Helly and Klee type intersection theorems for finitary connected paved spaces

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Abstract

In the present paper, the concept of $n-$ary and finitary connectedness is introduced, where $1-$ary connectedness coincides with the usual notion of (abstract) connectedness. Relationships between (n-ary) connectedness and an abstract concept of separation are studied. As applications, the classical intersection theorems of Helly, Klee, and others are obtained from the previous results by showing that the paving of closed convex resp. open convex subsets of a topological vector space are finitary connected.

Based on a general minimax theorem, an abstract separation theorem is proved, generalizing the classical separation theorem for convex compact subsets of a locally compact topological vector space. This theorem and other results on abstract separation can be used to derive fairly general results on finitary connectedness which can be applied to various types of (convex) topological spaces.

Keywords: connectedness; separation; convexity; Helly-type theorem

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0 Introduction

Many problems in pure and applied mathematics can be reduced to the following

Problem When does a system of subsets of a given set intersect?

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By a compactness argument this problem is usually reduced to finite systems.

A very simple property is the following:

If A and B are nonvoid closed (open) subsets of a topological space such that $A \cup B$ is connected, then A and B intersect.

This innocent looking observation is a useful tool for proving minimax theorems as was demonstrated in several papers by König, Simons, Kindler, and others (compare [23] for a survey). In this context, the notion of "connectedness" was lifted to an abstract level.

But, as Horvath remarks in [10], "connectedness is good enough to establish that a family of sets has the binary intersection property (any two sets of the family intersect), but does not allow the passage to the finite intersection property. One has to impose higher order connectedness properties to have the $n-$ary intersection property..."

1 Connectedness and separation

Let S be a nonvoid set and 2^S the power set of S. Then every nonvoid subset $\mathcal{P} \subset 2^S$ is called a *paving* in S and $(S, \mathcal{P})$ is a *paved space.* Especially, $\mathcal{E}(S)$ denotes the paving of all nonvoid finite subsets of S and $\mathcal{E}^n(S)$ is the paving of all subsets of S with n elements. The paving $\mathcal{P}^\top := \{T \subset S : T \cap P \in \mathcal{P} \ \forall P \in \mathcal{P}\}$ is the *transporter* of $\mathcal{P}$, and $\mathcal{P}^C$ is the paving of all complements $S \setminus P, \ P \in \mathcal{P}$.

The paving $\mathcal{P}$ is called

- $\cap_f-$*closed* ($\cup_f-$*closed*) iff $A, B \in \mathcal{P} \Longrightarrow A \cap B \in \mathcal{P}$ $(A \cup B \in \mathcal{P})$,
- $\cap_a-$*closed* iff $\bigcap_{R \in \mathcal{R}} R \in \mathcal{P}$ for all nonvoid $\mathcal{R} \subset \mathcal{P}$,
- a *lattice* iff it is $\cap_f-$closed and $\cup_f-$closed,
- *coherent* iff $A, B \in \mathcal{P}, \ A \cap B \neq \emptyset \Longrightarrow A \cup B \in \mathcal{P}$,
- *compact* iff every subpaving $\mathcal{R} \subset \mathcal{P}$ with the finite intersection property $\bigcap_{R \in \mathcal{F}} R \neq \emptyset \quad \forall \mathcal{F} \in \mathcal{E}(\mathcal{R})$ has the global intersection property $\bigcap_{R \in \mathcal{R}} R \neq \emptyset$, and a subset T of S is *compact* (*w.r.t.* $\mathcal{P}$) iff its *trace* $\mathcal{P} \cap T := \{P \cap T : P \in \mathcal{P}\}$ is compact,
- *upward filtrating* iff for all $A, B \in \mathcal{P}$ there exists a $C \in \mathcal{P}$ with $C \supset A \cup B$,

- an *alignment* iff $\{\emptyset, S\} \subset \mathcal{P}$, $\mathcal{P}$ is $\cap_a-$closed, and $\bigcup\{A : A \in \mathcal{A}\} \in \mathcal{P}$ for every totally ordered subpaving $\mathcal{A} \subset \mathcal{P}$.

Let $(S, \mathcal{P})$ be a paved space. For $k, n \in \mathbb{N} := \{1, 2, \ldots\}$ with $k \leq n$ we denote by $\mathcal{P}(n, k)$ the paving of all subsets $T \subset S$ where for all $\{P_0, \ldots, P_n\} \subset \mathcal{P}$ the relations

(1) $\bigcap_{j \in J} P_j \in \mathcal{P}$ for all nonvoid proper subsets J of $\{0, \ldots, n\}$,

(2) $T \subset \bigcup_{i=0}^{n} P_i$,

(3) $T \cap \bigcap_{j \in J} P_j \neq \emptyset$ for all subsets $J \in \mathcal{E}^k(\{0, \ldots, n\})$, and

(4) $T \cap \bigcap_{i=0}^{n} P_i = \emptyset$

cannot hold simultaneously.

In the above situation we set $\mathcal{J}^n$ for the paving of all nonvoid proper subsets of $\{0, \ldots, n\}$, $P_J := \bigcap_{j \in J} P_j$ for $J \in \mathcal{J}^n$, and $P_{-i} := P_{J_{-i}}$ for $J_{-i} := \{0, \ldots, n\} \setminus \{i\}$.

The subsets $T \in \mathcal{P}(n, n)$ will be called *$n-$ary connected (for $\mathcal{P}$)*. A subset is called *connected for $\mathcal{P}$* iff it is $1-$ary connected for $\mathcal{P}$.

A subset T is called *finitary connected (for $\mathcal{P}$)* iff it is $n-$ary connected (for $\mathcal{P}$) for every $n \in \mathbb{N}$.

If $\mathcal{K}$ is another paving in S, then we say that $\mathcal{K}$ is *$n-$ary/finitary connected for $\mathcal{P}$* if every $K \in \mathcal{K}$ has this property. Finally, in case $\mathcal{K} = \mathcal{P}$ we say that $\mathcal{P}$ is *$n-$ary/finitary connected.*

Remark 1 *Let $(S, \mathcal{P})$ be a paved space.*

a) $\mathcal{P}^\top \subset \mathcal{P}$ iff $S \in \mathcal{P}$, and $\mathcal{P}^\top \supset \mathcal{P}$ iff $\mathcal{P}$ is $\cap_f-$closed.

b) A subset T of S is connected for $\mathcal{P}$ iff it is connected for $\mathcal{P}^C$.
This property does not carry over to $n-$ary connectedness. For example, $S = \{1, 2, 3, 4, 5\}$ is $2-$ary connected for the paving $\mathcal{P} = \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}, \{2, 3, 4\}, \{1, 2\}, \{4, 5\}\}$ but not for $\mathcal{P}^C$.

c) $\mathcal{P} \subset \mathcal{P}(n, k)$ implies $\mathcal{P} \subset \mathcal{P}^\top(n, k)$ for $k \leq n$. For $\cap_f-$closed $\mathcal{P}$ the converse implication also holds.

Lemma 1 *Let $(S, \mathcal{P})$ be a paved space. Then the paving $\mathcal{D} = \{D \in 2^S : D$ is connected for $\mathcal{P}\}$ has the following properties.*

(i) $\mathcal{E}^1(S) \cup \{\emptyset\} \subset \mathcal{D}$.

(ii) $\mathcal{D}$ is coherent.

(iii) $\mathcal{D}$ *is closed w.r.t. upward filtrating unions (i.e.,* $\bigcup_{H\in\mathcal{H}} H \in \mathcal{D}$ *for every upward filtrating subpaving* $\mathcal{H} \subset \mathcal{D}$*).*

(iv) For $\mathcal{H} \subset \mathcal{D}$ *with* $\bigcap_{H\in\mathcal{H}} H \neq \emptyset$ *we have* $\bigcup_{H\in\mathcal{H}} H \in \mathcal{D}$*.*

Proof. (i) and (iii) are obvious.
(ii) Let $A, B \in \mathcal{D}$ with $A \cap B \neq \emptyset$. Suppose that $A \cup B \notin \mathcal{D}$. Then there exist sets $C, D \in \mathcal{P}$ with $A \cup B \subset C \cup D$, (α) $(A \cup B) \cap C \neq \emptyset$, (β) $(A \cup B) \cap D \neq \emptyset$, and $(A \cup B) \cap C \cap D = \emptyset$. Since A and B are connected for $\{C, D\}$, we have

$$\text{either} \quad A \cap C = \emptyset \quad \text{or} \quad A \cap D = \emptyset,$$

and

$$\text{either} \quad B \cap C = \emptyset \quad \text{or} \quad B \cap D = \emptyset.$$

But $A \cap C = \emptyset = B \cap C$ contradicts (α) and $A \cap D = \emptyset = B \cap D$ contradicts (β). Hence we have either $A \cap D = B \cap C = \emptyset$ or $A \cap C = B \cap D = \emptyset$. In both cases we arrive at $A \cap B = (A \cap B \cap C) \cup (A \cap B \cap D) = \emptyset$, a contradiction.
(iv) For finite $\mathcal{H}$ this follows from (ii) by induction. The general case follows together with (iii). □

Remark 2 *Pavings* $\mathcal{D}$ *satisfying conditions (i) and (iv) of Lemma 1 are called* connectivities. *We have seen in the above proof that conditions (ii) and (iii) imply (iv). It is easy to verify that, conversely, (iv) implies (ii) and (iii).*

Connectivities were introduced by K. Csásár in her study of abstract separation and connectedness [4] and, independently, by Matheron and Serra as a new approach in the analysis of digital images [22]. In the context of stochastic and algebraic independence, Matúš [20] introduced the notation of a C-family *which – in essence – is a paving* $\mathcal{D}$ *satisfying relations (i) and (ii).*

A *bipaved space* is a triplet $(S, \mathcal{P}, \mathcal{K})$ where $\mathcal{P}$ and $\mathcal{K}$ are pavings in S. We set $\mathcal{K} \sqcap \mathcal{P} := \{K \cap P : K \in \mathcal{K}, P \in \mathcal{P}\}$.

Given a bipaved space $(S, \mathcal{P}, \mathcal{K})$, we say that $\mathcal{K}$ *separates* $\mathcal{P}$ or $(S, \mathcal{P}, \mathcal{K})$ *is separated* iff for all $A, B \in \mathcal{P} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$ there exists a set $K \in \mathcal{K}$ with $A \cap K = B \cap K = \emptyset$ and $P \cap K \neq \emptyset$ for all $P \in \mathcal{P}$ with $P \supset A$ and $P \cap B \neq \emptyset$. A paved space $(S, \mathcal{P})$ resp. a paving $\mathcal{P}$ is *separated* iff $(S, \mathcal{P}, \mathcal{P})$ is separated.

