Hölder continuous homomorphisms between infinite-dimensional Lie groups are smooth

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Abstract. Let $f : G \to H$ be a homomorphism between smooth Lie groups modelled on Mackey complete, locally convex real topological vector spaces. We show that if f is Hölder continuous at 1, then f is smooth.

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Introduction

While specific examples of infinite-dimensional Lie groups have been studied extensively and are well understood, in the general theory of infinite-dimensional Lie groups even very fundamental questions are still open. Various important unsolved problems were recorded in the preprint [17] by John Milnor in 1982; most of them have resisted all attempts at a solution so far.[1] In the present article, we give a partial answer to Milnor's third problem: "Is a continuous homomorphism between Lie groups necessarily smooth ?" As our main result, we show that every *Hölder* continuous homomorphism is smooth. More precisely:

Main Theorem. *Let $f : G \to H$ be a homomorphism between Lie groups modelled on real locally convex spaces. If f is Hölder continuous at* 1 *and the modelling space of H is Mackey complete, then f is smooth.*

In particular, the Main Theorem applies to *Lipschitz* continuous homomorphisms. Milnor only considered Lie groups modelled on complete locally convex spaces. Mackey completeness is a very natural and useful weakened completeness condition [14].

A simple special case. The basic idea underlying our approach is most easily explained for one-parameter groups. It is helpful to keep this simplest special case in mind as a guideline also when dealing with the general case (which is much harder). Thus, consider a continuous homomorphism $\xi : \mathbb{R} \to G$ from $\mathbb{R}$ to a Lie group G modelled on a locally convex space E. Using a chart, we identify an open identity neighbourhood of G with an

[1] Milnor's second problem (does every closed subalgebra correspond to an immersed Lie subgroup ?) had in fact already been solved earlier by Omori [19] (in the negative). The other three main problems remain open. Various smaller problems mentioned in Milnor's preprint could be settled: A Lie group whose exponential map is a local diffeomorphism at 0 need not be of Campbell-Hausdorff type [20, §3.4.1]. A real or complex analytic Lie group need not be of Campbell-Hausdorff type [7, Rem. 4.7 (b)]. The complexification of an enlargible real Banach-Lie algebra need not be enlargible [10, Ex. VI.4]. A connected Lie group modelled on a locally convex space is abelian if and only if its Lie algebra is abelian [5, Prop. 22.15].

open 0-neighbourhood in E. Making use of the first order Taylor expansion

$$x^2 = 2x + R(x)$$

of the squaring map around the identity 0, for small $t \in \mathbb{R}$ we obtain $\xi(t) = \xi(\frac{1}{2}t)^2 = 2\xi(\frac{1}{2}t) + R(\xi(\frac{1}{2}t))$ and thus $\xi(\frac{1}{2}t) = \frac{1}{2}\xi(t) - \frac{1}{2}R(\xi(\frac{1}{2}t))$. Applying this formula twice yields

$$\xi(\tfrac{1}{4}t) \;=\; \tfrac{1}{2}\xi(\tfrac{1}{2}t) - \tfrac{1}{2}R(\xi(\tfrac{1}{4}t)) \;=\; \tfrac{1}{4}\xi(t) - \tfrac{1}{4}R(\xi(\tfrac{1}{2}t)) - \tfrac{1}{2}R(\xi(\tfrac{1}{4}t))\,.$$

Similarly, $\xi(2^{-n}t) = 2^{-n}\xi(t) - \sum_{k=1}^{n} 2^{k-n-1}R(\xi(2^{-k}t))$ for all $n \in \mathbb{N}$, by induction. After re-parametrizing ξ, we may assume that $t = 1$ can be chosen here. This gives

$$\frac{\xi(2^{-n})}{2^{-n}} \;=\; \xi(1) - \sum_{k=1}^{n} 2^{k-1}R(\xi(2^{-k})) \qquad \text{for all } n \in \mathbb{N}. \tag{1}$$

Now assume that ξ is Hölder continuous at 0, with Hölder exponent $\alpha \in \,]0,1]$. Then $\xi(2^{-k})$ is of order $\mathcal{O}(2^{-k\alpha})$ (as $k \to \infty$). A first order Taylor remainder being at most quadratic in the order of its argument, we see that $R(\xi(2^{-k}))$ is of order $\mathcal{O}(2^{-2k\alpha})$. Therefore the summands $2^{k-1}R(\xi(2^{-k}))$ in (1) are of order $\mathcal{O}(2^{(1-2\alpha)k})$. If $\alpha \in \,]\frac{1}{2},1]$, the preceding estimates show that $n \mapsto \sum_{k=1}^{n} 2^{k-1}R(\xi(2^{-k}))$ is a Mackey-Cauchy sequence in E and hence convergent if E is Mackey complete. Thus $\lim_{n\to\infty} \frac{\xi(2^{-n})}{2^{-n}}$ exists in E, and apparently this limit gives us a candidate for $\xi'(0)$. Of course, it remains to show that $\xi'(0)$ exists, and that existence of $\xi'(0)$ entails smoothness of ξ. Also, it remains to remove the requirement that $\alpha > \frac{1}{2}$ (but all of this can be done).

Organization of the paper. After a brief description of the setting of differential calculus used in the paper, in Section 1 we discuss various properties a mapping between open subsets of locally convex spaces (or manifolds) can have at a given point: Hölder continuity at x, total differentiability at x, and feeble differentiability (an auxiliary notion which we introduce for internal use). In Section 2, we show that C^1-homomorphisms between Lie groups modelled on real locally convex spaces are smooth (Lemma 2.1), and we show that a homomorphism is C^1 if it is totally (or merely feebly) differentiable at 1 (Lemma 2.2). Section 3 is devoted to the proof of the Main Theorem (Theorem 3.2). In view of the reduction steps already performed, the crucial point will be to deduce total differentiability at 1 from Hölder continuity at 1. Although our main result concerns real Lie groups, some of our considerations are not restricted to the real case and have been formulated more generally for complete valued fields. This enables us to show in Section 4 that Hölder continuous homomorphisms between p-adic Lie groups are C^1 (Theorem 4.1). Proofs for various auxiliary results, which are best taken on faith on a first reading, are compiled in two appendices.

Analogues in convenient differential calculus. In the subsequent paper [9], variants of the ideas presented here are used to show that every $\mathcal{L}ip^0$-homomorphism between Lie groups in the sense of convenient differential calculus (as in [14]) is smooth in the convenient sense. More generally, this conclusion holds for "conveniently Hölder" homomorphisms [9].

1 Basic definitions and facts

We compile and develop basic material. The proofs are recorded in Appendix A.

Differential calculus in topological vector spaces

We are working in the framework of differential calculus known as Keller's C^∞_c-theory [13] (going back to Michal and Bastiani), as used in [4], [11], [16], [17], [18] and generalized to a differential calculus over topological fields in [2]. We recall some of the basic ideas.

1.1 Let E be a real topological vector space, F be a real locally convex space, and $U \subseteq E$ be open. A map $f : U \to F$ is called C^1 if it is continuous, the directional derivative $df(x,y) := \frac{d}{dt}\big|_{t=0} f(x+ty)$ exists for all $x \in U$ and $y \in E$, and the mapping $df\colon U \times E \to F$ so obtained is continuous. Inductively, we say that f is C^{k+1} (for $k \geq 1$) if f is C^1 and $df\colon U \times E \to F$ is C^k. The map f is called C^∞ or *smooth* if it is C^k for all $k \in \mathbb{N}$.

1.2 If $f : E \supseteq U \to F$ as before is C^1, define $f^{[1]} : U^{[1]} \to F$ on the open set $U^{[1]} := \{(x,y,t) \in U \times E \times \mathbb{R} \colon x + ty \in U\} \subseteq E \times E \times \mathbb{R}$ via $f^{[1]}(x,y,t) = \frac{1}{t}(f(x+ty) - f(x))$ if $t \neq 0$, $f^{[1]}(x,y,0) := df(x,y)$. Then $f^{[1]}$ is continuous, because for small t we have the integral representation $f^{[1]}(x,y,t) = \int_0^1 df(x + sty, y)\, ds$, by the Mean Value Theorem. Furthermore, by definition of $f^{[1]}$,

$$f^{[1]}(x,y,t) = \frac{1}{t}(f(x+ty) - f(x)) \qquad \text{for all } (x,y,t) \in U^{[1]} \text{ such that } t \neq 0. \tag{2}$$

If, conversely, $f\colon U \to F$ is continuous and (2) holds for a continuous map $f^{[1]}\colon U^{[1]} \to F$, then f is C^1, with $df(x,y) = \lim_{t\to 0} t^{-1}(f(x+ty) - f(x)) = \lim_{t \to 0} f^{[1]}(x,y,t) = f^{[1]}(x,y,0)$.

The preceding characterization of C^1-maps is a useful tool for various purposes. Beyond the real case, the characterizing property just described can be used to *define* C^1-maps [2]:

1.3 Let E and F be (Hausdorff) topological vector spaces over a topological field $\mathbb{K}$ (which we always assume Hausdorff and non-discrete), and $U \subseteq E$ be open. Let $U^{[1]} := \{(x,y,t) \in U \times E \times \mathbb{K} \colon x + ty \in U\}$. A map $f\colon U \to F$ is called C^1 if it is continuous and there exists a (necessarily unique) continuous map $f^{[1]}\colon U^{[1]} \to F$ such that (2) holds. Inductively, f is called C^{k+1} for $k \in \mathbb{N}$ if f is C^1 and $f^{[1]}\colon U^{[1]} \to F$ is C^k. The map f is C^∞ or *smooth* if it is C^k for all $k \in \mathbb{N}$. We write $C^k_{\mathbb{K}}$ for C^k if we wish to emphasize the ground field.

