Algebras of complex analytic germs

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Abstract

Let X be a metrizable complex analytic manifold modelled on a locally convex space E, and let $K \subseteq X$ be compact. Let A be a normed unital algebra over $\mathbb{C}$. Let $\mathcal{O}(K, A)$ be the algebra of germs of complex analytic A-valued functions in K, topologized as the locally convex direct limit of the normed algebras of bounded complex analytic A-valued functions on open neighbourhoods of K in X.

Then $\mathcal{O}(K, A)$ is a locally m-convex Hausdorff algebra. If the unit group of A is open then so is the unit group of $\mathcal{O}(K, A)$. If A has finite dimension then $\mathcal{O}(K, A)$ is complete.[1] [2]

Introduction

This note contains the proof of the result stated in the abstract. For the case that $X = E$, this result can be collected from the literature (the more difficult steps can be taken from Dierolf and Wengenroth [9] and Mujica [16], cf. Glöckner [13]). Here, their methods are applied to the case that X is a manifold. Algebras of germs are encountered in the holomorphic functional calculus (e.g. [3, 4], Waelbroeck [18]). Moreover, the result provides a construction principle for algebras with continuous inversion, which have recently enjoyed renewed interest in the context of infinite-dimensional linear Lie groups (e.g. Glöckner [12]).

Metrizability seems to be essential for this result. For instance, if E is the weak*-dual of a locally convex complex vector space of uncountable dimension and $K \subseteq E$ is compact then multiplication in $\mathcal{O}(K, \mathbb{C})$ is not jointly continuous [3].

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1 Complex analytic functions

Let U be an open subset of the locally convex complex vector space E, and let F be a locally convex complex vector space. Following Bochnak and Siciak [5, Definition 5.6], we call a map $f\colon U \to F$ *complex analytic* if it is continuous and if, for every $x \in U$, there is a neighbourhood $V \subseteq U$ of x and a sequence $(\delta_x^n f)_{n\in\mathbb{N}_0}$ of continuous homogeneous polynomials from E into F with $\deg(\delta_x^n f) = n$ such that for each $y \in V$, the series $\sum_{n=0}^{\infty} \frac{1}{n!}(\delta_x^n f)(y-x)$ converges to $f(y)$. Here a homogeneous polynomial from E into F of degree $n \in \mathbb{N}_0$ is the composition of an n-linear map $E^n \to F$ with the diagonal embedding $E \hookrightarrow E^n$.

Let $f\colon U \to F$ be a map, and choose $x \in U$. For each $v \in E$, define a function g_v from an open zero-neighbourhood in $\mathbb{C}$ into F by $g_v(\zeta) := f(x + \zeta v)$. If f is complex analytic then g_v is complex analytic, and the homogeneous polynomial $\delta_x^n f$ is determined by the n-th derivative of g_v according to the formula $(\delta_x^n f)(v) = g_v^{(n)}(0)$. Conversely, if f is continuous and each g_v is differentiable then f is complex analytic, provided that F is sequentially complete (Bochnak and Siciak [5, Theorems 3.1 and 6.2]).

A complex analytic manifold modelled on E is defined in the usual way as a manifold with an atlas in which the coordinate changes are complex analytic.

2 Local m-convexity

Fix a metric d on the manifold X which is compatible with the topology. For $n \in \mathbb{N}$, let U_n be the union of those connected components of $\left\{x \in X;\ d(x,K) < \frac{1}{n}\right\}$ which meet K. Thus the family $(U_n)_{n\in\mathbb{N}}$ is a neighbourhood basis of K in X. Let $\mathcal{O}^\infty(U_n, A)$ be the algebra of bounded complex analytic A-valued functions on U_n, equipped with the supremum norm. Let $\mathcal{O}(K, A)$ be the algebra of germs of A-valued complex analytic functions in K. By compactness, every continuous A-valued function defined in a neighbourhood of K is bounded in a neighbourhood of K. Thus we give $\mathcal{O}(K, A)$ the locally convex direct limit topology with respect to the system formed by the spaces $\mathcal{O}^\infty(U_n, A)$ together with the restriction maps, which are injective by the Identity Theorem (Bochnak and Siciak [5, Proposition 6.6]). For $n \in \mathbb{N}$, let $B_n \subseteq \mathcal{O}^\infty(U_n, A)$ be the open unit ball, and let $g_n\colon \mathcal{O}^\infty(U_n, A) \to \mathcal{O}(K, A)$ denote the germ map. Note that $B_m|_{U_n} \subseteq B_n$ whenever $m \leq n$. Hence the images $g_n(B_n)$ form an ascending sequence. The set $\mathcal{U} := \left\{\operatorname{conv}\left(\bigcup_{n\in\mathbb{N}} \varepsilon_n g_n(B_n)\right);\ 0 < \varepsilon_n \leq 1\right\}$ is a basis of zero-neighbourhoods in $\mathcal{O}(K, A)$, see Bourbaki [7, II, § 4.4]. Each member $V \in \mathcal{U}$ is absolutely convex and satisfies $V \cdot V \subseteq V$. Therefore, the Minkowski functionals of the members of $\mathcal{U}$ form a family of sub-multiplicative seminorms on $\mathcal{O}(K, A)$ which defines the topology. The existence of such a family is the defining property of a locally m-convex algebra. (This proof is a specialization of arguments in Akkar and Nacir [1] and in Dierolf and Wengenroth [9].)

Note that inversion in a locally m-convex algebra is continuous on its domain. This is due to Michael [15]. Indeed, if σ is a sub-multiplicative semi-norm and h is an algebra element such that $\sigma(h) < 1$ and $1 + h$ is invertible then the equation $(1+h)^{-1} - 1 = -\big((1+h)^{-1} - 1\big)h - h$ implies that $\sigma\big((1+h)^{-1} - 1\big) \leq \frac{\sigma(h)}{1-\sigma(h)}$. We conclude that inversion is continuous at 1 and hence continuous with respect to σ.