Lemma 2 *Let* $(S, \mathcal{P})$ *be a paved space.*

a) *Let $T \in 2^S \setminus \{\emptyset\}$ such that 2^S separates $\mathcal{P} \cap T$ and $\mathcal{P} \cap T$ is upward filtrating. Then T is connected for $\mathcal{P}$.*

b) *If 2^S separates $\mathcal{P}$, then $\mathcal{P}$ is connected for $\mathcal{P}^\top$.*

Proof. a) Let sets $A, B \subset \mathcal{P}$ be given with $T \subset A \cup B$, $T \cap A \neq \emptyset$ and $T \cap B \neq \emptyset$. Assume that $T \cap A \cap B = \emptyset$. Then there exists a $K \in 2^S$ with $K \cap A \cap T = K \cap B \cap T = \emptyset$ such that for $C \in \mathcal{P}$ with $C \supset (A \cup B) \cap T$ we have $C \cap T \cap K \neq \emptyset$ in contradiction to $T \cap K = (T \cap K \cap A) \cup (T \cap K \cap B) = \emptyset$.
b) Part a) applies to $(S, \mathcal{P}^\top)$ and $T \in \mathcal{P}$ since $T \in \mathcal{P}^\top \cap T$. $\square$

The method of proof of the following Lemma goes back to Helly [9].

Lemma 3 *Let $(S, \mathcal{P}, \mathcal{L})$ be a bipaved space, let $T \in 2^S \setminus \{\emptyset\}$ such that $\mathcal{L}$ separates $\mathcal{P} \cap T$, and let $n \geq k \geq 2$. Then $\mathcal{L} \cap T \subset \mathcal{P}(n-1, k-1)$ implies $T \in \mathcal{P}(n, k)$.*

Proof. Let $\{P_0, \ldots, P_n\} \subset \mathcal{P}$ satisfy relations (1), (2), and (3). Assume that $T \cap P_{-0} \cap P_0 = \emptyset$. Since $\mathcal{L}$ separates $\mathcal{P} \cap T$, there exists an $L \in \mathcal{L}$ with $L \cap P_{-0} \cap T = L \cap P_0 \cap T = \emptyset$ such that

$$M \in \mathcal{P} \cap T,\ M \supset P_{-0} \cap T \text{ and } M \cap P_0 \cap T \neq \emptyset \text{ imply } M \cap L \neq \emptyset.$$

In particular, for every subset $J \in \mathcal{E}^{k-1}(\{1, \ldots, n\})$ we obtain $P_J \cap T \cap L \neq \emptyset$ by taking $M = P_J \cap T$. But $L \cap T \cap P_0 = \emptyset$ together with $T \subset \bigcup_{i=0}^n P_i$ implies $L \cap T \subset \bigcup_{i=1}^n P_i$, and with $\mathcal{L} \cap T \subset \mathcal{P}(n-1, k-1)$ we arrive at $\emptyset \neq L \cap T \cap \bigcap_{i=1}^n P_i = L \cap P_{-0} \cap T$, a contradiction. $\square$

Theorem 1 *Let S be a nonvoid set endowed with three pavings $\mathcal{K}, \mathcal{L}$ and $\mathcal{M}$ such that $\mathcal{K} \sqcap \mathcal{L} \subset \mathcal{K} \cup \{\emptyset\}$ and $\mathcal{L}$ separates every trace $\mathcal{M} \cap K, K \in \mathcal{K}$. Then $\mathcal{K}$ is finitary connected for $\mathcal{M}$ iff it is connected for $\mathcal{M}$.*

Proof. Let $n \in \mathbb{N} \setminus \{1\}$. Suppose that $\mathcal{K}$ is $(n-1)-$ary connected for $\mathcal{M}$. Then every $\mathcal{L} \cap K (\subset \mathcal{K} \cup \{\emptyset\}), K \in \mathcal{K}$, is also $(n-1)-$ary connected for $\mathcal{M}$. Hence, by Lemma 3, $\mathcal{K}$ is $n-$ary connected for $\mathcal{M}$. The assertion follows by induction. $\square$

Corollary 1 *Let $(S, \mathcal{P}, \mathcal{K})$ be a bipaved space such that $\mathcal{P}$ is upward filtrating and $\mathcal{K}^\top$ separates $\mathcal{K} \sqcap \mathcal{P}$. Then $\mathcal{K}$ is finitary connected for $\mathcal{P}$.*

Proof. By Lemma 2 a), $\mathcal{K}$ is connected for $\mathcal{P}$. Now apply Theorem 1 with $\mathcal{L} = \mathcal{K}^\top$ and $\mathcal{M} = \mathcal{P}$. □

Corollary 2 *Let $(S, \mathcal{P})$ be a paved space. If $\mathcal{P}^\top$ separates $\mathcal{P}$, then $\mathcal{P}$ is finitary connected for $\mathcal{P}^\top$.*

Proof. Corollary 1 applies to $(S, \mathcal{P}^\top, \mathcal{P})$ since $S \in \mathcal{P}^\top$. □

Let S be a nonvoid set, and let A, B be a pair of disjoint subsets of S. Then A, B is said to be *screened* with the subsets C, D of S iff $A \subset C \setminus D$, $B \subset D \setminus C$ and $C \cup D = S$. Let $\mathcal{P}$ and $\mathcal{K}$ be pavings in S. Then $\mathcal{P}$ is *screened* with $\mathcal{K}$ iff every pair of nonvoid disjoint subsets in $\mathcal{P}$ is screened with a pair of subsets in $\mathcal{K}$. In particular, a paving $\mathcal{P}$ is *normal* iff $\mathcal{P}$ is screened with $\mathcal{P}$.

Lemma 4 *Let $(S, \mathcal{P}, \mathcal{K})$ be a bipaved space such that $\mathcal{P}$ is screened with $\mathcal{K}$, and $\mathcal{P}$ is connected for $\mathcal{K}$. Then $\mathcal{K} \sqcap \mathcal{K}$ separates $\mathcal{P}$.*

Proof. Let $A, B \in \mathcal{P} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$. Let A, B be screened with $C, D \in \mathcal{K}$. Then for $K := C \cap D$ we have $A \cap K = B \cap K = \emptyset$, and for $P \in \mathcal{P}$ with $P \supset A$ and $P \cap B \neq \emptyset$ we have $P \cap C \supset A \neq \emptyset$ and $P \cap D \supset P \cap B \neq \emptyset$. Since P is connected for $\mathcal{K}$ we arrive at $P \cap K = P \cap C \cap D \neq \emptyset$. □

We note two instances where our concepts of connectedness and separation coincide:

Example 1 Let $(S, \mathcal{P})$ be a paved space such that $\mathcal{P}$ is $\cap_f$−closed. Then we have (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) for the following properties.

(a) $\mathcal{P}$ is separated.

(b) $\mathcal{P}$ is finitary connected.

(c) $\mathcal{P}$ is connected.

If $\mathcal{P}$ is normal, then the three conditions are equivalent.

Proof. (a) $\Longrightarrow$ (b) follows from Corollary 2 together with Remark 1 a) and c), and (b) $\Longrightarrow$ (c) is obvious. If $\mathcal{P}$ is normal, then (c) $\Longrightarrow$ (a) follows from Lemma 4 with $\mathcal{K} = \mathcal{P}$. □

Example 2 Let $(S, \mathcal{P})$ be a paved space such that $\mathcal{P}$ is a lattice. Then the following are equivalent:

(a) $\mathcal{P}$ is separated.

(b) $\mathcal{P}$ is finitary connected.

(c) $\mathcal{P}$ is connected.

(d) $\mathcal{P} \setminus \{\emptyset\}$ is a lattice.

Proof. (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a) is obvious, and (a) $\Longrightarrow$ (b) follows with Example 1. $\square$

Lemma 5 *Let S be a nonvoid set endowed with three pavings $\mathcal{K}, \mathcal{L}$, and $\mathcal{P}$, and let $k, n \in \mathbb{N}$ with $k \leq n$. Suppose that*

(i) $\mathcal{L} \subset \mathcal{P}(n,k)$, and

(ii) for every $A \in \mathcal{E}(S)$ and every $K \in \mathcal{K}$ with $K \supset A$ there exists an $L \in \mathcal{L}$ with $A \subset L \subset K$.

Then $\mathcal{K} \subset \mathcal{P}(n,k)$.
In case $n = k$ condition (ii) my be relaxed according to

(ii)' for every $A \in \mathcal{E}^{n+1}(S)$ and every $K \in \mathcal{K}$ with $K \supset A$ there exists an $L \in \mathcal{L}$ with $A \subset L \subset K$.

Proof. Let $T \in \mathcal{K}$ and $\{P_0, \ldots, P_n\} \subset \mathcal{P}$ satisfy relations (1), (2), and (3). Choose $s_J \in T \cap P_J, J \in \mathcal{E}^k := \mathcal{E}^k(\{0, \ldots, n\})$, and set $A := \{s_J : J \in \mathcal{E}^k\}$. In case $k = n$ we have card $A \leq n+1$, and in case card $A \leq n$ there exists a pair $J_1, J_2 \in \mathcal{E}^k$ with $J_1 \neq J_2$ and $s_{J_1} = s_{J_2}$, and we obtain $s_{J_1} \in T \cap P_{J_1} \cap P_{J_2} = T \cap \bigcap_{l=0}^n P_l$. Hence without loss of generality we may assume $A \in \mathcal{E}^{n+1}(S)$ in case $k = n$. Now choose L according to condition (ii), resp.condition (ii)' in case $k = n$, for A as above and $K = T$. Then $L \subset \bigcup_{i=0}^n P_i$ and $L \cap P_J \neq \emptyset$ for all $J \in \mathcal{E}^k$. Hence, $T \cap \bigcap_{i=0}^n P_i \supset L \cap \bigcap_{i=0}^n P_i \neq \emptyset$, since $L \subset \mathcal{P}(n,k)$. $\square$

Example 3 Let $\mathcal{P}$ be a normal paving in S containing $\mathcal{E}(S)$. Then the following holds.

a) $\mathcal{P} \subset \mathcal{P}(n,k)$ implies $\mathcal{P}^C \subset \mathcal{P}(n,k)$ for $n \geq k$.

b) If $\mathcal{P}$ is a lattice, then every $A \in \mathcal{P}$ which is $n-$ary connected for $\mathcal{P}$ is also $n-$ary connected for $\mathcal{P}^C$.

