Conveniently Hölder Homomorphisms are Smooth in the Convenient Sense

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Abstract. We show that every "conveniently Hölder" homomorphism between Lie groups in the sense of convenient differential calculus is smooth (in the convenient sense). In particular, every ℓip^0-homomorphism is smooth.

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Introduction

In a preprint from 1982, John Milnor formulated various absolutely fundamental open problems concerning infinite-dimensional Lie groups [11]. Our investigations are related to Milnor's question: *Is a continuous homomorphism between Lie groups necessarily smooth*?, which refers to smooth Lie groups modelled on complete locally convex spaces, based on smooth maps in the sense of Michal-Bastiani (Keller's C^∞_c-maps). While the answer to this question is still unknown, some progress has been made recently: every *Hölder* continuous homomorphism is smooth [9]. In particular, every Lipschitz continuous homomorphism is smooth. The goal of this article is to establish analogous results in the framework of infinite-dimensional analysis and Lie theory known as Convenient Differential Calculus (see [7], [10]). In this setting, a map is called ℓip^0 (or $\mathcal{L}ip^0$) if it takes smooth curves to Lipschitz curves. Instead of continuous maps, one considers ℓip^0-maps as the more adequate fundamental notion here, because the ℓip^0-property is a purely bornological concept. Our main result is the following (Theorem 9.1):

Main Theorem. *Let G and H be Lie groups in the sense of convenient differential calculus and $f\colon G \to H$ be a homomorphism. If f is ℓip^0, then f is smooth in the convenient sense. More generally, this conclusion remains valid if f is conveniently Hölder.*

Here, a map f is called *conveniently Hölder* if it is h_α for some $\alpha \in \,]0,1]$ in the sense that f takes smooth curves to Hölder continuous curves of Hölder exponent α (thus $h_1 = \ell ip^0$).

Strategy of proof. To establish our Main Theorem, the strategy is to encode smoothness of homomorphisms in a suitable differentiability property at the identity, which can be checked in the conveniently Hölder case. Our starting point is the (trivial) observation that every ℓip^1-homomorphism $f\colon G \to H$ is smooth in the convenient sense (Lemma 6.1). The proof then proceeds in two main steps:

- First, we show that a homomorphism $f\colon G \to H$ is ℓip^1 (and hence smooth) provided it is "curve differentiable" at 1.

- The difficult task, then, is to show that every conveniently Hölder homomorphism is curve differentiable at 1. What we actually establish is *bornological* curve differentiability at 1, a stronger (but more tangible) property.

We remark that smoothness of Hölder continuous homomorphisms in the setting of Keller's C^∞_c-theory is proved in [9] in two analogous steps. The appropriate notion of differentiability at a point used there is total differentiability. In the framework of convenient differential calculus, differentiability at point has not been considered much in the literature. It was therefore necessary to develop various new concepts. Curve differentiability and bornological curve differentiability, which we introduce here, serve us as efficient tools for the discussion of homomorphisms. Once the basic facts concerning these differentiability properties are established, Step 1 (as described above) is easily performed. The proof of Step 2 is rather technical and much more difficult. However, one central idea of the proof is easily explained on an informal level, and we describe it now. We recommend to keep this basic idea in the back of one's mind when reading Section 9. To shorten formulas and increase the readability, let us identify an open identity neighbourhood of H with a c^∞-open 0-neighbourhood $V \subseteq L(H)$ for the moment (such that 0 becomes the identity element of H). Likewise, we identify an open identity neighbourhood in G with a c^∞-open 0-neighbourhood $U \subseteq L(G)$, which we choose so small that $f(U) \subseteq V$.

The core idea. To establish bornological curve differentiability of f at 1, (among other things) we need to show that $(f \circ \gamma)'(0)$ exists, for each smooth curve $\gamma : \mathbb{R} \to U$ such that $\gamma(0) = 0$. We now explain how a candidate for $(f \circ \gamma)'(0)$ can be obtained. The idea is to exploit the first order Taylor expansion $x^2 y = 2x + y + R(x, y)$ of the map $(x, y) \mapsto x^2 y \in V \subseteq L(H)$ (defined on some c^∞-open $(0,0)$-neighbourhood in $V \times V$). For sufficiently small t, we have

$$\begin{aligned} f(\gamma(t)) &= f(\gamma(\tfrac{1}{2}t))^2\, f(\gamma(\tfrac{1}{2}t)^{-2}\gamma(t)) \\ &= 2\, f(\gamma(\tfrac{1}{2}t)) \;+\; f(\gamma(\tfrac{1}{2}t)^{-2}\gamma(t)) \;+\; R\big(f(\gamma(\tfrac{1}{2}t)),\, f(\gamma(\tfrac{1}{2}t)^{-2}\gamma(t))\big) \end{aligned}$$

and thus $2f(\gamma(\frac{1}{2}t)) = f(\gamma(t)) - f(\gamma(\frac{1}{2}t)^{-2}\gamma(t)) - R\big(f(\gamma(\frac{1}{2}t)), f(\gamma(\frac{1}{2}t)^{-2}\gamma(t))\big)$. Hence

$$\begin{aligned} 4f(\gamma(\tfrac{1}{4}t)) &= 2f(\gamma(\tfrac{1}{2}t)) - 2f(\gamma(\tfrac{1}{4}t)^{-2}\gamma(\tfrac{1}{2}t)) - 2R\big(f(\gamma(\tfrac{1}{4}t)), f(\gamma(\tfrac{1}{4}t)^{-2}\gamma(\tfrac{1}{2}t))\big) \\ &= f(\gamma(t)) - f(\gamma(\tfrac{1}{2}t)^{-2}\gamma(t)) - R\big(f(\gamma(\tfrac{1}{2}t)), f(\gamma(\tfrac{1}{2}t)^{-2}\gamma(t))\big) \\ &\qquad - 2f(\gamma(\tfrac{1}{4}t)^{-2}\gamma(\tfrac{1}{2}t)) - 2R\big(f(\gamma(\tfrac{1}{4}t)), f(\gamma(\tfrac{1}{4}t)^{-2}\gamma(\tfrac{1}{2}t))\big)\,, \end{aligned}$$

using the preceding formula twice. Repeating this argument, we obtain

$$\begin{aligned} \frac{f(\gamma(2^{-n}t))}{2^{-n}} &= f(\gamma(t)) \;-\; \sum_{k=0}^{n-1} 2^k \Big[f(\gamma(2^{-k-1}t)^{-2}\gamma(2^{-k}t)) \\ &\qquad\qquad + R\big(f(\gamma(2^{-k-1}t),\, f(\gamma(2^{-k-1}t)^{-2}\gamma(2^{-k}t))\big) \Big] \end{aligned} \tag{1}$$

for all $n \in \mathbb{N}$. Since $\frac{d}{ds}\big|_{s=0}\big(\gamma(\frac{1}{2}s)^{-2}\gamma(s)\big) = 0$, we have $\gamma(2^{-k-1}t)^{-2}\gamma(2^{-k}t) = \mathcal{O}(2^{-2k})$ here and hence $f(\gamma(2^{-k-1}t)^{-2}\gamma(2^{-k}t)) = \mathcal{O}(2^{-2\alpha k})$, as f is h_α. Thus $2^k f(\gamma(2^{-k-1}t)^{-2}\gamma(2^{-k}t)) =$

$\mathcal{O}(2^{-(2\alpha-1)k})$. Likewise, using that first order Taylor remainders are at most quadratic in the size of their argument (Lemma 8.1), we see that $2^k R\big(f(\gamma(2^{-k-1}t),\ f(\gamma(2^{-k-1}t)^{-2}\gamma(2^{-k}t)\big) = \mathcal{O}(2^{-(2\alpha-1)k})$ as well. If $\alpha > \frac{1}{2}$, the preceding estimates entail that the partial sums of the series in (1) form a Mackey-Cauchy sequence, which converges in $L(H)$ as the latter is assumed Mackey complete. Therefore $\lim_{n\to\infty} \frac{f(\gamma(2^{-n}t))}{2^{-n}}$ exists in $L(H)$, and hence so does $\lambda := \lim_{n\to\infty} \frac{f(\gamma(2^{-n}t))}{2^{-n}t}$. Now clearly λ gives us a candidate for $(f\circ\gamma)'(0)$. Of course, this rough outline has to be made more precise. Furthermore, a lot of work remains:

- One has to check that λ is independent of t, and that $(f\circ\gamma)'(0)$ really exists.
- It has to be shown that $(f\circ\gamma)'(0)$ only depends on $\gamma'(0)$, and that the mapping $L(G)\to L(H)$, $\gamma'(0)\mapsto (f\circ\gamma)'(0)$ is bounded linear.
- The general case $\alpha\in\,]0,1]$ has to be reduced to the case where $\alpha>\frac{1}{2}$.

To master the last and penultimate task, it is essential to discuss not only a single curve γ, but a whole family of curves $\eta_s := \eta(s,\bullet)$ for a smooth map $\eta\colon \mathbb{R}^2\to U$. Therefore most of the actual proof in Section 9 is formulated for η's instead of mere γ's.

Organization of the paper. In Section 1, we recall several basic definitions from convenient differential calculus, explain some notations, and compile and develop various basic facts for later use. We then define conveniently Hölder (h_α-) maps and characterize them by their behaviour on "bornologically compact" sets (Section 2), along the lines of the Lipschitz case treated in [7] (cf. also [6]). In Sections 3 and 5, we specify and discuss notions of differentiability at a point for curves and general mappings, respectively. Hölder differentiable curves and the corresponding mappings are defined and discussed in Section 4, as far as required for our purposes.[1] In Section 6, we carry out Step 1 of the proof of our Main Theorem (curve differentiability at 1 implies smoothness). Before we can carry out Step 2, further preparations are necessary: To enable the reduction from arbitrary α to $\alpha>\frac{1}{2}$, we characterize h_α-homomorphisms (Section 7) and in Section 8, we study the behavior of Taylor remainders on bornologically compact sets. In Section 9, the core of the article, we then complete the proof of our Main Theorem, based on the reduction steps and preparatory considerations carried out before. Various proofs (part of which are mere adaptations of the Lipschitz case, [7]) have been relegated to an appendix, and can be taken on faith on a first reading.

1 Preliminaries

This article is based on the Convenient Differential Calculus of Frölicher, Kriegl and Michor, and we presume familiarity with its basic ideas. Our main references are [7] and [10]. For the readers convenience, we briefly recall some of the basic concepts now, and explain our notation and terminology. We also prove various simple results, for later use.

[1] See [6] for the general theory of such maps, which parallels the familiar ℓip^k-case as in [7] or [10].

1.1 Given a locally convex (Hausdorff real topological vector) space E and absolutely convex, bounded subset $B \neq \emptyset$ of E, we let $E_B := \operatorname{span}(B) \subseteq E$ and make E_B a normed space with the Minkowski functional $\|.\|_B : E_B \to [0, \infty[$, $\|x\|_B := \inf\{r > 0 : x \in rB\}$ as the norm. Then the inclusion map $j : E_B \to E$ is continuous linear (see [4, Ch. III, §1, No. 5] for further information). A locally convex space E is called a *convenient vector space* (or: *Mackey complete*) if E_B is complete, for each absolutely convex, closed, bounded subset $B \neq \emptyset$ of E (see [10, Thm. 2.14] for alternative characterizations).

1.2 The c^∞-*topology* on a locally convex space E is the final topology on E with respect to the set of all smooth curves $\gamma : \mathbb{R} \to E$ (which are defined as expected). We write $c^\infty(E)$ for E, equipped with the c^∞-topology. A subset $U \subseteq E$ is called c^∞-*open* if it is open in $c^\infty(E)$; in this case, we write $c^\infty(U)$ for U, equipped with the topology induced by $c^\infty(E)$. We recall that the c^∞-topology is finer than the locally convex topology and can be properly finer; if E is metrizable, then $c^\infty(E) = E$. The c^∞-topology on a product $E \times F$ is finer then the product topology on $c^\infty(E) \times c^\infty(F)$, and can be properly finer.

1.3 Let E and F be convenient vector spaces and $f : U \to F$ be a map, where $U \subseteq E$ is c^∞-open. We call f *conveniently smooth* or a c^∞-*map* if $f \circ \gamma : \mathbb{R} \to F$ is a smooth curve, for each smooth curve $\gamma : \mathbb{R} \to E$ with image in U. If $f : U \to F$ is c^∞, then the iterated directional derivatives $d^k f(x, y_1, \ldots, y_k) := (D_{y_1} \cdots D_{y_k} f)(x)$ exist for any $(x, y_1, \ldots, y_k) \in U \times E^k$ and define a c^∞-map $d^k f : U \times E^k \to F$. For each $x \in U$, $f^{(k)}(x) := d^k f(x, \bullet) : E^k \to F$ is a bounded, symmetric, k-linear mapping, and the map $f^{(k)} : U \to \mathcal{L}^k(E, F)$ into the space of such mappings (equipped with its natural convenient vector topology) is c^∞. We abbreviate $df := d^1 f$, $f' := f^{(1)}$, $f'' := f^{(2)}$.

Lemma 1.4 *Let E be a convenient vector space and $B \subseteq E$ be a non-empty, absolutely convex, bounded subset. Then the inclusion map $j : E_B \to E$ is continuous as a map into E, equipped with the c^∞-topology.*

Proof. As recalled in **1.1**, j is a continuous linear map into E, equipped with its locally convex vector topology. Hence j is a bounded linear map and hence c^∞ [10, Cor. 2.11]. Being a c^∞-map, j is continuous as a map from $c^\infty(E_B)$ to $c^\infty(E)$ (this is immediate from the definition of the c^∞-topologies). Here $c^\infty(E_B) = E_B$ due the metrizability of E_B [10, Thm. 4.11 (1)]. The assertion follows. □

1.5 The manifolds and Lie groups of convenient differential calculus based on the above c^∞-maps (modelled on convenient vector spaces) will be referred to as c^∞-*manifolds*, resp., c^∞-*Lie groups* in this article. The c^∞-*topology* (or "natural topology") on a c^∞-manifold M is defined as the final topology with respect to the set of smooth curves in M (see [10, §27.4] for further information). If $f : M \to E$ is a c^∞-map from a c^∞-manifold to a convenient vector space E, identifying TE with $E \times E$ the tangent map attains the form $Tf = (f, df)$ for a unique c^∞-map $TM \to E$ denoted df.

