surjective if and only if the n-tuple x is linearly independent. If this is the case, we fix a linear section $s_x\colon \mathbb{C}^n \to E'$ for $\hat{x}$.

For an open subset $U \subseteq E'$, let $\mathcal{O}(U)$ be the algebra of holomorphic (i.e. analytic complex-valued) functions on U, and let $\mathcal{O}^\infty(U) \subseteq \mathcal{O}(U)$ be the subalgebra of bounded holomorphic functions on U.

Suppose that $U = \hat{x}^{-1}(V)$ for a linearly independent n-tuple $x \in E^n$ and an open subset $V \subseteq \mathbb{C}^n$. (Note that the subsets U of this kind form a basis of the weak*-topology on E'.) Then we have an injective algebra homomorphism $f \mapsto f \circ \hat{x}|_U \colon \mathcal{O}^\infty(V) \to \mathcal{O}^\infty(U)$. This homomorphism is in fact bijective, with inverse given by $g \mapsto g \circ s_x|_V$. To prove this, it suffices to show that every function $g \in \mathcal{O}^\infty(U)$ is constant on the fibres of $\hat{x}|_U$. Now if $\varphi, \psi \in U$ satisfy $\hat{x}(\varphi) = \hat{x}(\psi)$ then $\zeta \mapsto g(\varphi + \zeta(\psi - \varphi))\colon \mathbb{C} \to \mathbb{C}$ is a bounded entire function, so that Liouville's Theorem yields $g(\varphi) = g(\psi)$.

In particular, this observation implies that each uniform limit of bounded holomorphic functions on an arbitrary open subset $U \subseteq E'$ is holomorphic. In other words, the algebra $\mathcal{O}^\infty(U)$ with the supremum norm is a Banach algebra.

For a compact subset $K \subseteq E'$, let $\mathcal{O}(K)$ be the algebra of germs at K of holomorphic functions defined in open neighbourhoods of K in E'. By compactness, every element of $\mathcal{O}(K)$ is the germ of a bounded holomorphic function on some open neighbourhood of K. We topologize $\mathcal{O}(K)$ as the direct limit of the system of Banach algebras $(\mathcal{O}^\infty(U))_{U \in \mathcal{U}^\circ(K)}$ in the category of locally multiplicatively convex complex algebras, where $\mathcal{U}^\circ(K)$ denotes the set of open neighbourhoods of K in E'. This topology was introduced by Warner [51]. It can be described in several ways.

(a) It is the finest locally multiplicatively convex algebra topology such that the germ maps $\gamma_U\colon \mathcal{O}^\infty(U) \to \mathcal{O}(K)$ are continuous for all $U \in \mathcal{U}^\circ(K)$. Thus it has a basis consisting of all finite intersections $W_1 \cap \cdots \cap W_m$, where each W_j is open with respect to some locally multiplicatively convex algebra topology on $\mathcal{O}(K)$ which makes all the maps γ_U continuous.

(b) The topology of $\mathcal{O}(K)$ is described by all sub-multiplicative semi-norms σ on $\mathcal{O}(K)$ for which the compositions $\sigma \circ \gamma_U$ are continuous for all $U \in \mathcal{U}^\circ(K)$. In other words, a basis of zero-neighbourhoods in $\mathcal{O}(K)$ is given by the family of convex balanced absorbing subsets $W \subseteq \mathcal{O}(K)$ with $W \cdot W \subseteq W$ for which $\gamma_U^{-1}(W) \subseteq \mathcal{O}^\infty(U)$ is a neighbourhood of 0 for all $U \in \mathcal{U}^\circ(K)$.

(c) A basis of zero-neighbourhoods in $\mathcal{O}(K)$ is also given by all $W \subseteq \mathcal{O}(K)$ of the following form. Choose a zero-neighbourhood $W_U \subseteq \mathcal{O}^\infty(U)$ for each $U \in \mathcal{U}^\circ(K)$. Set $W_1 := \bigcup \gamma_U(W_U)$ and $W_2 := \bigcup_{n=1}^\infty W_1^n$, where W_1^n is inductively defined by $W_1^n = W_1^{n-1} \cdot W_1$. Thus $W_2 \cdot W_2 \subseteq W_2$. Let W be the convex balanced hull of W_2. Then W is a zero-neighbourhood by (b). Conversely, every zero-neighbourhood $W' \in \mathcal{O}(K)$ contains a zero-neighbourhood of the form described in this paragraph.

(d) Finally, the topology is the unique locally multiplicatively convex algebra topology which satisfies the universal property that an algebra homomorphism $\varphi\colon \mathcal{O}(K) \to A$ from $\mathcal{O}(K)$ into a locally multiplicatively convex algebra A over $\mathbb{C}$ is continuous if and only if the compositions $\varphi \circ \gamma_U$ are continuous for all $U \in \mathcal{U}^\circ(K)$. We will usually apply this property to commutative continuous inverse algebras A, which are locally multiplicatively convex by Turpin's result [44].

Note that if $K \subseteq U \subseteq E'$ with K compact and U open then there exist $n \in \mathbb{N}$, an n-tuple $x \in E^n$, and an open subset $V \subseteq \mathbb{C}^n$ such that $K \subseteq \hat{x}^{-1}(V) \subseteq U$, and we may assume that x is linearly independent. Therefore, the corresponding algebras $\mathcal{O}^\infty(\hat{x}^{-1}(V)) \cong \mathcal{O}^\infty(V)$ are cofinal in the directed system which defines $\mathcal{O}(K)$.

4.1 Proposition ($\mathcal{O}(K)$ is a continuous inverse algebra). *Let E be a locally convex complex vector space, let $K \subseteq E'$ be a compact subset of the weak*-dual of E, and topologize the algebra $\mathcal{O}(K)$ of germs of holomorphic functions in K as the locally multiplicatively convex direct limit of the Banach algebras of bounded holomorphic functions on open neighbourhoods of K in E'. Then $\mathcal{O}(K)$ is a Hausdorff space and a continuous inverse algebra.*

Proof. In order to prove the Hausdorff property, choose a linearly independent n-tuple $x \in E^n$, an open neighbourhood $V \subseteq \mathbb{C}^n$ of $\hat{x}(K)$, and a function $f \in \mathcal{O}^\infty(\hat{x}^{-1}(V))$ such that the germ of f in K does not vanish. We will separate f from 0 by an algebra homomorphism ψ from $\mathcal{O}(K)$ into the algebra $\mathbb{C}[[z]]$ of formal power series such that ψ is continuous with respect to the sequence of sub-multiplicative semi-norms

$$\sum_{j=0}^\infty a_j z^j \longmapsto \sum_{j=0}^m |a_j| \colon \mathbb{C}[[z]] \longrightarrow \mathbb{R}_0^+ \qquad (m \in \mathbb{N}_0).$$

Note that these semi-norms describe the product topology on $\mathbb{C}^{\mathbb{N}_0}$ if we identify a formal power series with its sequence of coefficients. Choose $\varphi \in K$ such that the germ of f in φ does not vanish. Considering the Taylor expansion of $f \circ s_x$ at $\hat{x}(\varphi) \in \mathbb{C}^n$, we find a vector $\lambda \in \mathbb{C}^n$ such that the holomorphic function $\zeta \mapsto f\big(s_x(\hat{x}(\varphi) + \zeta\lambda)\big)$, which is defined in a neighbourhood of 0 in $\mathbb{C}$, has non-vanishing germ in $0 \in \mathbb{C}$. Define $s\colon \mathbb{C} \to E'$, $\zeta \mapsto \varphi + \zeta s_x(\lambda)$. Note that all $\zeta \in \mathbb{C}$ satisfy $\hat{x}(s(\zeta)) = \hat{x}(\varphi) + \zeta\lambda$, so that the equation $f(s(\zeta)) = f\big(s_x(\hat{x}(\varphi) + \zeta\lambda)\big)$ holds for all $\zeta \in \mathbb{C}$ for which $f(s(\zeta))$ is defined. In particular, the germ of $f \circ s$ in $0 \in \mathbb{C}$ does not vanish. For each $U \in \mathcal{U}^\circ(K)$, define an algebra homomorphism $\psi_U\colon \mathcal{O}^\infty(U) \to \mathbb{C}[[z]]$ by assigning to $g \in \mathcal{O}^\infty(U)$ the Taylor expansion of $g \circ s$ at $0 \in \mathbb{C}$. The Cauchy integral formula implies that each ψ_U is continuous. As $\mathbb{C}[[z]]$ is a locally multiplicatively convex algebra, the family $(\psi_U)_{U \in \mathcal{U}^\circ(K)}$ induces a unique continuous algebra homomorphism $\psi\colon \mathcal{O}(K) \to \mathbb{C}[[z]]$ such that $\psi \circ \gamma_U = \psi_U$ holds for each $U \in \mathcal{U}^\circ(K)$. Since $\mathbb{C}[[z]]$ is a Hausdorff space and $\psi(f) \neq 0$, we conclude that 0 and f have disjoint neighbourhoods in $\mathcal{O}(K)$.