By [2, Prop. 7.4], the definitions of C^k-maps given in **1.1** and **1.3** are equivalent for maps into real locally convex spaces. Compositions of C^k-maps being C^k [2, Prop. 4.5], manifolds and (smooth) Lie groups modelled on topological $\mathbb{K}$-vector spaces can be defined in the usual way. For further information, see [18] (real case) and [2]. Examples of infinite-dimensional Lie groups over topological fields can be found in [8].

1.4 A *valued field* is a field $\mathbb{K}$, equipped with an absolute value $|.|\colon \mathbb{K} \to [0,\infty[$ (see [21]); we require furthermore that the absolute value be non-trivial (i.e., the corresponding metric defines a non-discrete topology on $\mathbb{K}$). Every valued field is, in particular, a topological field. A topological vector space E over a valued field $\mathbb{K}$ is called *polynormed* if its vector topology arises from a family of continuous seminorms $q\colon E \to [0,\infty[$. Thus polynormed vector spaces over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ are the usual locally convex spaces. We also write $\|.\|_q := q$, for better readability. Given $x \in E$ and $r > 0$, we let $B^q_r(x) := \{y \in E\colon \|y - x\|_q < r\}$ be the open q-ball of radius r around x.

Our studies hinge on Taylor's formula [2, Thm. 5.1]:

Proposition 1.5 *If $k \in \mathbb{N}$ and $f\colon E \supseteq U \to F$ is C^k, then there are continuous functions $a_j\colon U \times E \to F$ for $j = 1,\ldots,k$ and a continuous function $R_k\colon U^{[1]} \to F$ such that*

$$f(x+ty) - f(x) = \sum_{j=1}^{k} t^j a_j(x,y) \; + \; t^k R_k(x,y,t) \qquad \text{for all } (x,y,t) \in U^{[1]}$$

and $R_k(x,y,0) = 0$ for all $(x,y) \in U \times E$. The functions a_j and R_k are uniquely determined, $a_j(x,\bullet)$ is homogeneous of degree j, and $j!a_j(x,y) = d^j f(x,y,\ldots,y)$ for all $(x,y) \in U \times E$. □

Here $d^j f : U \times E^j \to F$ denotes the jth differential of f, defined in terms of iterated directional derivatives via $d^j f(x,y_1,\ldots,y_j) := (D_{y_1} \cdots D_{y_j} f)(x)$.

Lemma 1.6 *Let E and F be polynormed vector spaces over a valued field $\mathbb{K}$ and $f\colon U \to F$ be a C^2-map on an open subset $U \subseteq E$. Let $x_0 \in U$, q be a continuous seminorm on F, and $C > 0$. Then there exists a continuous seminorm p on E such that $B^p_2(x_0) \subseteq U$ and $\|f(x+y) - f(x) - df(x,y)\|_q = \|R_1(x,y,1)\|_q \leq C\,\|y\|^2_p$ for all $x \in B^p_1(x_0)$ and $y \in B^p_1(0)$.*

Hölder continuity at a point

Until **1.15**, $\mathbb{K}$ denotes a valued field.

Definition 1.7 Let E and F be polynormed $\mathbb{K}$-vector spaces, $x \in E$, $U \subseteq E$ be a neighbourhood of x, $f\colon U \to F$ be a map, and $\alpha \in\,]0,1]$. We say that f is *Hölder continuous of degree* (or Hölder exponent) α *at* x (for short: f is H_α at x) if, for every continuous seminorm q on F, there exist $\delta > 0$, $C > 0$ and a continuous seminorm p on E such that $B^p_\delta(x) \subseteq U$ and

$$\|f(y) - f(x)\|_q \leq C\,(\|y - x\|_p)^\alpha \qquad \text{for all } y \in B^p_\delta(x). \tag{3}$$

If f is H_1 at x, we also say that f is *Lipschitz continuous at* x. We say that f is *Hölder continuous at* x if f is H_α at x for some $\alpha \in\,]0,1]$.

Remark 1.8 Replacing p with $\max\left\{\delta^{-1}, C^{\frac{1}{\alpha}}\right\} \cdot p$, we can always achieve that $C = \delta = 1$.

Lemma 1.9 *For maps between subsets of polynormed $\mathbb{K}$-vector spaces, we have:*

(a) *If f is H_α at x then f is continuous at x.*

(b) *If $\alpha \geq \beta$ and f is H_α at x, then f is H_β at x.*

(c) *Any C^1-map is Lipschitz continuous at each point.*

(d) *If f is H_α at x and g is H_β at $f(x)$, then $g \circ f$ is $H_{\alpha \cdot \beta}$ at x.*

Definition 1.10 Let $f\colon M \to N$ be a map between $C^1_{\mathbb{K}}$-manifolds modelled on polynormed $\mathbb{K}$-vector spaces, and $\alpha \in \,]0,1]$. We say that f is *Hölder continuous of degree α at $x \in M$* (or briefly: F is H_α at x), if f is continuous at x and there are a chart $\phi : U_1 \to U$ of M around x and a chart $\psi\colon V_1 \to V$ of N around $f(x)$, such that $\phi(f^{-1}(V_1) \cap U_1) \to V$, $y \mapsto \psi(f(\phi^{-1}(y)))$ is H_α at $\phi(x)$. (This then holds for any choice of ϕ and ψ, by La. 1.9).

Notions of differentiability at a point

1.11 (Cf. [15, I, §3]). Let E and F be topological $\mathbb{K}$-vector spaces, $x \in E$, and $f\colon U \to F$ be a map defined on a neighbourhood U of x in E. The map f is called *totally differentiable at x* if there is a (necessarily unique) continuous linear map $f'(x)\colon E \to F$ such that

$$h\colon U - x \to F, \qquad h(y) := f(x+y) - f(x) - f'(x).y$$

is *tangent to* 0 in the sense that, for every 0-neighbourhood $W \subseteq F$, there is a 0-neighbourhood $V \subseteq E$ and a function $\theta\colon I \to \mathbb{K}$ defined on some 0-neighbourhood $I \subseteq \mathbb{K}$ such that $I \cdot V \subseteq U - x$, $\theta(t) = o(t)$ (i.e., $\theta(0) = 0$ and $\lim_{t\to 0} \theta(t)/t = 0$), and

$$h(tV) \subseteq \theta(t)W \qquad \text{for all } t \in I.$$

1.12 If E and F are polynormed, then h as before is tangent to 0 if and only if, for every continuous seminorm q on F, there exists a continuous seminorm p on E such that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $B^p_\delta(0) \subseteq U - x$ and

$$\|h(y)\|_q \leq \varepsilon \|y\|_p \qquad \text{for all } y \in B^p_\delta(0).$$

1.13 The Chain Rule holds: If $f : E \supseteq U \to F$ is totally differentiable at x and the map $g : F \supseteq V \to H$ is totally differentiable at $f(x)$ and $f(U) \subseteq V$, then $g \circ f : U \to H$ is totally differentiable at x, with $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$.

Lemma 1.14 *Let E and F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be an open subset, and $f\colon U \to F$ be a C^2-map. Then f is totally differentiable at each $x \in U$, and $f'(x) = df(x, \bullet)$.*

1.15 Given $r \in \mathbb{N} \cup \{\infty\}$, a map $f\colon M \to N$ between C^r-manifolds modelled on topological $\mathbb{K}$-vector spaces, and $x \in M$, we call f *totally differentiable at x* if f is continuous at x and there exist a chart $\phi\colon U_1 \to U$ of M around x and a chart $\psi\colon V_1 \to V$ of N around $f(x)$, such that $\phi(f^{-1}(V_1) \cap U_1) \to V$, $y \mapsto \psi(f(\phi^{-1}(y)))$ is totally differentiable at $\phi(x)$.[2]

[2] If $r \geq 2$, then the latter property is independent of the choice of charts, by the Chain Rule (the chart changes are C^2 and hence totally differentiable at each point by Lemma 1.14).

We find it convenient to work with a certain weaker differentiability property, which even makes sense over arbitrary topological fields:

1.16 Let E and F be topological vector spaces over a topological field $\mathbb{K}$, $U \subseteq E$ be open, $x \in U$, and $f : U \to F$ a continuous map. Let $A := \{(y,t) \in E \times \mathbb{K}^\times : x + ty \in U\}$ and $\widetilde{U}_x := A \cup (E \times \{0\}) \subseteq E \times \mathbb{K}$. We say that f is *feebly differentiable at* x if there is a (unique) continuous linear map $f'(x)\colon E \to F$ making the following map continuous:

$$\widetilde{f}_x\colon \widetilde{U}_x \to F, \qquad (y,t) \mapsto \begin{cases} \frac{f(x+ty)-f(x)}{t} & \text{if } t \neq 0 \\ f'(x).y & \text{if } t = 0. \end{cases}$$

Lemma 1.17 *Let E and F be topological vector spaces over a topological field $\mathbb{K}$, $U \subseteq E$ be open, $f : U \to F$ be a map, and $x \in U$. If f is C^1 or if $\mathbb{K}$ is a valued field, f is continuous on U and totally differentiable at x, then f is feebly differentiable at x.*

1.18 The Chain Rule holds for feebly differentiable maps: If $f : E \supseteq U \to F$ is feebly differentiable at x and $g\colon F \supseteq V \to H$ is feebly differentiable at $f(x)$ and $f(U) \subseteq V$, then $g \circ f\colon U \to H$ is feebly differentiable at x, with $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$.

1.19 A map $f\colon M \to N$ between C^1-manifolds modelled on topological $\mathbb{K}$-vector spaces is called *feebly differentiable at* $x \in M$ if it is continuous at x and $y \mapsto \psi(f(\phi^{-1}(y)))$ is feebly differentiable at $\phi(x)$ for charts ϕ and ψ as in **1.15**.

Cf. [1] for a comparative study of various differentiability properties at a point.