The next lemma is a variant of [10, Cor. 2.11]:

Lemma 1.6 *Let $\alpha\colon E \to F$ be a linear map between locally convex spaces. If $\alpha\circ\gamma\colon \mathbb{R}\to F$ is continuous at 0 for each smooth curve $\gamma\colon \mathbb{R}\to E$ such that $\gamma(0)=0$, then α is bounded.*

Proof. The proof is by contraposition. If the linear map α is not bounded, then there exists a bounded subset $X\subseteq E$ such that $\alpha(X)\subseteq F$ is not bounded. Thus $\|\alpha(X)\|_q$ is unbounded for some continuous seminorm $\|.\|_q$ on F, entailing that there exist elements $x_n\in X$ such that $\|\alpha(x_n)\|_q\geq n2^n$. Then the sequence $(2^{-n}x_n)_{n\in\mathbb{N}}$ in E converges fast to 0 (in the sense of [10, §2.8]). Hence, by the Special Curve Lemma [10, §2.8], there is a smooth curve $\gamma\colon\mathbb{R}\to E$ such that $\gamma(\frac{1}{n})=2^{-n}x_n$ for each $n\in\mathbb{N}$. Since $\|\alpha(\gamma(\frac{1}{n}))\|_q=2^{-n}\|\alpha(x_n)\|_q\geq n$ tends to ∞ as $n\to 0$, the curve $\alpha\circ\gamma$ is discontinuous at 0. □

The following useful fact is clear from the (somewhat sketchy) discussions in [1, end of §7]. For the convenience of the reader, a self-contained proof is given in Appendix A.

Lemma 1.7 *Let E and F be convenient vector spaces and $f: U\to F$ be a c^∞-map on a c^∞-open subset $U\subseteq E$. Then $U^{[1]}:=\{(x,y,t)\in U\times E\times\mathbb{R}\colon x+ty\in U\}$ is c^∞-open in $E\times E\times\mathbb{R}$, and the following map is c^∞:*

$$f^{[1]}\colon U^{[1]}\to F,\qquad f^{[1]}(x,y,t):=\begin{cases}\frac{f(x+ty)-f(x)}{t} & \text{if } t\neq 0\,;\\ df(x,y) & \text{if } t=0\,.\end{cases}$$ □

2 Conveniently Hölder maps

In this section, we define and study conveniently Hölder mappings. These generalize the $\mathcal{L}ip^0$-maps familiar from convenient differential calculus (see [7] or [10]), and can be discussed along similar lines (see also [6]). Throughout the following, $\alpha\in\,]0,1]$.

We begin with the definition of Hölder continuous maps on subsets of normed spaces, and describe their basic properties, for later use.

Definition 2.1 Let $(E,\|.\|)$ be a normed real vector space, F be a real locally convex space, and $U\subseteq E$. A map $f: U\to F$ is called *Hölder continuous of exponent α* (or H_α, for short) if, for every $x\in U$ and continuous seminorm $\|.\|_q\colon F\to[0,\infty[$ on F, there exists a neighbourhood V of x in U and $C\in[0,\infty[$ such that

$$\|f(z)-f(y)\|_q\leq C\,\|z-y\|^\alpha\quad\text{for all } y,z\in V. \tag{2}$$

H_1-maps are also called *Lipschitz continuous.*

Compare [9, App. B] for the case of open domains in locally convex spaces.

Lemma 2.2 *Let $\alpha,\beta\in\,]0,1]$.*

(a) *If $\alpha\geq\beta$, then any H_α-map is also H_β.*

(b) *If E_1,E_2 are normed spaces, $U_1\subseteq E_1$, $U_2\subseteq E_2$, $f\colon U_1\to U_2$ is H_α and $g\colon U_2\to F$ an H_β-map into a locally convex space F, then $g\circ f\colon U_1\to F$ is $H_{\alpha\beta}$.*

(c) *Every smooth curve* $\gamma\colon \mathbb{R} \to E$ *in a locally convex space* E *is* H_α.

Proof. (a) and (b) are obvious. Every smooth curve being Lipschitz continuous ([10, §1.2] or [9, La. B2]), (c) follows from (a). □

The special behaviour of Hölder maps on compact sets will be exploited extensively.

Lemma 2.3 *Let* $(E, \|.\|)$ *be a normed space,* $K \subseteq E$ *be compact and* $f : K \to F$ *be an* H_α*-map into a real locally convex space* F*. Then the following holds:*

(a) $W_K := \left\{ \frac{f(z)-f(y)}{\|z-y\|^\alpha} : y, z \in K,\, y \neq z \right\}$ *is bounded in* F.

(b) *There exists an absolutely convex, bounded subset* $D \neq \emptyset$ *of* F *such that* $f(K) \subseteq F_D$, $W_K \subseteq D$ *and* $\|f(z) - f(y)\|_D \leq \|z - y\|^\alpha$ *for all* $y, z \in K$*. In particular,* W_K *is bounded in* F_D*, and* $f\colon K \to F_D$ *is* H_α (*and thus continuous*).

Proof. (a) If we can show that $\lambda(W_K) \subseteq \mathbb{R}$ is bounded for each continuous linear functional $\lambda : F \to \mathbb{R}$, then W_K is bounded by Mackey's Theorem; here $\lambda \circ f$ is H_α (as is readily verified). Hence $F = \mathbb{R}$ without loss of generality. Since f is H_α and K is compact, we find $n \in \mathbb{N}$, $C > 0$ and open subsets $V_1, \ldots, V_n, \widetilde{V}_1, \ldots, \widetilde{V}_n$ of K such that $K = \bigcup_{j=1}^n V_j$, $\overline{V_j} \subseteq \widetilde{V}_j$ for $j \in \{1, \ldots, n\}$ (where $\overline{V_j}$ is the closure of V_j in K), and

$$|f(z) - f(y)| \leq C\,\|z - y\|^\alpha \quad \text{for all } j \in \{1, \ldots, n\} \text{ and } y, z \in \widetilde{V}_j.$$

Thus $X_j := \{\|z - y\|^{-\alpha}(f(z) - f(y))\colon y, z \in \widetilde{V}_j,\, y \neq z\}$ is bounded for each $j \in \{1, \ldots, n\}$. Also $Y_j := \{\|z - y\|^{-\alpha}(f(z) - f(y))\colon y \in \overline{V_j},\, z \in K \setminus \widetilde{V}_j\}$ is bounded, the sets $K_j := K \setminus \widetilde{V}_j$ and $\overline{V_j}$ being compact and disjoint. Hence $W_K \subseteq \bigcup_{j=1}^n X_j \cup \bigcup_{j=1}^n (Y_j \cup (-Y_j))$ is bounded.

(b) We may assume that $K \neq \emptyset$. The set W_K being bounded by (a), also the absolutely convex hull D of $W_K \cup f(K)$ is bounded, and clearly D has the desired properties. □

Lemma 2.4 *Let* $(E, \|.\|)$ *be a finite-dimensional normed vector space,* $U \subseteq E$ *be open, and* $f\colon U \to F$ *be a map into a real locally convex space* F*. Then the following holds:*

(a) f *is* H_α *if and only if every* $x \in U$ *has a neighbourhood* K *for which* $W_K \subseteq F$ (*as in Lemma* 2.3) *is bounded.*

(b) f *is* H_α *if and only if* $\lambda \circ f$ *is* H_α *for each continuous linear functional* $\lambda\colon E \to \mathbb{R}$.

(c) *If* f *is* H_α *and* $K \subseteq U$ *is compact, then there exists an absolutely convex, bounded subset* $B \neq \emptyset$ *of* F *such that* $f(K) \subseteq F_B$ *and* $f|_K\colon K \to F_B$ *is* H_α.

(d) *If* f *is* H_α*, then* f *is continuous as a map into* $c^\infty(F)$.

Proof. (a) If f is H_α and $x \in U$, then W_K is bounded for every compact neighbourhood $K \subseteq U$ of x, by Lemma 2.3. Conversely, assume that every $x \in U$ has a neighbourhood $K \subseteq U$ such that W_K is bounded. Let $\|.\|_q$ be a continuous seminorm on F. Then $\|W_K\|_q \subseteq [0, C]$ for some $C \in [0, \infty[$ and thus $\|f(z) - f(y)\|_q \leq C\,\|z - y\|$ for all $y, z \in K$, showing that f is H_α.

(b) If any $\lambda \circ f$ is H_α, then W_K is weakly bounded and hence bounded, for any compact neighbourhood $K \subseteq U$ of a given element $x \in U$. Thus f is H_α. The converse is trivial.

(c) is a special case of Lemma 2.3 (b).

(d) Given $x \in U$, we choose a compact neighbourhood $K \subseteq U$ of x. Then $f|_K^{F_B}$ is continuous for B as in (c), and hence so is $f|_K^{c^\infty(F)}$, by Lemma 1.4. □

Remark 2.5 Lemma 2.4 (a) becomes false for every infinite-dimensional normed space $(E, \|.\|)$. Indeed, for any such E, there exists a smooth map $g\colon E \to \mathbb{R}$ which is unbounded on the unit ball $B_1(0) \subseteq E$ (see [2, La. 2.3]). Then also $f\colon E \to \mathbb{R}^{\mathbb{N}}$, $f(x) := (g(nx))_{n\in\mathbb{N}}$ is smooth and hence Lipschitz continuous in the sense of Definition 2.1, by [9, La. B2 (e)]. By construction, f is unbounded on any 0-neighbourhood $K \subseteq E$. Hence also $W_K := \{\|z - y\|^{-1}(f(z) - f(y))\colon y, z \in K, y \neq z\} \subseteq \mathbb{R}^{\mathbb{N}}$ is unbounded for any K.

Definition 2.6 Let E be a convenient vector space. A map $f : U \to F$ from a c^∞-open subset $U \subseteq E$ to a convenient vector space F is called *conveniently Hölder with exponent* α (or an h_α*-map*, for short) if $f \circ \gamma$ is an H_α-curve, for each smooth curve $\gamma\colon \mathbb{R} \to U$. The h_1-maps will also be called *conveniently Lipschitz* (or ℓip^0, for short).[2]

Remark 2.7 For $f\colon E \supseteq U \to F$ as before, we have:

(a) f is h_α if and only if $\lambda \circ f$ is h_α for each continuous linear functional λ on F, as a consequence of Lemma 2.4 (b).

(b) If f is h_α and $\gamma\colon I \to U$ is a c^∞-curve defined on an open subset $I \subseteq \mathbb{R}$, then $f \circ \gamma$ is H_α. Indeed, given $t_0 \in I$, a smooth cut-off function $\chi : \mathbb{R} \to I$ can be used to create a smooth curve $\gamma \circ \chi\colon \mathbb{R} \to U$ which coincides with γ on a neighbourhood J of t_0. Then $f \circ \gamma|_J = f \circ (\gamma \circ \chi)|_J$ is H_α. Trivial facts like this one will be used frequently in the following, without mention.

We record some immediate consequences of Lemma 2.2:

Lemma 2.8 *In the situation of Definition* 2.6*, we have:*

(a) *If f is h_α, then f is h_β for each $\beta \in \,]0, \alpha]$.*

(b) *If f is c^∞, then f is h_α for each $\alpha \in \,]0, 1]$.* □

[2] In [10], the notation $\mathcal{L}ip^0$ is used.

The following lemma is a variant of [7, Thm. 1.4.2 and Thm. 4.3.8], and can be proved analogously. For completeness, the proof is given in Appendix A. Recall that a subset K of a locally convex space E is called *bornologically compact* if there exists an absolutely convex, bounded subset $B \neq \emptyset$ of E such that $K \subseteq E_B$ and K is compact in E_B. It is essential for our purposes that h_α-maps are H_α on bornologically compact sets (and are, indeed, characterized by this property):

Lemma 2.9 *Let E and F be convenient vector spaces, $U \subseteq E$ be c^∞-open, and $f\colon U \to F$ be a map. Then the following conditions are equivalent:*

(a) *f is h_α.*

(b) *For each absolutely convex, bounded subset $B \neq \emptyset$ of E and compact set $K \subseteq E_B \cap U$, the map $f|_K \colon E_B \supseteq K \to F$ is H_α (and thus Lemma 2.3 applies to $f|_K$).* □

For $f\colon \mathbb{R}^n \to \mathbb{R}$, the next lemma is covered by [3, Thm. 2].