Let $W \subseteq \mathcal{O}(K)$ be the union of the images of the open unit balls in $\mathcal{O}^\infty(U)$ under the germ maps γ_U, where U ranges over $\mathcal{U}^\circ(K)$. Then W is convex, balanced, and absorbing, and $W \cdot W \subseteq W$. Hence W is a neighbourhood of 0 in $\mathcal{O}(K)$. Since $1 + W$ is contained in the unit group $\mathcal{O}(K)^\times$, we conclude that $\mathcal{O}(K)^\times$ is a neighbourhood of 1 and hence open in $\mathcal{O}(K)$. As inversion is continuous with respect to every sub-multiplicative semi-norm (Michael [30, 2.8]), this proves that $\mathcal{O}(K)$ is a continuous inverse algebra. $\square$

The question whether $\mathcal{O}(K)$ is complete (or at least Mackey complete) is left open except in the case that the dimension of E is countable, in which a positive answer will be given in Remark 4.6. I do not know any positive result about the completeness of the direct limit of an uncountable directed system of vector spaces. Countable direct limits of Banach spaces are complete if the connecting maps are embeddings (Bourbaki [13, II, § 4, prop. 9]) or compact (Floret [16, § 7.4]). Bourbaki [11, III, § 1, exerc. 2] gives an example of an uncountable direct limit which is not complete although the directed system consists of embeddings between Banach spaces. Raĭkov [38, § 6] observed that every direct limit of Banach spaces can be written as the direct limit of a system in which the connecting maps are compact (see Floret [16, § 6.7]).

One could try to circumvent this problem by considering the completion of $\mathcal{O}(K)$. Unfortunately, it seems to be unknown whether the completion of a continuous inverse algebra is always a continuous inverse algebra.

As a substitute for completeness, the holomorphic functional calculus works for the algebra $\mathcal{O}(K)$. Indeed, let $f \in \mathcal{O}(K)^n$. Then $f(K) \subseteq \mathrm{Sp}_{\mathcal{O}(K)}(f)$ because the Gelfand spectrum $\Gamma_{\mathcal{O}(K)}$ contains the evaluations in points of K. Hence for all $g \in \mathcal{O}(\mathrm{Sp}_{\mathcal{O}(K)}(f))$, one can form $g \circ f \in \mathcal{O}(K)$.

If $f \in \mathcal{O}(K)$ is a single element then $f \in \mathcal{O}(K)^\times$ if and only if $0 \notin f(K)$, so that $\mathrm{Sp}_{\mathcal{O}(K)}(f) = f(K)$. The analogue of the last statement for the joint spectrum of an n-tuple in $\mathcal{O}(K)$ already fails if $E = \mathbb{C}^2$ and K is a Hartogs figure (see Range [37, II.2]). However, this analogue is contained in Proposition 4.5 below under the hypothesis that K is equicontinuous and rationally convex. The latter condition means that for every $\varphi \in E' \setminus K$, there is a polynomial function $p\colon E' \to \mathbb{C}$ such that $p(\varphi) \notin p(K)$.

4.2 Lemma (Approximation by rational polyhedra). *Let E be a locally convex complex vector space, let $K \subseteq E'$ be an rationally convex equicontinuous closed subset, and let $U \subseteq E'$ be an open neighbourhood of K. Then there exist a finite subset $F \subseteq E$ and a finite set P of complex-valued polynomial functions on E' such that*

$$K \subseteq \big\{\varphi \in E';\ \forall\, x \in F\colon |\varphi(x)| < 1 \text{ and } \forall\, p \in P\colon |p(\varphi)| > 1\big\} \quad \text{and}$$
$$\big\{\varphi \in E';\ \forall\, x \in F\colon |\varphi(x)| \leq 1 \text{ and } \forall\, p \in P\colon |p(\varphi)| \geq 1\big\} \subseteq U.$$

In particular, the finite set $P' := P \cup \{\hat{x};\ x \in F\}$ of polynomial functions on E' satisfies $\bigcap_{p \in P'} p^{-1}\big(p(K)\big) \subseteq U$.

Note that an equicontinuous closed subset of E' is compact by the Alaoglu–Bourbaki Theorem (Schaefer [41, III.4.3]). Conversely, if E is a Fréchet space then every weak*-compact subset of E' is equicontinuous by the Banach–Steinhaus Theorem (Rudin [40, 2.6]).

Proof. Since K is equicontinuous, the polar

$$K^\circ = \{y \in E;\ \forall\, \psi \in K\colon |\psi(y)| \leq 1\}$$

of K is a neighbourhood of 0 in E. Hence the bipolar $K^{\circ\circ}$ is an equicontinuous closed subset of E' and therefore compact. Note that $K^{\circ\circ}$ is the closed convex balanced hull of K by the Bipolar Theorem (see, for instance, Jarchow [28, 8.2.2].)

Choose a number $n \in \mathbb{N}$, an n-tuple $y \in E^n$, and an open subset $V \subseteq \mathbb{C}^n$ such that $K \subseteq \hat{y}^{-1}(V) \subseteq U$. Set $K_1 := K^{\circ\circ} \setminus \hat{y}^{-1}(V)$. For each $\psi \in K_1$, there is a polynomial function $p_\psi\colon E' \to \mathbb{C}$ such that $p_\psi(\psi) \notin p_\psi(K)$, and we may assume that $p_\psi(\psi) = 0$ and $p_\psi(K) \subseteq \{\zeta \in \mathbb{C};\ |\zeta| \geq 2\}$. Thus $\{\varphi \in E';\ |p_\psi(\varphi)| < 1\}$ is an open neighbourhood of ψ. By compactness of K_1, there is a finite subset $F_1 \in K_1$ such that

$$K_1 \subseteq \bigcup_{\psi \in F_1} \{\varphi \in E';\ |p_\psi(\varphi)| < 1\}.$$

Set $P := \{p_\psi;\ \psi \in F_1\}$. Then $K \subseteq \{\varphi \in E';\ \forall\, p \in P\colon |p(\varphi)| > 1\}$.

Enlarging n and extending y if necessary, we may assume that for every $\psi \in F_1$, there is a polynomial $q_\psi\colon \mathbb{C}^n \to \mathbb{C}$ such that $p_\psi = q_\psi \circ \hat{y}$. Set

$$C := \{\zeta \in \mathbb{C}^n;\ \forall\, \psi \in F_1\colon |q_\psi(\zeta)| \geq 1\}.$$

If $\varphi \in K^{\circ\circ}$ satisfies $\hat{y}(\varphi) \in C$ then $\varphi \notin K_1$ and hence $\varphi \in \hat{y}^{-1}(V)$. Thus $C \cap \hat{y}(K^{\circ\circ}) \subseteq V$. Since the compact convex balanced set $\hat{y}(K^{\circ\circ})$ and the closed set $C \backslash V$ are disjoint, we may choose a compact convex neighbourhood $K_2 \subseteq \mathbb{C}^n$ of $\hat{y}(K^{\circ\circ})$ which does not meet $C \setminus V$. For each boundary point $\zeta \in \partial K_2$, there is a linear functional $\psi_\zeta \in (\mathbb{C}^n)'$ such that $|\psi_\zeta(\zeta)| > 1$ and $\psi_\zeta\big(\hat{y}(K^{\circ\circ})\big) \subseteq \{\eta \in \mathbb{C};\ |\eta| < 1\}$. By compactness of ∂K_2, there is a finite subset $F_2 \subseteq \partial K_2$ such that

$$\partial K_2 \subseteq \bigcup_{\zeta \in F_2} \{\eta \in \mathbb{C}^n;\ |\psi_\zeta(\eta)| > 1\}.$$

Thus the set $\{\eta \in \mathbb{C}^n;\ \forall\, \zeta \in F_2 \colon |\psi_\zeta(\eta)| \leq 1\}$ is a convex neighbourhood of $\hat{y}(K^{\circ\circ})$ which does not meet ∂K_2 and hence is contained in the interior of K_2. For each $\zeta \in F_2$, there is a unique $x_\zeta \in E$ such that $\hat{x}_\zeta = \psi_\zeta \circ \hat{y}$. Set $F := \{x_\zeta;\ \zeta \in F_2\}$. Then $K \subseteq \{\varphi \in E';\ \forall\, x \in F \colon |\varphi(x)| < 1\}$.

It remains to prove that

$$\big\{\varphi \in E';\ \forall\, x \in F \colon |\varphi(x)| \leq 1 \text{ and } \forall\, p \in P \colon |p(\varphi)| \geq 1\big\} \subseteq U.$$

Let φ be an element of the set on the left-hand side. All $\zeta \in F_2$ satisfy $1 \geq |\varphi(x_\zeta)| = |\psi_\zeta(\hat{y}(\varphi))|$, so that $\hat{y}(\varphi) \in K_2$. All $\psi \in F_1$ satisfy $1 \leq |p_\psi(\varphi)| = |q_\psi(\hat{y}(\varphi))|$, so that $\hat{y}(\varphi) \in C$. Hence $\hat{y}(\varphi) \in C \cap K_2 \subseteq V$, and we conclude that $\varphi \in \hat{y}^{-1}(V) \subseteq U$. $\square$

4.3 Remark. Let E be a locally convex complex vector space, let $K \subseteq E'$ be an *polynomially* convex equicontinuous closed subset, and let $U \subseteq E'$ be an open neighbourhood of K. Then there is a finite set F of complex-valued polynomial functions on E' such that

$$\begin{aligned} K \subseteq \big\{\varphi \in E';\ \forall\, p \in P \colon |p(\varphi)| < 1\big\} \quad \text{and} \\ \big\{\varphi \in E';\ \forall\, p \in P \colon |p(\varphi)| \leq 1\big\} \subseteq U. \end{aligned}$$

The proof of this fact is a small modification of the preceding proof of Lemma 4.2. This result, applied to the Gelfand spectrum of a continuous inverse algebra over $\mathbb{C}$, entails Corollary 3.7. This illustrates the Arens–Calderón trick.