Proof. a) Let $A \in \mathcal{E}(S)$ and $K \in \mathcal{P}^C$ with $A \subset K$. Then there exists a screening $C, D \in \mathcal{P}$ for $A, S \setminus K$ which implies $A \subset C \subset K$. Now apply Lemma 5 with $\mathcal{L} = \mathcal{P}$ and $\mathcal{K} = \mathcal{P}^C$.
b) We first show by induction that for $\{A, A_0, \ldots, A_n\} \subset \mathcal{P}$ with $A \subset \bigcup_{i=0}^n (S \setminus A_i)$ there exist sets $B_0, \ldots, B_n \in \mathcal{P}$ with $B_i \subset S \setminus A_i$ and $A = \bigcup_{i=0}^n B_i$.
In case $n = 0$ take $B_0 = A$. Suppose that the assertion is true for $n = k - 1$, and let $\{A, A_0, \ldots, A_k\} \subset \mathcal{P}$ with $A \cap \bigcap_{i=0}^k A_i = \emptyset$. Since $\mathcal{P}$ is normal and $\cap_f-$closed, there exist sets $C, D \in \mathcal{P}$ with $C \cup D = S$, $A \cap A_0 \subset C \setminus D$ and $\bigcap_{i=1}^k A_i \subset D \setminus C$. In particular, $A \cap C \cap \bigcap_{i=1}^k A_i = \emptyset$ and $B_0 := D \cap A \subset A \setminus A_0$. By the induction hypothesis there exist sets $B_1, \ldots, B_k \in \mathcal{P}$ with $B_i \cap A_i = \emptyset$ and $A \cap C = \bigcup_{i=1}^k B_i$. Finally, $\bigcup_{i=0}^k B_i = (D \cap A) \cup (C \cap A) = A$.
Now let $A \in \mathcal{P}$ be $n-$ary connected for $\mathcal{P}$. Suppose that conditions (2) and (3) with $k = n$ are satisfied for $\{T, P_0, P_1, \ldots, P_n\} = \{A, S \setminus A_0, \ldots, S \setminus A_n\}$ with $\{A_0, \ldots, A_n\} \subset \mathcal{P}$. By (3) there exist $s_i \in A \cap (S \setminus A_j)$, $j \neq i$. Now choose $B_0, \ldots, B_n$ as above. Then for $C_i = B_i \cup (\{s_0, \ldots, s_n\} \setminus \{s_i\})$ we have $A = \bigcup_{i=0}^n C_i$, $C_i \subset A \setminus A_i$, and $s_i \in C_{-i}, i \in \{0, \ldots, n\}$, and we obtain $A \cap \bigcap_{i=0}^n (S \setminus A_i) \supset \bigcap_{i=0}^n C_i \neq \emptyset$, i.e. condition (4) is violated. □

For further use, the following lemma is formulated more general than required in section 4 below.

Lemma 6 *Let $(S, \mathcal{F}, \mathcal{G})$ be a bipaved space such that $\mathcal{F}$ is $\cup_f-$closed, $\mathcal{G}$ is $\cap_f-$closed, and the following regularity condition is satisfied.*

(5) $\forall t \in V \in \mathcal{G}\ \exists U \in \mathcal{G}, F \in \mathcal{F} : t \in U \subset F \subset V$

Let K be a nonvoid subset of S which is compact w.r.t. $\mathcal{G}^C$.

a) Let $\{G_0, \ldots, G_n\} \subset \mathcal{G}$ with $K \subset \bigcup_{i=0}^n G_i$, and let $s_i \in K \cap G_{-i}, i \in I := \{0, \ldots, n\}$ be given. Then there exist subsets $F_i \in \mathcal{F}$ with $K \subset \bigcup_{i=0}^n F_i$, $F_i \subset G_i$ and $s_i \in F_{-i}, i \in I$.

b) If $\mathcal{P}$ is a $\cap_f-$closed subpaving of S such that K is $n-$ary connected for $\mathcal{P}$, then K is also $n-$ary connected for every subpaving $\mathcal{H} \subset \mathcal{G}$ satisfying

(6) $\forall F \in \mathcal{F}, H \in \mathcal{H}, F \subset H\ \exists P \in \mathcal{P} : F \cap K \subset P \subset H.$

Proof. a) For $t \in G := \bigcup_{i=0}^n G_i$ and $V_t := \bigcap\{G_i : t \in G_i, i \in I\} (\in \mathcal{G})$ choose $U_t \in \mathcal{G}$ and $F_t \in \mathcal{F}$ with

$$t \in U_t \subset F_t \subset V_t$$

Then $G = \bigcup\{U_t : t \in G\}$. Since K is compact w.r.t $\mathcal{G}^C$, there exists a set $H \in \mathcal{E}(G)$ such that $K \subset \bigcup_{z\in H} U_z$. Without loss of generality we may assume $\{s_0, \dots, s_n\} \subset H$. For $i \in I$ the sets

$$F_i := \bigcup\{F_t : U_t \subset G_i, t \in H\}(\in \mathcal{F})$$

are subsets of G_i because $t \in U_t \subset G_i$ implies $F_t \subset V_t \subset G_i$.
Let $t \in K$. Then there exists a $z \in H$ with $t \in U_z$ and an $i \in I$ with $z \in G_i$. Now $t \in U_z \subset F_z \subset V_z \subset G_i$ implies $t \in F_i$, i.e., we have $K \subset \bigcup_{i\in I} F_i$. Finally, for $i \in I$,

$$s_i \in U_{s_i} \subset F_{s_i} \subset V_{s_i} \subset G_k \text{ for all } k \in I \text{ with } s_i \in G_k$$

together with $s_i \in G_{-i}$ yields $s_i \in F_{-i}, i \in I$.
b) This is an easy consequence of a). □

The following result is an abstract version of a minimax theorem due to Terkelsen [26]:

Lemma 7 *Let $(S, \mathcal{K})$ be a paved space such that $\mathcal{K}$ is compact, connected, and $\cap_f-$closed. Let G be a nonvoid family of functions $g : S \to \mathbb{R}$ with the properties*

(i) $\{g \le \lambda\} \in \mathcal{K}, \lambda \in \mathbb{R}$, *and*

(ii) $\forall g, h \in \mathsf{G}\ \exists k \in \mathsf{G} : 2k \ge g + h$

Then there exists an $\hat{s} \in S$ with $\sup_{g\in\mathsf{G}} g(\hat{s}) = \sup_{g\in\mathsf{G}} \inf_{s\in S} g(s)$.

Proof. For $\alpha > \gamma := \sup_{g\in\mathsf{G}} \inf_{s\in S} g(s)$ we have

$$S(\mathsf{H}) := \bigcap_{h\in\mathsf{H}} \{h \le \alpha\} \in \mathcal{K},\ \mathsf{H} \in \mathcal{E}(\mathsf{G})$$

Let $\mathsf{F} = \{f_1, f_2\} \in \mathcal{E}^2(\mathsf{G})$ and $\mathsf{H} \in \mathcal{E}(\mathsf{G})$ such that $S(\mathsf{H} \cup \{f_i\}) \ne \emptyset,\ i \in \{1, 2\}$, and $S(\mathsf{H}) \subset S(\{f_1\}) \cup S(\{f_2\})$. Since $\mathcal{K}$ is connected, we obtain $S(\mathsf{H} \cup \mathsf{F}) \ne \emptyset$, which means that S is $\Gamma-$connected [12] w.r.t. $\Gamma = (\mathsf{G}, S, a)$ with $a(g, s) = g(s)$, and assumption (ii) means that Γ is $\varphi-$concave [12] w.r.t. the arithmetic mean $\varphi(\sigma, \tau) = \frac{1}{2}(\sigma + \tau)$. Moreover every $g \in \mathsf{G}$ is bounded from below since $\{g \le -n\} \ne \emptyset\ \forall n \in \mathbb{N}$ would imply $\{g = -\infty\} \ne \emptyset$. Hence, the assumptions of [12];Theorem 1 are satisfied, and we obtain $\inf_{s\in S} \max_{h\in\mathsf{H}} h(s) \le \gamma$ for every finite $\mathsf{H} \subset \mathsf{G}$. Therefore, the system $\mathcal{R}$ of sets $\{g \le \beta\},\ g \in \mathsf{G},\ \beta \in \mathbb{R},\ \beta > \gamma$, has the finite intersection property. By compactness of $\mathcal{K}$ there exists an $\hat{s} \in \bigcap_{R\in\mathcal{R}} R = \bigcap_{g\in\mathsf{G}}\{g \le \gamma\}$. □

Lemma 8 *Let $(S, \mathcal{P})$ be a paved space, and let F be a linear space of functions $f : S \to \mathbb{R}$ such that $\{f \leq \alpha\} \in \mathcal{P}^\top$ for all $f \in \mathsf{F}$, $\alpha \in \mathbb{R}$. Then for $Y, Z \in 2^S \setminus \{\emptyset\}$ with compact connected traces $\mathcal{P}^\top \cap Y$ and $\mathcal{P}^\top \cap Z$ the following are equivalent:*

(a) $\forall (y, z) \in Y \times Z \; \exists f \in \mathsf{F} : f(y) \neq f(z)$.

(b) $\exists f \in \mathsf{F} : \min_{z \in Z} f(z) > \max_{y \in Y} f(y)$.