Lemma 2.10 *Let E be a finite-dimensional normed vector space, $U \subseteq E$ be open, and $f\colon U \to F$ be a map into a convenient vector space F. Then f is h_α if and only if f is H_α.*

Proof. If f is H_α, then $f \circ \gamma$ is H_α for each smooth curve $\gamma\colon \mathbb{R} \to U$ (Lemma 2.2), and thus f is h_α. If, conversely, f is h_α, using that every $x \in U$ has a compact neighbourhood, we deduce from Lemmas 2.9, 2.3 and 2.4 (a) that f is H_α. □

Lemma 2.11 *Let E, F and H be convenient vector spaces, $U \subseteq E$ and $V \subseteq F$ be c^∞-open, $f\colon U \to V$ be an h_α-map, and $g\colon V \to H$ be an h_β-map, where $\alpha, \beta \in\,]0,1]$. Then $g \circ f\colon U \to H$ is $h_{\alpha\beta}$.*

Proof. Let $\gamma\colon \mathbb{R} \to U$ be a smooth curve. Then $\eta := f \circ \gamma\colon \mathbb{R} \to V$ is H_α. Given $t_0 \in \mathbb{R}$, set $I := [t_0 - 1, t_0 + 1]$; there exists an absolutely convex, bounded subset $B \neq \emptyset$ of F such that $\eta(I) \subseteq F_B$ and $\eta|_I : I \to F_B$ is H_α (Lemma 2.4 (c)). Hence $K := \eta(I) \subseteq V$ is compact in F_B. By Lemma 2.3 (b), there exists an absolutely convex, bounded subset $D \neq \emptyset$ of H such that $g(K) \subseteq H_D$ and $g|_K\colon F_B \supseteq K \to H_D$ is H_β. By Lemma 2.2 (b), the composition $g|_K \circ (f \circ \eta|_I)$ is $H_{\alpha\beta}$ as a map into H_D. The inclusion map $H_D \to H$ being continuous linear and thus H_1, we deduce that $g \circ (f \circ \eta|_I)$ is $H_{\alpha\beta}$ also as a map into H, using Lemma 2.2 (b). Thus $g \circ f \circ \gamma$ is locally $H_{\alpha\beta}$ and hence $H_{\alpha\beta}$. Hence $g \circ f$ is $h_{\alpha\beta}$. □

Lemma 2.12 *Let E_j be convenient vector spaces for $j \in \{1,2,3,4\}$, and $U_j \subseteq E_j$ be c^∞-open subsets for $j \in \{1,2,3\}$. Let $\phi : U_1 \to U_2$ and $\psi : U_3 \to E_4$ be c^∞-maps, and $f\colon U_2 \to U_3$ be h_α. Then $\psi \circ f \circ \phi\colon U_1 \to E_4$ is h_α.*

Proof. c^∞-maps being h_1 by Lemma 2.8 (b), the assertion follows from Lemma 2.11. □

Definition 2.13 Let M be a c^∞-manifold modelled on a convenient vector space E. A curve $\gamma: I \to M$ is called H_α if it is continuous with respect to the c^∞-topology on M and, for every $t \in I$, there exists a chart $\phi: U \to V \subseteq E$ of M around $\gamma(t)$ such that $\phi \circ \gamma|_{\gamma^{-1}(U)}: \gamma^{-1}(U) \to E$ is H_α. This then holds for all charts, by Lemma 2.12.

Definition 2.14 A map $f: M \to N$ between c^∞-manifolds is *conveniently Hölder with exponent* α (or an h_α*-map*) if $f \circ \gamma: \mathbb{R} \to N$ is an H_α-curve for each smooth curve $\gamma: \mathbb{R} \to M$. If f is h_α for some α, we say that f is *conveniently Hölder*.

Remark 2.15 Any h_α-map $f: M \to N$ is continuous with respect to the c^∞-topologies on M and N. Indeed, the topology on M being final with respect to the set of smooth curves in M, this follows from the observation that $f \circ \gamma$ is an H_α-curve and hence continuous with respect to the c^∞-topology on N, for each smooth curve $\gamma: \mathbb{R} \to M$.

3 Differentiability properties of curves at a point

In this section, we fix our terminology concerning differentiability of curves at a given point and prove a variant of the Chain Rule, which enables us to give a meaning to pointwise differentiability of a curve with values in a manifold.

Definition 3.1 Let E be a convenient vector space, $\gamma: I \to E$ be a map, defined on a subset $I \subseteq \mathbb{R}$, and $t \in I$ be a cluster point of I.

(a) γ is called *differentiable at* t if

$$\gamma'(t) := \lim_{s \to t} \frac{\gamma(s) - \gamma(t)}{s - t} \tag{3}$$

(where $s \in I \setminus \{t\}$) exists in E, equipped with its locally convex vector topology.

(b) We say that γ is *bornologically differentiable at* t if there exists a neighbourhood $J \subseteq I$ of t and an absolutely convex, bounded subset $B \neq \emptyset$ of E such that $\gamma(J) \subseteq E_B$ and the limit (3) exists in E_B.[3]

If γ is bornologically differentiable at t, then apparently γ is continuous at t as a map into E_B, and γ is differentiable at t (cf. **1.1**). Beside the standard case where I is an interval, we shall encounter the situation where $t = 0$ and $I = \{0\} \cup \{2^{-n}: n \in \mathbb{N}\}$.

Lemma 3.2 *Suppose that $\gamma: I \to E$ as before is bornologically differentiable at t and $f: U \to F$ a c^∞-map from an open neighbourhood $U \subseteq c^\infty(E)$ of $\gamma(t)$ to a convenient vector space F. Then $J := \gamma^{-1}(U)$ is a neighbourhood of t in I, and $f \circ \gamma|_J: J \to F$ is differentiable at t, with derivative $(f \circ \gamma)'(t) = f'(\gamma(t)).\gamma'(t)$.*

[3]In other words, the net of difference quotients is Mackey convergent to $\gamma'(t)$, cf. [10, La. 1.6].

Proof. Choose a neighbourhood $J \subseteq I$ of t and an absolutely convex, bounded subset $B \neq \emptyset$ of E as in Definition 3.1 (b). Thus $\gamma(J) \subseteq E_B$, and $\xi\colon J \to E_B$,

$$\xi(s) := \begin{cases} \frac{\gamma(s)-\gamma(t)}{s-t} & \text{if } s \in J \setminus \{t\} \\ \gamma'(t) & \text{if } s = t \end{cases}$$

is continuous at t as a map into E_B. This entails that $\gamma|_J$ is continuous at t as a map into E_B and hence also as a map into $c^\infty(E)$ (Lemma 1.4). Therefore $\gamma(J) \subseteq U$ without loss of generality, after shrinking J. We have

$$\frac{f(\gamma(s)) - f(\gamma(t))}{s-t} = \frac{f\Big(\gamma(t) + (s-t)\frac{\gamma(s)-\gamma(t)}{s-t}\Big) - f(\gamma(t))}{s-t} = f^{[1]}\big(\gamma(t),\, \xi(s),\, s-t\big)$$

for all $s \in J \setminus \{t\}$, where $f^{[1]} : U^{[1]} \to F$ is as in Lemma 1.7. Here $J \to E \times E \times \mathbb{R}$, $s \mapsto (\gamma(t), \xi(s), s-t)$ is continuous at t as a map into $E_B \times E_B \times \mathbb{R} = (E \times E \times \mathbb{R})_{B\times B\times[-1,1]}$ and hence also continuous at t as a map into $c^\infty(E \times E \times \mathbb{R})$, by Lemma 1.4. Since $f^{[1]}$ is c^∞ (see Lemma 1.7) and hence continuous with respect to the c^∞-topologies, we infer that the limit $(f \circ \gamma)'(t) = \lim_{s\to t} f^{[1]}(\gamma(t), \xi(s), s-t) = f^{[1]}(\gamma(t), \xi(t), 0) = f'(\gamma(t)).\gamma'(t)$ exists in $c^\infty(F)$ and hence also in F, and satisfies the required identity. □

Definition 3.3 Let $t \in \mathbb{R}$ and $\gamma\colon I \to M$ be a map from a neighbourhood $I \subseteq \mathbb{R}$ of t to a c^∞-manifold M modelled on a convenient vector space E. We say that γ is *invariantly differentiable at* t if the following conditions are satisfied:

(a) γ is continuous at t (with respect to the c^∞-topology on M).

(b) $\phi \circ \gamma\colon \gamma^{-1}(U) \to E$ is differentiable at t, for every chart $\phi\colon U \to V \subseteq E$ of M around $\gamma(t)$.

(c) For any charts ϕ and ψ of M around $\gamma(t)$, we have $(\psi \circ \gamma)'(t) = A.(\phi \circ \gamma)'(t)$ with $A := (\psi \circ \phi^{-1})'(\phi(\gamma(t)))$.

Property (c) ensures that there is a uniquely determined element $\gamma'(t) \in T_{\gamma(t)}M$ such that $(d\phi)(\gamma'(t)) = (\phi \circ \gamma)'(t)$, for every chart ϕ of M around $\gamma(t)$.

Remark 3.4 Let M, E and $\gamma\colon I \to M$ be as before. If γ is continuous at t and $\phi \circ \gamma$ is bornologically differentiable at t for some chart ϕ of M around $\gamma(t)$, then γ is invariantly differentiable at t, as a consequence of Lemma 3.2. Hence $\gamma'(t) \in TM$ makes sense.

4 Hölder differentiable curves and h^1_α-maps

In this section, we define k-times Hölder differentiable (H^k_α-) curves and conveniently Hölder differentiable (h^1_α-) maps. Beyond the Lipschitz case, H^k_α-curves will only play a role for $k = 1$ (or $k = 0$). We therefore refrain from discussing H^k_α-curves (and h^k_α-maps) for general k (cf. [6] for this), and focus on the case $k = 1$. As to higher order differentiability, the standard results on ℓip^k-curves and maps from [7] and [10] are sufficient for us.

Definition 4.1 Let E be a convenient vector space and $I \subseteq \mathbb{R}$ be open. A map $\gamma\colon I \to E$ is called an H^1_α*-curve* if it is differentiable at each $t \in I$ and both γ and $\gamma' : I \to E$ are H_α. Recursively, we say that γ is an H^{k+1}_α-curve if γ is H^1_α and γ' is an H^k_α-curve. The H^k_1-curves are also called ℓip^k-curves (cf. [10, p. 9]).

Lemma 4.2 *Let $\gamma\colon I \to E$ be an H^1_α-curve. Then the following holds:*

(a) *γ is bornologically differentiable at each $t \in I$.*

(b) *If $U \subseteq c^\infty(E)$ is an open neighbourhood of $\gamma(I)$ and $f : U \to F$ a c^∞-map to a convenient vector space F, then $f \circ \gamma : I \to F$ is an H^1_α-curve, and $(f \circ \gamma)'(t) = df(\gamma(t), \gamma'(t))$.*

(c) *If $\eta\colon \mathbb{R} \to I$ is smooth, then $\gamma \circ \eta$ is an H^1_α-curve.*

Proof. (a) (cf. [10, §1.7]). Given $t \in I$, let $J \subseteq I$ be a compact interval with t in its interior. By Lemma 2.3 (b), there exists an absolutely convex, bounded subset $B \neq \emptyset$ of E such that $\gamma'(J) \subseteq E_B$, $\gamma(J) \subseteq E_B$ and $\|\gamma'(r) - \gamma'(s)\|_B \leq |r - s|^\alpha$ for all $r, s \in J$. Then $\big\|\frac{\gamma(t+s)-\gamma(t)}{s} - \gamma'(t)\big\|_B = \big\| \int_0^1 \big(\gamma'(t + rs) - \gamma'(t)\big)\, dr\big\|_B \leq |s|^\alpha$ for each $s \in (J - t) \setminus \{0\}$, as $\|\gamma'(t + rs) - \gamma'(t)\|_B \leq |rs|^\alpha \leq |s|^\alpha$ for each r. Thus $\gamma'(t) = \lim_{s\to 0} s^{-1}(\gamma(t + s) - \gamma(t))$ in E_B.

(b) Fix $t \in I$. Since γ is bornologically differentiable at t (by (a)), Lemma 3.2 shows that $f\circ\gamma$ is differentiable at t, with $(f\circ\gamma)' = df\circ(\gamma, \gamma')$. Note that $f\circ\gamma$ and $(f\circ\gamma)' = df\circ(\gamma, \gamma')$ are h_α and hence H_α, by Lemmas 2.12 and 2.10. Hence $f \circ \gamma$ is H^1_α.

(c) By Lemmas 2.12 and 2.10, $\gamma\circ\eta$ is H_α. Both γ and η being C^1, also the composition $\gamma\circ\eta$ is C^1, with $(\gamma\circ\eta)'(t) = \eta'(t)\cdot\gamma'(\eta(t))$ (cf. [8, Prop. 1.12]). Thus $(\gamma\circ\eta)' = m\circ(\eta', \gamma'\circ\eta)$ is a composition of the continuous bilinear (and hence smooth) scalar multiplication map $m\colon \mathbb{R} \times E \to E$ and an H_α-curve, and thus $(\gamma \circ \eta)'$ is H_α (see Lemmas 2.12 and 2.10). □

Definition 4.3 Let M be a c^∞-manifold modelled on a convenient vector space E, and $I \subseteq \mathbb{R}$ be open. A map $\gamma : I \to M$ is called an ℓip^k-curve (resp., an H^1_α-curve) if it is continuous with respect to the c^∞-topology on M and, for every $t \in I$, there exists a chart $\phi\colon U \to V \subseteq E$ of M around $\gamma(t)$ such that $\phi \circ \gamma|_{\gamma^{-1}(U)} : \gamma^{-1}(U) \to E$ is ℓip^k (resp., H^1_α). This then holds for all charts, by [10, Cor. 12.9] (resp., Lemma 4.2 (b)).

Lemma 4.4 *A map $\gamma\colon I \to M$ from an open set $I \subseteq \mathbb{R}$ to a c^∞-manifold is an H^1_α-curve if and only γ is invariantly differentiable at each $t \in I$ and the curve $\gamma' : I \to TM$ is H_α.*

Proof. If γ is an H^1_α-curve, then γ is continuous (see Remark 2.15). Given $t \in I$, we let $\phi\colon U \to V \subseteq E$ be a chart of M around $\gamma(t)$. Then $\phi \circ \gamma|_J$ is an H^1_α-curve on $J := \gamma^{-1}(U)$ and thus bornologically differentiable at t (see Lemma 4.2 (a)), whence γ is invariantly differentiable at t (Remark 3.4). If, conversely, γ is invariantly differentiable at each t and γ' an H_α-curve, then γ is continuous and for t, ϕ and J as before $(\phi \circ \gamma|_J)' = d\phi \circ \gamma'|_J$ is H_α (by Lemmas 2.12 and 2.10), entailing that $\phi \circ \gamma|_J$ is an H^1_α-curve. □

Definition 4.5 A map $f : M \to N$ between c^∞-manifolds is called h^1_α (resp., ℓip^k) if $f \circ \gamma\colon \mathbb{R} \to N$ is an H^1_α-curve (resp., a ℓip^k-curve), for each smooth curve $\gamma\colon \mathbb{R} \to M$.

5 Curve differentiability of maps at a given point

In this section, we define and discuss differentiability properties of mappings at a given point. "Bornological curve differentiability," which we introduce here, is a bornological variant of the classical notion of Hadamard differentiability. It is a well-chosen concept in the sense that, on the one hand, bornological curve differentiability at 1 is a sufficiently strong property to ensure smoothness of a homomorphism. On the other hand, it is a sufficiently weak differentiability property, in the sense that we shall manage to establish it for conveniently Hölder homomorphisms. We also introduce a notion of "curve differentiability," as a technical tool. This is a weaker and much more illusive concept.