2 Homomorphisms between Lie groups

We prove preparatory results concerning differentiability properties of homomorphisms.

Lemma 2.1 *Let $f\colon G \to H$ be a $C^1_{\mathbb{K}}$-homomorphism between Lie groups over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, where H is modelled on a locally convex space. Then f is $C^\infty_{\mathbb{K}}$.*

Proof. We show that f is C^k for each $k \in \mathbb{N}$, by induction. By hypothesis, f is C^1. Using the trivialization $\tau_G\colon G \times L(G) \to TG$, $\tau_G(g, X) := T_1\lambda_g(X)$ (where $\lambda_g\colon G \to G$, $x \mapsto gx$ denotes left translation by g) and the corresponding trivialization $\tau_H\colon H \times L(H) \to TH$, the tangent map Tf can be expressed as

$$Tf = \tau_H \circ (f \times L(f)) \circ (\tau_G)^{-1} . \tag{4}$$

Since τ_G and τ_H are C^∞-diffeomorphisms and the continuous linear map $L(f)$ is smooth, (4) shows that if f is C^k, then so is Tf. But then f being a C^1-map into a manifold modelled on a *locally convex* space with Tf of class C^k, the map f is C^{k+1} (cf. [2, Prop. 7.4]). □

Lemma 2.2 *Let $f : G \to H$ be a homomorphism between Lie groups modelled on topological vector spaces over a topological field $\mathbb{K}$. Assume that f is feebly differentiable at 1 (this is the case if $\mathbb{K}$ is a valued field and f is totally differentiable at 1). Then f is $C^1_{\mathbb{K}}$. If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and the modelling space of H is locally convex, then f is $C^\infty_{\mathbb{K}}$.*

Proof. We let $\phi\colon U_1 \to U \subseteq L(H)$ be a chart of H around 1, such that $\phi(1) = 0$. There exists an open identity neighbourhood $V_1 \subseteq U_1$ such that $V_1V_1 \subseteq U_1$; let $V := \phi(V_1)$. Then

$$\mu\colon V \times V \to U, \qquad \mu(x, y) := x * y := \phi(\phi^{-1}(x)\phi^{-1}(y))$$

expresses multiplication on H in local coordinates. Let $\psi\colon P_1 \to P \subseteq L(G)$ be a chart of G such that $f(P_1) \subseteq U_1$, $0 \in P$ and $\psi(1) = 0$; let Q_1 and S_1 be open identity neighbourhoods in G such that $Q_1Q_1 \subseteq P_1$, $f(Q_1) \subseteq V_1$, $S_1 = (S_1)^{-1}$, and $S_1S_1 \subseteq Q_1$. Then $Q := \psi(Q_1)$ and $S := \psi(S_1)$ are open 0-neighbourhoods in $L(G)$. Define $\iota : S \to S$, $\iota(x) := x^{-1} := \psi(\psi^{-1}(x)^{-1})$ and $\nu\colon Q \times Q \to P$, $\nu(x, y) := x * y := \psi(\psi^{-1}(x)\psi^{-1}(y))$. Then

$$g := \phi \circ f|_{P_1}^{U_1} \circ \psi^{-1}\colon P \to U$$

maps 0 to 0 and is continuous (since f is continuous, being a homomorphism which is continuous at one point). Furthermore, g is feebly differentiable at 0 by hypothesis (resp., Lemma 1.17). For $(x, y, t) \in S^{]1[} := \{(x, y, t) \in S^{[1]} : t \neq 0\}$, we have

$$\begin{aligned}
t^{-1}(g(x + ty) - g(x)) &= t^{-1}(g(x) * g(x^{-1} * (x + ty)) - g(x)) \\
&= t^{-1}(g(x) * (0 + tt^{-1}g(x^{-1} * (x + ty))) - g(x) * 0) \\
&= \mu^{[1]}((g(x), 0),\ (0, t^{-1}g(x^{-1} * (x + ty))),\ t) \\
&= \mu^{[1]}((g(x), 0),\ (0, t^{-1}g(th(x, y, t))),\ t) \\
&= \mu^{[1]}((g(x), 0),\ (0, \widetilde{g}_0(h(x, y, t), t)),\ t)
\end{aligned}$$

where $h : S^{[1]} \to L(G)$, $h(x, y, t) := \nu^{[1]}((x^{-1}, x),\ (0, y),\ t)$ is continuous, and so is the map $\widetilde{g}_0 : \widetilde{P}_0 \to L(H)$ (defined as in **1.16**). Note that $F : S^{[1]} \to L(H)$, $F(x, y, t) := \mu^{[1]}((g(x), 0),\ (0, \widetilde{g}_0(h(x, y, t), t)),\ t)$ makes sense on all of $S^{[1]}$. The map F is continuous and, by the preceding, we have $F(x, y, t) = \frac{1}{t}(g(x + ty) - g(x))$ for all $(x, y, t) \in S^{]1[}$. Thus $g|_S$ is C^1, with $(g|_S)^{[1]} = F$. Hence $f|_{S_1}$ is C^1 and hence so is f on all of G, by [6, La. 3.1]. If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $L(H)$ is locally convex, this entails that f is C^∞ (Lemma 2.1). □

3 Hölder continuous homomorphisms are smooth

In this section, which is the core of the article, we establish the main result.

Definition 3.1 A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological vector space E over a valued field $\mathbb{K}$ is called *Mackey-Cauchy* if there exists a bounded subset $B \subseteq E$ and elements $\mu_{n,m} \in \mathbb{K}$ such that $x_n - x_m \in \mu_{n,m}B$ for all $n, m \in \mathbb{N}$ and $\mu_{n,m} \to 0$ as both $n, m \to \infty$ (cf. [14, p. 14]). We say that E is *Mackey complete* if every Mackey-Cauchy sequence in E is convergent (cf. [14, La. 2.2]).

Theorem 3.2 *Let $f : G \to H$ be a homomorphism between smooth Lie groups modelled on locally convex, real topological vector spaces. If the modelling space of H is Mackey complete and f is Hölder continuous at 1, then f is smooth.*

Proof. By hypothesis, f is H_α at 1 for some $\alpha \in]0,1]$. The proof proceeds in two steps. The first goal is to show that if $\alpha \in]0, \frac{1}{2}]$, then f also is $H_{\frac{3}{2}\alpha}$ at 1. Since the Hölder exponent can be improved repeatedly, this means that f actually is H_α at 1 with $\alpha \in]\frac{1}{2}, 1]$. Having achieved this, the second goal will be to show that f is totally differentiable at 1 and hence smooth, by Lemma 2.2.

For the moment, we only know that $\alpha \in]0,1]$. We let $\phi\colon U_1 \to U \subseteq L(H)$ be a chart of H around 1, such that $\phi(1) = 0$. There exist open, symmetric[3] identity neighbourhoods $V_1 \subseteq U_1$ and $W_1 \subseteq V_1$ such that $V_1V_1 \subseteq U_1$ and $W_1W_1 \subseteq V_1$; let $V := \phi(V_1)$ and $W := \phi(W_1)$. Then

$$\mu\colon V \times V \to U, \qquad \mu(x,y) := x * y := \phi(\phi^{-1}(x)\phi^{-1}(y))$$

expresses the multiplication of H in local coordinates. Products of more than two elements are formed from left to right; for example, $x{*}y{*}z := (x{*}y){*}z$. Of course, $(x{*}y){*}z = x{*}(y{*}z)$ whenever both products are defined (and likewise for products of more than three factors). Since $0 * 0 = 0$ and $\mu'(0,0).(x,y) = x + y$, the map

$$\sigma\colon W \times W \to U, \quad \sigma(x,y) := x * x * y$$

satisfies $\sigma(0,0) = 0$ and $\sigma'(0,0)(u,v) = 2u + v$ for $u, v \in L(H)$. Hence, using the Taylor expansion of σ about $(0,0)$, we have

$$\sigma(x,y) = 2x + y + R(x,y) \qquad \text{for all } x, y \in W,$$

where $R(x,y) := R_1((0,0),(x,y),1)$ (cf. Proposition 1.5). Let $\psi\colon P_1 \to P \subseteq L(G)$ be a chart of G around 1, such that $f(P_1) \subseteq U_1$ and $\psi(1) = 0$; let $Q_1 \subseteq P_1$ and $B_1 \subseteq Q_1$ be symmetric identity neighbourhoods such that $Q_1Q_1 \subseteq P_1$, $B_1B_1 \subseteq Q_1$, $f(Q_1) \subseteq V_1$, and $f(B_1) \subseteq W_1$. Set $Q := \psi(Q_1)$ and $B := \psi(B_1)$. Define $\iota : Q \to Q$, $\iota(x) := x^{-1} := \psi(\psi^{-1}(x)^{-1})$ and $\nu\colon Q \times Q \to P$, $\nu(x,y) := x * y := \psi(\psi^{-1}(x)\psi^{-1}(y))$. Then

$$g := \phi \circ f|_{P_1}^{U_1} \circ \psi^{-1}\colon P \to U$$

is continuous, maps 0 to 0, and is H_α at 0.

We now adapt the ideas explained in the Introduction for the special case of one-parameter groups to the present, fully general situation. To this end, let $A \subseteq B$ be a balanced, open 0-neighbourhood such that $A * A \subseteq B$, $\iota(A) * \iota(A) \subseteq B$, and $\iota(A) * \iota(A) * A \subseteq g^{-1}(W)$. We abbreviate $(\frac{1}{2}x)^{-2} := \iota(\frac{1}{2}x) * \iota(\frac{1}{2}x)$ for $x \in A$ and define

$$h\colon A \to W, \quad h(x) := g((\tfrac{1}{2}x)^{-2} * x)\,. \tag{5}$$

[3] Recall that an identity neighbourhood X is *symmetric* if $X = X^{-1}$.