Proof. (a) $\Longrightarrow$ (b): Observe that $\{f \leq \max[\inf_{z\in Z} f(z) + \frac{1}{n}, -n]\} \cap Z \in (\mathcal{P}^\top \cap Z) \setminus \{\emptyset\}, n \in \mathbb{N}$, implies $\{f = \inf_{z\in Z} f(z)\} \cap Z \neq \emptyset$, i.e., $\min_{z\in Z} f(z)$ and $\max_{z\in Z} f(z)$ exist for $f \in \mathsf{F}$.
We now fix a $z \in Z$ and set $g_{f,z}(y) = f(z) - f(y), y \in Y, f \in \mathsf{F}$. Then (a) implies $\sup_{f\in\mathsf{F}} g_{f,z}(y) > 0$ for all $y \in Y$. From Lemma 7, applied to $(Y, \mathcal{P}^\top \cap Y)$ and $\mathsf{G} = \{g_{f,z} : f \in \mathsf{F}\}$, we obtain $\sup_{f\in\mathsf{F}} \min_{y\in Y} g_{f,z}(y) > 0$. A second application of Lemma 7 to the functions $h_f, f \in \mathsf{F}$, with $h_f(z) = f(z) - \max_{y\in Y} f(y), z \in Z$, yields

$$\sup_{f\in\mathsf{F}}[\min_{z\in Z} f(z) - \max_{y\in Y} f(y)] = \sup_{f\in\mathsf{F}} \min_{z\in Z} h_f(z) =$$

$$\min_{z\in Z} \sup_{f\in\mathsf{F}} h_f(z) = \min_{z\in Z} \sup_{f\in\mathsf{F}} \min_{y\in Y} g_{f,z}(y) > 0.$$

(b) $\Longrightarrow$ (a) is obvious. □

Let $(S, \mathcal{P})$ be a paved space and F a family of real-valued functions on S. Then we say that F *separates $\mathcal{P}$ pointwise* iff

$$\forall A, B \in \mathcal{P} \setminus \{\emptyset\}, A \cap B = \emptyset, \; \forall a \in A, b \in B \; \exists f \in \mathsf{F} : \; f(a) \neq f(b)$$

and F is *point separating* iff F separates 2^S pointwise, i.e., $\forall \{s, t\} \in \mathcal{E}^2(S) \; \exists f \in \mathsf{F} : \; f(s) \neq f(t)$.

Theorem 2 *Let $(S, \mathcal{P})$ be a paved space with compact and connected $\mathcal{P}$, and let F be a linear space of real-valued functions on S separating $\mathcal{P}$ pointwise, such that $\{f \leq \alpha\} \in \mathcal{P}^\top$ for all $f \in \mathsf{F}$, $\alpha \in \mathbb{R}$. Let $\mathcal{K}$ denote the paving of all sets $\{f = \alpha\}$, $f \in \mathsf{F}$, $\alpha \in \mathbb{R}$, and let $\mathcal{L}$ be the paving of all sets $\{\beta < f < \alpha\}$, $f \in \mathsf{F}$, $\beta < \alpha$. Then $\mathcal{K}$ and $\mathcal{L}$ both separate $\mathcal{P}$, and $\mathcal{P}$ is finitary connected for $\mathcal{P}^\top$.*

Proof. 1. We first show that $\mathcal{K}$ and $\mathcal{L}$ separate $\mathcal{P}$. Let $A, B \in \mathcal{P} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$. Then by Lemma 8 there exists an $f \in \mathsf{F}$ and $\alpha, \beta \in \mathbb{R}$ with $f(a) > \alpha > \beta > f(b)$ for all $a \in A, b \in B$. Let $P \in \mathcal{P}$ with $P \supset A$ and

$P \cap B \neq \emptyset$.
For $K = \{f = \alpha\}$ we have $A \cap K = B \cap K = \emptyset$. From $C := \{f \geq \alpha\} \cap P \supset A \neq \emptyset$, $D := \{f \leq \alpha\} \cap P \supset P \cap B \neq \emptyset$, $\{C, D\} \subset \mathcal{P}$, and $C \cup D = P$ it follows that $K \cap P = C \cap D \neq \emptyset$, since $\mathcal{P}$ is connected.
For $L = \{\beta < f < \alpha\}$ we have $A \cap L = B \cap L = \emptyset$. Here $P \subset S = \{f < \alpha\} \cup \{f > \beta\}$ together with $\{f > \beta\} \cap P \supset A \neq \emptyset$ and $\{f < \alpha\} \cap P \supset B \cap P \neq \emptyset$ implies $P \cap L = P \cap \{f < \alpha\} \cap \{f > \beta\} \neq \emptyset$ since, by Remark 1 b) and c), $\mathcal{P}$ is connected for the paving $(\mathcal{P}^\top)^C$ which contains the sets $\{f < \alpha\}$ and $\{f > \beta\}$.
2. Since $\mathcal{P}^\top$ is $\cap_f$-closed, we have $\mathcal{K} \subset \mathcal{P}^\top$. By Corollary 2 together with 1. it follows that $\mathcal{P}$ is finitary connected for $\mathcal{P}^\top$. $\square$

2 Segments and hulls

A *segment space* is a pair $(S, \langle \cdot, \cdot \rangle)$ where S is a nonvoid set and $\langle \cdot, \cdot \rangle : S \times S \to 2^S$ is a set–valued map with $\langle s, t \rangle \supset \{s, t\}$ for all $s, t \in S$.

$\langle \cdot, \cdot \rangle$ is called a *segment function* for S. A subset $T \subset S$ is *convex* iff $\{s, t\} \subset T \;\Rightarrow\; \langle s, t \rangle \subset T$.
Examples of segment spaces can be found in abundance in the books of Coppel [3] and van de Vel [28].

A *paved segment space* is is a triplet $(S, \mathcal{P}, \langle \cdot, \cdot \rangle)$ where $(S, \mathcal{P})$ is a paved space and $\langle \cdot, \cdot \rangle$ is a segment function for S. An *interval* is a segment $\langle s, t \rangle$ that is connected (for $\mathcal{P}$). An *interval space* is a paved segment space where every segment is an interval.

Every segment space $(S, \langle \cdot, \cdot \rangle)$ gives rise to a hull operator $\langle \cdot \rangle : 2^S \to 2^S$ according to

$$\langle D \rangle := \bigcap \{C \in \mathcal{C} : C \supset D\}$$

where $\mathcal{C}$ denotes the paving of all convex subsets of S.

For a paving $\mathcal{K}$ in S we set $\langle \mathcal{K} \rangle := \{\langle K \rangle, \; K \in \mathcal{K}\}$. Especially, $\langle \mathcal{E}(S) \rangle$ is the paving of all *polytopes* in S. Obviously, $\mathcal{C}$ is an alignment and therefore $\langle 2^S \rangle \subset \mathcal{C}$.

The standard example is the following.

Example 4 In the following, every vector space S will be endowed with the *standard segment function*

$$[s, t] := \{\lambda s + (1 - \lambda) t : 0 \leq \lambda \leq 1\}, \;\; s, t, \in S$$

Here a subset T is convex in the segment space $(S, [\cdot, \cdot])$ iff it is convex in the ordinary sense. The induced hull operation yields $[D]$, the convex hull of D, and for $A = \{t_1, \ldots, t_n\} \in \mathcal{E}(S)$ we have $[A] = \{\sum_{i=1}^n \lambda_i t_i : \lambda_i \geq 0, 1 \leq i \leq n,\ \sum_{i=1}^n \lambda_i = 1\}$.
If S is a topological vector space, and if $\mathcal{P}$ is the paving of all open (closed) subsets of S, then $(S, \mathcal{P}, [\cdot, \cdot])$ is an interval space.

Lemma 9 *In an interval space $(S, \mathcal{P}, \langle \cdot, \cdot \rangle)$ the following holds.*

a) Every set $P*Q := \bigcup_{x \in P, y \in Q} \langle x, y \rangle, \quad P, Q \in 2^S \setminus \{\emptyset\}$, is connected for $\mathcal{P}$.

b) Every convex subset is connected for $\mathcal{P}$.

c) Let the paving $\mathcal{K}$ be screened with $\mathcal{P}$, and let every $K \in \mathcal{K}$ be convex. Then $\mathcal{P} \sqcap \mathcal{P}$ separates $\mathcal{K}$.

Proof. a) $P*Q$ is the upward filtrating union of the sets $A*B, A \in \mathcal{E}(P), B \in \mathcal{E}(Q)$. Hence, by Lemma 1 (iii), it is sufficient to show that every set $A*B, A, B \in \mathcal{E}(S)$, is connected. Every set $\{z\} * B = \bigcup_{y \in B} \langle z, y \rangle, B \in \mathcal{E}(S)$, is connected by Lemma 1 (iv), since $z \in \bigcap_{y \in B} \langle z, y \rangle$. Suppose that $A * B$ is connected for some pair $A, B \in \mathcal{E}(S)$. Then for $z \in S \setminus A$ we have $(A \cup \{z\}) * B = (A * B) \cup (\{z\} * B)$. Again this set is connected, since $(A * B) \cap (\{z\} * B) \supset B \neq \emptyset$. The assertion follows by induction.
b) If P is convex, then $P = P * P$ is connected according to a).
c) By b), $\mathcal{K}$ is connected for $\mathcal{P}$, and with Lemma 4 the assertion follows. □

Example 5 Every paved space $(S, \mathcal{P})$ gives rise to an *intrinsic segment function*

$$\langle s, t \rangle_{\mathcal{P}} := \bigcap \{C \in \mathcal{P} : C \supset \{s, t\}\}, s, t \in S \text{ with } \bigcap \emptyset = S.$$

Here every $P \in \mathcal{P}$ is convex, and the implication (a) $\Longrightarrow$ (b) holds for the conditions

(a) $(S, \mathcal{P}, \langle \cdot, \cdot \rangle_{\mathcal{P}})$ is an interval space.

(b) $\mathcal{P}$ is connected.

If $\mathcal{P}$ is $\cap_a$–closed and if $\forall s, t \in S\ \exists P \in \mathcal{P} : \{s, t\} \subset P$, then the segments are convex, and (a) and (b) are equivalent.

Proof. (a) $\Longrightarrow$ (b): Let $\{P, P_1, P_2\} \subset \mathcal{P}$ with $P \subset P_1 \cup P_2$, and let $x \in P \cup P_1$ and $y \in P \cup P_2$. Then $\langle x, y \rangle_{\mathcal{P}} \subset P \subset P_1 \cup P_2$. But $\langle x, y \rangle_{\mathcal{P}}$ is connected for $\mathcal{P}$, and we obtain $\emptyset \neq \langle x, y \rangle_{\mathcal{P}} \cap P_1 \cap P_2 \subset P \cap P_1 \cap P_2$.
(b) $\Longrightarrow$ (a): This follows from $\langle s, t \rangle_{\mathcal{P}} \in \mathcal{P}$. □

Lemma 10 *Let $(S, \mathcal{P}, \mathcal{L})$ be a bipaved space and $\langle \cdot, \cdot \rangle$ a segment function for S such that every set in $\mathcal{P}$ is convex. Suppose that*

(7) *for all $A, B \in \mathcal{P} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$ there exists an $L \in \mathcal{L}$ such that $A \cap L = B \cap L = \emptyset$ and $\forall y \in B \; \exists x \in A : \langle x, y \rangle \cap L \neq \emptyset$.*

Then $\mathcal{L}$ separates $\mathcal{P}$.