4.4 Lemma (Approximation by rational functions). *Let E be a locally convex complex vector space, and let $K \subseteq E'$ be an rationally convex equicontinuous closed subset. Let $f \in \mathcal{O}(K)$ and $\varepsilon > 0$. Then there are polynomial functions s and t on E' with $0 \notin t(K)$ such that $\big\|\big(f - \frac{s}{t}\big)\big|_K\big\|_\infty < \varepsilon$.*

The proof of this lemma was inspired by Rossi's work [39] on meromorphic convexity.

Proof. Choose a linearly independent n-tuple $y \in E^n$, an open neighbourhood $V \subseteq \mathbb{C}^n$ of $\hat{y}(K)$, and a holomorphic function $g \in \mathcal{O}(V)$ such that f is the germ of $g \circ \hat{y}$ in K. Set $U := \hat{y}^{-1}(V)$, and choose a finite subset $F = \{x_1, \dots, x_l\} \subseteq E$ and a finite set $P = \{p_1, \dots, p_m\}$ of complex-valued polynomial functions on E' as in Lemma 4.2. Enlarging n and extending y if necessary, we may assume that every $p \in P$ has the form $q_p \circ \hat{y}$ for some polynomial q_p on $\mathbb{C}^n$ and that F is contained in the linear span of $\{y_1, \dots, y_n\}$. We may still assume that the n-tuple y is linearly independent, so that $\hat{y} \colon E \to \mathbb{C}^n$ is surjective. Under these hypotheses, we have a well-defined complex analytic map

$$\begin{aligned} h \colon \mathbb{C}^n &\longrightarrow \mathbb{C}^{l+m+n} \\ \hat{y}(\varphi) &\longmapsto \big(\hat{x}_1(\varphi), \dots, \hat{x}_l(\varphi), p_1(\varphi), \dots, p_m(\varphi), \hat{y}_1(\varphi), \dots, \hat{y}_n(\varphi)\big), \end{aligned}$$

where $\varphi \in E'$. Note that h is a closed embedding. The complex analytic manifold

$$Y := \Big\{ \zeta \in \mathbb{C}^{l+m+n};\ |\zeta_1| < 1, \dots, |\zeta_l| < 1, |\zeta_{l+1}| > 1, \dots, |\zeta_{l+m}| > 1 \Big\}$$

is a product of open subsets of $\mathbb{C}$ and hence a Stein manifold (see, for instance, Range [37, II.3.8]). We claim that $W := h^{-1}(Y) \subseteq V$. Indeed, if $\zeta \in W$ then $\zeta = \hat{y}(\varphi)$ for some $\varphi \in E'$, and we have $|\varphi(x)| < 1$ for all $x \in F$ and $|p(\varphi)| > 1$ for all $p \in P$. Hence $\varphi \in \hat{y}^{-1}(V)$ by the choice of F and P, and we conclude that $\zeta \in V$. Similarly, we find that $\hat{y}(K) \subseteq W$. The image $h(W) = Y \cap \operatorname{im}(h)$ is a closed submanifold of Y. Since Y is a Stein manifold, every holomorphic function on $h(W)$ has a holomorphic extension to Y (Gunning and Rossi [24, VIII.A.18]). Hence there exists $k \in \mathcal{O}(Y)$ such that $k \circ h|_W = g|_W$. By Laurent extension (see Range [37, II.1.4]), there is a Laurent polynomial

$$r \in \mathbb{C}\big[\zeta_1, \dots, \zeta_l, \zeta_{l+1}, \zeta_{l+1}^{-1}, \dots, \zeta_{l+m}, \zeta_{l+m}^{-1}, \zeta_{l+m+1}, \dots, \zeta_{l+m+n}\big]$$

such that $|k(\zeta) - r(\zeta)| < \varepsilon$ holds for every $\zeta \in h(\hat{y}(K))$. In other words, every $\varphi \in K$ satisfies $|f(\varphi) - (r \circ h \circ \hat{y})(\varphi)| < \varepsilon$. Since none of the polynomial functions $p \in P$ has a zero in K, the rational function $r \circ h \circ \hat{y}$ on E' can be written as a quotient of two polynomials such that the denominator has no zero in K. $\square$

4.5 Proposition (Gelfand spectrum of $\mathcal{O}(K)$). *Let E be a locally convex complex vector space, and let $K \subseteq E'$ be an rationally convex equicontinuous closed subset of the weak*-dual of E. Then the map*

$$\Phi \colon K \longrightarrow \Gamma_{\mathcal{O}(K)}, \quad \varphi \longmapsto \big(f \mapsto f(\varphi)\big),$$

which maps $\varphi \in K$ to the evaluation in φ, is a homeomorphism. Its inverse is given by $\Phi^{-1}(\chi)(x) = \chi(\hat{x})$ for $\chi \in \Gamma_{\mathcal{O}(K)}$ and $x \in E$. In particular, every n-tuple $f \in \mathcal{O}(K)^n$ satisfies $\mathrm{Sp}_{\mathcal{O}(K)}(f) = f(K)$.

Note that the hypotheses of the proposition are satisfied if E is a continuous inverse algebra over $\mathbb{C}$ and $K \subseteq E'$ is its Gelfand spectrum.

Proof. The map Φ is continuous because $\varphi \mapsto f(\varphi)\colon K \to \mathbb{C}$ is continuous for every $f \in \mathcal{O}(K)$. It is injective because the elements of $\mathcal{O}(K)$, and indeed those of the form $\hat{x}$ for $x \in E$, separate the points of K.

Choose $\chi \in \Gamma_{\mathcal{O}(K)}$, and define a linear functional $\varphi\colon E \to \mathbb{C}$, $x \mapsto \chi(\hat{x})$. In order to see that φ is continuous, let $U \subseteq \mathbb{C}$ be a zero-neighbourhood. By equicontinuity of K, there is a zero-neighbourhood $V \subseteq E$ such that $\psi(V) \subseteq U$ holds for every $\psi \in K$. Hence every $x \in V$ satisfies

$$\varphi(x) = \chi(\hat{x}) \in \mathrm{Sp}_{\mathcal{O}(K)}(\hat{x}) = \hat{x}(K) \subseteq U.$$

Thus φ is continuous. Every polynomial function from E' into $\mathbb{C}$ has the form $p \circ \hat{x}$ for some $x \in E^n$ and some polynomial $p\colon \mathbb{C}^n \to \mathbb{C}$. The calculation

$$(p \circ \hat{x})(\varphi) = p\big(\chi(\hat{x}_1), \ldots, \chi(\hat{x}_n)\big) = \chi(p \circ \hat{x}) \in \mathrm{Sp}_{\mathcal{O}(K)}(p \circ \hat{x}) = (p \circ \hat{x})(K)$$

shows that $\varphi \in K$. Therefore, we may define a map

$$\Psi\colon \Gamma_{\mathcal{O}(K)} \longrightarrow K, \quad \chi \longmapsto \big(x \mapsto \chi(\hat{x})\big).$$

This map is continuous because $\chi \mapsto \chi(\hat{x})\colon \Gamma_{\mathcal{O}(K)} \to \mathbb{C}$ is continuous for every $x \in E$. If $\varphi \in K$ and $x \in E$ then

$$\Psi\big(\Phi(\varphi)\big)(x) = \Phi(\varphi)(\hat{x}) = \hat{x}(\varphi) = \varphi(x),$$

whence $\Psi(\Phi(\varphi)) = \varphi$.

It remains to prove that $\Phi \circ \Psi$ is the identity map on $\Gamma_{\mathcal{O}(K)}$. Let $\chi \in \Gamma_{\mathcal{O}(K)}$ and $f \in \mathcal{O}(K)$. We have to prove that $\Phi(\Psi(\chi))(f) = \chi(f)$. Choose $\varepsilon > 0$. Lemma 4.4 yields polynomial functions s and t on E' with $0 \notin t(K)$ such that $\big\| \big(f - \frac{s}{t}\big)\big|_K \big\|_\infty < \varepsilon$. The calculation

$$\chi(f) - \chi\big(\tfrac{s}{t}\big) = \chi\big(f - \tfrac{s}{t}\big) \in \mathrm{Sp}_{\mathcal{O}(K)}\big(f - \tfrac{s}{t}\big) = \big(f - \tfrac{s}{t}\big)(K)$$

shows that $\big|\chi(f) - \chi\big(\frac{s}{t}\big)\big| < \varepsilon$, and similarly $\big|\Phi(\Psi(\chi))(f) - \Phi(\Psi(\chi))\big(\frac{s}{t}\big)\big| < \varepsilon$. There exist an n-tuple $x \in E^n$ and polynomials p and q on $\mathbb{C}^n$ such that $s = p \circ \hat{x}$ and $t = q \circ \hat{x}$. We calculate

$$\begin{aligned}\Phi\big(\Psi(\chi)\big)\left(\frac{s}{t}\right) &= \frac{s}{t}\big(\Psi(\chi)\big) = \left(\frac{p}{q} \circ \hat{x}\right)\big(\Psi(\chi)\big) = \frac{p}{q}\big(\chi(\hat{x}_1), \ldots, \chi(\hat{x}_n)\big) \\ &= \frac{p\big(\chi(\hat{x}_1), \ldots, \chi(\hat{x}_n)\big)}{q\big(\chi(\hat{x}_1), \ldots, \chi(\hat{x}_n)\big)} = \frac{\chi(p \circ \hat{x})}{\chi(q \circ \hat{x})} = \chi\left(\frac{p \circ \hat{x}}{q \circ \hat{x}}\right) = \chi\left(\frac{s}{t}\right).\end{aligned}$$