Let E and F be convenient vector spaces, $U \subseteq E$ be c^∞-open, $x \in U$, and $f\colon U \to F$.

Definition 5.1 f is called *curve differentiable at x* if the conditions (a)–(d) are satisfied:

(a) $f \circ \gamma$ is differentiable at 0, for every smooth curve $\gamma\colon \mathbb{R} \to U$ such that $\gamma(0) = x$.

(b) There exists a (necessarily unique) bounded linear map $f'(x) : E \to F$ such that $(f \circ \gamma)'(0) = f'(x).\gamma'(0)$, for every γ as in (a).

(c) f is continuous at x with respect to the c^∞-topologies on U and F.

(d) For every convenient vector space H and c^∞-map $g : V \to H$ defined on an open neighbourhood V of $f(x)$ in $c^\infty(F)$, and every smooth curve $\gamma : \mathbb{R} \to f^{-1}(V)$, the curve $g \circ f \circ \gamma$ is differentiable at 0, with $(g \circ f \circ \gamma)'(0) = g'(f(x)).f'(x).\gamma'(0)$.

We call f *bornologically curve differentiable at x* if $f \circ \gamma$ is bornologically differentiable at 0 for every γ as in (a), and conditions (b) and (c) are satisfied.

Of course, if f is curve differentiable at x, then analogues of (a), (b) and (d) hold if $\gamma\colon I \to U$ is defined on an open 0-neighbourhood $I \subseteq \mathbb{R}$ only (cf. Remark 2.7 (b)). Also note that (c) holds if f is conveniently Hölder, by Remark 2.15.

Remark 5.2 Let f as before be curve differentiable at x, and g be as in (d).

(a) If $h\colon W \to f^{-1}(V)$ is a c^∞-map defined on a c^∞-open subset of a convenient vector space and $w \in W$ such that $h(w) = x$, then $(g \circ f \circ h \circ \gamma)'(0) = (g \circ f \circ (h \circ \gamma))'(0) = g'(f(x)).f'(x).(h \circ \gamma)'(0) = g'(f(x)).f'(x).h'(w).\gamma'(0)$ for any smooth curve $\gamma\colon \mathbb{R} \to W$ such that $\gamma(0) = w$, exploiting condition (d) from Definition 5.1.

(b) For f, g, h and w as before, we readily deduce from Definition 5.1 (d), Part (a) of the present remark and the Chain Rule for c^∞-maps that $g \circ f \circ h : W \to H$ is curve differentiable at w, with $(g \circ f \circ h)'(w) = g'(f(x)) \circ f'(x) \circ h'(w)$.

From Lemma 3.2, we immediately deduce:

Lemma 5.3 *If $f\colon U \to F$ (as before) is bornologically curve differentiable at x, then f is curve differentiable at x.* □

Definition 5.4 Let M and N be c^∞-manifolds modelled on E, resp., F, and $x \in M$. A map $f\colon M \to N$ is called *curve differentiable at* x (resp., *bornologically curve differentiable at* x), if there exists a chart $\phi\colon U_1 \to U \subseteq E$ of M around x and a chart $\psi\colon V_1 \to V \subseteq F$ of N around $f(x)$ such that $f(U_1) \subseteq V_1$ and $\psi \circ f \circ \phi^{-1}$ is curve differentiable (resp., bornologically curve differentiable) at $\phi(x)$.

Remark 5.5 Let f (as before) be curve differentiable at x. The the following holds:

(a) f is continuous at x with respect to the c^∞-topologies on M and N.

(b) In the situation of Definition 5.4, $\psi \circ f \circ \phi^{-1}$ is curve differentiable at $\phi(x)$ for *every* choice of charts (use Remark 5.2 (b)).

(c) If $\gamma\colon I \to M$ is a c^∞-curve on an open 0-neighbourhood $I \subseteq \mathbb{R}$ such that $\gamma(0) = x$, then $f \circ \gamma$ is invariantly differentiable at 0 (in view of Part (a) of the present remark and Part (d) of the definition of curve differentiability, applied to f in local coordinates). Furthermore, $(f \circ \gamma)'(0)$ only depends on $\gamma'(0)$ (as a consequence of Definition 5.1 (b)). Hence $T_x f : T_x M \to T_{f(x)} N$, $(T_x f).\gamma'(0) := (f \circ \gamma)'(0)$ is well defined. As a consequence of Definition 5.1 (b), $T_x f$ is bounded linear.

(d) Remark 5.2 (b) readily entails: If Y, Z are c^∞-manifolds, $g : N \to Z$ as well as $h : Y \to M$ are c^∞-maps, and $y \in Y$ is an element such that $h(y) = x$, then $g \circ f \circ h\colon Y \to Z$ is curve differentiable at y, with $T_y(g \circ f \circ h) = T_{f(x)} g \circ T_x f \circ T_y h$.

6 Pointwise differentiable homomorphisms are smooth

We now perform Step 1 of the programme outlined in the Introduction: curve differentiability at 1 implies smoothness for homomorphisms. A simple observation will be used:

Lemma 6.1 *Let $f\colon G \to H$ be a ℓip^1-homomorphism between Lie groups in the sense of convenient differential calculus. Then f is a c^∞-map.*

Proof. We show that f is of ℓip^k for each $k \in \mathbb{N}$, by induction. By hypothesis, f is ℓip^1; it therefore gives rise to a tangent map $Tf\colon TG \to TH$. With respect to the left trivializations of TG and TH, the map Tf corresponds to $f \times L(f)\colon G \times L(G) \to H \times L(H)$. Here f is ℓip^k, and the linear map $L(f)$ is ℓip^0 (cf. [10, Thm. 12.8]), hence bounded (Lemma 1.6) and thus smooth [10, Cor. 2.11]. Hence Tf is ℓip^k. Now f being ℓip^1 with Tf a ℓip^k-map, the map f is ℓip^{k+1} (cf. [10, Thm. 12.8]). □

Proposition 6.2 *Let $f\colon G \to H$ be a conveniently Hölder homomorphism between c^∞-Lie groups. If f is curve differentiable at 1, then f is smooth.*

Proof. Being conveniently Hölder, f is h_α for some $\alpha \in \,]0,1]$. Given $x \in G$ we have $f = \lambda^H_{f(x)} \circ f \circ \lambda^G_{x^{-1}}$ (with left translation maps as indicated), because f is a homomorphism. The map f being curve differentiable at 1, using Remark 5.5 (d), we deduce from the latter formula that f is curve differentiable at x, with

$$T_x f = T_1\lambda^H_{f(x)} \circ T_1 f \circ T_x \lambda^G_{x^{-1}}\,. \tag{4}$$

Now let $\gamma\colon \mathbb{R} \to G$ be a smooth curve. Given $t \in \mathbb{R}$, the curve $f \circ \gamma$ is invariantly differentiable at t because f is curve differentiable at $\gamma(t)$ (cf. Remark 5.5 (c)). By (4), we have $(f\circ\gamma)'(t) = (T_{\gamma(t)}f).\gamma'(t) = (T_1\lambda^H_{f(\gamma(t))} \circ T_1 f \circ T_{\gamma(t)}\lambda^G_{\gamma(t)^{-1}}).\gamma'(t)$. Using the bounded linear (and hence c^∞-) map $A := T_1 f\colon L(G) \to L(H)$, the Maurer-Cartan form $\omega_G\colon TG \to L(G)$, $T_xG \ni v \mapsto (T_x\lambda^G_{x^{-1}})(v)$ (which is c^∞), and the c^∞-map $h\colon H \times L(H) \to TH$ defined via $h(x,v) := (T_1\lambda^H_x)(v)$, the preceding formula can be rewritten in the form

$$(f\circ\gamma)'(t) \;=\; h\big(f(\gamma(t)), A.\omega_G(\gamma'(t))\big) \qquad \text{for all } t \in \mathbb{R}\,.$$

The curve $\zeta := A \circ \omega_G \circ \gamma' \colon \mathbb{R} \to L(H)$ is smooth and thus H_α. Furthermore, $\eta := f\circ\gamma\colon \mathbb{R} \to H$ is H_α, as f is h_α. Hence $(\eta,\zeta)\colon \mathbb{R} \to H \times L(H)$ is H_α and thus $(f\circ\gamma)' \;=\; h\circ(\eta,\zeta)$ is H_α, by Lemmas 2.10 and 2.12. Now Lemma 4.4 shows that f is h^1_α and hence ℓip^0, using that h^1_α-curves in $L(H)$ are C^1 and hence ℓip^0. We may therefore take $\alpha = 1$ in the preceding considerations, showing that f is $h^1_1 = \ell ip^1$ and hence smooth (Lemma 6.1). $\square$

7 Testing at 1 whether a homomorphism is h_α

In this section, we explain how the h_α-property of a homomorphism can be characterized by a suitable property at the identity element. In the next section, this characterization will be used to show that an h_β-homomorphism, where $\beta \in \,]0,\frac{1}{2}]$, is also h_α for $\alpha := \frac{3}{2}\beta$.

Lemma 7.1 *A homomorphism $f : G \to H$ between c^∞-Lie groups is h_α if and only if f is continuous with respect to the c^∞-topologies and for every c^∞-map $\theta\colon \mathbb{R}^2 \to G$ with $\theta(s,0) = 1$ for all $s \in \mathbb{R}$, there exists a chart $\phi : U \to V \subseteq L(H)$ of H around 1, an open 0-neighbourhood $I \subseteq \mathbb{R}$ such that $f(\theta(I\times I)) \subseteq U$, and an absolutely convex, bounded subset $B \neq \emptyset$ of $L(H)$ such that*

(a) *$\phi(f(\theta(I^2))) \subseteq L(H)_B$;*

(b) *$\xi := \phi\circ f\circ\theta|_{I^2}\colon I^2 \to L(H)_B$ is continuous;*

(c) *There exists $K \geq 0$ such that $\|\xi(s,t) - \xi(s,0)\|_B \leq K|t|^\alpha$, for all $s,t \in I$.*

This then holds for any choice of the chart ϕ.

Proof. The final assertion is easily established using Lemma 2.9; we omit the details. The necessity of the condition is apparent (cf. Remark 2.15, Lemma 2.10, Lemma 2.3).

7.2 Conversely, assume now that f satisfies the described condition. We have to show that $f \circ \gamma$ is H_α, for each c^∞-curve $\gamma : \mathbb{R} \to G$. This will hold if $f \circ \gamma$ is H_α on some open neighbourhood of each given $t_0 \in \mathbb{R}$. Because $(f \circ \gamma)(t) = f(\gamma(t_0)) f(\gamma(t_0)^{-1}\gamma(t)) = (\lambda_{f(\gamma(t_0))} \circ f \circ \zeta)(t - t_0)$, where $\zeta : \mathbb{R} \to G$, $\zeta(t) := \gamma(t_0)^{-1}\gamma(t + t_0)$ is a c^∞-curve, after replacing γ with ζ it actually suffices to assume that $t_0 = 0$ and $\gamma(t_0) = 1$.

7.3 Pick a chart $\phi : U \to V \subseteq L(H)$ of H around 1 such that $\phi(1) = 0$. The group multiplication $m : H \times H \to H$ being c^∞, $M_1 := \{(x, y) \in U \times U : xy \in U\}$ is open in $H \times H$ (equipped with the c^∞-topology). Then $M := (\phi \times \phi)(M_1)$ is c^∞-open in $L(H) \times L(H)$, and $\mu := \phi \circ m|_{M_1}^U \circ (\phi^{-1} \times \phi^{-1})|_M^{M_1} : M \to V$ is c^∞.

7.4 The c^∞-maps $\theta : \mathbb{R}^2 \to G$, $\theta(s, t) := \gamma(s)^{-1}\gamma(s + t)$ and $\bar{\theta} : \mathbb{R}^2 \to G$, $\bar{\theta}(s, t) := \gamma(t)$ satisfy $\theta(s, 0) = \bar{\theta}(s, 0) = 1$ for all $s \in \mathbb{R}$. Using the hypotheses on f (and the final assertion of the lemma), we find an open 0-neighbourhood $I \subseteq \mathbb{R}$, $K_1 \geq 0$ and an absolutely convex, bounded subset $B \neq \emptyset$ of $L(H)$ such that $f(\theta(I^2)), f(\bar{\theta}(I^2)) \subseteq U$, $\phi(f(\theta(I^2))), \phi(f(\bar{\theta}(I^2))) \subseteq L(H)_B$, and both of the maps $\xi := \phi \circ f \circ \theta|_{I^2} : I^2 \to L(H)_B$ and $\bar{\xi} := \phi \circ f \circ \bar{\theta}|_{I^2} : I^2 \to L(H)_B$ are continuous and $\|\xi(s, t)\|_B \leq K_1 |t|^\alpha$ for all $s, t \in I$. Then also the map $\bar{\xi} \times \xi : I^2 \times I^2 \to c^\infty(L(H) \times L(H))$ is continuous (cf. Lemma 1.4). After shrinking I, we may hence assume that $\bar{\xi}(I^2) \times \xi(I^2) \subseteq M$.