Proof. For $A, B \in \mathcal{P} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$ choose $L \in \mathcal{L}$ according to (7). Let $P \in \mathcal{P}$ with $P \supset A$ and $P \cap B \neq \emptyset$. Choose $y \in P \cap B$ arbitrarily and take $x \in A$ such that $\langle x, y \rangle \cap L \neq \emptyset$. Then $\{x, y\} \subset P$ implies $\emptyset \neq \langle x, y \rangle \cap L \subset P \cap L$ since P is convex. □

Example 6 Let $(S, \mathcal{P}, \langle \cdot, \cdot \rangle)$ be a paved segment space and F a family of real–valued functions on S such that every image $f(\langle s, t \rangle)$, $s, t \in S$, is convex, and for all $A, B \in \mathcal{P} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$ there exists an $f \in \mathsf{F}$ and a $\lambda \in \mathbb{R}$ such that

$$f(a) > \lambda > f(b) \quad \forall a \in A, b \in B$$

Then relation (7) holds for the paving $\mathcal{L}$ of sets $\{f = \lambda\}$, $\lambda \in \mathbb{R}$.

Theorem 3 *Let S be a nonvoid set endowed with three pavings $\mathcal{P}, \mathcal{K}$, and $\mathcal{L}$ and a segment function $\langle \cdot, \cdot \rangle$. Suppose that*

(i) $\mathcal{K} \sqcap \mathcal{L} \subset \mathcal{K} \cup \{\emptyset\}$,

(ii) every set in $\mathcal{K} \sqcap \mathcal{P}$ is convex, and

(iii) condition (7) with $\mathcal{P}$ replaced by $\mathcal{K} \sqcap \mathcal{P}$ is satisfied.

Then $\mathcal{K}$ is finitary connected for $\mathcal{P}$ iff it is connected for $\mathcal{P}$.

Proof. From Lemma 10 it follows that $\mathcal{L}$ separates $\mathcal{K} \sqcap \mathcal{P}$. Now apply Theorem 1 with $\mathcal{M} = \mathcal{P}$. □

Corollary 3 *Let $(S, \mathcal{P}, \langle \cdot, \cdot \rangle)$ be a paved segment space such that every $P \in \mathcal{P}$ is convex, and relation (7) holds with $\mathcal{L} = \mathcal{P}^\top$. Then $\mathcal{P}$ is finitary connected for $\mathcal{P}^\top$. If moreover $\mathcal{P}$ is upward filtrating, then also $\mathcal{P}^\top$ is finitary connected for $\mathcal{P}$.*

Proof. By Lemma 10, $\mathcal{P}^\top$ separates $\mathcal{P}$, and Corollary 2 implies that $\mathcal{P}$ is finitary connected for $\mathcal{P}^\top$. Now let $\mathcal{P}$ be upward filtrating. Then it follows from Lemma 2 a) that $\mathcal{P}^\top$ is connected for $\mathcal{P}$, and from Theorem 3, applied to $\mathcal{K} = \mathcal{L} = \mathcal{P}^\top$, it follows that $\mathcal{P}^\top$ is finitary connected for $\mathcal{P}$. □

Remark 3 *Let $(S, \mathcal{P}, \langle\cdot,\cdot\rangle)$ be a paved segment space such that every $P \in \mathcal{P}$ is convex. Then for $n \in \mathbb{N}$ the following are equivalent.*

(a) The paving $\mathcal{C}$ of all convex subsets of S is $n-$ary connected for $\mathcal{P}$.

(b) $\langle \mathcal{E}^{n+1}(S) \rangle$ is $n-$ary connected for $\mathcal{P}$.

Proof. Apply Lemma 5 with $k = n, \mathcal{K} = \mathcal{C}$ and $\mathcal{L} = \langle \mathcal{E}^{n+1}(S) \rangle$. □

Theorem 4 *Let $(S, \mathcal{P}, \langle\cdot,\cdot\rangle)$ be a paved segment space with $\cap_f-$closed $\mathcal{P}$. Let $\mathcal{B}$ denote the paving of all subsets of S which are contained in some polytope, and let $\mathcal{C}$ be the paving of all convex subsets of S. Suppose that $\mathcal{C} \cap \mathcal{P}$ separates $\mathcal{B} \cap \mathcal{C} \cap \mathcal{P}$, and $\langle \mathcal{E}(S) \rangle \subset \mathcal{P}$. Then $\mathcal{C}$ is finitary connected for $\mathcal{C} \cap \mathcal{P}$.*

Proof. Let $\mathcal{Q} := \mathcal{B} \cap \mathcal{C} \cap \mathcal{P}$. Then $\mathcal{Q}^\top (\supset \mathcal{C} \cap \mathcal{P})$ separates $\mathcal{Q}$. By Corollary 2, the paving of polytopes $\langle \mathcal{E}(S) \rangle$ $(\subset \mathcal{Q})$ is finitary connected for $\mathcal{C} \cap \mathcal{P}$ and, by Remark 3, $\mathcal{C}$ is finitary connected for $\mathcal{C} \cap \mathcal{P}$. □

3 Helly and Klee type intersection theorems

Let $n \in \mathbb{N}$. A paved space $(S, \mathcal{P})$ will be called a *K_n-space* (resp. an *H_n-space*) and $\mathcal{P}$ is a *K_n-paving* (resp. an *H_n-paving*) iff for all $\{P_0, \ldots, P_n\} \subset \mathcal{P}$ the relations

(8) $\bigcap_{j \in J} P_J \in \mathcal{P} \setminus \{\emptyset\}$ for all nonvoid proper subsets J of $\{0, \ldots, n\}$

(9) $\bigcup_{i=0}^n P_i \in \mathcal{P}$, and

(10) $\bigcap_{i=0}^n P_i = \emptyset$

(resp. (8) and (10)) cannot hold simultaneously, and $\mathcal{P}$ is a *Klee paving* [13] iff it is a *K_n-paving* for every $n \in \mathbb{N}$.

Remark 4 *For a paving $\mathcal{P}$ in S and for $n \in \mathbb{N}$ the implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) hold for the following conditions.*

(a) $\mathcal{P}$ is $n-$ary connected.

(b) $\mathcal{P}$ is a K_n-paving.

(c) $\mathcal{P}$ is $n-$ary connected for $\mathcal{P}^\top$.

In particular, every finitary connected paving is a Klee paving.
If $\mathcal{P}$ is $\cap_f-$closed, then the three conditions are equivalent.

Lemma 11 *Let $(S, \mathcal{P})$ be a paved space, and let $n \in \mathbb{N}$. Then we have (a) $\Longleftrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) for the properties*

(a) $\mathcal{P}$ is an H_n-paving.

(b) For all $\{P_0, \ldots, P_m\} \subset \mathcal{P}$ with $m \geq n$ the relations

(i) $\bigcap_{j \in J} P_j \neq \emptyset$ for all subsets $J \in \mathcal{E}^n(\{0, \ldots, m\})$,
(ii) $\bigcap_{j \in J} P_j \in \mathcal{P}$ for all nonvoid proper subsets J of $\{0, \ldots, m\}$, and
(iii) $\bigcap_{i=0}^m P_i = \emptyset$

cannot hold simultaneously.

(c) $\mathcal{P}^\top \subset \mathcal{P}(m, n)$ for all $m \geq n$.

(d) $S \in \mathcal{P}(n+1, n)$.

In case $\{\emptyset, S\} \subset \mathcal{P}$ the four conditions are equivalent.

Proof. (a) $\Longrightarrow$ (b): Suppose that $\{P_0, \ldots, P_m\} \subset \mathcal{P}$ with $m > n$ satisfies conditions (i), (ii), and (iii). Then without loss of generality there exists an $l \in \{n, \ldots, m\}$ with $\bigcap_{i=0}^l P_i = \emptyset$ and $Q_j := \bigcap_{i=0, i \neq j}^l P_i \neq \emptyset$ for all $j \in \{0, \ldots, l\}$. Set $P_i' := P_i \cap \bigcap_{j=n+1}^l P_j, i \in \{0, \ldots, n\}$. Then we have $\bigcap_{i=0, i\neq j}^n P_i' = Q_j \neq \emptyset, j \in \{0, \ldots, n\}$, i.e., condition (8) holds for $\{P_0', \ldots, P_n'\} (\subset \mathcal{P})$. Now $\bigcap_{i=0}^l P_i = \bigcap_{i=0}^n P_i' \neq \emptyset$ leads to a contradiction.
(b) $\Longrightarrow$ (c): Let $T \in \mathcal{P}^\top$ and $\{P_0, \ldots, P_m\} \subset \mathcal{P}$ such that conditions (1), (2), and (3) are satisfied with n and k replaced by m and n. Then relations (i) and (ii) hold with P_i replaced by $P_i \cap T (\in \mathcal{P})$, and from (b) we infer $\bigcap_{i=0}^m P_i \cap T \neq \emptyset$.
(c) $\Longrightarrow$ (d) and (b) $\Longrightarrow$ (a) are obvious.
(d) $\Longrightarrow$ (a): Let (d) be satisfied, and let $\{\emptyset, S\} \subset \mathcal{P}$. Suppose that for $\{P_0, \ldots, P_n\} \subset \mathcal{P}$ relations (8) and (10) are satisfied. Then $S = \bigcup_{i=0}^{n+1} P_i$ with

$P_{n+1} := S$, $S \cap \bigcap_{j \in J} P_j \neq \emptyset$ for all $J \in \mathcal{E}^n(\{0, \ldots, n+1\})$ and $\bigcap_{j \in J} P_j \in \mathcal{P}$ for all $J \in \mathcal{J}^{n+1}$ together with $S \in \mathcal{P}(n+1, n)$ implies $\bigcap_{i=0}^{n} P_i = \bigcap_{i=0}^{n+1} P_i \neq \emptyset$, a contradiction. □

Remark 5 *Let $\mathcal{P}$ be a H_n-paving in S. Then $\mathcal{P}$ is a K_n-paving, and $\mathcal{P} \cup \{\emptyset, S\}$ is an H_l-paving for every $l \geq n$.*

Example 7 (Discrete Helly-Klee Theorem, cf. [13, 14]) Let S be a finite set with m elements. Then for $n \in \mathbb{N}$ the following are equivalent:

(a) $n \geq m$.