Thus $|\Phi(\Psi(\chi))(f) - \chi(f)| < 2\varepsilon$. Since this holds for arbitrary $\varepsilon > 0$, we conclude that $\Phi(\Psi(\chi))(f) = \chi(f)$. $\square$

4.6 Remark. One could also topologize $\mathcal{O}(K)$ as the direct limit of the algebras $\mathcal{O}^\infty(U)$ in the category of locally convex vector spaces. This topology is finer than the one considered before. The construction of a basis of zero-neighbourhoods which is analogous to (c) in the description of the locally multiplicatively convex direct limit topology above (see Bourbaki [13, II, § 4.4]) is simpler, and the universal property applies to all linear maps into locally convex complex vector spaces rather than to algebra homomorphisms only. If E' is metrizable, which holds if and only if the dimension of E is at most countable, then $\mathcal{O}(K)$ in its locally convex direct limit topology is a complete continuous inverse algebra over $\mathbb{C}$ and hence locally multiplicatively convex, see [5] or Glöckner [21]. Therefore, the two direct limit topologies coincide for metrizable E'. This implies that $\mathcal{O}(K)$ is a Silva space and hence complete (cf. Floret [16, § 7.4]). However, if E has uncountable dimension, we will prove in Section 5 that multiplication in $\mathcal{O}(K)$ is not jointly continuous with respect to the locally convex direct limit topology. According to Turpin [44], this implies that inversion in $\mathcal{O}(K)$ is not continuous either. For this reason, we will always consider $\mathcal{O}(K)$ in its locally multiplicatively convex direct limit topology.

5 Discontinuity of multiplication in $\mathcal{O}(K)$

When we introduced the algebra $\mathcal{O}(K)$ for a compact subset K of the weak*-dual of a locally convex vector space E in Section 4, we announced that its multiplication is no longer continuous if its topology is replaced by the direct limit topology in the category of locally convex complex vector spaces, provided that the dimension of E is uncountable. This section is devoted to the proof of this fact.

5.1 Theorem. *Let E be a locally convex complex vector space of uncountable dimension, let $K \subseteq E'$ be a compact subset of the weak*-dual of E, and topologize the algebra $\mathcal{O}(K)$ of germs of holomorphic functions in K as the locally convex direct limit of the Banach algebras of bounded holomorphic functions on open neighbourhoods of K in E'. Then multiplication in $\mathcal{O}(K)$ is not jointly continuous.*

This situation arises in many relevant examples. For instance, the Baire Category Theorem implies that every infinite-dimensional Fréchet space has uncountable dimension. Note, however, that a real vector space of countable dimension is complete with respect to its finest locally convex topology (cf. Bourbaki [13, II, § 4.6]), i.e. if it is topologized as the locally convex direct limit of its finite-dimensional subspaces.

Note that the weak*-topology is slightly more general than it might seem. Indeed, let $E \times F \to \mathbb{C}$ be a non-degenerate bilinear pairing of complex vector spaces. Then F, equipped with the E-topology, is the weak*-dual of E, equipped with the F-topology (cf. Rudin [40, 3.10]).

Proof of Theorem 5.1. By translation invariance, we may assume that $0 \in K$. Choose a basis $B \subseteq E$ of the vector space E. This gives rise to a linear topological embedding $E' \hookrightarrow \mathbb{C}^B$, $\varphi \mapsto (x \mapsto \varphi(x))$. Let $\mathcal{F}$ be the set of finite subsets of B. If $F \in \mathcal{F}$ then the linear map

$$\hat{F}\colon E' \longrightarrow \mathbb{C}^F, \quad \varphi \longmapsto \big(x \mapsto \hat{x}(\varphi) = \varphi(x)\big),$$

which is the composition of the above embedding $E' \hookrightarrow \mathbb{C}^B$ with the restriction projection $\mathbb{C}^B \to \mathbb{C}^F$, is surjective by the Hahn–Banach Separation Theorem. Like in Section 4, we fix a linear section $s_F\colon \mathbb{C}^F \to E'$ for $\hat{F}$.

Equip $\mathbb{C}^F$ with the maximum metric. For each $F \in \mathcal{F}$ and each $n \in \mathbb{N}$, set

$$V_{F,n} := \left\{\zeta \in \mathbb{C}^F;\ d(\zeta, \hat{F}(K)) < \tfrac{1}{n}\right\}$$

and $U_{F,n} := \hat{F}^{-1}(V_{F,n})$. Note that $V_{F,n}$ contains the open polycylinder with centre 0 and polyradius $\big(\frac{1}{n}, \dots, \frac{1}{n}\big)$ because $0 \in K$. All $F, F' \in \mathcal{F}$ and $n \in \mathbb{N}$ satisfy $U_{F \cup F',n} \subseteq U_{F,n} \cap U_{F',n}$. The sets $U_{F,n}$ form a neighbourhood basis of K in E'. As we saw at the beginning of Section 4, the map

$$f \longmapsto f \circ \hat{F}|_{U_{F,n}}\colon \mathcal{O}^\infty(V_{F,n}) \longrightarrow \mathcal{O}^\infty(U_{F,n})$$

is an isomorphism with inverse $g \mapsto g \circ s_F|_{V_{F,n}}$. Let $A_{F,n} \subseteq A := \mathcal{O}(K)$ be the subalgebra of germs of functions in $\mathcal{O}^\infty(U_{F,n})$, and let $B_{F,n} \subseteq A_{F,n}$ be the open unit ball with respect to the supremum norm on $\mathcal{O}^\infty(U_{F,n})$.

The locally convex direct limit topology on A can be described as follows (see Bourbaki [13, II, § 4.4]). Since the $U_{F,n}$ form a neighbourhood basis of K in E', the algebras $A_{F,n}$ are cofinal in the directed system which defines A. For every function $h\colon \mathcal{F} \times \mathbb{N} \to \mathbb{R}^+$, set

$$W_h := \operatorname{conv} \bigcup \{h(F,n)\ B_{F,n};\ F \in \mathcal{F},\ n \in \mathbb{N}\}\,.$$

The sets W_h form a basis of zero-neighbourhoods in A.

Since B is an uncountable set, a lemma due to Bisgaard [6] yields a function $g\colon B \times B \to \mathbb{R}^+$ with the property that for every function $f\colon B \to \mathbb{R}^+$, there is some $(x,y) \in B^2$ such that $g(x,y) < f(x) \cdot f(y)$. Bisgaard's proof shows that one may assume $x \neq y$. (Instead of analysing the proof, one could apply Bisgaard's Lemma to the function $x \mapsto \min\{f(x), \sqrt{g(x,x)}\}$ in the place of f.) Define

$$h\colon \mathcal{F} \times \mathbb{N} \longrightarrow \mathbb{R}^+, \quad (F,n) \longmapsto \frac{1}{n^2}\ \min\big\{g(x,y);\ (x,y) \in F^2\big\}\,,$$

and let W_h be the zero-neighbourhood in A defined in terms of h as above. Let $k\colon \mathcal{F}\times\mathbb{N}\to\mathbb{R}^+$ be arbitrary. We will show that $W_k\cdot W_k\not\subseteq W_h$, which implies that multiplication in A is not continuous. Define

$$f\colon B\longrightarrow\mathbb{R}^+,\quad x\longmapsto\frac{k(\{x\},1)}{2\,\sup\left\{|\zeta|;\,\zeta\in V_{\{x\},1}\right\}}\,,$$

so that every $x\in B$ satisfies

$$f(x)\cdot\hat{x}\in k(\{x\},1)\;B_{\{x\},1}\subseteq W_k.$$

Bisgaard's Lemma yields $x,y\in B$ with $x\neq y$ such that $g(x,y)<f(x)\cdot f(y)$. We claim that the germ

$$u:=f(x)\cdot\hat{x}\cdot f(y)\cdot\hat{y}\in W_k\cdot W_k$$

does not belong to W_h. Suppose, to the contrary, that u can be written as a convex combination $u=\sum_{j=1}^{l}\lambda_j u_j$ with $\lambda_j\geq 0$ and $\sum_{j=1}^{l}\lambda_j=1$, where

$$u_j\in h(F_j,n_j)\;B_{F_j,n_j}\text{ for some }F_j\in\mathcal{F}\text{ and some }n_j\in\mathbb{N}.$$

Set $F:=\{x,y\}\cup\bigcup_{j=1}^{l}F_j$. Observe that

$$\hat{F}(U_{F_j,n_j})=s_F^{-1}(U_{F_j,n_j})=\left\{\zeta\in\mathbb{C}^F;\,(\zeta_z)_{z\in F_j}\in V_{F_j,n_j}\right\}$$

holds for all $j\in\{1,\ldots,l\}$. Extending the germs u and u_j to holomorphic functions in a natural way, we may define holomorphic functions on neighbourhoods of 0 in $\mathbb{C}^F$ by

$$\tilde{u}:=\frac{1}{g(x,y)}\;u\circ s_F\in\mathcal{O}^\infty\big(s_F^{-1}(U_{\{x,y\},1})\big),$$
$$\tilde{u}_j:=\frac{1}{g(x,y)}\;u_j\circ s_F\in\mathcal{O}^\infty\big(s_F^{-1}(U_{F_j,n_j})\big).$$

Note that $\tilde{u}(\zeta)=c\,\zeta_x\zeta_y$ with $c:=\frac{f(x)\cdot f(y)}{g(x,y)}>1$. Moreover, if $\{x,y\}\subseteq F_j$ then

$$\big\|\tilde{u}_j\big\|_\infty<\frac{h(F_j,n_j)}{g(x,y)}\leq\frac{1}{n_j^2}\,.$$