We have $g(x) = g((\frac{1}{2}x)^2 * (\frac{1}{2}x)^{-2} * x) = g(\frac{1}{2}x)^2 * g((\frac{1}{2}x)^{-2} * x) = \sigma(g(\frac{1}{2}x), g((\frac{1}{2}x)^{-2}x)) = 2g(\frac{1}{2}x) + g((\frac{1}{2}x)^{-2}x) + R(g(\frac{1}{2}x), g((\frac{1}{2}x)^{-2}x))$ for $x \in A$ and hence

$$g(\tfrac{1}{2}x) = \tfrac{1}{2}g(x) - \tfrac{1}{2}h(x) - \tfrac{1}{2}R(g(\tfrac{1}{2}x), h(x))\,, \tag{6}$$

with h as in (5). Since also $\frac{1}{2}x \in A$, likewise $g(\frac{1}{4}x) = \frac{1}{2}g(\frac{1}{2}x) - \frac{1}{2}h(\frac{1}{2}x) - \frac{1}{2}R(g(\frac{1}{4}x), h(\frac{1}{2}x))$. Inserting the right hand side of (6) for $g(\frac{1}{2}x)$ here, we arrive at

$$g(\tfrac{1}{4}x) = \tfrac{1}{4}g(x) - \tfrac{1}{4}h(x) - \tfrac{1}{4}R(g(\tfrac{1}{2}x), h(x)) - \tfrac{1}{2}h(\tfrac{1}{2}x) - \tfrac{1}{2}R(g(\tfrac{1}{4}x), h(\tfrac{1}{2}x))\,.$$

Proceeding in this way, we obtain

$$g(2^{-n}x) = 2^{-n}g(x) - \sum_{k=1}^{n} 2^{-n+k-1}\left(h(2^{1-k}x) + R(g(2^{-k}x), h(2^{1-k}x))\right) \tag{7}$$

for all $n \in \mathbb{N}_0$, by induction. Hence

$$2^n g(2^{-n}x) = g(x) - \sum_{k=1}^{n} 2^{k-1}\left(h(2^{1-k}x) + R(g(2^{-k}x), h(2^{1-k}x))\right) \tag{8}$$

for all $x \in A$ and $n \in \mathbb{N}_0$. The following lemma provides estimates on the summands in (8); later, these estimates will be used to show that the series is summable (see (18)).

Lemma 3.3 *Let q be a continuous seminorm on $L(H)$. Then there exists a continuous seminorm p on $L(G)$ such that $B_1^p(0) \subseteq A$,*

$$\|h(x) + R(g(\tfrac{1}{2}x), h(x))\|_q \leq \|x\|_p^{2\alpha} \qquad \textit{for all } x \in B_1^p(0), \tag{9}$$

and $\|g(x)\|_q \leq (\|x\|_p)^\alpha$ for all $x \in B_1^p(0)$.

Proof. As a consequence of Lemma 1.6, there exists a continuous seminorm r on $L(H)$ such that $B_1^r(0) \subseteq W$ and

$$\|R(y,z)\|_q \leq \tfrac{1}{2}(\max\{\|y\|_r, \|z\|_r\})^2 \quad \text{for all } y, z \in B_1^r(0)\,; \tag{10}$$

after replacing r with $r + q$, we may assume that $r \geq q$. Since g is H_α at 0, there is a continuous seminorm s on $L(G)$ such that $B_1^s(0) \subseteq P$, $g(B_1^s(0)) \subseteq B_1^r(0)$, and

$$\|g(x)\|_r \leq \tfrac{1}{2}(\|x\|_s)^\alpha \qquad \text{for all } x \in B_1^s(0). \tag{11}$$

We now consider the smooth map $j : A \to Q$, $j(x) := (\frac{1}{2}x)^{-2} * x$. Then $j(0) = 0$ and $j'(0) = 0$, entailing that there exists a continuous seminorm p on $L(G)$ such that $B_1^p(0) \subseteq A$, $j(B_1^p(0)) \subseteq B_1^s(0)$, and

$$\|j(x)\|_s \leq (\|x\|_p)^2 \qquad \text{for all } x \in B_1^p(0) \tag{12}$$

(cf. Lemma 1.6); we may assume that $p \geq s$. Then $\|h(x)\|_q \leq \|h(x)\|_r = \|g(j(x))\|_r \leq \frac{1}{2}(\|j(x)\|_s)^\alpha \leq \frac{1}{2}(\|x\|_p)^{2\alpha}$ for all $x \in B_1^p(0)$, by (11) and (12). Also $\|g(\frac{1}{2}x)\|_r \leq \frac{1}{2}(\|\frac{1}{2}x\|_s)^\alpha \leq (\|x\|_s)^\alpha \leq (\|x\|_p)^\alpha$ and $\|h(x)\|_r \leq \frac{1}{2}(\|x\|_p)^{2\alpha} \leq (\|x\|_p)^\alpha$, whence $\|R(g(\frac{1}{2}x), h(x))\|_q \leq \frac{1}{2}(\|x\|_p)^{2\alpha}$, by (10). Using the preceding estimates, we obtain $\|h(x) + R(g(\frac{1}{2}x), h(x))\|_q \leq \|h(x)\|_q + \|R(g(\frac{1}{2}x), h(x))\|_q \leq \frac{1}{2}(\|x\|_p)^{2\alpha} + \frac{1}{2}(\|x\|_p)^{2\alpha} = (\|x\|_p)^{2\alpha}$ for all $x \in B_1^p(0)$. Thus (9) holds. We also have $\|g(x)\|_q \leq \|g(x)\|_r \leq \frac{1}{2}(\|x\|_s)^\alpha \leq (\|x\|_s)^\alpha \leq (\|x\|_p)^\alpha$. □

Lemma 3.4 *If f is H_α at 1 with $\alpha \in \,]0, \frac{1}{2}]$, then f also is $H_{\frac{3}{2}\alpha}$ at 1.*

Proof. Given a continuous seminorm q on $L(H)$, we let p be as in Lemma 3.3. In the following, we show that

$$\|g(y)\|_q \leq K\, 2^{\frac{3}{2}\alpha}(\|y\|_p)^{\frac{3}{2}\alpha} \quad \text{for all } y \in B_1^p(0), \tag{13}$$

for a suitable constant $K \in [0, \infty[$. Thus g will be $H_{\frac{3}{2}\alpha}$ at 0, and hence f will be $H_{\frac{3}{2}\alpha}$ at 1.

Using (7) and the estimates from Lemma 3.3, we obtain

$$\begin{aligned}
\|g(2^{-n}x)\|_q &\leq 2^{-n}\|g(x)\|_q + \sum_{k=1}^{n} 2^{-n+k-1}\|h(2^{1-k}x) + R(g(2^{-k}x), h(2^{1-k}x))\|_q \\
&\leq 2^{-n} + \sum_{k=1}^{n} 2^{-n+k-1}(\|2^{1-k}x\|_p)^{2\alpha} \leq 2^{-n} + \sum_{k=1}^{n} 2^{-n+k-1}2^{2\alpha-2\alpha k} \\
&= \left(2^{-(1-\frac{3}{2}\alpha)n} + 2^{2\alpha-1}2^{-(1-\frac{3}{2}\alpha)n}\sum_{k=1}^{n} 2^{(1-2\alpha)k}\right) 2^{-\frac{3}{2}\alpha n}
\end{aligned} \tag{14}$$

for all $x \in B_1^p(0)$ and $n \in \mathbb{N}_0$. Since $\lim_{n\to\infty} 2^{-(1-\frac{3}{2}\alpha)n} = 0$, there is $K_1 \in [0, \infty[$ such that $2^{-(1-\frac{3}{2}\alpha)n} \leq K_1$ for all $n \in \mathbb{N}_0$. The summation formula for the finite geometric series yields

$$\sum_{k=1}^{n} 2^{(1-2\alpha)k} = \frac{2^{(1-2\alpha)(n+1)} - 2^{1-2\alpha}}{2^{1-2\alpha} - 1} \leq \frac{2^{(1-2\alpha)(n+1)}}{2^{1-2\alpha} - 1} = c\, 2^{(1-2\alpha)n}$$

with $c := \frac{2^{1-2\alpha}}{2^{1-2\alpha}-1}$. We therefore obtain the following estimates for the second term in (14):

$$2^{2\alpha-1}2^{-(1-\frac{3}{2}\alpha)n}\sum_{k=1}^{n} 2^{(1-2\alpha)k} \leq c\, 2^{2\alpha-1}2^{-(1-\frac{3}{2}\alpha)n}2^{(1-2\alpha)n} \leq K_2\, 2^{-\frac{1}{2}\alpha n} \leq K_2$$

for all $n \in \mathbb{N}_0$, with $K_2 := c\, 2^{2\alpha-1}$. Using the estimates just established, (14) yields

$$\|g(2^{-n}x)\|_q \leq K\, 2^{-\frac{3}{2}\alpha n} \qquad \text{for all } x \in B_1^p(0) \text{ and } n \in \mathbb{N}_0, \tag{15}$$

with $K := K_1 + K_2$. Then (13) holds with K as just defined. To see this, let $y \in B_1^p(0)$. If $\|y\|_p = 0$, then $\|g(y)\|_q \leq \|y\|_p^\alpha = 0 \leq K 2^{\frac{3}{2}\alpha}\|y\|_p^{\frac{3}{2}\alpha}$, as desired. If $\|y\|_p > 0$, then there exists $n \in \mathbb{N}_0$ such that $2^{-n-1} \leq \|y\|_p < 2^{-n}$. Thus $2^{-n} \leq 2\|y\|_p$. Since $y = 2^{-n}x$ with $x := 2^n y \in B_1^p(0)$, (15) yields

$$\|g(y)\|_q = \|g(2^{-n}x)\|_q \leq K(2^{-n})^{\frac{3}{2}\alpha} \leq K(2\|y\|_p)^{\frac{3}{2}\alpha} = K2^{\frac{3}{2}\alpha}(\|y\|_p)^{\frac{3}{2}\alpha},$$

whence (13) also holds if $\|y\|_p > 0$. This completes the proof of Lemma 3.4. □

If $\alpha \in]0, \frac{1}{2}]$, there exists $k \in \mathbb{N}$ such that $(\frac{3}{2})^{k-1}\alpha \leq \frac{1}{2}$ and $\beta := (\frac{3}{2})^k\alpha \in]\frac{1}{2}, 1]$. Repeated application of Lemma 3.4 shows that f is H_β at 1. After replacing α with β, we may assume throughout the following that $\alpha \in]\frac{1}{2}, 1]$.