(b) 2^S is $n-$ary connected.

(c) 2^S is a K_n-paving.

(d) 2^S is an H_n-paving.

Proof. (a) $\Longrightarrow$ (d): Let $\{P_0, \ldots, P_m\} \subset 2^S$ and $x_i \in P_{-i}, i \in \{0, \ldots, m\}$. Then we have $x_i = x_j$ for some pair $i \neq j$, and we arrive at $x_i \in P_{-i} \cap P_{-j} = \bigcap_{l=0}^{m} P_l$. With Remark 5 the assertion follows.
(d) $\Longrightarrow$ (c) $\iff$ (b) follows from Remarks 5 and 4.
(b) $\Longrightarrow$ (a): Suppose that $n < m$. Then for $A = \{x_0, \ldots, x_n\} \in \mathcal{E}^{n+1}(S)$ we have $A = \bigcup_{i=0}^{n}(A \setminus \{x_i\})$, $x_i \in \bigcap_{j=0, j\neq i}^{n}(A \setminus \{x_j\})$, $i \in \{0, \ldots, n\}$ and $\bigcap_{i=0}^{n}(A \setminus \{x_i\}) = \emptyset$, i.e., A is not $n-$ary connected for 2^S. □

Lemma 12 *Let $(S, \mathcal{P}, \mathcal{K})$ be a separated bipaved space, and let $n \in \mathbb{N}$.*

a) If every trace $\mathcal{P} \cap K, K \in \mathcal{K}$, is an H_n-paving, then $\mathcal{P}$ is an $H_{n+1}-$paving.

b) If every trace $\mathcal{P} \cap K, K \in \mathcal{K}$, is a K_n-paving, then $\mathcal{P}$ is a $K_{n+1}-$paving.

Proof. a) Without loss of generality we may assume $\{\emptyset, S\} \subset \mathcal{P}$. By Lemma 11 we have $K \in (\mathcal{P} \cap K)(n+1, n), K \in \mathcal{K}$, which implies $\mathcal{K} \subset \mathcal{P}(n+1, n)$. From Lemma 3, applied to $T = S$, we obtain $S \in \mathcal{P}(n+2, n+1)$, and with Lemma 11 the assertion follows.
b) This follows from [13];Proposition 1. □

Theorem 5 *Let $(S, \mathcal{P})$ be a paved space with compact and connected $\mathcal{P}$, and let $\{f_1, \ldots, f_n\}$ be a finite family of real-valued functions on S separating $\mathcal{P}$ pointwise such that $\{f \leq \alpha\} \in \mathcal{P}^\top$ for all $\alpha \in \mathbb{R}$ and all f in the linear hull of $\{f_1, \ldots, f_n\}$. Then $\mathcal{P}$ is an $H_{n+1}-$paving.*

Proof. We proceed by induction. The assertion is true for $n = 1$:

By Theorem 2, $\mathcal{K} = \{\{f_1 = \alpha\} : \alpha \in \mathbb{R}\}$ separates $\mathcal{P}$. On the other hand, every trace $\mathcal{P} \cap K, K \in \mathcal{K}$, is easily seen to be an H_1-paving, since f_1 separates $\mathcal{P}$ pointwise. Hence, by Lemma 12, $\mathcal{P}$ is an H_2-paving.

Suppose that the theorem is true for $n = k$. Let $\{f_1, \ldots, f_k, f_{k+1}\}$ satisfy the assumptions of the theorem. Let $K = \{f = \alpha\}$ with $f = \sum_{i=1}^{k+1} \gamma_i f_i \neq 0$. Without loss of generality we may assume $\gamma_{n+1} \neq 0$, which implies that $\{f_1, \ldots, f_k, f\}$ separates $\mathcal{P}$ pointwise. Since $K \in \mathcal{P}^\top$, the set of restrictions $\{f_1|K, \ldots, f_k|K\}$ separates $\mathcal{P} \cap K$ pointwise. Therefore, all assumptions of the theorem are satisfied for $(K, \mathcal{P} \cap K)$ and $\{f_1|K, \ldots, f_k|K\}$, and it follows that the trace $\mathcal{P} \cap K$ is an $H_{k+1}-$paving. Now from Theorem 2 together with Lemma 12 it follows that $\mathcal{P}$ is an $H_{k+2}-$paving. □

Lemma 13 *Let $(S, \mathcal{P}, \mathcal{L})$ be a bipaved space such that the following holds.*

(i) $\mathcal{L}$ is a $\cap_f-$closed H_n-paving, and

(ii) for every $A \in \mathcal{E}^n(S)$ and every $P \in \mathcal{P}$ with $P \supset A$ there exists an $L \in \mathcal{L}$ with $A \subset L \subset P$.

Then $\mathcal{P}$ is an H_n-paving.

Proof. Let $\{P_0, \ldots, P_n\} \subset \mathcal{P}$ satisfy relation (8). Choose $s_i \in P_{-i}$. In case $A := \{s_0, \ldots, s_n\} \notin \mathcal{E}^{n+1}(S)$ there exists a pair $s_i = s_j, i \neq j$, and we have $\bigcap_{l=0}^n P_l = P_{-i} \cap P_{-j} \neq \emptyset$. Otherwise there exist sets $L_i \in \mathcal{L}$ with $A \setminus \{s_i\} \subset L_i \subset P_i$, and with (i) we arrive at $\bigcap_{l=0}^n P_l \supset \bigcap_{l=0}^n L_l \neq \emptyset$. □

The following is classical.

Example 8 (Helly's Theorem [9]). Let $C_1, \ldots, C_n$, $n \geq d + 1$, be convex subsets of the Euclidean space $\mathbb{R}^d$ such that $\bigcap_{j \in J} C_j \neq \emptyset$ for all $J \in \mathcal{E}^{d+1}(\{1, \ldots, n\})$. Then $\bigcap_{i=1}^n C_i \neq \emptyset$.

Proof. By Theorem 5, applied to the projections $f_i(x_1, \ldots, x_i, \ldots, x_d) = x_i$, $i \leq d$, the paving of all compact convex subsets is an $H_{d+1}-$paving. By Lemma 13, the paving of all convex subsets is an $H_{d+1}-$paving as well. With Lemma 11 "(a) $\Longrightarrow$ (b)" the assertion follows. □

Remark 6 *Example 8 can be used to derive Carathéodory's theorem [6] which implies [5] that, in $\mathbb{R}^n$, the convex hull of a compact subset is compact.*

Example 9 a) Every connected, coherent, and $\cap_f$-closed paving $\mathcal{P}$ is an H_2-paving.

b) Every connected H_2-paving $\mathcal{P}$ is a Klee paving.

c) Let $(S, \mathcal{P})$ be a paved space and T a nonvoid subset of S such that $\mathcal{P} \cap T$ is an H_2-paving. Then T is finitary connected for $\mathcal{P}$ iff it is connected for $\mathcal{P}$.

d) A $\cap_f$-closed H_2-paving is finitary connected iff it is connected.

e) Let $\mathcal{P}$ be a coherent paving such that $\mathcal{D} = \{D \in \mathcal{P} : D$ is connected for $\mathcal{P}\}$ is $\cap_f$-closed. Then $\mathcal{D}$ is a finitary connected H_2-paving.

Proof. a) Let $\{P_0, P_1, P_2\} \subset \mathcal{P}$ such that $P_i \cap P_j \neq \emptyset$, $i \neq j$. Then $(P_0 \cap P_1) \cup (P_0 \cap P_2) = P_0 \cap (P_1 \cup P_2) \in \mathcal{P}$ implies $P_0 \cap P_1 \cap P_2 \neq \emptyset$.
b) By Remark 4, $\mathcal{P}$ is a K_1-paving, and by Remark 5, $\mathcal{P}$ is a H_n-paving, and therefore a K_n-paving for all $n \geq 2$.
c) Let T be connected for $\mathcal{P}$, let $\{P_0, \ldots, P_n\} \subset \mathcal{P}$, $T \subset \bigcup_{i=0}^{n} P_i$, $T \cap P_{-i} \neq \emptyset$ $\forall i \in \{0, \ldots, n\}$, and $P_J \in \mathcal{P}$, $J \in \mathcal{J}^n$.
In case $n = 1$ we have $T \cap P_0 \cap P_1 \neq \emptyset$ since T is connected for $\mathcal{P}$. In case $n \geq 2$ we have $T \cap P_k \cap P_l \supset T \cap P_{-i} \neq \emptyset$ for $i \notin \{k, l\}$ which, by Lemma 11, implies $\bigcap_{i=0}^{n} P_i \cap T \neq \emptyset$ since $\mathcal{P} \cap T$ is an H_2-paving.
d) This follows from c).
e) From Lemma 1 it follows that $\mathcal{D}$ is coherent, and with a) and d) the assertion follows. □

Theorem 6 *Let $(S, \mathcal{P}, \langle \cdot, \cdot \rangle)$ be a paved segment space, $\mathcal{C}$ the paving of all convex subsets of S and $\mathcal{S}$ the paving of all segments in S. Suppose that*

(i) $\mathcal{S} \subset \mathcal{P}$,

(ii) every trace $\mathcal{S} \cap P$, $P \in \mathcal{P}$, is compact,

(iii) $\mathcal{P}$ is a $\cap_f$-closed H_2-paving, and

(iv) $\mathcal{E}^1(S)$ is screened with $\mathcal{P} \cap \mathcal{C}$.

Then $\mathcal{P}$ is screened with $\mathcal{C} \cap \mathcal{P}$, and the following are equivalent.

(a) $\mathcal{P}$ is connected.

(b) $\mathcal{P}$ is connected for $\mathcal{C} \cap \mathcal{P}$.