If $x\notin F_j$ then $\left\{\zeta_x;\,\zeta\in s_F^{-1}(U_{F_j,n_j})\right\}=\mathbb{C}$, whence $\tilde{u}_j$ does not depend on ζ_x by Liouville's Theorem. Similarly, if $y\notin F_j$ then $\tilde{u}_j$ does not depend on ζ_y. Hence all $j\in\{1,\ldots,l\}$ satisfy

$$\big\|\tilde{u}_j\big\|_\infty<\frac{1}{n_j^2}\qquad\text{or}\qquad\frac{\partial^2\tilde{u}_j}{\partial\zeta_x\partial\zeta_y}(0)=0.$$

As the germs of $\tilde{u}$ and of $\sum_{j=1}^{l}\lambda_j\tilde{u}_j$ at $0\in\mathbb{C}^F$ coincide, we have reached a contradiction to the following lemma. $\square$

5.2 Lemma. *Let $l, m \in \mathbb{N}$ with $m \geq 2$, and let $n_1, \dots, n_l \in \mathbb{N}$. For each $j \in \{1, \dots, l\}$, let $f_j \in \mathcal{O}^\infty\big(\frac{1}{n_j} D^m\big)$, where $D := \{\zeta \in \mathbb{C};\ |\zeta| < 1\}$ denotes the open unit disc, and assume that*

$$\|f_j\|_\infty < \frac{1}{n_j^2} \qquad \text{or} \qquad \frac{\partial^2 f_j}{\partial \zeta_1 \partial \zeta_2}(0) = 0.$$

Let $\lambda_1, \dots, \lambda_l \geq 0$ with $\sum_{j=1}^l \lambda_j = 1$, let $c \in \mathbb{C}$, and assume that

$$c\,\zeta_1 \zeta_2 = \sum_{j=1}^{l} \lambda_j f_j(\zeta)$$

holds for all sufficiently small $\zeta \in \mathbb{C}^m$. Then $|c| < 1$.

Proof. The equation in the lemma holds, in particular, for all sufficiently small $\zeta \in \mathbb{C}^m$ with $\zeta_3 = \zeta_4 = \dots = \zeta_m = 0$. Therefore, it suffices to treat the case that $m = 2$, which we will now assume.

Fix $j \in \{1, \dots, l\}$. The function f_j has a Taylor series expansion

$$f_j(\zeta) = \sum_{k \in \mathbb{N}_0{}^2} a_k \zeta_1^{k_1} \zeta_2^{k_2} \qquad \left(\zeta \in \tfrac{1}{n_j} D^2\right),$$

and the coefficients $a_k \in \mathbb{C}$ satisfy the Cauchy estimates $|a_k| \leq \|f_j\|_\infty \cdot n_j^{k_1 + k_2}$ (see, for instance, Range [37, Chapter 1]). In particular, the hypotheses on f_j imply that the coefficient $a_{(1,1)} = \frac{\partial^2 f_j}{\partial \zeta_1 \partial \zeta_2}(0)$ satisfies $|a_{(1,1)}| < 1$. Define holomorphic functions by

$$\begin{aligned}
g_j &: \tfrac{1}{n_j} D \longrightarrow \mathbb{C}, \quad \zeta \longmapsto \sum_{k=1}^{\infty} a_{(k,0)} \zeta^{k-1}, \\
h_j &: \tfrac{1}{n_j} D \longrightarrow \mathbb{C}, \quad \zeta \longmapsto \sum_{k=1}^{\infty} a_{(0,k)} \zeta^{k-1}, \\
k_j &: \tfrac{1}{n_j} D^2 \longrightarrow \mathbb{C}, \quad \zeta \longmapsto \sum_{k \in \mathbb{N}^2} a_k \zeta_1^{k_1 - 1} \zeta_2^{k_2 - 1},
\end{aligned}$$

so that

$$f_j(\zeta) = f_j(0) + \zeta_1 g_j(\zeta_1) + \zeta_2 h_j(\zeta_2) + \zeta_1 \zeta_2 k_j(\zeta)$$

and $|k_j(0)| = |a_{(1,1)}| < 1$.

We obtain the equation

$$c\zeta_1\zeta_2 = \sum_{j=1}^{l} \lambda_j f_j(0) + \zeta_1 \sum_{j=1}^{l} \lambda_j g_j(\zeta_1) + \zeta_2 \sum_{j=1}^{l} \lambda_j h_j(\zeta_2) + \zeta_1\zeta_2 \sum_{j=1}^{l} \lambda_j k_j(\zeta)$$

for all sufficiently small $\zeta \in \mathbb{C}^2$. Setting the variables equal to 0 in turn, we find that the first three sums on the right-hand side vanish. Thus $c = \sum_{j=1}^{l} \lambda_j k_j(\zeta)$. For $\zeta = 0$, we conclude that

$$|c| = \left|\sum_{j=1}^{l} \lambda_j k_j(0)\right| \leq \sum_{j=1}^{l} \lambda_j |k_j(0)| < 1,$$

which was to be proved. □

6 Holomorphic functions on neighbourhoods of the Gelfand spectrum

6.1 Theorem (The functional calculus for holomorphic functions on neighbourhoods of the Gelfand spectrum). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. Then there is a unique continuous unital algebra homomorphism $\Xi_A\colon \mathcal{O}(\Gamma_A) \to A$ with $\Xi_A\big((\hat{a})^{\sim}\big) = a$ for every $a \in A$.*

Here $(\hat{a})^{\sim}$ denotes the germ of $\hat{a}$ in Γ_A.

Proof. Assume that $\Xi_A\colon \mathcal{O}(\Gamma_A) \to A$ is a homomorphism with the required properties. For each $n \in \mathbb{N}$ and each open subset $U \subseteq \mathbb{C}^n$, define

$$\theta_U\colon \mathcal{O}(U) \times A_U \longrightarrow A, \quad (f, a) \longmapsto \Xi_A\big((f \circ \hat{a})^{\sim}\big).$$

Choose $a \in A_U$. Let $V \subseteq \mathbb{C}^n$ be a relatively compact open neighbourhood of $\mathrm{Sp}(a)$ such that $\overline{V} \subseteq U$. If $f \in \mathcal{O}(U)$ then $f \circ \hat{a}|_{\hat{a}^{-1}(V)} \in \mathcal{O}^{\infty}(\hat{a}^{-1}(V))$. This shows that $f \mapsto (f \circ \hat{a})^{\sim}\colon \mathcal{O}(U) \to \mathcal{O}(\Gamma_A)$ is continuous. Hence $\theta_U(\cdot, a)\colon \mathcal{O}(U) \to A$ is a continuous unital algebra homomorphism, and it maps the j-th coordinate function to a_j.

Let $m \leq n$, and let $V \subseteq \mathbb{C}^n$ and $U \subseteq \mathbb{C}^m$ be open sets with $\mathrm{pr}_{m,n}(V) \subseteq U$. If $f \in \mathcal{O}(U)$ and $a \in A_V \subseteq A^n$ then

$$\begin{aligned}
\theta_V\big(f \circ \mathrm{pr}_{m,n}|_V, (a_1, \ldots, a_n)\big) &= \Xi_A\Big(\big(f \circ \mathrm{pr}_{m,n} \circ (\hat{a}_1, \ldots, \hat{a}_n)\big)^{\sim}\Big) \\
&= \Xi_A\Big(\big(f \circ (\hat{a}_1, \ldots, \hat{a}_m)\big)^{\sim}\Big) \\
&= \theta_U\big(f, (a_1, \ldots, a_m)\big).
\end{aligned}$$

The uniqueness of the n-variable holomorphic functional calculus (Theorem 3.8) yields $\theta_U = \Theta_{A,U}$ for each open subset U of some $\mathbb{C}^n$. Since every element of $\mathcal{O}(\Gamma_A)$ is the germ of $f \circ \hat{a}$ for suitable $U \subseteq \mathbb{C}^n$ and $(f, \hat{a}) \in \mathcal{O}(U) \times A_U$, the equation

$$\Xi_A\big((f \circ \hat{a})^{\sim}\big) = \Theta_{A,U}(f, a) \tag{1}$$

proves uniqueness of Ξ_A.

We will use this equation in order to define Ξ_A. Let $U \subseteq A'$ be an open neighbourhood of Γ_A. Choose $m \in \mathbb{N}$, a linearly independent m-tuple $a \in A^m$, and a bounded open subset $V \subseteq \mathbb{C}^m$ such that $\Gamma_A \subseteq \hat{a}^{-1}(V) \subseteq U$. Let $s_a \colon \mathbb{C}^m \to A'$ be a linear section for $\hat{a}$. Define a continuous unital algebra homomorphism

$$\xi_U \colon \mathcal{O}^\infty(U) \longrightarrow A, \quad f \longmapsto \Theta_{A,V}(f \circ s_a|_V, a).$$