In the remainder of the proof, we show that g is totally differentiable at 0. The main point is to construct a candidate Λ for the derivative $g'(0)$. We first construct $\lambda = \Lambda|_A$ on the 0-neighbourhood $A \subseteq L(G)$ (from above).

Lemma 3.5 *The limit $\lambda(x) := \lim_{n\to\infty} \frac{g(2^{-n}x)}{2^{-n}}$ exists in $L(H)$, for each $x \in A$. For each continuous seminorm q on $L(H)$, the convergence of $\frac{g(2^{-n}x)}{2^{-n}}$ in $(L(H), \|.\|_q)$ is locally uniform in x. The map $\lambda\colon A \to L(H)$ is continuous.*

Proof. Fix $x_0 \in A$. Given a continuous seminorm q on $L(H)$, we let p be as in Lemma 3.3. There is $N \in \mathbb{N}$ such that $2^{-N}\|x_0\|_p < 1$. Then $S := B_{2^N}^p(0) \cap A$ is an open neighbourhood of x_0 in A such that $2^{-N}S \subseteq B_1^p(0) \subseteq A$. Abbreviate $C := 2^{2\alpha N}$ and $K := \frac{C2^{2\alpha-1}}{1-2^{-(2\alpha-1)}}$. Let $M \geq N$. For all $m, n \geq M$ (where $m \geq n$, say), using (8) we obtain for all $x \in S$:

$$\begin{aligned}
\left\|2^m g(2^{-m}x) - 2^n g(2^{-n}x)\right\|_q &= \left\|\sum_{k=n+1}^{m} 2^{k-1}\left(h(2^{1-k}x) + R(g(2^{-k}x), h(2^{1-k}x))\right)\right\|_q \\
&\leq \sum_{k=n+1}^{m} 2^{k-1}\|h(2^{1-k}x) + R(g(2^{-k}x), h(2^{1-k}x))\|_q \\
&\leq \underbrace{\|x\|_p^{2\alpha}}_{\leq C} \sum_{k=n+1}^{m} 2^{k-1}2^{2\alpha(1-k)} \leq C\,2^{2\alpha-1}\sum_{k=n+1}^{m} 2^{-(2\alpha-1)k} \qquad (16)\\
&\leq K\cdot(2^{-(2\alpha-1)})^{n+1} \leq K\cdot(2^{-(2\alpha-1)})^{M+1}, \qquad (17)
\end{aligned}$$

using (9) to pass to the third line, then using that $2^{-(2\alpha-1)} < 1$ since $\alpha \in]\frac{1}{2}, 1]$. Here, the final expression tends to 0 as $M \to \infty$, uniformly in $x \in S$.

By the preceding considerations, $(2^n g(2^{-n}x_0))_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $L(H)$ in particular. Hence, if $L(H)$ is sequentially complete, the limit

$$\lambda(x_0) := \lim_{n\to\infty} 2^n g(2^{-n}x_0) = g(x_0) - \sum_{k=1}^{\infty} 2^{k-1}(h(2^{1-k}x_0) + R(g(2^{-k}x_0), h(2^{1-k}x_0))) \qquad (18)$$

exists in $L(H)$. As we shall presently see, the limit also exists when $L(H)$ is Mackey complete. Assuming the validity of this claim for the moment, letting $m \to \infty$ in the lines before (17) we obtain $\big\|\lambda(x) - 2^n g(2^{-n}x)\big\|_q \leq K \cdot (2^{-(2\alpha-1)})^{M+1}$ for all $n \geq M$. Hence $\big\|\lambda(x) - 2^n g(2^{-n}x)\big\|_q \to 0$ uniformly in $x \in S$, proving the second assertion of the lemma. The preceding also entails that λ is continuous.

To complete the proof, it only remains to prove our claim that the limit (18) exists. Since $L(H)$ is Mackey complete, we only need to show that $(v_n)_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence, where $v_n := 2^n g(2^{-n}x_0)$. To this end, pick $a \in \,]2^{-(2\alpha-1)}, 1[$ and define $r_{n,m} := a^{\min\{n,m\}+1}$. Then $r_{n,m} \to 0$ as both $n, m \to \infty$, and

$$v_n - v_m \in r_{n,m}\,\Omega \qquad \text{for all } n, m \in \mathbb{N}\,,$$

where $\Omega := \{r_{n,m}^{-1}(v_n - v_m) : n, m \in \mathbb{N}\}$. If we can show that Ω is bounded in E, then $(v_n)_{n\in\mathbb{N}}$ will be Mackey-Cauchy. To prove boundedness, assume that q is a continuous seminorm on $L(H)$. Let p, N and K be as before. For all $n, m \in \mathbb{N}$, we have, abbreviating $\ell := \max\{N+1, \min\{n,m\}+1\}$:

$$\begin{aligned}
\|r_{n,m}^{-1}(v_n - v_m)\|_q &\leq a^{-\min\{n,m\}-1} \sum_{k=\min\{n,m\}+1}^{\max\{n,m\}} 2^{k-1}\|h(2^{1-k}x_0) + R(g(2^{-k}x_0), h(2^{1-k}x_0))\|_q \\
&\leq C_q + a^{-\min\{n,m\}-1} \sum_{k=\ell}^{\max\{n,m\}} 2^{k-1}\|h(2^{1-k}x_0) + R(g(2^{-k}x_0), h(2^{1-k}x_0))\|_q \\
&\leq C_q + a^{-\min\{n,m\}-1} K (2^{-(2\alpha-1)})^{\ell} \\
&\leq C_q + K(a^{-1}2^{-(2\alpha-1)})^{\min\{n,m\}+1} \leq C_q + K\,,
\end{aligned}$$

where $C_q := a^{-N-1}\sum_{k=2}^{N} 2^{k-1}\|h(2^{1-k}x_0) + R(g(2^{-k}x_0), h(2^{1-k}x_0))\|_q$ is an upper bound for the sum of all terms with $k \leq N$, for which we do not have estimates available. Passing to the third line, we tackled the summands with $k > N$ as in the proof of (17). The final inequality holds because $a^{-1}2^{-(2\alpha-1)} < 1$, by the choice of a. Thus $\|v\|_q \leq C_q + K$ for all $v \in \Omega$, entailing that Ω is indeed bounded. □

Before we can prove that λ extends to a continuous linear map, we need another technical result analogous to Lemma 3.3.

Let $Z \subseteq A$ be an open 0-neighbourhood such that $Z + Z \subseteq A$. We define $j\colon Z \times Z \to Q$, $j(x,y) := y^{-1} * x^{-1} * (x+y)$. Then $j(Z \times Z) \subseteq g^{-1}(W)$. The map $\tau\colon W \times W \times W \to U$, $\tau(x,y,z) := x * y * z$ is smooth, with $\tau(0,0,0) = 0$ and $\tau'(0,0,0)(u,v,w) = u + v + w$ for all $u, v, w \in L(H)$. Let $\widetilde{R}_1\colon (W \times W \times W)^{[1]} \to L(H)$ be the first order Taylor remainder of τ. Abbreviating $D(x,y,z) := \widetilde{R}_1((0,0,0), (x,y,z), 1)$, we then have

$$\tau(x,y,z) = x + y + z + D(x,y,z) \qquad \text{for all } x, y, z \in W. \tag{19}$$

Lemma 3.6 *For every continuous seminorm q on $L(H)$, there is a continuous seminorm p on $L(G)$ such that $B_1^p(0) \subseteq Z$ and*

$$\big\|\, g(j(x,y)) + D\big(g(x),\, g(y),\, g(j(x,y))\big)\,\big\|_q \leq (\max\{\|x\|_p, \|y\|_p\})^{2\alpha} \quad \text{for all } x, y \in B_1^p(0).$$

Proof. There exists a continuous seminorm r on $L(H)$ such that $B_1^r(0) \subseteq W$,

$$\|D(x,y,z)\|_q \leq \tfrac{1}{2}(\max\{\|x\|_r, \|y\|_r, \|z\|_r\})^2 \quad \text{for all } x,y,z \in B_1^r(0)\,, \tag{20}$$

and $r \geq q$ (cf. Lemma 1.6). Since g is H_α at 0, there exists a continuous seminorm s on $L(G)$ such that $B_1^s(0) \subseteq P$, $g(B_1^s(0)) \subseteq B_1^r(0)$, and

$$\|g(x)\|_r \leq \tfrac{1}{2}(\|x\|_s)^\alpha \qquad \text{for all } x \in B_1^s(0). \tag{21}$$

Since j is smooth, $j(0,0) = 0$ and $j'(0,0) = 0$, there exists a continuous seminorm p on $L(G)$ such that $B_1^p(0) \subseteq Z$, $j(B_1^p(0) \times B_1^p(0)) \subseteq B_1^s(0)$, and

$$\|j(x,y)\|_s \leq (\max\{\|x\|_p, \|y\|_p\})^2 \qquad \text{for all } x,y \in B_1^p(0) \tag{22}$$