(c) *$\mathcal{P}$ is separated.*

(d) *$\mathcal{C} \cap \mathcal{P}$ separates $\mathcal{P}$.*

(e) *$\mathcal{P}$ is finitary connected.*

Proof. 1. We adopt an argument from [27], p. 21:
Let $A, B \in \mathcal{P} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$, and let $z \in A$. By (i), (ii), and (iii) there exists a point $y \in B \cap \bigcap_{b \in B} \langle z, b \rangle$, since the sets of the paving $\{\langle z, b \rangle : b \in B\} \cup \{B\}$ intersect pairwise. Similarly there exists a point $x \in A \cap \bigcap_{a \in A} \langle y, a \rangle$.
Choose convex sets C, $D \in \mathcal{P} \cap \mathcal{C}$ screening $\{x\}$ and $\{y\}$. Assume that there exists an $x' \in D \cap A$. Then $x \in \langle y, x' \rangle \subset D$ leads to a contradiction. Hence, $A \subset C \setminus D$. Similarly, $y' \in C \cap B$ implies $y \in \langle z, y' \rangle \subset C$, another contradiction, and we obtain $B \subset D \setminus C$.
2. (a) $\Longrightarrow$ (b) and (d) $\Longrightarrow$ (c) are obvious, (a) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (e) follows from Example 1, and (b) $\Longrightarrow$ (d) follows from Lemma 4. □

Corollary 4 *Let $(S, \mathcal{P}, \langle \cdot, \cdot \rangle)$ be an interval space such that the assumptions (i) – (iv) of Theorem 6 are satisfied. Assume moreover that every set in $\mathcal{P}$ is convex. Then $\mathcal{P}$ is finitary connected.*

Proof. By Theorem 6 the paving $\mathcal{P}$ is screened with $\mathcal{C} \cap \mathcal{P} = \mathcal{P}$, and by Lemma 9 c) $\mathcal{P} = \mathcal{P} \sqcap \mathcal{P}$ separates $\mathcal{P}$. With Theorem 6 "(c) $\Longrightarrow$ (e)" the assertion follows. □

Example 10 Let $(S, \langle \cdot, \cdot \rangle)$ be a segment space. Then we have (a) $\Longrightarrow$ (b) for the following conditions.

(a) $(S, \langle \cdot, \cdot \rangle)$ is *modular*, i.e., $\langle x, y \rangle \cap \langle y, z \rangle \cap \langle z, x \rangle \neq \emptyset$ for all $\{x, y, z\} \in \mathcal{E}^3(S)$.

(b) The paving $\mathcal{C}$ of all convex subsets of S is an H_2−paving.

If every segment is convex, then (a) and (b) are equivalent.
A lattice L is modular iff the segment space $(L, \langle \cdot, \cdot \rangle)$ with

$$\langle s, t \rangle = \{x \in L : (s \wedge x) \vee (t \wedge x) = x = (s \vee x) \wedge (t \vee x)\}, \ s, t \in L$$

is modular ([2]; Proposition 1.6). An abundance of further examples of modular segment spaces can be found in [2], [28], and [29].

4 Topological connectedness

If S is a topological space, then we denote by $\mathcal{F}(S)$, $\mathcal{G}(S)$, $\mathcal{K}(S)$ and $\mathcal{C}(S)$ the pavings of all closed, open, compact, and connected subsets, respectively. Here the empty set is considered to be connected. Of course, S is normal (i.e., T_4 but not neccessarily T_1) iff $\mathcal{F}(S)$ is normal, the paving $\mathcal{K}(S)\cap\mathcal{F}(S)$ is always compact, and $\mathcal{F}(S)$ is compact iff S is compact. A subset is connected iff it is (1-ary) connected for $\mathcal{F}(S)$ or for $\mathcal{G}(S)$, respectively. In particular, the pavings $\mathcal{C}(S)\cap\mathcal{F}(S)$ and $\mathcal{C}(S)\cap\mathcal{G}(S)$ are connected, but $\mathcal{C}(S)$ need not be connected. If S is normal, then by Lemma 4 the paving $\mathcal{F}(S)$ separates $\mathcal{F}(S)\cap\mathcal{C}(S)$.

Example 11 For a topological space S the following are equivalent.

(a) $\mathcal{F}(S)\subset\mathcal{C}(S)$.

(b) $\mathcal{F}(S)$ is connected.

(c) $\mathcal{F}(S)$ is finitary connected.

(d) $\mathcal{F}(S)$ is separated.

(e) $\mathcal{F}(S)$ is an H_1-paving.

(f) $\mathcal{F}(S)$ is a Klee paving.

(g) $\mathcal{F}(S)\setminus\{\emptyset\}$ is $\cap_f-$closed.

A topological space with these properties is called *ultraconnected* [25].
A similar result with $\mathcal{F}(S)$ replaced by $\mathcal{G}(S)$ holds for *hyperconnected* [25] topological spaces.

Proof. (a) $\iff$ (b) and (e) $\iff$ (g) are obvious, (b) $\iff$ (c) $\iff$ (d) $\iff$ (g) follows from Example 2, and (c) $\iff$ (f) follows from Remark 4. □

Example 12 Let S be a topological space. Then every $\cap_f-$closed normal paving $\mathcal{P}$ in S with $\mathcal{P}\subset\mathcal{C}(S)\cap\mathcal{F}(S)$ or $\mathcal{P}\subset\mathcal{C}(S)\cap\mathcal{G}(S)$ is finitary connected. If moreover $\mathcal{P}$ contains $\mathcal{E}(S)$, then $\mathcal{P}^C$ is also finitary connected for $\mathcal{P}$.

Proof. This follows from Examples 1 and 3 a). □

Example 13 Let S be a regular topological space (i.e., a closed subset and a disjoint singleton possess disjoint neigborhoods), and let K be a compact subset of S. If K is $n-$ary connected for $\mathcal{F}(S)$, then K is $n-$ary connected for $\mathcal{G}(S)$.

Proof. Apply Lemma 6 b) with $\mathcal{F} = \mathcal{P} = \mathcal{F}(S)$ and $\mathcal{H} = \mathcal{G} = \mathcal{G}(S)$. Observe that regularity of S is equivalent with condition (5). $\square$

Since, for $n > 1$, $n-$ary connectedness for $\mathcal{F}(S)$ is quite a strong condition, we present a more sophisticated consequence of Lemma 6.

Theorem 7 *Let S be a regular topological space endowed with a segment function $\langle \cdot, \cdot \rangle$, and let $\mathcal{C}$ denote the paving of all convex subsets of S. Suppose that*

(i) $\mathcal{C} \cap \mathcal{F}(S)$ separates $\mathcal{C} \cap \mathcal{F}(S) \cap \mathcal{K}(S)$, and

(ii) $\langle \mathcal{E}(S) \rangle \subset \mathcal{F}(S) \cap \mathcal{K}(S)$.

Then $\mathcal{C}$ is finitary connected for $\mathcal{C} \cap \mathcal{F}(S)$.
If moreover

(iii) for all $F \in \mathcal{F}(S) \cap \mathcal{K}(S)$ and $G \in \mathcal{C} \cap \mathcal{G}(S)$ with $F \subset G$ there exists a $C \in \mathcal{C} \cap \mathcal{F}(S)$ with $F \subset C \subset G$,

then $\mathcal{C}$ is also finitary connected for $\mathcal{C} \cap \mathcal{G}(S)$.

Proof. We set $\mathcal{F} = \mathcal{F}(S), \mathcal{G} = \mathcal{G}(S), \mathcal{H} = \mathcal{C} \cap \mathcal{G}(S)$, and $\mathcal{P} = \mathcal{C} \cap \mathcal{F}(S)$. By Theorem 4, conditions (i) and (ii) imply that $\mathcal{C}$ (and therefore $\langle \mathcal{E}(S) \rangle$) is finitary connected for $\mathcal{P}$.
Since (iii) implies condition (6) for every $K \in \mathcal{F}(S) \cap \mathcal{K}(S) (\supset \langle \mathcal{E}(S) \rangle)$, $\langle \mathcal{E}(S) \rangle$ is finitary connected for $\mathcal{H}$ according to Lemma 6 b). From Remark 3 the second assertion follows. $\square$

In the following all topological vector spaces are assumed to be Hausdorff.

Theorem 8 *Let S be a topological vector space. Then the paving $\mathcal{C}$ of all convex subsets of S is finitary connected for $\mathcal{C} \cap \mathcal{F}(S)$ and for $\mathcal{C} \cap \mathcal{G}(S)$.*

Proof. According to Remark 3 it is sufficient to show that every polytope is finitary connected for $\mathcal{C} \cap \mathcal{F}(S)$ and for $\mathcal{C} \cap \mathcal{G}(S)$. Therefore we may assume S to be finite dimensional. By Theorem 2 (or by the classical separation theorem) $\mathcal{C} \cap \mathcal{F}(S)$ separates $\mathcal{C} \cap \mathcal{K}(S)$. Together with Remark 6 it follows that the assumptions (i), (ii), and (iii) of Theorem 7 are satisfied. □

The "closed version" of the following example is due to Kołodziejczyk [17]:

Example 14 Let $C_0, \ldots, C_n$ be closed (open) convex subsets of a topological vector space S such that there exist points $s_i \in C_{-i},\ i \in \{0, \ldots, n\}$, with $K \subset \bigcup_{i=0}^{n} C_i$ for the convex hull K of $\{s_0, \ldots, s_n\}$. Then $K \cap \bigcap_{i=0}^{n} C_i \neq \emptyset$.

Proof. This follows from Theorem 8. □

The "closed version" of the following example is implicitely contained in [24]; 3.3, and the open version generalizes [1]; Theorem 3.

Example 15 Let S be a topological vector space, T a convex subset of S and $C_0, \ldots, C_n$ closed (open) convex subsets of S such that $T \subset \bigcup_{i=0}^{n} C_i$ and $T \cap C_{-i} \neq \emptyset$ for every $i \in \{0, \ldots, n\}$. Then $T \cap \bigcap_{i=0}^{n} C_i \neq \emptyset$.

Proof. The assumption of Example 14 is satisfied for arbitrary $s_i \in T \cap C_{-i}$. □

As a special case of Example 14 or 15 we obtain Klee's intersection theorem:

Example 16 (Klee [15], [16]). Let $C_0, \ldots, C_n$ be closed (open) convex subsets of a topological vector space S such that $\bigcup_{i=0}^{n} C_i$ is convex and $C_{-i} \neq \emptyset$ for every $i \in \{0, \ldots, n\}$. Then $\bigcap_{i=0}^{n} C_i \neq \emptyset$.