We claim that ξ_U does not depend on the choice of m, a, V, and s_a. Different choices of s_a lead to the same composition $f \circ s_a|_V$ because every function $f \in \mathcal{O}^\infty(U)$ is constant on the fibres $\hat{a}^{-1}(\zeta)$ for $\zeta \in V$. The choice of V does not matter since if $V' \subseteq \mathbb{C}^m$ is a bounded open subset with $\Gamma_A \subseteq \hat{a}^{-1}(V') \subseteq U$ then $f \circ s_a|_V$ and $f \circ s_a|_{V'}$ coincide on the neighbourhood $V \cap V'$ of $\mathrm{Sp}(a)$. Choose $n \in \mathbb{N}$, a linearly independent n-tuple $b \in A^n$, and a bounded open subset $W \subseteq \mathbb{C}^n$ such that $\Gamma_A \subseteq \hat{b}^{-1}(W) \subseteq U$, and let $s_b \colon \mathbb{C}^n \to A'$ be a linear section for $\hat{b}$. In order to prove that $\Theta_{A,V}(f \circ s_a|_V, a) = \Theta_{A,W}(f \circ s_b|_W, b)$, we may assume that $\hat{b}^{-1}(W) \subseteq \hat{a}^{-1}(V)$. The set $\hat{b}^{-1}(W)$ contains a translate of $\ker(\hat{b})$, and the image of this affine subspace under $\hat{a}$ is contained in V and hence bounded. This shows that $\ker(\hat{b}) \subseteq \ker(\hat{a})$. Hence there is a linear map $\lambda \colon \mathbb{C}^n \to \mathbb{C}^m$ such that $\hat{a} = \lambda \circ \hat{b}$. The Hahn–Banach Theorem shows that the matrix $(l_{jk})_{jk}$ of λ with respect to the standard bases satisfies $a_j = \sum_k l_{jk} b_k$ whenever $1 \leq j \leq m$. This implies that $\Theta_{A,\mathbb{C}^n}(\lambda_j, b) = a_j$. Let $\varphi \in \hat{b}^{-1}(W)$. Then $f(\varphi) = (f \circ s_b \circ \hat{b})(\varphi)$ and $f(\varphi) = (f \circ s_a \circ \hat{a})(\varphi) = (f \circ s_a \circ \lambda \circ \hat{b})(\varphi)$. Hence $f \circ s_b|_W = f \circ s_a \circ \lambda|_W$. Compatibility of the n-variable holomorphic functional calculus with composition of analytic maps (Theorem 3.11) allows us to conclude that

$$\begin{aligned} \Theta_{A,W}(f \circ s_b|_W, b) &= \Theta_{A,W}(f \circ s_a \circ \lambda|_W, b) \\ &= \Theta_{A,V}\Big(f \circ s_a|_V, \big(\Theta_{A,W}(\lambda_j|_W, b)\big)_{j=1,\dots,m}\Big) \\ &= \Theta_{A,V}(f \circ s_a|_V, a). \end{aligned}$$

Hence the definition of ξ_U is indeed independent of all our choices. Moreover, if $U' \subseteq A'$ is an open neighbourhood of Γ_A with $U' \subseteq U$ then the same argument shows that every $f \in \mathcal{O}^\infty(U)$ satisfies $\xi_{U'}(f|_{U'}) = \xi_U(f)$. Hence the system $(\xi_U)_U$ induces a continuous unital algebra homomorphism Ξ_A from the direct limit $\mathcal{O}(\Gamma_A)$ into A. Finally, let $a \in A \setminus \{0\}$, so that $\hat{a} \colon A' \to \mathbb{C}$ is surjective, and let $V \subseteq \mathbb{C}$ be a bounded neighbourhood of $\mathrm{Sp}(a)$. Then $\Xi_A\big((\hat{a})^\sim\big) = \Theta_{A,V}(\mathrm{id}_V, a) = a$. □

6.2 Theorem (Naturality of Ξ_A). *Let $\varphi \colon A \to B$ be a unital homomorphism of Mackey-complete commutative continuous inverse algebras over $\mathbb{C}$, and let $\varphi^* \colon B' \to A'$ be the adjoint map. If $f \in \mathcal{O}(\Gamma_A)$ then*

$$\varphi\big(\Xi_A(f)\big) = \Xi_B(f \circ \varphi^*).$$

Note that the germ $f \circ \varphi^*$ is defined because $\varphi^*(\Gamma_B) \subseteq \Gamma_A$.

Proof. Choose $n \in \mathbb{N}$, an n-tuple $a \in A^n$, an open neighbourhood $U \subseteq \mathbb{C}^n$ of $\mathrm{Sp}_A(a)$, and a function $g \in \mathcal{O}(U)$ such that $f = (g \circ \hat{a})^{\sim}$. Then Equation (1) in the proof of Theorem 6.1 and naturality of the n-variable holomorphic functional calculus with respect to algebra homomorphisms (Theorem 3.9) yield

$$\begin{aligned}\Xi_B(f \circ \varphi^*) &= \Xi_B\big((g \circ \hat{a} \circ \varphi^*)^{\sim}\big) = \Xi_B\Big(\big(g \circ \varphi^{\times n}(a)^{\wedge}\big)^{\sim}\Big) \\ &= \Theta_{B,U}\big(g, \varphi^{\times n}(a)\big) = \varphi\big(\Theta_{A,U}(g,a)\big) = \varphi\Big(\Xi_A\big((g \circ \hat{a})^{\sim}\big)\Big) = \varphi\big(\Xi_A(f)\big)\end{aligned}$$

as required. ☐

6.3 Corollary (Compatibility of Ξ_A with characters). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. If $f \in \mathcal{O}(\Gamma_A)$ and $\chi \in \Gamma_A$ then $\chi\big(\Xi_A(f)\big) = f(\chi)$. In particular, $\mathrm{Sp}\big(\Xi_A(f)\big) = f(\Gamma_A)$.*

Proof. In order to understand $\Xi_{\mathbb{C}}$, we identify $\mathbb{C}$ with its own dual, using multiplication as the pairing. Then $\Gamma_{\mathbb{C}} = \{1\}$ and $\chi^*(\zeta) = \zeta \cdot \chi$. The germ at 1 of a holomorphic function g defined on an open neighbourhood of 1 in $\mathbb{C}$ can be identified with the power series expansion of g at 1. We have $\Xi_{\mathbb{C}}(1) = 1$ and $\Xi_{\mathbb{C}}\big(\widetilde{\mathrm{id}_{\mathbb{C}}}\big) = \Xi_{\mathbb{C}}\big((\hat{1})^{\sim}\big) = 1$. Continuity of $\Xi_{\mathbb{C}}$ implies that

$$\Xi_{\mathbb{C}}\left(\left(\zeta \mapsto \sum_{n=0}^{\infty} a_n(\zeta-1)^n\right)^{\sim}\right) = a_0$$

or, in other words, that $\Xi_{\mathbb{C}}(\tilde{g}) = g(1)$. We conclude from Theorem 6.2 that

$$\chi\big(\Xi_A(f)\big) = \Xi_{\mathbb{C}}\big((f \circ \chi^*)^{\sim}\big) = f(\chi^*(1)) = f(\chi).$$ ☐

6.4 Proposition (Compatibility of Ξ_A with composition). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. Let $f \in \big(\mathcal{O}(\Gamma_A)\big)^n$, let $U \subseteq \mathbb{C}^n$ be an open neighbourhood of $f(\Gamma_A)$, and let $g \in \mathcal{O}(U)$. Then*

$$\Xi_A(g \circ f) = \Theta_{A,U}\Big(g, \big(\Xi_A(f_1), \ldots, \Xi_A(f_n)\big)\Big).$$

Proof. Choose a number $m \in \mathbb{N}$, an m-tuple $a \in A^m$, an open neighbourhood $V \subseteq \mathbb{C}^m$ of $\mathrm{Sp}(a)$, and a complex analytic map $h\colon V \to U$ such that $f = (h \circ \hat{a})^{\sim}$. Then Equation (1) in the proof of Theorem 6.1 and compatibility of the n-variable holomorphic functional calculus with composition of analytic maps (Theorem 3.11) show that

$$\begin{aligned}\Xi_A(g \circ f) &= \Xi_A\big((g \circ h \circ \hat{a})^{\sim}\big) = \Theta_{A,V}(g \circ h, a) \\ &= \Theta_{A,U}\Big(g, \big(\Theta_{A,V}(h_j, a)\big)_{j=1,\ldots,n}\Big) = \Theta_{A,U}\Big(g, \big(\Xi_A(f_j)\big)_{j=1,\ldots,n}\Big),\end{aligned}$$

which is what we had to prove. ☐

7 The Implicit Function Theorem

Let A be a Mackey-complete continuous inverse algebra over $\mathbb{C}$. Let $U \subseteq \mathbb{C}^m \times \mathbb{C}^n$ be open, let $F\colon U \to \mathbb{C}^n$ be complex analytic, and let $a \in A^m$. We would like to find an n-tuple $b \in A^n$ such that $\mathrm{Sp}(a,b) \subseteq U$ and $F[a,b] = 0$. A necessary condition for the existence of a solution $b \in A^n$ is the existence of a continuous function $g\colon \Gamma_A \to \mathbb{C}^n$ such that $F(\hat{a}(\chi), g(\chi))$ is defined and vanishes for all $\chi \in \Gamma_A$. Indeed, the Gelfand transform of a solution is such a function. We prove that this condition is also sufficient if $\partial_2 F(\hat{a}(\chi), g(\chi); \cdot) \in \mathrm{End}(\mathbb{C}^n)$ is invertible for all $\chi \in \Gamma_A$. Under this hypothesis, there is exactly one solution $b \in A^n$ which satisfies $\hat{b}|_{\Gamma_A} = g$.

When $n = 1$ and A is a Banach algebra, this is due to Arens and Calderón [3] (for polynomial equations) and to Gamelin [17]. We combine the infinite-dimensional holomorphic functional calculus with recent results on implicitly defined functions from locally convex vector spaces into Banach spaces due to Hiltunen [26, 27] and Glöckner [22] in order to obtain a proof of the n-variable result. Even in the one-variable situation, this proof seems particularly short and transparent.