(cf. Lemma 1.6); we may assume that $p \geq s$. Then $\|g(j(x,y))\|_q \leq \|g(j(x,y))\|_r \leq \frac{1}{2}(\|j(x,y)\|_s)^\alpha \leq \frac{1}{2}(\max\{\|x\|_p, \|y\|_p\})^{2\alpha}$ for all $x,y \in B_1^p(0)$, by (21) and (22). Furthermore, $\|g(x)\|_r \leq \frac{1}{2}(\|x\|_s)^\alpha \leq (\|x\|_s)^\alpha \leq (\|x\|_p)^\alpha$, likewise $\|g(y)\|_r \leq (\|y\|_p)^\alpha$, and $\|g(j(x,y))\|_r \leq \frac{1}{2}(\max\{\|x\|_p, \|y\|_p\})^{2\alpha} \leq (\max\{\|x\|_p, \|y\|_p\})^\alpha$, entailing that $\|D(g(x), g(y), g(j(x,y)))\|_q \leq \frac{1}{2}(\max\{\|x\|_p, \|y\|_p\})^{2\alpha}$, by (20). We now obtain $\|g(j(x,y)) + D(g(x), g(y), g(j(x,y)))\|_q \leq (\max\{\|x\|_p, \|y\|_p\})^{2\alpha}$ for all $x,y \in B_1^p(0)$, using the triangle inequality. □

Lemma 3.7 *There exists a continuous linear map* $\Lambda \colon L(G) \to L(H)$ *such that* $\lambda(x) = \Lambda(x)$ *for all* $x \in A$.

Proof. If we can show that

$$\lambda(x+y) = \lambda(x) + \lambda(y) \qquad \text{for all } x,y \in A \text{ such that } x+y \in A, \tag{23}$$

then, by [12, Cor. A.2.27], the continuous map λ extends to a continuous homomorphism of groups $\Lambda \colon L(G) \to L(H)$. Being a continuous homomorphism between real topological vector spaces, Λ will be continuous linear.

To prove (23), fix $x,y \in A$ such that $x+y \in A$. There is $n_0 \in \mathbb{N}$ such that $2^{-n}x \in Z$ and $2^{-n}y \in Z$ for all $n \geq n_0$. For any such n, (19) shows that

$$\begin{aligned} g(2^{-n}(x+y)) &= g(2^{-n}x + 2^{-n}y) \\ &= g(2^{-n}x) * g(2^{-n}y) * g((2^{-n}y)^{-1} * (2^{-n}x)^{-1} * (2^{-n}x + 2^{-n}y)) \\ &= g(2^{-n}x) * g(2^{-n}y) * g(j(2^{-n}x, 2^{-n}y)) \\ &= g(2^{-n}x) + g(2^{-n}y) + r_n\,, \end{aligned}$$

where $r_n := g(j(2^{-n}x, 2^{-n}y)) + D(g(2^{-n}x), g(2^{-n}y), g(j(2^{-n}x, 2^{-n}y)))$. Thus

$$2^n g(2^{-n}(x+y)) - 2^n g(2^{-n}x) - 2^n g(2^{-n}y) = 2^n r_n \qquad \text{for all } n \geq n_0. \tag{24}$$

Note that the left hand side of (24) converges to $\lambda(x+y) - \lambda(x) - \lambda(y)$ as $n \to \infty$. Hence $\lambda(x+y) = \lambda(x) + \lambda(y)$ will hold if we can show that $2^n r_n \to 0$ in $L(H)$ as $n \to \infty$. To this end, given a continuous seminorm q on $L(H)$, let p be as in Lemma 3.6. There is $n_1 \geq n_0$ such that $2^{-n}x, 2^{-n}y \in B_1^p(0)$ for all $n \geq n_1$. For any such n, the cited lemma yields $\|2^n r_n\|_q = 2^n \|r_n\|_q \leq 2^n (\max\{\|2^{-n}x\|_p, \|2^{-n}y\|_p\})^{2\alpha} \leq \left(2^{-(2\alpha-1)}\right)^n \cdot (\max\{\|x\|_p, \|y\|_p\})^{2\alpha}$, which tends to 0 as $n \to \infty$. Thus $2^n r_n \to 0$. □

Lemma 3.8 *g is totally differentiable at* 0*, with* $g'(0) = \Lambda$.

Proof. Given a continuous seminorm q on $L(H)$, Lemma 3.3 provides a continuous seminorm p on $L(G)$ such that $B_1^p(0) \subseteq A$ and (9) holds. Choosing $n := 0$ and letting $m \to \infty$ in the first half of (16), we find that

$$\|\Lambda(x) - g(x)\|_q \leq c\|x\|_p^{2\alpha} \quad \text{for all } x \in B_1^p(0),$$

where $c := 2^{2\alpha-1} \sum_{k=1}^{\infty} 2^{-(2\alpha-1)k} < \infty$. Since $2\alpha - 1 > 0$, given $\varepsilon > 0$, there exists $\rho \in \,]0,1]$ such that $c\rho^{2\alpha-1} \leq \varepsilon$. Then $B_\rho^p(0) \subseteq A$, and for each $x \in B_\rho^p(0)$ we have

$$\|g(x) - g(0) - \Lambda(x)\|_q = \|g(x) - \Lambda(x)\|_q \leq c\|x\|_p^{2\alpha-1}\|x\|_p \leq c\rho^{2\alpha-1}\|x\|_p \leq \varepsilon\|x\|_p \,.$$

Hence g is totally differentiable at 0, with $g'(0) = \Lambda$. This completes the proof of Lemma 3.8. □

Having proved Lemma 3.8, also Theorem 3.2 is now fully established. □

Note that Lemma 3.4 does not make use of the Mackey completeness of $L(H)$. Beyond the real case (and independent of Mackey completeness of $L(H)$), we still have:

Proposition 3.9 *Let $\mathbb{K}$ be a valued field, $\alpha \in \,]0,1]$, and $f\colon G \to H$ be a homomorphism between Lie groups modelled on polynormed $\mathbb{K}$-vector spaces. Then f is Hölder continuous of degree α at* 1 *if and only if f is Hölder continuous of degree α.*

See Appendix B for the precise definitions and the proof.

4 Homomorphisms between *p*-adic Lie groups

We now formulate a (slightly weaker) analogue of Theorem 3.2 for p-adic Lie groups. The proof carries over rather directly, whence we only indicate the most important changes.

Theorem 4.1 *Let $f\colon G \to H$ be a homomorphism between smooth Lie groups modelled on polynormed $\mathbb{Q}_p$-vector spaces. If f is Hölder continuous at* 1 *and the modelling space of H is Mackey complete, then f is $C^1_{\mathbb{Q}_p}$.*

Proof. By hypothesis, f is H_α at 1 for some $\alpha \in]0,1]$. We let $\phi\colon U_1 \to U \subseteq L(H)$ be a chart of H around 1, such that $\phi(1) = 0$. There exist open, symmetric identity neighbourhoods $V_1 \subseteq U_1$ and $W_1 \subseteq V_1$ such that $V_1V_1 \subseteq U_1$ and $(W_1)^{2p+1} := \underbrace{W_1W_1\cdots W_1}_{2p+1} \subseteq V_1$; let $V := \phi(V_1)$ and $W := \phi(W_1)$. Define

$$\mu\colon V\times V \to U, \qquad \mu(x,y) := x*y := \phi(\phi^{-1}(x)\phi^{-1}(y))\,.$$

Then the k-fold products $x_1*x_2*\cdots*x_k$ are defined (and contained in V), for all $k \leq 2p+1$, $x_1,\ldots,x_k \in W$, and every choice of brackets in this product. The map $\sigma\colon W\times W \to U$, $\sigma(x,y) := x^p * y$ satisfies $\sigma(0,0) = 0$ and $\sigma'(0,0)(u,v) = pu + v$ for $u, v \in L(H)$. The Taylor expansion around $(0,0)$ yields

$$\sigma(x,y) = px + y + R(x,y) \qquad \text{for all } x, y \in W,$$

where $R(x,y) := R_1((0,0),(x,y),1)$. Let $\psi\colon P_1 \to P \subseteq L(G)$ be a chart of G around 1, such that $f(P_1) \subseteq U_1$ and $\psi(1) = 0$; let $Q_1 \subseteq P_1$ and $B_1 \subseteq Q_1$ be symmetric identity neighbourhoods such that $Q_1Q_1 \subseteq P_1$, $f(Q_1) \subseteq V_1$, $(B_1)^{2p+1} \subseteq Q_1$, and $f(B_1) \subseteq W_1$. Set $Q := \psi(Q_1)$ and $B := \psi(B_1)$. Define $\nu\colon Q\times Q \to P$, $\nu(x,y) := x*y := \psi(\psi^{-1}(x)\psi^{-1}(y))$. Then $g := \phi \circ f|_{P_1}^{U_1} \circ \psi^{-1}\colon P \to U$ is continuous, maps 0 to 0, and is H_α at 0. Let $A \subseteq B$ be a balanced, open 0-neighbourhood such that $g(x^{-p} * px) \in W$ for all $x \in A$. We define

$$h\colon A \to W, \quad h(x) := g(x^{-p} * px)\,. \tag{25}$$

For $x \in A$, we have $g(px) = g(x^p * (x^{-p} * px)) = g(x)^p * g(x^{-p} * px) = \sigma(g(x), h(x)) = pg(x) + h(x) + R(g(x), h(x))$. Likewise, $g(p^2x) = pg(px) + h(px) + R(g(px), h(px)) = p^2g(x) + ph(x) + pR(g(x), h(x)) + h(px) + R(g(px), h(px))$ and similarly

$$g(p^nx) = p^ng(x) + \sum_{k=1}^{n} p^{n-k}\left(h(p^{k-1}x) + R(g(p^{k-1}x), h(p^{k-1}x))\right) \tag{26}$$

for all $x \in A$ and $n \in \mathbb{N}_0$, by induction. Hence

$$\frac{g(p^nx)}{p^n} = g(x) + \sum_{k=1}^{n} p^{-k}\left(h(p^{k-1}x) + R(g(p^{k-1}x), h(p^{k-1}x))\right) \tag{27}$$

for all $x \in A$ and $n \in \mathbb{N}_0$. As in the proof of Lemma 3.3, we see:

Lemma 4.2 *Let q be a continuous seminorm on $L(H)$. Then there exists a continuous seminorm b on $L(G)$ such that $B_1^b(0) \subseteq A$,*

$$\|h(x) + R(g(x), h(x))\|_q \leq \|x\|_b^{2\alpha} \qquad \textit{for all } x \in B_1^b(0),$$

and $\|g(x)\|_q \leq (\|x\|_b)^\alpha$ for all $x \in B_1^b(0)$. □

Using Lemma 4.2, we obtain by a simple adaptation of the proof of Lemma 3.4 (where now $p \in \mathbb{Q}_p$ with $|p| = p^{-1}$ plays the role of $\frac{1}{2} \in \mathbb{R}$):

Lemma 4.3 *If f is H_α at 1 with $\alpha \in]0, \frac{1}{2}]$, then f also is $H_{\frac{3}{2}\alpha}$ at 1.* □

By the preceding, we may assume now that $\alpha \in]\frac{1}{2}, 1]$.

Lemma 4.4 *The limit $\lambda(x) := \lim_{n\to\infty} \frac{g(p^n x)}{p^n}$ exists in $L(H)$, for each $x \in A$. For each continuous seminorm q on $L(H)$, the convergence of $\frac{g(p^n x)}{p^n}$ in $(L(H), \|.\|_q)$ is locally uniform in x. The map $\lambda\colon A \to L(H)$ is continuous.*

Proof. The arguments from the real case are easily adapted. To prove that $v_n := p^{-n} g(p^n x_0)$ is a Mackey-Cauchy sequence for $x_0 \in A$, pick $0 < \theta \in \mathbb{Q}$ such that $p^{-\theta} \in]p^{-(2\alpha-1)}, 1[$; set $r_{n,m} := p^{[\theta(\min\{n,m\}+1)]} \in \mathbb{Q}_p$, where $[.]$ is the Gauss bracket (integer part). Thus $|r_{n,m}| = p^{-[\theta(\min\{n,m\}+1)]} \to 0$ as $n, m \to \infty$. Now complete the proof as above. □

Let $Z \subseteq A$ be an open 0-neighbourhood such that $Z + Z \subseteq A$ and $Z^{-1} * Z^{-1} * (Z + Z) \subseteq A$. We define $j : Z \times Z \to A$, $j(x, y) := y^{-1} * x^{-1} * (x + y)$. Then $j(Z \times Z) \subseteq g^{-1}(W)$. The map $\tau\colon W \times W \times W \to U$, $\tau(x, y, z) := x * y * z$ is smooth, with $\tau(0, 0, 0) = 0$ and $\tau'(0, 0, 0)(u, v, w) = u + v + w$ for all $u, v, w \in L(H)$. Let $\widetilde{R}_1 : (W \times W \times W)^{[1]} \to L(H)$ be the first order Taylor remainder of τ. Then $\tau(x, y, z) = x + y + z + D(x, y, z)$ for all $x, y, z \in W$, with $D(x, y, z) := \widetilde{R}_1((0, 0, 0), (x, y, z), 1)$. Lemma 3.6 carries over:

Lemma 4.5 *For every continuous seminorm q on $L(H)$, there is a continuous seminorm b on $L(G)$ such that $B^b_1(0) \subseteq Z$ and*

$$\left\| g(j(x,y)) + D\big(g(x), g(y), g(j(x,y))\big) \right\|_q \le (\max\{\|x\|_b, \|y\|_b\})^{2\alpha} \quad \text{for all } x, y \in B^b_1(0). \square$$

Lemma 4.6 *λ extends to a continuous $\mathbb{Q}_p$-linear map $\Lambda\colon L(G) \to L(H)$.*

Proof. The proof of Lemma 3.7 is easily adapted. □

In view of Lemma 2.2, Theorem 4.1 follows from the next lemma, whose proof directly parallels that of Lemma 3.8:

Lemma 4.7 *g is totally differentiable at 0, with $g'(0) = \Lambda$.* □

A Proofs for the auxiliary results from Section 1

In this appendix, we prove the results stated without proof in Section 1. Not all techniques from the real case carry over to general valued fields $\mathbb{K}$, whence some of the proofs may look slightly unfamiliar. In particular, given an element x of a polynormed $\mathbb{K}$-vector space E and a continuous seminorm q on E such that $\|x\|_q > 0$, there need not be an element $r \in \mathbb{K}$ such that $\|rx\|_q = 1$. As a substitute for normalization, we shall frequently fix an element $a \in \mathbb{K}^\times$ such that $|a| < 1$, and consider $a^{-k}x$ where $k \in \mathbb{Z}$ is chosen such that $|a|^{k+1} \le \|x\|_q < |a|^k$.

Proof of Lemma 1.6. We use the second order Taylor expansion of f,

$$f(x+ty)-f(x)-tdf(x,y)=t^2a_2(x,y)+t^2\,R_2(x,y,t)\quad\text{for }(x,y,t)\in U^{[1]}.$$

Since $R_2(x_0,0,0)=0$ and $a_2(x_0,0)=0$, there exists $\rho\in\,]0,1]$ and a continuous seminorm s on E such that $B^s_{2\rho}(x_0)\subseteq U$,

$$\|R_2(x,y,t)\|_q\le 1\qquad\text{for all }x\in B^s_\rho(x_0),\ y\in B^s_\rho(0)\text{ and }|t|<\rho,$$

and $\|a_2(x,y)\|_q\le 1$ for all $x\in B^s_\rho(x_0)$ and $y\in B^s_\rho(0)$. Pick $a\in\mathbb{K}^\times$ such that $|a|<1$; define $\delta:=\rho^2|a|<\rho$, $c:=2/(\rho|a|)^2$, and $p:=\max\{\frac{1}{\rho},\sqrt{\frac{c}{C}}\}s$. Let $x\in B^p_1(x_0)$ and $y\in B^p_1(0)$; then $x\in B^s_\delta(x_0)$ and $y\in B^s_\delta(0)$. If $\|y\|_s>0$, there exists $k\in\mathbb{Z}$ such that $|a|^{k+1}\le\rho^{-1}\|y\|_s<|a|^k$. Then $\|a^{-k}y\|_s<\rho$ and $|a^k|\le|a|^{-1}\rho^{-1}\|y\|_s<\rho$. If $\|y\|_s=0$, let $\varepsilon\in\,]0,\rho[$ and choose $k\in\mathbb{N}$ so large that $|a|^k<\rho$ and $2|a|^{2k}<\varepsilon$. Then, in either case,

$$f(x+y)-f(x)=f(x+a^ka^{-k}y)-f(x)=df(x,y)+a^{2k}a_2(x,a^{-k}y)+a^{2k}R_2(x,a^{-k}y,a^k)$$

with $r:=\|a^{2k}a_2(x,a^{-k}y)+a^{2k}R_2(x,a^{-k}y,a^k)\|_q\le|a|^{2k}(\|a_2(x,a^{-k}y)\|_q+\|R_2(x,a^{-k}y,a^k)\|_q)$ $\le 2|a|^{2k}$. If $\|y\|_s>0$, the preceding formula shows that $r\le 2|a|^{-2}\rho^{-2}\|y\|_s^2=c\|y\|_s^2\le C\|y\|_p^2$. If $\|y\|_s=0$, we have $r<\varepsilon$ and thus $r=0\le C\|y\|_p^2$, as ε was arbitrary. Hence $\|f(x+y)-f(x)-df(x,y)\|_q\le C\|y\|_p^2$ for all $x\in B^p_1(x_0)$ and $y\in B^p_1(0)$. □

Proof of Lemma 1.9. (a) and (b) are trivial; (c) follows from Lemma B.2 (a) and (e).

(d) Let E, F and H be polynormed $\mathbb{K}$-vector spaces, $U\subseteq E$ and $V\subseteq F$ be open, $x\in U$ and $f\colon U\to F$, $g\colon V\to H$ be maps such that $f(U)\subseteq V$, f is H_α at x, and g is H_β at $f(x)$. Given a continuous seminorm q on H, there exists a continuous seminorm p on F such that $B^p_1(f(x))\subseteq V$ and $\|g(z)-g(f(x))\|_q\le\|z-f(x)\|_p^\beta$ for all $z\in B^p_1(f(x))$. There is a continuous seminorm r on E such that $B^r_1(0)\subseteq U$ and $\|f(y)-f(x)\|_p\le\|y-x\|_r^\alpha\le 1$ for all $y\in B^r_1(x)$. Then $\|g(f(y))-g(f(x))\|_q\le\|f(y)-f(x)\|_p^\beta\le\|y-x\|_r^{\alpha\beta}$ for all $y\in B^r_1(0)$. □

Proof of 1.12. *If h is tangent to* 0, let q be a continuous seminorm on F. For $W:=B^q_1(0)$ we then find V and $\theta:I\to\mathbb{K}$ as in **1.11**. We may assume that V is balanced and $I=B_r(0)\subseteq\mathbb{K}$ for some $r>0$. There exists a continuous seminorm p on E such that $B^p_1(0)\subseteq V$ and $B^p_1(0)\subseteq U-x$. Replacing V with $B^p_1(0)$, we may assume that $V=B^p_1(0)$. Fix $a\in\mathbb{K}^\times$ such that $|a|<1$. Given $\varepsilon>0$, there exists $\delta\in\,]0,1]$ such that $\frac{|\theta(t)|}{|t|}<\varepsilon|a|$ if $|t|<\delta$. Then $B^p_\delta(0)\subseteq B^p_1(0)\subseteq U-x$. Let $y\in B^p_\delta(0)$; we claim that $\|h(y)\|_q\le\varepsilon\|y\|_p$. If $\|y\|_p=0$, then $t^{-1}y\in V$ for each $0\neq t\in I$, whence $h(y)=h(t(t^{-1}y))\in\theta(t)W$ and thus $\|h(y)\|_q\le|\theta(t)|$. Hence $\|h(y)\|_q=0\le\varepsilon\|y\|_p$. If $\|y\|_p>0$, then there is $k\in\mathbb{N}_0$ such that $|a|^{k+1}\le\|y\|_p<|a|^k$. Set $t:=a^k$. Then $t^{-1}y\in V$ and thus $h(y)=h(t(t^{-1}y))\in\theta(t)W$, whence $h(y)=\theta(t)w$ with $w\in W$. Hence $\|h(y)\|_q=|\theta(t)|\cdot\|w\|_q\le|\theta(t)|\le\varepsilon\,|a|\,|t|\le\varepsilon\|y\|_p$.