Example 17 Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$. For $s, t \in \mathbb{S}^1$ with $s \neq -t$ let $\langle s, t \rangle$ denote the minor arc joining s and t, and let $\langle s, -s \rangle = \mathbb{S}^1, s \in \mathbb{S}^1$. Then $(\mathbb{S}^1, \mathcal{F}(\mathbb{S}^1), \langle \cdot, \cdot \rangle)$ and $(\mathbb{S}^1, \mathcal{G}(\mathbb{S}^1), \langle \cdot, \cdot \rangle)$ are interval spaces. Let $\mathcal{C}$ be the paving of all convex subsets of $\mathbb{S}^1$. By Lemma 10, $\mathcal{K} := \{\{s, -s\} : s \in \mathbb{S}^1\}$ separates $\mathcal{C} \cap \mathcal{F}(\mathbb{S}^1)$ and $\mathcal{C} \cap \mathcal{G}(\mathbb{S}^1)$. Obviously, every trace $(\mathcal{C} \cap \mathcal{F}(\mathbb{S}^1)) \cap K, K \in \mathcal{K}$, is an H_2-paving, and therefore, by Lemma 12, $\mathcal{C} \cap \mathcal{F}(\mathbb{S}^1)$ is an H_3-paving. Since $\langle \mathcal{E}(\mathbb{S}^1) \rangle \subset \mathcal{C} \cap \mathcal{F}(\mathbb{S}^1)$, it follows from Lemma 13 that $\mathcal{C}$ is an H_3-paving.

Obviously, there exist closed (or open) convex sets C_0, C_1, C_2 intersecting pairwise with $C_0 \cap C_1 \cap C_2 = \emptyset$ and $C_0 \cup C_1 \cup C_2 = \mathbb{S}^1$. Hence, $\mathbb{S}^1$ is neither $2-$ary connected for $\mathcal{C} \cap \mathcal{F}(\mathbb{S}^1)$ nor for $\mathcal{C} \cap \mathcal{G}(\mathbb{S}^1)$. Together with Remarks 4 and 5 we obtain for $n \in \mathbb{N}$:

$$\mathcal{C} \text{ is an } H_n-\text{paving} \iff \mathcal{C} \setminus \{\mathbb{S}^1\} \text{ is an } H_n-\text{paving} \iff$$

$$\mathcal{C} \text{ is a } K_n-\text{paving} \iff \mathcal{C} \text{ is } n-\text{ary connected} \iff n \geq 3.$$

From Theorem 8 it follows easily that $\mathcal{C} \setminus \{\mathbb{S}^1\}$ is finitary connected for the pavings $\mathcal{C} \cap \mathcal{F}(\mathbb{S}^1)$ and $\mathcal{C} \cap \mathcal{G}(\mathbb{S}^1)$. In particular, the pavings $(\mathcal{C} \cap \mathcal{F}(\mathbb{S}^1)) \setminus \{\mathbb{S}^1\}$ and $(\mathcal{C} \cap \mathcal{G}(\mathbb{S}^1)) \setminus \{\mathbb{S}^1\}$ are finitary connected.

Example 18 Let $\mathbb{Z}$ be the set of integers endowed with the segments $\langle s, t \rangle := \{z \in \mathbb{Z} : \min\{s, t\} \leq z \leq \max\{s, t\}\},\ s, t \in \mathbb{Z}$, and let $\mathcal{C}$ be the paving of all convex subsets of $\mathbb{Z}$. Obviously, $(\mathbb{Z}, \langle \cdot, \cdot \rangle)$ is modular, and therefore, by Example 10, $\mathcal{C}$ is an H_2-paving. The set $\mathbb{Z}$ can be endowed with a topology such that $\mathcal{C} = \mathcal{C}(S)$ [19]. As a neigborhood base of open sets one can take

$$G_n = \begin{cases} \{n\} & : \text{ if } n \text{ is even} \\ \{n-1, n, n+1\} & : \text{ if } n \text{ is odd} \end{cases}$$

A second topology is obtained by interchanging the words "even" and "odd". From Example 9 c) it follows that $\mathcal{C}$ is finitary connected for $\mathcal{C} \cap \mathcal{F}(S)$ and for $\mathcal{C} \cap \mathcal{G}(S)$.

A *tree-like space* is a connected Hausdorff space S in which every two points x and y can be separated by a third point z, i.e., x and y belong to different connected components of $S \setminus \{z\}$.

Example 19 For a locally connected tree-like space S the following holds.

(a) $\mathcal{C}(S)$ is $\cap_a-$closed [30] and therefore an alignment according to Lemma 1 (iii). By Example 9 a), $\mathcal{C}(S) \cap \mathcal{F}(S)$ is an H_2-paving.

(b) ([30], [27];2.10) For $\{x, y\} \in \mathcal{E}^2(S)$ the segment

$$[x, y] := \{x, y\} \cup \{z \in S : z \text{ separates } x \text{ from } y\}$$

is compact, and it is the smallest connected set containing x and y. In particular, $(S, \mathcal{F}(S), [\cdot, \cdot])$ and $(S, \mathcal{G}(S), [\cdot, \cdot])$ are interval spaces, and $[\cdot, \cdot] = \langle \cdot, \cdot \rangle_{\mathcal{C}(S)}$ is the intrinsic segment function in $(S, \mathcal{C}(S))$.

(c) By Lemma 9 b) together with (b) a subset is convex iff it is connected, and the polytopes are of the form $[A] = \bigcup_{x,y \in A}[x, y],\ A \in \mathcal{E}(S)$. In particular, the polytopes are compact.

(d) By (a), (c), and Lemma 13, $\mathcal{C}(S)$ is an H_2-paving, and by Example 10 the segment space $(S, [\cdot, \cdot])$ is modular. With Example 9 d) it follows that $\mathcal{C}(S) \cap \mathcal{F}(S)$ and $\mathcal{C}(S) \cap \mathcal{G}(S)$ are finitary connected. Together with (b), (c), and Remark 3 it follows that $\mathcal{C}(S)$ is finitary connected for $\mathcal{C}(S) \cap \mathcal{F}(S)$.

(e) It is easy to see ([27]; p.25) that $\mathcal{E}^1(S)$ is screened with $\mathcal{C}(S) \cap \mathcal{F}(S)$. Hence, by Theorem 6 together with (a) and (b), $\mathcal{C}(S) \cap \mathcal{F}(S)$ is normal and separated.

Every metric space (S, d) can be endowed with the *geodesic segments*

$$\langle x, y \rangle_d = \{s \in S : d(x, s) + d(s, y) = d(x, y)\}, \; x, y \in S.$$

Here the convex subsets will be called *$d-$convex*, and a function $f : S \to \mathbb{R}$ is *$d-$affine* provided that

$$f(s_0)d(s_1, s_2) = f(s_1)d(s_0, s_2) + f(s_2)d(s_0, s_1)$$

for all $s_0, s_1, s_2 \in S$ with $s_0 \in \langle s_1, s_2 \rangle_d$.

Let $\mathcal{C}_d$ denote the paving of all $d-$convex subsets of S and let A_d^* denote the linear space of all continuous d–affine functions $f : S \to \mathbb{R}$.

A metric space (S, d) is called *Menger–convex* iff $\langle x, y \rangle_d \setminus \{x, y\} \neq \emptyset$ for all $x, y \in S$ with $x \neq y$. Every nonvoid convex subset of a normed linear space is Menger–convex w.r.t. the induced metric. In [8] various examples of Menger–convex metric spaces in hyperbolic geometry can be found. A classical example is the Poincaré disc.

Example 20 Let (S, d) be a complete Menger–convex metric space. Then, by a theorem of Menger [21, 7], for every pair $s, t \in S$ there exists an isometry $\varphi : [0, d(s, t)] \to S$ with $\varphi(0) = s$ and $\varphi(d(s, t)) = t$. Now $\varphi([0, d(s, t)]) \subset \langle s, t \rangle_d$ implies $\mathcal{C}_d \subset \mathcal{C}(S)$. In particular, the paving $\mathcal{C}_d \cap \mathcal{K}(S)$ is (compact and) connected.
For $f \in \mathsf{A}_d^*$ and $\alpha \in \mathbb{R}$ we have $\{f \leq \alpha\} \in \mathcal{C}_d \cap \mathcal{F}(S)$ and $\{f < \alpha\} \in \mathcal{C}_d \cap \mathcal{G}(S)$. If A_d^* separates points, then it follows with Theorem 2 that $\mathcal{C}_d \cap \mathcal{F}(S)$ and $\mathcal{C}_d \cap \mathcal{G}(S)$ both separate $\mathcal{C}_d \cap \mathcal{K}(S)$, and that $\mathcal{C}_d \cap \mathcal{K}(S)$ is finitary connected for $\mathcal{C}_d \cap \mathcal{F}(S)$.

The segment function $\langle \cdot, \cdot \rangle_d$ is modular iff every triple of pairwise intersecting closed balls has a common point ([28]; p.134, [29]; p.32). In this case, by Example 10, $\mathcal{C}_d$ is an H_2-paving.

A metric space (S, d) is *hyperconvex* iff any family $\{B(x_i, r_i)\}$ of closed balls in S satisfying $d(x_i, x_j) \leq r_i + r_j$ has a nonvoid intersection.

The Nachbin–Kelley–Goodner–Hasumi theorem ([18]; p. 92) states that a Banach space is hyperconvex iff it is linearly isomorphic to some $C(\Omega)$, where Ω is a stonian space. In particular, l_∞ is hyperconvex.

Example 21 Let (S, d) be a hyperconvex metric space, and let $\mathcal{A}$ denote the paving of arbitrary intersections of closed balls (the paving of 'admissible' subsets of S). Then the following holds.

(a) Obviously, $\mathcal{A}$ is a compact $\cap_a-$closed H_2-paving. In particular, the intrinsic segment function $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ is modular.

(b) (S,d) is complete and Menger–convex, and every subspace $(T,d), T \in \mathcal{A}$, is hyperconvex [11], and therefore $\mathcal{A} \subset \mathcal{C}_d$ by Example 20.

(c) By (a), (b), and by Examples 5 and 9 d), the paving $\mathcal{A}$ is finitary connected, and $(S,\mathcal{A},\langle\cdot,\cdot\rangle_{\mathcal{A}})$ is an interval space.

5 Concluding remark

Our results on finitary connectedness and separation can be applied in many fields of pure and applied mathematics. Among others, they can be used to obtain intersection theorems, KKM-type theorems, coincidence theorems, minimax theorems, existence theorems for an Euler characteristic, etc. These topics will be treated elsewhere.

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