7.1 Proposition (Analytic extension of implicit functions). *Let X be a locally convex vector space over $\mathbb{C}$, and let Y be a complex Banach space. Let $U \subseteq X \times Y$ be open, and let $F\colon U \to Y$ be complex analytic. Let $K \subseteq X$ be compact. Let $g\colon K \to Y$ be a continuous map such that for all $x \in K$, the relations $(x, g(x)) \in U$ and $F(x, g(x)) = 0$ hold, and the continuous linear endomorphism $\partial_2 F(x, g(x); \cdot)$ of Y is invertible. Then there is an open neighbourhood $U' \subseteq U$ of the graph of g and a complex analytic function $h\colon \mathrm{pr}_1(U') \to Y$ such that the graph of h is $U' \cap F^{-1}(0)$.*

In other words, for every $x \in \mathrm{pr}_1(U')$, the equation $F(x,y) = 0$ has a unique solution $(x,y) \in U'$, and this solution depends complex analytically on x. (Here $\mathrm{pr}_1\colon X \times Y \to X$ is the canonical projection.)

Proof. For each $x \in K$, the Implicit Function Theorem as given by Glöckner [22, 2.3] yields open neighbourhoods $V_x \subseteq X$ of x and $W_x \subseteq Y$ of $g(x)$ and a complex analytic function $h_x\colon V_x \to W_x$ such that $V_x \times W_x \subseteq U$, and the graph of h_x is $(V_x \times W_x) \cap F^{-1}(0)$. We may assume that $g(V_x \cap K) \subseteq W_x$, so that the functions g and h_x agree on $V_x \cap K$, and that V_x is convex.

By Lebesgue's Lemma, we may choose an open convex balanced neighbourhood V of 0 in X such that for every $x \in K$, there is a $y \in K$ such that $x + 3V \subseteq V_y$. For each $x \in K$, the set $V'_x := V_x \cap (x + V)$ is an open convex neighbourhood of x. Choose $x, y \in K$, and assume that $V'_x \cap V'_y \neq \emptyset$. Then $y + V \subseteq x + 3V$, and we find $z \in K$ such that $V'_x \cup V'_y \subseteq V_z$. The set $\{x' \in V'_x;\ h_x(x') = h_z(x')\}$ is both open and closed in V'_x, and it is not empty because it contains x. Hence the functions h_x and h_z agree on V'_x.

Similarly, the functions h_y and h_z agree on V'_y. Therefore, the functions h_x and h_y agree on the intersection $V'_x \cap V'_y$.

Set $V' := \bigcup_{x\in K} V'_x$. The preceding paragraph shows that we may define an analytic function $h\colon V' \to Y$ by $h(x') = h_x(x')$ if $x' \in V'_x$. This function and the open neighbourhood $U' := \bigcup_{x\in K} V'_x \times W_x$ of the graph of g have the required properties. $\square$

7.2 Theorem (Analytic equations in continuous inverse algebras). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. Let $U \subseteq \mathbb{C}^m \times \mathbb{C}^n$ be open, and let $F\colon U \to \mathbb{C}^n$ be complex analytic. Let $a \in A^m$, and let $g\colon \Gamma_A \to \mathbb{C}^n$ be continuous. For all $\chi \in \Gamma_A$, suppose that $(\hat{a}(\chi), g(\chi)) \in U$ and $F(\hat{a}(\chi), g(\chi)) = 0$ and that $\partial_2 F(\hat{a}(\chi), g(\chi); \cdot) \in \operatorname{End}(\mathbb{C}^n)$ is invertible. Then there is a unique n-tuple $b \in A^n$ such that $\hat{b}|_{\Gamma_A} = g$ and $\Theta_{A,U}(F, (a, b)) = 0$.*

Proof. Set $U' := (\hat{a} \times \mathrm{id}_{\mathbb{C}^n})^{-1}(U) \subseteq A' \times \mathbb{C}^n$, and define

$$F'\colon U' \longrightarrow \mathbb{C}^n, \quad (\varphi, \zeta) \longmapsto F\big(\hat{a}(\varphi), \zeta\big).$$

Proposition 7.1 yields an open neighbourhood $V \subseteq A'$ of Γ_A and a complex analytic function $h\colon V \to \mathbb{C}^n$ such that $h|_{\Gamma_A} = g$, and every $\varphi \in V$ satisfies $(\varphi, h(\varphi)) \in U'$ and $0 = F'(\varphi, h(\varphi)) = F(\hat{a}(\varphi), h(\varphi))$. Set

$$b := \Xi_A(\tilde{h}) = \big(\Xi_A(\tilde{h}_1), \ldots, \Xi_A(\tilde{h}_n)\big) \in A^n.$$

If $\chi \in \Gamma_A$ then $\hat{b}(\chi) = \chi^{\times n}(b) = h(\chi) = g(\chi)$ by Corollary 6.3. Moreover, Proposition 6.4 shows that

$$\begin{aligned}
&\Theta_{A,U}\big(F, (a, b)\big) \\
&\quad = \Theta_{A,U}\Big(F, \Big(\Xi_A\big((\hat{a}_1)^\sim\big), \ldots, \Xi_A\big((\hat{a}_m)^\sim\big), \Xi_A\big(\tilde{h}_1\big), \ldots, \Xi_A\big(\tilde{h}_n\big)\Big)\Big) \\
&\qquad\qquad\qquad\qquad = \Xi_A\Big(\big(F \circ (\hat{a}, h)\big)^\sim\Big) = \Xi_A(0) = 0.
\end{aligned}$$

This proves the existence of a solution $b \in A^n$.

Let $c \in A^n$ be an n-tuple with $\hat{c}|_{\Gamma_A} = g$ and $\Theta_{A,U}(F, (a, c)) = 0$. Set $r := c - b$, and let $W \subseteq \mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^n$ be the set of all (ζ, ξ, η) such that U contains the line segment $[(\zeta, \xi), (\zeta, \xi + \eta)]$. Then W is an open neighbourhood of $\mathrm{Sp}(a, b, r) = \mathrm{Sp}(a, b) \times \{0\}$. For $j, k, l \in \{1, \ldots, n\}$, let $e_k, e_l \in \mathbb{C}^n$ be the standard basis vectors, and define complex analytic maps

$$\begin{aligned}
G_{jk}\colon U &\longrightarrow \mathbb{C}, & (\zeta, \xi) &\longmapsto \partial_2 F_j(\zeta, \xi; e_k) \qquad \text{and} \\
H_{jkl}\colon W &\longrightarrow \mathbb{C}, & (\zeta, \xi, \eta) &\longmapsto \int_0^1 (1-t)\, \partial_2^2 F_j(\zeta, \xi + t\eta; e_k, e_l)\, dt.
\end{aligned}$$

Here $\partial_2^2 F_j$ is the derivative of the map $\xi \mapsto \partial_2 F_j(\zeta, \xi; \eta)$. By the Taylor Formula, all $(\zeta, \xi, \eta) \in W$ satisfy

$$F_j(\zeta, \xi + \eta) - F_j(\zeta, \xi) = \sum_{k=1}^{n} G_{jk}(\zeta, \xi)\eta_k + \sum_{k,l=1}^{n} H_{jkl}(\zeta, \xi, \eta)\eta_k \eta_l.$$

Applying $\Theta_{A,W}$ to both sides of this equation at (a, b, r), we find that $0 = \sum_{k=1}^{n} m_{jk} r_k$, where

$$m_{jk} = G_{jk}[a, b] + \sum_{l=1}^{n} r_l H_{jkl}[a, b, r].$$

Let $M \in \mathrm{M}_n(A)$ be the $n \times n$ matrix with entries m_{jk}. For all $\chi \in \Gamma_A$,

$$\begin{aligned}\chi(\det M) &= \det\big(\chi(m_{jk})\big)_{j,k\in\{1,\ldots,n\}} = \det\big(\chi(G_{jk}[a, b])\big)_{j,k\in\{1,\ldots,n\}} \\ &= \det\big(G_{jk}(\hat{a}(\chi), \hat{b}(\chi))\big)_{j,k\in\{1,\ldots,n\}} = \det \partial_2 F(\hat{a}(\chi), g(\chi); \cdot) \neq 0.\end{aligned}$$

Hence $\det M \in A^\times$, which implies that M is an invertible element of $\mathrm{M}_n(A)$. We conclude that $r = 0$. □

7.3 Corollary (The Šilov Idempotent Theorem [42]). *Let A be a Mackey-complete commutative continuous inverse algebra over $\mathbb{C}$. Let $K \subseteq \Gamma_A$ be compact and open. Then there is a unique idempotent $e \in A$ such that $\hat{e}|_{\Gamma_A}$ is the characteristic function of K.*

Proof. Apply Theorem 7.2 with $m = 0$, $n = 1$, $U = \mathbb{C}$, and $F(\zeta) = \zeta^2 - \zeta$. □

One can easily prove Šilov's Theorem directly from Theorem 6.1. Indeed, choose disjoint open neighbourhoods $U, V \subseteq A'$ of K and of $\Gamma_A \setminus K$. Define $f \in \mathcal{O}(U \cup V)$ by $f|_U \equiv 1$ and $f|_V \equiv 0$. Then $e = \Xi_A(\tilde{f})$ is an idempotent, and Corollary 6.3 shows that $\chi(e) = f(\chi)$ holds for every $\chi \in \Gamma_A$.

If $e' \in A$ is another idempotent with the required property then $e' - e = (e')^2 - e^2 = (e' - e)(e' + e)$, whence $(e' - e)(e' + e - 1) = 0$. Since the element $e' + e - 1$ has non-vanishing Gelfand transform, it is invertible, and we conclude that $e' = e$.