The norm on A induces a sub-multiplicative semi-norm σ on $\mathcal{O}(K, A)$ by $\sigma\big(\tilde{f}\big) := \|f|_K\|_\infty$. The compositions $\sigma \circ g_n$ of σ with the germ maps are continuous, so that σ is continuous. Assume that the unit group $A^\times$ of A is open. Then the unit group of $\mathcal{O}(K, A)$ equals the set of germs $\tilde{f}$ such that $f(K) \subseteq A^\times$. Choose $r > 0$ such that the open r-ball around 1 in A is contained in $A^\times$. Then the open r-ball around 1 in $\mathcal{O}(K, A)$ with respect to σ consists of invertible elements. We conclude that $\mathcal{O}(K, A)^\times$ is a neighbourhood of 1 and hence open in $\mathcal{O}(K, A)$.

3 Separation

Choose $x \in K$, let $\varphi\colon V \to E$ be a chart of X with $\varphi(x) = 0$, and let $v \in E$. For each $k \in \mathbb{N}_0$, we obtain a well-defined map

$$\mathcal{O}(K, A) \longrightarrow A\colon \tilde{f} \longmapsto \big(\delta_0^k(f \circ \varphi^{-1})\big)(v) = \frac{k!}{2\pi i} \oint_{|\zeta|=\varepsilon} \frac{f\big(\varphi^{-1}(\zeta v)\big)}{\zeta^{k+1}}\, d\zeta.$$

The integral on the right-hand side exists in the completion of A if $\varepsilon > 0$ is sufficiently small. In this case, the equation was proved by Bochnak and Siciak [5, Theorem 3.1 and Corollary 3.1]. It implies that the integral actually belongs to A and that the map is continuous. Thus if the germ of $f \in \mathcal{O}^\infty(U_n, A)$ belongs to the closure of $\{0\}$ in $\mathcal{O}(K, A)$ then $\delta_0^k(f \circ \varphi^{-1}) = 0$ holds for each $k \in \mathbb{N}_0$, so that f vanishes in a neighbourhood of x. We conclude that $\mathcal{O}(K, A)$ is a Hausdorff algebra. (This argument was adapted from Glöckner [13].)

4 Completeness

For the following proof of completeness, we have to assume that the dimension of A is finite. As a first consequence, the algebra A is complete, which implies that the algebras $\mathcal{O}^\infty(U_n, A)$ are Banach algebras (Bochnak and Siciak [5, Proposition 6.5]). Local compactness of A will be used below.

For each $n \in \mathbb{N}$, we claim that the closed unit ball $\overline{B_n}$ of $\mathcal{O}^\infty(U_n, A)$ is an equicontinuous set of functions. Choose $x \in U_n$. Let $\varphi\colon V \to E$

be a chart of X with $V \subseteq U_n$ and $\varphi(x) = 0$ such that $\varphi(V) \subseteq E$ is a balanced neighbourhood of 0. Choose $f \in \overline{B_n}$ and $v \in \varphi(V)$, and define an A-valued complex analytic function g on the open unit disc $\{\zeta \in \mathbb{C};\ |\zeta| < 1\}$ by $g(\zeta) := f\big(\varphi^{-1}(\zeta v)\big)$. For $\zeta \in \mathbb{C}$ with $|\zeta| \leq \frac{1}{4}$, Cauchy's Formula

$$g(\zeta) - g(0) = \frac{1}{2\pi i} \oint_{|\xi|=\frac{1}{2}} g(\xi) \left(\frac{1}{\xi - \zeta} - \frac{1}{\xi} \right) d\xi$$

implies that $\|g(\zeta) - g(0)\| \leq 4|\zeta|$. Hence if $0 < \varepsilon \leq 1$ then $\varphi^{-1}\big(\frac{\varepsilon}{4}\varphi(V)\big)$ is a neighbourhood of ε-equicontinuity of x for $\overline{B_n}$.

We equip the space $\mathcal{O}(U_n, A)$ of all A-valued complex analytic functions on U_n with the compact-open topology. Then $\overline{B_n}$ is closed in $\mathcal{O}(U_n, A)$. Since the closed unit ball in A is compact, the Arzela–Ascoli Theorem (see, for instance, Dugundji [11, XII.6.4]) yields that $\overline{B_n}$ is a compact subset of $\mathcal{O}(U_n, A)$.

Let $\mathcal{T}$ be the locally convex direct limit topology on $\mathcal{O}(K, A)$ with respect to the directed system formed by the spaces $\mathcal{O}(U_n, A)$. The argument in Section 3 shows that this is a Hausdorff topology, and each closed unit ball $\overline{B_n}$ has compact image in $\mathcal{O}(K, A)$ with respect to $\mathcal{T}$. In this situation, Mujica [16] has proved that the original topology on $\mathcal{O}(K, A)$ is complete (another reference is Mujica and Nachbin [17]).

Mujica himself applied this abstract argument in order to prove completeness of $\mathcal{O}(K, \mathbb{C})$ in the case that K is a compact subspace of a Fréchet space E. Thus he simplified Dineen's solution [10] of the problem, which had been open for some time. Few completeness results are known beyond this situation. For example, $\mathcal{O}(K, F)$ is complete if both $E \supseteq K$ and F are Banach spaces (Chae [8]), or if E is a Fréchet space and F is a Banach space which is complemented in its bidual (Bonet, Damański and Mujica [6]); see also Bierstedt, Bonet and Peris [2] and Nguyen [14].

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