7.5 Let $A \subseteq I$ be a compact 0-neighbourhood and $J \subseteq \mathbb{R}$ be an open 0-neighbourhood such that $J - J \subseteq A$. We claim that $\phi \circ f \circ \gamma|_J : J \to L(H)$ is H_α (where $\phi(f(\gamma(t))) = \bar{\xi}(0, t)$). To see this, let $\|.\|_p$ be a continuous seminorm on $L(H)$. The set $C := (\bar{\xi} \times \xi)(A^2 \times A^2) \subseteq (L(H)_B)^2 = (L(H)^2)_{B^2}$ is compact. Since μ is c^∞ and thus ℓip^0, using Lemmas 2.9 and 2.3 (a) we find $K_2 \in [0, \infty[$ such that $\|\mu(u) - \mu(v)\|_p \leq K_2 \|u - v\|_{B^2}$ for all $u, v \in C$. Given $r, t \in J \subseteq A$, we have $s := r - t \in A$. Using that $f(\gamma(t)) = \phi^{-1}(\bar{\xi}(0, t))$, $f(\gamma(t)^{-1}\gamma(t+s)) = \phi^{-1}(\xi(t, s))$ and $f(1) = \phi^{-1}(\xi(t, 0))$, we obtain

$$\begin{aligned}
\phi(f(\gamma(r))) - \phi(f(\gamma(t))) &= \phi(f(\gamma(t + s))) - \phi(f(\gamma(t))) \\
&= \phi(f(\gamma(t)) f(\gamma(t)^{-1}\gamma(t + s))) - \phi(f(\gamma(t)) f(1)) \\
&= \mu(\bar{\xi}(0, t), \xi(t, s)) - \mu(\bar{\xi}(0, t), \xi(t, 0))\,.
\end{aligned}$$

We now deduce that $\|\phi(f(\gamma(r))) - \phi(f(\gamma(t)))\|_p = \|\mu(\bar{\xi}(0, t), \xi(t, s)) - \mu(\bar{\xi}(0, t), \xi(t, 0))\|_p \leq K_2 \|\xi(t, s) - \xi(t, 0)\|_B = K_2 \|\xi(t, s)\|_B \leq K_1 K_2 |s|^\alpha = K_1 K_2 |r - t|^\alpha$.

Thus $\phi \circ f \circ \gamma|_J : J \to L(H)$ is H_α and hence so is $f \circ \gamma|_J$, which completes the proof. □

8 Estimates on Taylor remainders

We study the behaviour of first order Taylor remainders on bornologically compact sets.

Lemma 8.1 *Let E and F be convenient vector spaces, $U \subseteq E$ be c^∞-open and $f\colon U \to F$ be c^∞; define the "first order Taylor remainder" $\rho : U \times U \to F$ of f via $\rho(x,y) := f(y) - f(x) - f'(x).(y-x)$. Let $C \neq \emptyset$ be an absolutely convex, bounded subset of E and $K \subseteq E_C \cap U$ be compact. Then there exists an absolutely convex, bounded subset $B \neq \emptyset$ of F such that $f(K) \subseteq F_B$, $\rho(K \times K) \subseteq F_B$, the restrictions $f|_K : E_C \supseteq K \to F_B$ and $\rho|_{K^2}\colon (E_C)^2 \supseteq K \times K \to F_B$ are Lipschitz continuous, and*

$$\|\rho(x,y)\|_B \leq (\|x-y\|_C)^2 \quad \textit{for all } x, y \in K. \tag{5}$$

Proof. After replacing C with its closure in E, we may assume that C is the closed unit ball in $(E_C, \|.\|_C)$. The maps f and ρ being c^∞ and hence ℓip^0, Lemma 2.9 provides an absolutely convex, bounded subset $B_0 \neq \emptyset$ of F such that $f(K) \subseteq F_{B_0}$, $\rho(K \times K) \subseteq F_{B_0}$, and such that both $f|_K : E_C \supseteq K \to F_{B_0}$ and $\rho|_{K^2} : (E_C)^2 \supseteq K^2 \to F_{B_0}$ are Lipschitz continuous. Since $U \cap E_C$ is open in E_C (see Lemma 1.4) and K compact, we find $\varepsilon > 0$ such that $K + \varepsilon C \subseteq U$. Then the set $K_\varepsilon := \{(x,y) \in K^2 \colon \|x-y\|_C \leq \varepsilon\}$ is compact in $(E_C)^2$, and $[x,y] := \{x + t(y-x)\colon t \in [0,1]\} \subseteq U$, for any $(x,y) \in K_\varepsilon$. Hence, by Taylor's Formula [7, Prop. 4.4.18], we have

$$\rho(x,y) = \int_0^1 (1-t) f''(x + t(y-x))(y-x, y-x)\, dt \quad \text{for all } (x,y) \in K_\varepsilon. \tag{6}$$

The set $\widetilde{K} := \bigcup_{(x,y)\in K_\varepsilon} [x,y] \subseteq U$ is compact in E_C. Now $f''\colon U \to \mathcal{L}^2(E,F)$ being c^∞, the image $f''(\widetilde{K})$ is compact in $\mathcal{L}^2(E,F)$, equipped with the c^∞-topology. Hence $f''(\widetilde{K})$ is also compact in the locally convex space $\mathcal{L}^2(E,F)$ and thus bounded. The trilinear evaluation map $\mathrm{ev}\colon \mathcal{L}^2(E,F) \times E \times E \to F$ being c^∞ (cf. [10, Cor. 3.13 (1)]) and thus bounded [10, La. 5.5], we see that the image $\mathrm{ev}(f''(\widetilde{K}) \times C \times C)$ is bounded in F and hence contained in an absolutely convex, bounded subset $B \neq \emptyset$ of F:

$$\mathrm{ev}(f''(\widetilde{K}) \times C \times C) \subseteq B\,. \tag{7}$$

After increasing B, we may assume that B is closed and $B_0 \subseteq B$. As a consequence of (7), we have $d^2 f(\widetilde{K} \times (K-K) \times (K-K)) \subseteq F_B$; since $\widetilde{K}$ is compact, after increasing B further we may assume that and $d^2 f$ is Lipschitz continuous as a map from $\widetilde{K} \times (K-K)^2 \subseteq (E_C)^3$ to F_B (Lemma 2.9). This entails that the integrands in (6) are continuous as maps into the Banach space F_B. Hence, the integral also exists in F_B, and clearly the F_B-valued integral coincides with the F-valued integral $\rho(x,y)$ (equality can be tested with linear functionals in F'). Now [10, Cor. 2.6 (4)] (or [8, La. 1.7]) implies that

$$\|\rho(x,y)\|_B \leq \sup_{t\in[0,1]} \|f''(x+t(y-x))(y-x,y-x)\|_B \leq \|y-x\|_C^2 \quad \text{for all } (x,y) \in K_\varepsilon, \tag{8}$$

where (7) was used to get the second inequality. To complete the proof, note that $D := \{(x,y) \in K^2 : \|x-y\|_C \geq \varepsilon\}$ is compact in $(E_C)^2$, whence $\rho(D)$ is compact in F_{B_0} and hence bounded in F. Let $B_1 := \mathrm{absconv}(\varepsilon^{-2}\rho(D))$ be the absolutely convex hull of $\varepsilon^{-2}\rho(D)$, and replace B with $\mathrm{absconv}(B \cup B_1)$. Then (8) remains valid, and furthermore $\|\rho(x,y)\|_B = \varepsilon^2 \|\varepsilon^{-2}\rho(x,y)\|_B \leq \varepsilon^2 \leq \|(x,y)\|_C^2$ for all $(x,y) \in D$. Since $K^2 = K_\varepsilon \cup D$, the preceding estimate and (8) show that (5) holds. □

9 Conveniently Hölder homomorphisms are smooth

Having completed all necessary preparations, we are now ready to prove the main result.

Theorem 9.1 *Let $f : G \to H$ be a homomorphism between Lie groups in the sense of convenient differential calculus. If f is conveniently Hölder, then f is a c^∞-map. In particular, f is c^∞ if f is ℓip^0.*

Proof. By hypothesis, f is h_α for some $\alpha \in \,]0,1]$. We choose a chart $\phi\colon U_1 \to U \subseteq L(G)$ of G around 1 and a chart $\psi : V_1 \to V \subseteq L(H)$ of H around 1 such that $f(U_1) \subseteq V_1$, $\phi(1) = 0$, and $\psi(1) = 0$. Then $g := \psi \circ f \circ \phi^{-1}\colon U \to V \subseteq L(H)$ is h_α, and $g(0) = 0$.

9.2 The map $\bar\sigma\colon V_1 \times V_1 \to H$, $\bar\sigma(x,y) := x^2 y$ being c^∞, the preimage $S_1 := \bar\sigma^{-1}(V_1)$ is c^∞-open in $V_1 \times V_1$. Then $S := (\psi\times\psi)(S_1)$ is a c^∞-open $(0,0)$-neighbourhood in $L(H)\times L(H)$, and $\sigma := \psi\circ\bar\sigma\circ(\psi^{-1}\times\psi^{-1})|_S\colon S \to V$ is c^∞. The first order Taylor expansion of σ around $(0,0)$ gives

$$\sigma(x,y) = 2x + y + R(x,y) \quad \text{for all } (x,y) \in S \subseteq L(H)\times L(H), \tag{9}$$

where $R\colon S \to L(H)$, $R(x,y) := \rho_\sigma((0,0),(x,y))$ with $\rho_\sigma : S\times S \to L(H)$ the first order Taylor remainder of σ (as in Lemma 8.1).

9.3 The map $\bar\tau\colon U_1\times U_1 \to G$, $\bar\tau(x,y) := x^{-2}y := x^{-1}x^{-1}y$ being c^∞, the set $W_1 := \bar\tau^{-1}(U_1)$ is c^∞-open in $U_1 \times U_1$. Then $W := (\phi\times\phi)(W_1)$ is a c^∞-open $(0,0)$-neighbourhood in $L(G)\times L(G)$, and $\tau := \phi\circ\bar\tau\circ(\phi^{-1}\times\phi^{-1})|_W\colon W \to U$ is c^∞. We have $\tau(0,0) = 0$ and

$$d\tau((0,0),(u,v)) = -2u + v \qquad \text{for all } u,v \in L(G). \tag{10}$$

9.4 Let $\ell \in \{1,2\}$, $\delta \in \,]0,\infty]$ and $\eta\colon B(\delta) \to U \subseteq L(G)$ be a c^∞-map on $B(\delta) := \,]-\delta,\delta[^\ell \subseteq \mathbb{R}^\ell$, such that $\eta(s,0) = 0$ for all $s \in \,]-\delta,\delta[^{\ell-1} \subseteq \mathbb{R}^{\ell-1}$ (if $\ell = 1$, we identify $\mathbb{R}$ with $\mathbb{R}^0\times\mathbb{R}$ here and in the following, to unify notation. Thus, the argument s has to be ignored if $\ell = 1$, and what we require is $\eta(0) = 0$). Then $\theta := \phi^{-1}\circ\eta\colon B(\delta) \to U_1 \subseteq G$ is c^∞. The map $\zeta : B(\delta) \to L(G)\times L(G)$, $\zeta(s,t) := (\eta(s,\tfrac12 t),\eta(s,t))$ being c^∞ and hence continuous into $c^\infty(L(G)\times L(G))$, we find $\delta_1 \in \,]0,\delta]$ with $\zeta(B(\delta_1)) \subseteq W$. Then $\kappa : B(\delta_1) \to U$, $\kappa(s,t) := \tau(\eta(s,\tfrac12 t),\eta(s,t))$ is c^∞.

9.5 The map $\chi\colon B(\delta_1) \to H\times H$, $\chi(s,t) := \big(f(\theta(s,\tfrac12 t)),\, f(\theta(s,\tfrac12 t)^{-2}\theta(s,t))\big)$ being h_α, we find $\delta_2 \in \,]0,\delta_1]$ such that $\chi(B(\delta_2)) \subseteq S_1$. We define $\omega\colon B(\delta_2) \to S$, $\omega := (\psi\times\psi)\circ\chi|_{B(\delta_2)}$. Then $\omega(s,t) = \big(g(\eta(s,\tfrac12 t)),\, g(\tau(\eta(s,\tfrac12 t),\eta(s,t)))\big) = \big(g(\eta(s,\tfrac12 t)),\, g(\kappa(s,t))\big)$ and

$$\sigma(\omega(s,t)) \;=\; \sigma\big(g(\eta(s,\tfrac12 t)),\, g(\tau(\eta(s,\tfrac12 t),\eta(s,t)))\big) \;=\; g(\eta(s,t)) \;\text{ for all } (s,t) \in B(\delta_2), \tag{11}$$

as $\psi^{-1}\big(\sigma(g(\eta(s,\tfrac12 t)),\, g(\tau(\eta(s,\tfrac12 t),\eta(s,t))))\big) \;=\; f(\theta(\tfrac12 s,t))^2 f\big(\theta(s,\tfrac12 t)^{-2}\theta(s,t)\big) \;=\; f(\theta(s,t)) = \psi^{-1}\big(g(\eta(s,t))\big)$.

9.6 Combining (11) and (9) yields $g(\eta(s,t)) = 2g(\eta(s,\frac{1}{2}t)) + g(\kappa(s,t)) + R(\omega(s,t))$, whence

$$g(\eta(s,\tfrac{1}{2}t)) \;=\; \tfrac{1}{2}g(\eta(s,t)) - \tfrac{1}{2}g(\kappa(s,t)) - \tfrac{1}{2}R(\omega(s,t)) \quad \text{for all } (s,t) \in B(\delta_2). \tag{12}$$

Since $(s,\frac{1}{2}t) \in B(\delta_2)$, applying (12) twice we see that

$$\begin{aligned} g(\eta(s,\tfrac{1}{4}t)) &= \tfrac{1}{2}g(\eta(s,\tfrac{1}{2}t)) - \tfrac{1}{2}g(\kappa(s,\tfrac{1}{2}t)) - \tfrac{1}{2}R(\omega(s,\tfrac{1}{2}t)) \\ &= \tfrac{1}{4}g(\eta(s,t)) - \tfrac{1}{4}g(\kappa(s,t)) - \tfrac{1}{4}R(\omega(s,t)) - \tfrac{1}{2}g(\kappa(s,\tfrac{1}{2}t)) - \tfrac{1}{2}R(\omega(s,\tfrac{1}{2}t))\,. \end{aligned}$$

Similarly, by a simple induction

$$g(\eta(s,2^{-n}t)) = 2^{-n}g(\eta(s,t)) - \sum_{k=0}^{n-1} 2^{k-n}\Big(g(\kappa(s,2^{-k}t)) + R(\omega(s,2^{-k}t))\Big) \tag{13}$$

for all $(s,t) \in B(\delta_2)$ and $n \in \mathbb{N}$. Thus, for all $(s,t) \in B(\delta_2)$ and $n \in \mathbb{N}$:

$$\frac{g(\eta(s,2^{-n}t))}{2^{-n}} = g(\eta(s,t)) - \sum_{k=0}^{n-1} 2^{k}\Big(g(\kappa(s,2^{-k}t)) + R(\omega(s,2^{-k}t))\Big)\,. \tag{14}$$

Choose $\delta_3 \in \,]0,\delta_2[$ such that $\delta_3 \le 1$; then the closure $K := \overline{B(\delta_3)} \subseteq B(\delta_2)$ is compact.