This is the result for which Šilov originally developed his early version of the n-variable functional calculus. Like Šilov's Theorem, many of the numerous and important consequences of the holomorphic functional calculus for commutative Banach algebras carry over to Mackey-complete commutative continuous inverse algebras with only minor changes in the proofs.

References

[1] G. R. Allan, *A note on the holomorphic functional calculus in a Banach algebra*, Proc. Amer. Math. Soc. **22** (1969), 77–81. MR 40 #405

[2] Richard Arens, *Linear topological division algebras*, Bull. Amer. Math. Soc. **53** (1947), 623–630. MR 9,6a

[3] Richard Arens and Alberto P. Calderón, *Analytic functions of several Banach algebra elements*, Ann. of Math. (2) **62** (1955), 204–216. MR 17,177c

[4] Andrée Bastiani, *Applications différentiables et variétés différentiables de dimension infinie*, J. Analyse Math. **13** (1964), 1–114 (French). MR 31 #1540

[5] Harald Biller, *Algebras of complex analytic germs*, Preprint 2331, Fachbereich Mathematik, Technische Universität Darmstadt, April 2004, http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/shadow/pp2331.html.

[6] Torben Maack Bisgaard, *The topology of finitely open sets is not a vector space topology*, Arch. Math. (Basel) **60** (1993), no. 6, 546–552. MR 94g:46013

[7] Bruce Blackadar, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR 99g:46104

[8] Jacek Bochnak and Józef Siciak, *Analytic functions in topological vector spaces*, Studia Math. **39** (1971), 77–112. MR 47 #2365

[9] Frank F. Bonsall and John Duncan, *Complete normed algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete 80, Springer, Berlin 1973.

[10] J.-B. Bost, *Principe d'Oka, K-théorie et systèmes dynamiques non commutatifs*, Invent. Math. **101** (1990), no. 2, 261–333. MR 92j:46126

[11] N. Bourbaki, *Éléments de mathématique. Fasc. XIII. Livre VI: Intégration. Chapitres 1, 2, 3 et 4: Inégalités de convexité, Espaces de Riesz, Mesures sur les espaces localement compacts, Prolongement d'une mesure, Espaces L^p*, Deuxième édition revue et augmentée. Actualités Scientifiques et Industrielles, No. 1175, Hermann, Paris, 1965. MR 36 #2763

[12] ______, *Éléments de mathématique. Fasc. XXXII. Théories spectrales. Chapitre I: Algèbres normées. Chapitre II: Groupes localement compacts commutatifs*, Actualités Scientifiques et Industrielles, No. 1332, Hermann, Paris, 1967. MR 35 #4725

[13] Nicolas Bourbaki, *Éléments de mathématique. Fasc. XV. Livre V: Espaces vectoriels topologiques. Chapitre I: Espaces vectoriels topologiques sur un corps valué. Chapitre II: Ensembles convexes et espaces localement convexes*, Actualités Scientifiques et Industrielles, No. 1189. Deuxième édition revue et corrigée, Hermann, Paris, 1966. MR 34 #3277

[14] Alain Connes, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. (1985), no. 62, 257–360. MR 87i:58162

[15] I. G. Craw, *A condition equivalent to the continuity of characters on a Fréchet algebra*, Proc. London Math. Soc. (3) **22** (1971), 452–464. MR 44 #4524

[16] Klaus Floret, *Lokalkonvexe Sequenzen mit kompakten Abbildungen*, J. Reine Angew. Math. **247** (1971), 155–195 (German). MR 44 #4478

[17] Theodore W. Gamelin, *Uniform algebras*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1969. MR 53 #14137

[18] Helge Glöckner, *Algebras whose groups of units are Lie groups*, Studia Math. **153** (2002), no. 2, 147–177.

[19] ______, *Infinite-dimensional Lie groups without completeness restrictions*, Geometry and analysis on finite- and infinite-dimensional Lie groups (Będlewo, 2000), Banach Center Publ., vol. 55, Polish Acad. Sci., Warsaw, 2002, pp. 43–59. MR 1 911 979

[20] ______, *Lie groups of germs of analytic mappings*, Preprint 2250, Fachbereich Mathematik, Technische Universität Darmstadt, 2002, to appear in V. Turaev and T. Wurzbacher, Proceedings of RCP 70 (Strasbourg 2002);
http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/shadow/pp2250.html.

[21] ______, *Lie groups of germs of analytic mappings*, Infinite Dimensional Groups and Manifolds in Mathematics and Quantum Physics, Proceedings of the 70ème R.C.P., Strasbourg (V. Turaev and T. Wurzbacher, eds.), 2002, to appear.

[22] ______, *Implicit functions from topological vector spaces to Banach spaces*, Preprint 2271, Fachbereich Mathematik, Technische Universität Darmstadt, 2003,
http://arXiv.org/abs/math.GM/0303320.

[23] Bernhard Gramsch, *Relative Inversion in der Störungstheorie von Operatoren und Ψ-Algebren*, Math. Ann. **269** (1984), no. 1, 27–71. MR 86j:47065

[24] Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1965. MR 31 #4927

[25] Richard S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 1, 65–222. MR 83j:58014

[26] Seppo Hiltunen, *Implicit functions from locally convex spaces to Banach spaces*, Studia Math. **134** (1999), no. 3, 235–250. MR 2000f:58019

[27] ______, *A Frobenius theorem for locally convex global analysis*, Monatsh. Math. **129** (2000), no. 2, 109–117. MR 2000m:58009

[28] Hans Jarchow, *Locally convex spaces*, B. G. Teubner, Stuttgart, 1981, Mathematische Leitfäden. [Mathematical Textbooks]. MR 83h:46008

[29] Andreas Kriegl and Peter W. Michor, *The convenient setting of global analysis*, American Mathematical Society, Providence, RI, 1997. MR 98i:58015

[30] Ernest A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc., **1952** (1952), no. 11, 79. MR 14,482a

[31] J. Milnor, *Remarks on infinite-dimensional Lie groups*, Relativity, groups and topology, II (Les Houches, 1983), North-Holland, Amsterdam, 1984, pp. 1007–1057. MR 87g:22024

[32] Karl-Hermann Neeb, *Central extensions of infinite-dimensional Lie groups*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 5, 1365–1442. MR 1 935 553

[33] ______, *Universal central extensions of Lie groups*, Acta Appl. Math. **73** (2002), no. 1-2, 175–219, The 2000 Twente Conference on Lie Groups (Enschede). MR 1 926 500

[34] ______, *Locally convex root graded Lie algebras*, Travaux mathématiques (Univ. du Luxembourg) **XIV** (2003), 25–120.

[35] Kiyoshi Oka, *Sur les fonctions analytiques de plusieurs variables*, Iwanami Shoten, Tokyo, 1961. MR 24 #A2048

[36] N. Christopher Phillips, *K-theory for Fréchet algebras*, Internat. J. Math. **2** (1991), no. 1, 77–129. MR 92e:46143

[37] R. Michael Range, *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics, vol. 108, Springer-Verlag, New York, 1986. MR 87i:32001

[38] D. A. Raĭkov, *Completely continuous spectra of locally convex spaces*, Trudy Moskov. Mat. Obšč. **7** (1958), 413–438. MR 20 #6642

[39] Hugo Rossi, *Holomorphically convex sets in several complex variables*, Ann. of Math. (2) **74** (1961), 470–493. MR 24 #A3310

[40] Walter Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York 1973.

[41] Helmut H. Schaefer, *Topological vector spaces*, Springer-Verlag, New York, 1971, Third printing corrected, Graduate Texts in Mathematics, Vol. 3. MR 49 #7722

[42] G. E. Šilov, *On the decomposition of a commutative normed ring into a direct sum of ideals*, Amer. Math. Soc. Transl. (2) **1** (1955), 37–48. MR 17,512c

[43] Joseph L. Taylor, *Topological invariants of the maximal ideal space of a Banach algebra*, Advances in Math. **19** (1976), no. 2, 149–206. MR 53 #14134

[44] Philippe Turpin, *Une remarque sur les algèbres à inverse continu*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A1686–A1689. MR 42 #3571

[45] Lucien Waelbroeck, *Le calcul symbolique dans les algèbres commutatives*, C. R. Acad. Sci. Paris **238** (1954), 556–558. MR 17,512g

[46] ———, *Le calcul symbolique dans les algèbres commutatives*, J. Math. Pures Appl. (9) **33** (1954), 147–186 (French). MR 17,513c

[47] ———, *Les algèbres à inverse continu*, C. R. Acad. Sci. Paris **238** (1954), 640–641 (French). MR 17,513a

[48] ———, *Topological vector spaces and algebras*, Springer-Verlag, Berlin, 1971, Lecture Notes in Mathematics, Vol. 230. MR 57 #7098

[49] ———, *The holomorphic functional calculus and non-Banach algebras*, Algebras in analysis (Proc. Instructional Conf. and NATO Advanced Study Inst., Birmingham, 1973), Academic Press, London, 1975, Notes elaborated with the collaboration of Peter Ludvik, pp. 187–251. MR 55 #8798

[50] ———, *The holomorphic functional calculus as an operational calculus*, Spectral theory (Warsaw, 1977), Banach Center Publ., vol. 8, PWN, Warsaw, 1982, pp. 513–552. MR 85g:46061

[51] Seth Warner, *Inductive limits of normed algebras*, Trans. Amer. Math. Soc. **82** (1956), 190–216. MR 18,52e

[52] W. Żelazko, *Concerning entire functions in B_0-algebras*, Studia Math. **110** (1994), no. 3, 283–290. MR 95j:46057