Lemma 9.7 *There exists an absolutely convex, bounded subset $B_1 \neq \emptyset$ of $L(G)$ such that $\kappa(K) \subseteq L(G)_{B_1}$, $\kappa|_K \colon K \to L(G)_{B_1}$ is Lipschitz continuous, and $\|\kappa(s,t)\|_{B_1} \le t^2$ for all $(s,t) \in K$. In particular, $K_1 := \kappa(K)$ is compact in $L(G)_{B_1}$.*

Proof. Let $\rho_\kappa \colon B(\delta_1)^2 \to L(G)$ be the first order Taylor remainder of κ. By Lemma 8.1, there exists an absolutely convex, bounded subset $B_1 \neq \emptyset$ of $L(G)$ such that $\kappa(K) \subseteq L(G)_{B_1}$, the map $\kappa|_K \colon K \to L(G)_{B_1}$ is Lipschitz continuous, $\rho_\kappa(K \times K) \subseteq L(G)_{B_1}$, and $\|\rho_\kappa(x,y)\|_{B_1} \le (\|x-y\|_\infty)^2$ for all $x,y \in K$ (where $\|.\|_\infty$ is the maximum norm). Note that $\frac{\partial \kappa}{\partial s}(s,0) = 0$ for all $s \in \,]-\delta_1,\delta_1[^{\ell-1}$ because $\kappa(\bullet,0) = 0$. Furthermore, $\frac{\partial \kappa}{\partial t}(s,0) = -2 \cdot \frac{1}{2} \cdot \frac{\partial \eta}{\partial t}(s,0) + \frac{\partial \eta}{\partial t}(s,0) = 0$ by the Chain Rule (cf. (10)). Hence $\kappa'(s,0) = 0$ for all s, whence the Taylor expansion around $(s,0)$ yields $\kappa(s,t) = \kappa(s,t) - \kappa(s,0) = \rho_\kappa((s,t),(s,0))$ for all $(s,t) \in K$. Therefore $\|\kappa(s,t)\|_{B_1} = \|\rho_\kappa((s,t),(s,0))\|_{B_1} \le \|(s,t)-(s,0)\|_\infty^2 = t^2$. $\square$

9.8 By Lemmas 2.9 and 2.3, there is an absolutely convex, bounded subset $B_2 \neq \emptyset$ of $L(H)$ such that $g(K_1) \subseteq L(H)_{B_2}$ and $\|g(y) - g(x)\|_{B_2} \le (\|y-x\|_{B_1})^\alpha$ for all $x,y \in K_1$.

9.9 The map ω being h_α and K compact, there exists an absolutely convex, bounded subset $D \subseteq L(H) \times L(H)$ such that $\omega(K) \subseteq (L(H)^2)_D$ and $\|\omega(y) - \omega(x)\|_D \le (\|y-x\|_\infty)^\alpha$ for all $x,y \in K$. Then $K_2 := \omega(K) \subseteq S$ is compact in $(L(H)^2)_D$.

9.10 Now Lemma 8.1 provides an absolutely convex, bounded subset $B_3 \neq \emptyset$ of $L(H)$ such that $R(K_2) \subseteq L(H)_{B_3}$, $R|_{K_2} \colon K_2 \to L(H)_{B_3}$ is Lipschitz continuous, and $\|R(x)\|_{B_3} \le (\|x\|_D)^2$ for all $x \in K_2$. After replacing B_2 and B_3 with an absolutely convex, bounded superset $B \subseteq L(H)$ (e.g., $B = \operatorname{absconv}(B_2 \cup B_3)$), we may assume that $B_2 = B_3 = B$.

9.11 Let $\rho_\eta\colon B(\delta)\times B(\delta)\to L(G)$ be the first order Taylor remainder of η. As η is c^∞, Lemma 8.1 provides an absolutely convex, bounded subset $B_4\neq\emptyset$ of $L(G)$ such that $\eta(K)\subseteq L(G)_{B_4}$, $\eta|_K\colon K\to L(G)_{B_4}$ is Lipschitz continuous, $\rho_\eta(K\times K)\subseteq L(G)_{B_4}$, and

$$\|\rho_\eta(x,y)\|_{B_4}\leq(\|y-x\|_\infty)^2\quad\text{for all } x,y\in K. \tag{15}$$

Then g being h_α and $\eta(K)\subseteq L(G)_{B_4}\cap U$ being compact, after increasing B we may assume that $g(\eta(K))\subseteq B\subseteq L(H)_B$ and

$$\|g(y)-g(x)\|_B\leq(\|y-x\|_{B_4})^\alpha\quad\text{for all } x,y\in\eta(K) \tag{16}$$

(Lemmas 2.9 and 2.3). Then $g\circ\eta|_K\colon K\to L(H)_B$ is H_α (Lemma 2.2 (b)). After passing to the closure, w.l.o.g. B is closed in $L(H)$ and so $(L(H)_B,\|.\|_B)$ is a Banach space (**1.1**).

We deduce estimates on the summands in (14) (or multiples thereof) now.

9.12 We have $\|g(\kappa(s,2^{-k}t))\|_B=\|g(\kappa(s,2^{-k}t))-g(\kappa(s,0))\|_B\leq\|\kappa(s,2^{-k}t)-\kappa(s,0)\|_{B_1}^\alpha=\|\kappa(s,2^{-k}t)\|_{B_1}^\alpha\leq 2^{-2\alpha k}|t|^{2\alpha}$ for any $(s,t)\in K$ and $k\in\mathbb{N}_0$, by **9.8** and Lemma 9.7.

9.13 For any $(s,t)\in K$ and $k\in\mathbb{N}_0$, we have $\|R(\omega(s,2^{-k}t))\|_B\leq\|\omega(s,2^{-k}t)\|_D^2=\|\omega(s,2^{-k}t)-\omega(s,0)\|_D^2\leq 2^{-2\alpha k}|t|^{2\alpha}$, by **9.10** and **9.9**.

Lemma 9.14 *If $\alpha\in\,]0,\frac{1}{2}]$, then f is also h_β for $\beta:=\frac{3}{2}$.*

Proof. Let $\ell=2$. For $\theta:\mathbb{R}^2\to G$ as in Lemma 7.1, there is $\delta\in\,]0,\infty]$ such that $\theta(B(\delta))\subseteq U_1$. Then **9.4**–**9.13** apply to $\eta:=\phi\circ\theta|_{B(\delta)}$. Let $I:=\,]-\delta_3/2,\delta_3/2[$. Since $\psi\circ f\circ\theta|_{B(\delta)}=g\circ\eta$, we have $\psi(f(\theta(I^2)))\subseteq g(\eta(K))\subseteq L(H)_B$ and $\psi\circ f\circ\theta|_{I^2}=g\circ\eta|_{I^2}\colon I^2\to L(H)_B$ is continuous (see **9.11**). Thus conditions (a) and (b) of Lemma 7.1 are satisfied. To verify condition (c), let $0\neq t\in I$ and $s\in I$. There is $n\in\mathbb{N}$ with $\frac{1}{2}\delta_3<|2^nt|\leq\delta_3$. Let $t^*:=2^nt$; then $(s,t^*)\in K$ and thus

$$\begin{aligned}
\|g(\eta(s,t)\|_B &= \|g(\eta(s,2^{-n}t^*))\|_B\\
&\leq 2^{-n}\|g(\eta(s,t^*))\|_B+\sum_{k=0}^{n-1}2^{k-n}\Big(\|g(\kappa(s,2^{-k}t^*))\|_B+\|R(\omega(s,2^{-k}t^*))\|_B\Big)\\
&\leq 2^{-n}+2\sum_{k=0}^{n-1}2^{k-n}2^{-2\alpha k}|t^*|^{2\alpha}=2^{-n}+2|t^*|^{2\alpha}2^{-n}\sum_{k=0}^{n-1}2^{(1-2\alpha)k}\\
&\leq 2^{-n}+2|t^*|^{2\alpha}2^{-n}\sum_{k=0}^{n-1}2^{(1-\frac{3}{2}\alpha)k}=2^{-n}+\frac{2|t^*|^{2\alpha}2^{-n}}{2^{1-\frac{3}{2}\alpha}-1}(2^{(1-\frac{3}{2}\alpha)n}-1)\\
&\leq 2^{-\frac{3}{2}n}+\frac{2|t^*|^{2\alpha}2^{-n}}{2^{1-\frac{3}{2}\alpha}-1}2^{(1-\frac{3}{2}\alpha)n}\leq c_1(2^{-n})^{\frac{3}{2}\alpha}<c_1(2/\delta_3)^{\frac{3}{2}\alpha}|t|^{\frac{3}{2}\alpha}=c_2|t|^{\frac{3}{2}\alpha}
\end{aligned}$$

with $c_1:=1+\frac{2(\delta_3)^{2\alpha}}{2^{1-\frac{3}{2}\alpha}-1}$ and $c_2:=c_1(2/\delta_3)^{\frac{3}{2}\alpha}$. Here, we used (13) to pass to the second line, then $g(\eta(K))\subseteq B$ (see **9.11**), **9.12** and **9.13** to pass to the third. We deduce that

$\|\psi(f(\theta(s,t)))-\psi(f(\theta(s,0)))\|_B = \|\psi(f(\theta(s,t)))\|_B = \|g(\eta(s,t)\|_B \leq c_2|t|^{\frac{3}{2}}$ for all $(s,t) \in I^2$ (for $t=0$, this is trivial). Thus all conditions of Lemma 7.1 hold, and thus f is $h_{\frac{3}{2}\alpha}$. $\square$

After replacing α with $\beta := \frac{3}{2}\alpha$ and repeating this process if necessary, we see that f is H_α with $\alpha \in]\frac{1}{2}, 1]$. Thus, we may assume throughout the following that $\alpha \in]\frac{1}{2}, 1]$. We retain the setting and notations from **9.4–9.13**.

Lemma 9.15 *For each $(s,t) \in K$ such that $t \neq 0$, the limit*

$$\lambda_\eta(s,t) := \lim_{n\to\infty} \frac{g(\eta(s,2^{-n}t))}{2^{-n}t} \tag{17}$$

exists in $(L(H)_B, \|.\|_B)$. The convergence of $\frac{g(\eta(s,2^{-n}t))}{2^{-n}t}$ in $L(H)_B$ is uniform for (s,t) in $K^\times := \{(s,t) \in K : t \neq 0\}$. The map $\lambda_\eta : K^\times \to L(H)_B$ is continuous, and

$$\lambda_\eta(s,t) = \lambda_\eta(s, 2^{-m}t) \quad \text{for all } (s,t) \in K^\times \text{ and } m \in \mathbb{Z} \text{ such that } (s,2^{-m}t) \in K^\times. \tag{18}$$

Proof. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $c\,2^{-(2\alpha-1)N} \leq \varepsilon$, where $c := \frac{2(\delta_3)^{2\alpha-1}}{1-2^{-(2\alpha-1)}}$. Using (14), **9.12**, **9.13** and the summation formula for the geometric series, we obtain for all $(s,t) \in K^\times$ and all $n,m \in \mathbb{N}$ such that $m \geq n \geq N$ the following estimates:

$$\begin{aligned}
\left\|\frac{g(\eta(s,2^{-m}t))}{2^{-m}t} - \frac{g(\eta(s,2^{-n}t))}{2^{-n}t}\right\|_B &\leq |t|^{-1}\sum_{k=n}^{m-1} 2^k\Big(\|g(\kappa(s,2^{-k}t))\|_B + \|R(\omega(s,2^{-k}t))\|_B\Big) \\
&\leq |t|^{-1}\sum_{k=n}^{m-1} 2\cdot 2^{-k}\cdot 2^{-2\alpha k}|t|^{2\alpha} = 2|t|^{2\alpha-1}\sum_{k=n}^{m-1} 2^{-(2\alpha-1)k} \\
&\leq 2(\delta_3)^{2\alpha-1}\sum_{k=n}^{m-1} 2^{-(2\alpha-1)k} \leq c\,2^{-(2\alpha-1)n} \leq c\,2^{-(2\alpha-1)N} \leq \varepsilon.
\end{aligned}$$

Thus $\big(\frac{g(\eta(s,2^{-n}t))}{2^{-n}t}\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(L(H)_B, \|.\|_B)$, and hence the limit in (17) exists. Letting $m \to \infty$ in the preceding inequalities, we see that $\|\frac{g(\eta(s,2^{-n}t))}{2^{-n}t} - \lambda_\eta(s,t)\|_B \leq \varepsilon$ for all $(s,t) \in K^\times$ and $n \geq N$. Hence $\mu_n \to \lambda_\eta$ uniformly, where $\mu_n : K^\times \to L(H)_B$, $\mu_n(s,t) := \frac{g(\eta(s,2^{-n}t))}{2^{-n}t}$. Each μ_n being continuous as a map into $L(H)_B$ (cf. **9.11**), so is the uniform limit λ_η. To prove the final assertion, we may assume that $m \geq 0$ (otherwise, interchange the roles of t and $2^{-m}t$). Then $\lambda_\eta(s,t) = \lim_{n\to\infty} \frac{g(\eta(s,2^{-n}t))}{2^{-n}t} = \lim_{n\to\infty} \frac{g(\eta(s,2^{-(n+m)}t))}{2^{-(n+m)}t} = \lim_{n\to\infty} \frac{g(\eta(s,2^{-n}2^{-m}t))}{2^{-n}2^{-m}t} = \lambda_\eta(s,2^{-m}t)$. $\square$

We now specialize to $\ell = 1$ for the rest of the proof (with the exception of Lemma 9.21). Thus $\gamma := \eta :]-\delta,\delta[\to U \subseteq L(G)$ is a c^∞-curve, with $\gamma(0) = 0$.

Lemma 9.16 *If $\gamma'(0) = 0$, then $\lambda_\gamma(t) = 0$ for all $t \in K^\times = [-\delta_3, \delta_3] \setminus \{0\}$.*

Proof. If $\gamma'(0) = 0$, then $\gamma(t) = \rho_\gamma(0,t)$ for each $t \in K$ (where ρ_γ is the first order Taylor remainder of γ). Hence $\|g(\gamma(t))\|_B \leq \|\gamma(t)\|^\alpha_{B_4} = \|\rho_\gamma(0,t)\|^\alpha_{B_4} \leq |t|^{2\alpha}$, using (15) and (16).

Therefore $\|t^{-1}g(\gamma(t))\|_B \leq |t|^{2\alpha-1} \to 0$ as $t \to 0$ and thus $\lim_{t\to 0} \frac{g(\gamma(t))}{t} = 0$ in $L(H)_B$, entailing that $\lambda_\gamma(t) = \lim_{n\to\infty} \frac{g(\gamma(2^{-n}t))}{2^{-n}t} = 0$ for each $t \in K^\times$. □

To emphasize their dependence on γ, we now write δ_γ, $\delta_{3,\gamma}$, θ_γ and $K^\times_\gamma$ for δ, δ_3, θ and $K^\times$.

Lemma 9.17 *Assume that $\gamma_1 \colon \mathbb{R} \supseteq B(\delta_{\gamma_1}) \to U$ and $\gamma_2 \colon \mathbb{R} \supseteq B(\delta_{\gamma_2}) \to U$ are c^∞-curves such that $\gamma_1(0) = \gamma_2(0) = 0$ and $\gamma_1'(0) = \gamma_2'(0)$. Then $\lambda_{\gamma_1}(t) = \lambda_{\gamma_2}(t)$ for all $t \in K^\times_{\gamma_1} \cap K^\times_{\gamma_2}$.*

Proof. There is $\delta \in \,]0, \min\{\delta_{\gamma_1}, \delta_{\gamma_2}\}]$ such that the c^∞-curve $\xi : B(\delta) \to G$, $\xi(t) := \theta_{\gamma_1}(t)\theta_{\gamma_2}(t)^{-1}$ has image in U_1. Then $\gamma := \phi\circ\xi \colon B(\delta) \to U$ is a c^∞-curve such that $\gamma(0) = 0$ and $\gamma'(0) = \gamma_1'(0) - \gamma_2'(0) = 0$. Hence $\lambda_\gamma(t) = 0$ for all $t \in K^\times_\gamma$, by Lemma 9.16. The group multiplication and inversion in H being c^∞, the set $M_1 := \{(x, y) \in V_1 \times V_1 : xy^{-1} \in V_1\}$ is open in $V_1 \times V_1 \subseteq H \times H$ (equipped with the c^∞-topology). Thus $M := (\psi \times \psi)(M_1)$ is c^∞-open in $L(H) \times L(H)$, and the map $\mu : M \to V$, $\mu(x, y) := \psi(\psi^{-1}(x)\psi^{-1}(y)^{-1})$ is c^∞. There is $\varepsilon \in \,]0, \min\{\delta_{3,\gamma_1}, \delta_{3,\gamma_2}, \delta_{3,\gamma}\}]$ such that $(g \circ \gamma_1, g \circ \gamma_2)(B(\varepsilon)) \subseteq M$. Set $J := \{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$. Fix $t \in B(\varepsilon)$ such that $t \neq 0$. Then $h_\gamma : J \to L(H)$, $h_\gamma(r) := \frac{g(\gamma(rt))}{t}$ and the analogous functions h_{γ_1} and $h_{\gamma_2} \colon J \to L(H)$ are bornologically differentiable at 0, with $h_\gamma'(0) = \lambda_\gamma(t)$, $h_{\gamma_1}'(0) = \lambda_{\gamma_1}(t)$ and $h_{\gamma_1}'(0) = \lambda_{\gamma_1}(t)$ (see Lemma 9.15). Since $h_\gamma = t^{-1}\mu \circ (th_{\gamma_1}, th_{\gamma_2})$, we deduce from Lemma 3.2 that

$$\begin{aligned} 0 &= \lambda_\gamma(t) = h_\gamma'(0) = t^{-1}d\mu\big((0,0),\, (th_{\gamma_1}'(0), th_{\gamma_2}'(0))\big) = d\mu\big((0,0),\, (\lambda_{\gamma_1}(t), \lambda_{\gamma_2}(t))\big) \\ &= \lambda_{\gamma_1}(t) - \lambda_{\gamma_2}(t) \end{aligned}$$

and thus $\lambda_{\gamma_1}(t) = \lambda_{\gamma_2}(t)$. Now let $t \in K^\times_{\gamma_1} \cap K^\times_{\gamma_2}$. There is $n \in \mathbb{N}$ such that $|2^{-n}t| < \varepsilon$. Then $0 \neq 2^{-n}t \in B(\varepsilon)$ and hence $\lambda_{\gamma_1}(t) = \lambda_{\gamma_1}(2^{-n}t) = \lambda_{\gamma_2}(2^{-n}t) = \lambda_{\gamma_2}(t)$ by the special case just treated and (18). □

As before, $\gamma \colon]{-\delta_\gamma}, \delta_\gamma[\to U \subseteq L(G)$ is any c^∞-curve with $\gamma(0) = 0$.

Lemma 9.18 $\lambda_\gamma(t_1) = \lambda_\gamma(t_2)$ *holds, for any $t_1, t_2 \in K^\times$.*

Proof. By continuity of λ_γ (Lemma 9.15), it suffices to show that $\lambda_\gamma(t_1) = \lambda_\gamma(t_2)$ for all $t_1, t_2 \in K^\times_\gamma \cap \mathbb{Q}$. Given such t_1, t_2, there exist $m_1, m_2 \in \mathbb{Z} \setminus \{0\}$ such that $m_1t_1 = m_2t_2$. For $\delta \in \,]0, 1]$ sufficiently small, the c^∞-curve $\xi \colon B(\delta) \to G$, $\xi(t) := \theta_\gamma(t_1t)^{m_1}\theta_\gamma(t_2t)^{-m_2}$ has image in U_1; we define $\eta := \phi \circ \xi : B(\delta) \to U$. Then $\eta(0) = 1$ and $\eta'(0) = m_1t_1\gamma'(0) - m_2t_2\gamma'(0) = 0$ and hence $\lambda_\eta(t) = 0$ for all $t \in K^\times_\eta$, by Lemma 9.16. Define $M_1 := \{(x, y) \in V_1 \times V_1 : x^{m_1}y^{-m_2} \in V_1\}$, $M := (\psi \times \psi)(M_1)$ and $\mu : M \to V$, $\mu(x, y) := \psi(\psi^{-1}(x)^{m_1}\psi^{-1}(y)^{-m_2})$. Then $(g(\gamma(t_1t)), g(\gamma(t_2t))) \in M$ for all $t \in B(\delta)$ and $g(\eta(t)) = \mu(g(\gamma(t_1t)), g(\gamma(t_2t)))$ for all $t \in B(\delta)$. Pick $m \in \mathbb{N}$ such that $2^{-m} \in K^\times_\eta$, and fix $t := 2^{-m}$. We consider the three mappings $h_1, h_2, h_3 \colon J \to L(H)$ on $J := \{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$ given by $h_1(r) := \frac{g(\gamma(rtt_1))}{tt_1}$, $h_2(r) := \frac{g(\gamma(rtt_2))}{tt_2}$ and $h_3(r) := \frac{g(\eta(rt))}{t}$, respectively. Since $h_3(r) = t^{-1}\mu(tt_1h_1(r), tt_2h_2(r))$ for all $r \in J$, $h_1'(0) = \lambda_\gamma(tt_1) = \lambda_\gamma(t_1)$, $h_2'(0) = \lambda_\gamma(tt_2) = \lambda_\gamma(t_2)$ and $h_3'(0) = \lambda_\eta(t) = 0$, we deduce as in the proof of Lemma 9.17 that $0 = \lambda_\eta(t) = t^{-1}d\mu\big((0,0),\, (tt_1\lambda_\gamma(t_1), tt_2\lambda_\gamma(t_2))\big) = m_1t_1\lambda_\gamma(t_1) - m_2t_2\lambda_\gamma(t_2) = m_1t_1\big(\lambda_\gamma(t_1) - \lambda_\gamma(t_2)\big)$. Hence $\lambda_\gamma(t_1) = \lambda_\gamma(t_2)$ indeed. □

By Lemma 9.18, $\Lambda(\gamma) := \lambda_\gamma(t_0)$ for $t_0 \in K^\times_\gamma$ is well defined, independent of the choice of t_0.

Lemma 9.19 *Let $\gamma_1 : B(\delta_{\gamma_1}) \to U \subseteq L(G)$ and $\gamma_2 : B(\delta_{\gamma_2}) \to U$ be c^∞-curves such that $\gamma_1(0) = \gamma_2(0)$. If $\gamma_1'(0) = \gamma_2'(0)$, then $\Lambda(\gamma_1) = \Lambda(\gamma_2)$.*

Proof. For small $t_0 \neq 0$, we have $\Lambda(\gamma_1) = \lambda_{\gamma_1}(t_0) = \lambda_{\gamma_2}(t_0) = \Lambda(\gamma_2)$, by Lemma 9.17. □

Given $v \in L(G)$, we choose a c^∞-curve $\gamma\colon B(\delta) \to U$ for some $\delta \in]0, \infty]$ such that $\gamma(0) = 0$ and $\gamma'(0) = v$, and set $A.v := \Lambda(\gamma) \in L(H)$. Lemma 9.19 implies that $A.v$ is well defined, independent of γ.

Lemma 9.20 *The map $A\colon L(G) \to L(H)$ is linear.*

Proof. *A is additive.* Given $v_1, v_2 \in L(G)$, we choose $\delta \in]0, \infty]$ and c^∞-curves $\gamma_j : B(\delta) \to U$ for $j \in \{1, 2\}$ such that $\gamma_j(0) = 0$ and $\gamma_j'(0) = v_j$. After shrinking δ, we may assume that $\xi : B(\delta) \to G$, $\xi(t) := \phi^{-1}(\gamma_1(t))\phi^{-1}(\gamma_2(t))$ has image in U_1; we define $\gamma : B(\delta) \to U$, $\gamma(t) := \phi(\xi(t))$. Then $\gamma(0) = 0$ and $\gamma'(0) = \gamma_1'(0) + \gamma_2'(0) = v_1 + v_2$. Define $M_1 := \{(x, y) \in V_1 \times V_1 : xy \in V_1\}$, $M := (\psi \times \psi)(M_1)$ and $\mu : M \to V$, $\mu(x, y) := \psi(\psi^{-1}(x)\psi^{-1}(y))$. Then $\mathrm{im}(g\circ\gamma_1, g\circ\gamma_2) \subseteq M$ and $g(\gamma(t)) = \mu(g(\gamma_1(t)), g(\gamma_2(t)))$ for all $t \in B(\delta)$. Let $t \in K_\gamma^\times \cap K_{\gamma_1}^\times \cap K_{\gamma_2}^\times$. As in the proof of Lemma 9.17, we conclude that $\lambda_\gamma(t) = d\mu\big((0, 0), (\lambda_{\gamma_1}(t), \lambda_{\gamma_2}(t))\big) = \lambda_{\gamma_1}(t) + \lambda_{\gamma_2}(t)$. Thus $A(v_1 + v_2) = \Lambda(\gamma) = \lambda_\gamma(t) = \lambda_{\gamma_1}(t) + \lambda_{\gamma_2}(t) = \Lambda(\gamma_1) + \Lambda(\gamma_2) = A(v_1) + A(v_2)$.

Homogeneity. Let $v \in L(G)$ and $a \in \mathbb{R}$. If $a = 0$, then $A(av) = A(0) = 0 = aA(v)$, using that $A(0) = 0$ as A is a homomorphism of additive groups. If $a \neq 0$, pick a c^∞-curve $\gamma\colon \mathbb{R} \to U$ such that $\gamma(0) = 0$ and $\gamma'(0) = v$. Then $\gamma_a\colon \mathbb{R} \to U$, $\gamma_a(t) := \gamma(at)$ is a c^∞-curve with $\gamma_a(0) = 0$ and $\gamma_a'(0) = av$. There is $t \in K_{\gamma_a}^\times$ such that $at \in K_\gamma^\times$. Then $aA(v) = a\Lambda(\gamma) = a\lambda_\gamma(at) = \lim_{n\to\infty} \frac{g(\gamma(2^{-n}at))}{2^{-n}t} = \lim_{n\to\infty} \frac{g(\gamma_a(2^{-n}t))}{2^{-n}t} = \lambda_{\gamma_a}(t) = \Lambda(\gamma_a) = A(av)$. □

Lemma 9.21 *The linear map $A\colon L(G) \to L(H)$ is bounded.*

Proof. By Lemma 1.6, it suffices to show that $A \circ \gamma$ is continuous at 0 for each c^∞-curve $\gamma\colon \mathbb{R} \to L(G)$ such that $\gamma(0) = 0$. Given such γ, there is $\delta > 0$ with $]-\delta, \delta[\cdot\gamma(]-\delta, \delta[) \subseteq U$. We define $\eta\colon \mathbb{R}^2 \supseteq B(\delta) \to U$, $\eta(s, t) := t\gamma(s)$ and $\eta_s\colon]-\delta, \delta[\to U$, $\eta_s(t) := \eta(s, t) = t\gamma(s)$ for $s \in]-\delta, \delta[$. Fix $t_0 \in]0, \delta_{3,\eta}[$. Then $\eta_s'(0) = \gamma(s)$ for all $s \in [-\delta_{3,\eta}, \delta_{3,\eta}]$, and apparently we can choose $\delta_{3,\eta_s} = \delta_{3,\eta}$. Thus $A(\gamma(s)) = \Lambda(\eta_s) = \lambda_\eta(s, t_0)$, which is continuous in s as a map into $L(H)_B$ (see Lemma 9.15) and hence also as a map into $L(H)$, as desired. □

Lemma 9.22 *For every c^∞-curve $\gamma\colon \mathbb{R} \to U$ such that $\gamma(0) = 0$, the composition $g \circ \gamma$ is bornologically differentiable at 0, with $(g \circ \gamma)'(0) = A.\gamma'(0)$.*

Proof. We re-use the notations from **9.4–9.13** and Lemma 9.15 for $\eta := \gamma$. Since $g(\gamma(K)) \subseteq L(H)_B$ and $A.\gamma'(0) = \Lambda(\gamma) \in L(H)_B$, we only need to show that $\lim_{n\to\infty} \frac{g(\gamma(t_n))}{t_n} = \Lambda(\gamma)$ in $L(H)_B$, for each sequence $(t_n)_{n\in\mathbb{N}}$ in $K^\times$ such that $t_n \to 0$. We may assume that $|t_n| \leq \frac{1}{2}\delta_3$ for all n. For each n, there exists a unique $m_n \in \mathbb{N}$ such that $t_n^* := 2^{m_n}t_n$ has absolute value $\frac{1}{2}\delta_3 < |t_n^*| \leq \delta_3$. Define $\mu_n\colon K^\times \to L(H)_B$, $\mu_n(t) := \frac{g(\gamma(2^{-n}t))}{2^{-n}t}$ for $n \in \mathbb{N}$, as in the proof of Lemma 9.15. Since $\mu_n \to \lambda_\gamma$ uniformly by Lemma 9.15, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|\mu_n(t) - \lambda_\gamma(t)\|_B \leq \varepsilon$ for all $n \geq n_0$ and $t \in K^\times$. Since $m_n \to \infty$ as $n \to \infty$, there exists $n_1 \in \mathbb{N}$ such that $m_n \geq n_0$ for all $n \geq n_1$. For each $n \geq n_1$, we then have $\big\|\frac{g(\gamma(t_n))}{t_n} - \Lambda(\gamma)\big\|_B = \big\|\frac{g(\gamma(2^{-m_n}t_n^*))}{2^{-m_n}t_n^*} - \lambda_\gamma(t_n^*)\big\|_B = \|\mu_{m_n}(t_n^*) - \lambda_\gamma(t_n^*)\|_B \leq \varepsilon$. Thus $\lim_{n\to\infty} \frac{g(\gamma(t_n))}{t_n} = \Lambda(\gamma)$ in $L(H)_B$ indeed. □

By Lemma 9.21 and Lemma 9.22, $g = \psi \circ f \circ \phi^{-1}$ is bornologically curve differentiable at 0. Therefore g is curve differentiable at 0 (Lemma 5.3), and hence f is curve differentiable (and bornologically curve differentiable) at 1. As a consequence, f is c^∞ (Lemma 6.2). This completes the proof of Theorem 9.1. □

A Proofs for Lemma 1.7 and Lemma 2.9

Proof of Lemma 1.7. Since $\mu : U \times E \times \mathbb{R} \to E$, $\mu(x, y, t) := x + ty$ is a c^∞-map and hence continuous with respect to the c^∞-topologies, the preimage $U^{[1]} = \mu^{-1}(U)$ is c^∞-open in $U \times E \times \mathbb{R}$. Let $\gamma = (\gamma_1, \gamma_2, \tau) \colon \mathbb{R} \to U^{[1]}$ be a smooth curve, with coordinates $\gamma_1 : \mathbb{R} \to U$, $\gamma_2 : \mathbb{R} \to E$ and $\tau : \mathbb{R} \to \mathbb{R}$, respectively. Then $I := \tau^{-1}(\mathbb{R} \setminus \{0\})$ is an open subset of $\mathbb{R}$. To see that $f^{[1]}$ is c^∞, we have to show that $f^{[1]} \circ \gamma$ is smooth. Clearly $f^{[1]} \circ \gamma|_I$ is smooth, because this function is composed of c^∞-maps: $f^{[1]}(\gamma(t)) = \tau(t)^{-1}\big(f(\gamma_1(t)+\tau(t)\gamma_2(t)) - f(\gamma_1(t))\big)$ for $t \in I$. Now assume that $t_0 \in \mathbb{R}\setminus I$; thus $\tau(t_0) = 0$. The map $h \colon \mathbb{R}^2 \to E$, $h(t, s) := \gamma_1(t) + s\tau(t)\gamma_2(t)$ being c^∞, with $h(t_0, s) = \gamma_1(t_0) \in U$ for all s, we see that $h^{-1}(U)$ is an open neighbourhood of $\{t_0\} \times [-1, 2]$ in $\mathbb{R}^2$. We therefore find an open neighbourhood $J \subseteq \mathbb{R}$ of t_0 such that $J \times [-1, 2] \subseteq h^{-1}(U)$. Then $J \times \,]-1, 2[\, \to F$, $(t, s) \mapsto df\big(\gamma_1(t) + s\tau(t)\gamma_2(t),\ \gamma_2(t)\big)$ is smooth, and we have

$$f^{[1]}(\gamma(t)) = \int_0^1 df\Big(\gamma_1(t) + s\,\tau(t)\gamma_2(t),\ \gamma_2(t)\Big)\, ds \quad \text{for } t \in J.$$

Indeed, this formula is obvious if $\tau(t) = 0$; if $\tau(t) \neq 0$, it follows from the fundamental theorem of calculus [10, Cor. 2.6 (6)]. Being given by a parameter-dependent integral with smooth integrand, $f^{[1]} \circ \gamma|_J \colon J \to F$ is smooth (cf. [10, Prop. 3.15] or [1, La. 7.5]). □

To facilitate a proof of Lemma 2.9, we first need to establish a variant of [7, Prop. 4.3.3]:

Lemma A.1 *Let E and F be convenient vector spaces, $U \subseteq E$ be c^∞-open, and $f \colon U \to F$ be h_α. Then $f \circ \gamma \colon \mathbb{R} \to F$ is H_α, for every Lipschitz continuous curve $\gamma \colon \mathbb{R} \to U$.*

Proof. In view of Lemma 2.4 (b), we may assume that $F = \mathbb{R}$. The proof is by contraposition. Thus, assume that $f \circ \eta$ is not H_α for some Lipschitz continuous curve $\eta : \mathbb{R} \to U$. Then there exists $t_0 \in \mathbb{R}$ such that $\Big\{\frac{f(\eta(s))-f(\eta(t))}{|s-t|^\alpha} : s, t \in I,\ s \neq t\Big\}$ is unbounded for any neighbourhood I of t_0. After translations, without loss of generality $t_0 = 0$ and $\eta(t_0) = 0$. For $n \in \mathbb{N}$, we choose $s_n, t_n \in [-2^{-2n}, 2^{-2n}]$ such that $t_n \neq s_n$ and

$|s_n - t_n|^{-\alpha}|f(\eta(s_n)) - f(\eta(t_n))| \geq n2^{\alpha n}$. We abbreviate $\sigma_n := 2^n s_n$, $\tau_n := 2^n t_n$ and define $\eta_n\colon \mathbb{R} \to E$, $\eta_n(t) := \eta(t_n) + (t - t_n)\frac{\eta(s_n)-\eta(t_n)}{s_n-t_n}$ and $\gamma_n\colon \mathbb{R} \to E$, $\gamma_n(t) := \eta_n(2^{-n}t)$. Then

$$\frac{|f(\gamma_n(\sigma_n)) - f(\gamma_n(\tau_n))|}{|\sigma_n - \tau_n|^\alpha} = \frac{|f(\eta(s_n)) - f(\eta(t_n))|}{2^{n\alpha}|s_n - t_n|^\alpha} \geq n\,.$$

Furthermore, $|\sigma_n| = 2^n|s_n| \leq 2^{-n}$ and likewise $|\tau_n| \leq 2^{-n}$. We claim that the sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{R}, E)$ is fast falling, in the sense of [7, Defn. 4.2.14]. By [7, Prop. 4.2.16] (or [10, Cor. 12.3]), we only need to show that $(\lambda(\gamma_n(t)))_{n\in\mathbb{N}}$ is fast falling in $\mathbb{R}$, for each $t \in [-1, 1]$ and each bounded linear functional λ on E. Since η is Lipschitz continuous, so is $\lambda \circ \eta : \mathbb{R} \to \mathbb{R}$. Using Lemma 2.3 (a), we therefore find $K \in [0, \infty[$ such that $|\lambda(\eta(r)) - \lambda(\eta(s))| \leq K\,|r - s|$ for all $r, s \in [-1, 1]$. For $t \in [-1, 1]$, we obtain

$$\begin{aligned}
|\lambda(\gamma_n(t))| &= |\lambda(\eta_n(2^{-n}t))| = \left|\lambda(\eta(t_n)) + (2^{-n}t - t_n)\frac{\lambda(\eta(s_n) - \eta(t_n))}{s_n - t_n}\right| \\
&\leq |\lambda(\eta(t_n)) - \lambda(\eta(0))| + |2^{-n}t - t_n| \cdot \left|\frac{\lambda(\eta(s_n) - \eta(t_n))}{s_n - t_n}\right| \\
&\leq K\,|t_n| + (2^{-n} + |t_n|)\,K \leq (2^{-2n} + 2^{-n} + 2^{-2n})\,K\,,
\end{aligned}$$

which is fast falling in $\mathbb{R}$ as $n \to \infty$ (passing to the second line, we used that $\lambda(\eta(0)) = 0$). Hence indeed $(\gamma_n)_{n\in\mathbb{N}}$ is fast falling in $C^\infty(\mathbb{R}, E)$. Applying the General Curve Lemma [7, Prop. 4.2.15] (or [10, 12.2]) with $\varepsilon_n := 2^{-n}$, we get a smooth curve $\gamma\colon \mathbb{R} \to E$ and a convergent sequence $(r_n)_{n\in\mathbb{N}}$ of reals, with limit $r := \lim_{n\to\infty} r_n$, such that $\gamma(r) = 0$ and $\gamma(r_n + t) = \gamma_n(t)$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ such that $|t| \leq \varepsilon_n$. Since $\gamma(r) = 0 = \eta(0) \in U$, the set $J := \gamma^{-1}(U)$ is an open neighbourhood of r, and thus $\gamma|_J : J \to U$ is a smooth curve in U. There is $N \in \mathbb{N}$ such that $r_n + \tau_n, r_n + \sigma_n \in J$ for all $n \geq N$. For any such n,

$$\frac{|f(\gamma(r_n + \sigma_n) - f(\gamma(r_n + \tau_n))|}{|\sigma_n - \tau_n|^\alpha} = 2^{-\alpha n}\frac{|f(\eta(s_n)) - f(\eta(t_n))|}{|s_n - t_n|^\alpha} \geq n\,,$$

entailing that $\left\{\frac{f(\gamma(s))-f(\gamma(t))}{|s-t|^\alpha} : s, t \in W,\ s \neq t\right\} \subseteq \mathbb{R}$ is unbounded for each neighbourhood $W \subseteq J$ of r. Hence $f \circ \gamma|_J$ is not H_α and hence f is not h_α (cf. Remark 2.7 (b)). □

Proof of Lemma 2.9. The implication "(b)⇒(a)" can be proved like Lemma 2.11 (and we shall not use it). "(a)⇒(b)": The proof is by contraposition; we assume that (b) is false and so $f|_K : K \to F$ is not H_α for some K. Then $\lambda \circ f|_K$ is not H_α for some continuous linear functional $\lambda\colon F \to \mathbb{R}$ (cf. Lemma 2.4 (b)). If we can show that $\lambda \circ f$ is not h_α, then neither is f (Remark 2.7 (a)). Hence $F = \mathbb{R}$ without loss of generality. As we assume that $f|_K$ is not H_α, for each $n \in \mathbb{N}$ we find elements $x_n, y_n \in K$ such that $\|y_n - x_n\|_B < 1/n^2$ and $|f(y_n) - f(x_n)| \geq n(\|y_n - x_n\|_B)^\alpha$. Using that K is compact and metrizable, after passing to subsequences we may assume that both x_n and y_n converge to some $x \in K$, and $\|y_n - x\|_B, \|x_n - x\|_B < 1/n^2$. We now consider the curve $\gamma\colon \mathbb{R} \to E_B$ defined as follows: $\gamma(t) := x_1$ if $t \leq 0$; γ runs with constant velocity $\frac{y_1-x_1}{\|y_1-x_1\|_B}$ from x_1 to y_1 if $t_1 := 0 \leq t \leq \|y_1 - x_1\|_B =: s_1$; γ runs with constant velocity $\frac{x_2-y_1}{\|x_2-y_1\|_B}$ from y_1 to x_2 if

$s_1 \leq t \leq s_1 + \|x_2 - y_1\|_B =: t_2$, and so on. Since $t_\infty := \sum_{n=1}^\infty \|y_n - x_n\|_B + \sum_{n=1}^\infty \|x_{n+1} - y_n\|_B$ is finite, $\gamma(t)$ tends to x as t increases towards t_∞; so we define $\gamma(t) := x$ for $t \geq t_\infty$. By construction, we have $\|\gamma(s) - \gamma(t)\|_B \leq |s - t|$ for all $s, t \in \mathbb{R}$, and thus γ is Lipschitz continuous. Since $\gamma(t_\infty) \in K \subseteq U$, the map $\gamma \colon \mathbb{R} \to E_B$ is continuous, and $U \cap E_B$ is open in E_B, we deduce that $J := \gamma^{-1}(U)$ is an open neighbourhood of t_∞ in $\mathbb{R}$. There is $N \in \mathbb{N}$ such that $s_n, t_n \in J$ for all $n \geq N$. For any such n, we have $\|\gamma(s_n) - \gamma(t_n)\|_B = \|y_n - x_n\|_B = |s_n - t_n|$ and hence $|f(\gamma(s_n)) - f(\gamma(t_n))| = |f(y_n) - f(x_n)| \geq n(\|y_n - x_n\|_B)^\alpha = n|s_n - t_n|^\alpha$. Since $t_n, s_n \to t_\infty$, this implies that the set $\left\{\frac{f(\gamma(s)) - f(\gamma(t))}{|s-t|^\alpha} : s, t \in W, s \neq t\right\} \subseteq \mathbb{R}$ is unbounded for each neighbourhood $W \subseteq J$ of t_∞, whence $f \circ \gamma|_J$ is not H_α. Therefore, by Lemma A.1, f is not h_α. The proof is complete. □

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