Conversely, assume that the condition from **1.12** is satisfied. Given a 0-neighbourhood $W\subseteq F$, there exists a continuous seminorm q on F such that $B^q_1(0)\subseteq W$. We choose a continuous seminorm p on E as described in **1.12**. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}^\times$ such that $\lim_{n\to\infty}a_n=0$. For each n, there exists $\delta_n>0$ such that $B^p_{\delta_n}(0)\subseteq U-x$ and

$\|h(y)\|_q \leq |a_n| \cdot \|y\|_p$ for all $y \in B^p_{\delta_n}(0)$. We may assume that $\delta_1 > \delta_2 > \cdots$ and $\lim_{n\to\infty} \delta_n = 0$. Now set $V := B^p_1(0)$ and define $\theta\colon I \to \mathbb{K}$ on $I := B_{\delta_1}(0) \subseteq \mathbb{K}$ via $\theta(0) = 0$, $\theta(t) := a_n t$ if $|t| \in [\delta_{n+1}, \delta_n[$. Then $\theta(t) = o(t)$ and $IV \subseteq B^p_{\delta_1}(0) \subseteq U - x$. Furthermore, $h(tv) \subseteq \theta(t)W$ for $t \in I$ and $v \in V$: This is trivial if $t = 0$, and also if $t \neq 0$ and $\|v\|_p = 0$, because then $\|h(tv)\|_q \leq |a_n| \cdot \|tv\|_p = 0$ (for any n) and hence $h(tv) \in q^{-1}(\{0\}) \subseteq \theta(t)W$. Otherwise, $0 < \|tv\|_p \in [\delta_{n+1}, \delta_n[$ for some n and thus $\|h(tv)\|_q \leq |a_n| \cdot \|tv\|_q = |ta_n| \cdot \|v\|_q < |ta_n|$, whence $h(tv) \in B^q_{|ta_n|}(0) = ta_n B^q_1(0) = \theta(t)B^q_1(0) \subseteq \theta(t)W$. Hence h is tangent to 0. □

Proof of 1.13. Let f, g be as in **1.13**, and $W_1 \subseteq H$ be a 0-neighbourhood. There is a balanced 0-neighbourhood $W \subseteq H$ such that $W + W \subseteq W_1$. As g is totally differentiable at $f(x)$ and $g'(f(x))$ continuous linear, we find balanced 0-neighbourhoods $P_1 \subseteq F$, $I \subseteq \mathbb{K}$ and a map $\theta : I \to \mathbb{K}$ which is $o(t)$, such that $g'(f(x)).P_1 \subseteq W$, $IP_1 \subseteq V - f(x)$, and $h_2(tP_1) \subseteq \theta(t)W$ for $t \in I$, where $h_2 : V - f(x) \to H$, $h_2(z) = g(f(x)+z) - g(f(x)) - g'(f(x)).z$. There is a balanced 0-neighbourhood $P \subseteq F$ such that $P + P \subseteq P_1$.

Define $h_1 : U - x \to F$, $h_1(y) := f(x+y) - f(x) - f'(x).y$. There are 0-neighbourhoods $Q \subseteq E$, $J \subseteq \mathbb{K}$ and a map $\xi\colon J \to \mathbb{K}$ which is $o(t)$, such that $JQ \subseteq U - x$, $f'(x).Q \subseteq P$, and $h_1(tQ) \subseteq \xi(t)P$. After shrinking I and J, we may assume that $I = J$ and $\left|\frac{\xi(t)}{t}\right| \leq 1$ for all $0 \neq t \in I$. Define $\eta\colon I \to \mathbb{K}$ via $\eta(t) := \theta(t)$ if $|\theta(t)| \geq |\xi(t)|$, $\eta(t) := \xi(t)$ if $|\theta(t)| < |\xi(t)|$. Define $A := g'(f(x)) \circ f'(x)$ and $h\colon U - x \to H$, $h(y) := g(f(x+y)) - g(f(x)) - A.y$. Then

$$\begin{aligned} h(y) &= g(f(x) + f'(x).y + h_1(y)) - g(f(x)) - A.y \\ &= g(f(x)) + g'(f(x)).z + h_2(z) - g(f(x)) - A.y \\ &= g'(f(x)).h_1(y) + h_2(f'(x).y + h_1(y)), \end{aligned}$$

where $z := f'(x).y + h_1(y)$. Let $t \in I$ and $y \in Q$. Then $h_1(ty) \in \xi(t)P \subseteq \xi(t)P_1 \subseteq \eta(t)P_1$ as P_1 is balanced, and thus $g'(f(x)).h_1(ty) \in \eta(t)W$. Furthermore, $f'(x).ty \in tP$ and $h_1(ty) \in \xi(t)P \subseteq tP$ (as $|\xi(t)| \leq |t|$), whence $f'(x).ty + h_1(ty) \in t(P + P) \subseteq tP_1$ and thus $h_2(f'(x).ty + h_1(ty)) \in \theta(t)W \subseteq \eta(t)W$, using that W is balanced. Hence $h(ty) = g'(f(x)).h_1(ty) + h_2(f'(x).ty + h_1(ty)) \in \eta(t)(W + W) \subseteq \eta(t)W_1$, and thus $h(tQ) \subseteq \eta(t)W_1$. We have shown that h is tangent to 0; the assertions follow. □

Proof of Lemma 1.14. We consider the second order Taylor expansion of f:

$$f(x + tv) = f(x) + t\,df(x, v) + t^2 a_2(x, v) + t^2 R_2(x, v, t) \qquad \text{for all } (x, v, t) \in U^{[1]} \tag{28}$$

(see Proposition 1.5). Fix $x \in U$. The map $f'(x) := df(x, \bullet) : E \to F$ being continuous linear, to establish total differentiability of f at x we only need to show that

$$h\colon U - x \to F, \quad h(y) := f(x + y) - f(x) - f'(x).y$$

is tangent to 0. To this end, let W be a 0-neighbourhood in F. There exists a 0-neighbourhood $W_1 \subseteq F$ such that $W_1 + W_1 \subseteq W$. As $R_2(x, 0, 0) = 0$ and R_2 is continuous, there is a 0-neighbourhood $V \subseteq E$ and a 0-neighbourhood $I \subseteq \mathbb{K}$ such that $(x, v, t) \in U^{[1]}$ and $R_2(x, v, t) \in W_1$ for all $v \in V$ and $t \in I$. Since a_2 is continuous and $a_2(x, 0) = 0$, after shrinking V we may assume that furthermore $a_2(x, v) \in W_1$ for all $v \in V$. Define

$$\theta\colon I \to \mathbb{K}, \quad \theta(t) := t^2 .$$

Then $\theta(t) = o(t)$. For each $t \in I$ and $y \in tV$, say $y = tv$ with $v \in V$, we have

$$h(y) = t^2(a_2(x,v) + R_2(x,v,t)) \in t^2(W_1 + W_1) \subseteq t^2 W = \theta(t)W\,,$$

using (28). Hence h is indeed tangent to 0. □

Proof of Lemma 1.17. If f is C^1, set $f'(x) := df(x, \bullet)$. Then $\widetilde{f}(y,t) = f^{[1]}(x,y,t)$ is continuous.

Now assume that $\mathbb{K}$ is a valued field, f is continuous on U and totally differentiable at x. Define $\widetilde{f}_x \colon \widetilde{U}_x \to E$ as in **1.16**, using the total differential $f'(x)$. Since f is continuous, so is $\widetilde{f}_x|_A$. By a theorem of Bourbaki and Dieudonné [3, Exerc. 3.2 A (b)], the map $\widetilde{f}_x$ is continuous if its restriction $\widetilde{f}_x|_{A\cup\{(y,0)\}}$ is continuous for each $y \in E$. This will hold if we can show that $\widetilde{f}_x(y_\alpha, t_\alpha) \to \widetilde{f}_x(y,0)$ for each net (y_α, t_α) in A converging to $(y,0)$ for some $y \in E$. To see that this condition is satisfied, let $W_1 \subseteq F$ be a 0-neighbourhood. There is a balanced 0-neighbourhood $W \subseteq F$ such that $W + W \subseteq W_1$. Since f is totally differentiable at x, there exists an open 0-neighbourhood $V \subseteq E$ and a function $\theta \colon I \to \mathbb{K}$ on some 0-neighbourhood in $\mathbb{K}$ such that $I \cdot V \subseteq U - x$ holds, $\theta(t) = o(t)$, and

$$f(x + sv) \in f(x) + sf'(x).v + \theta(s)W \qquad \text{for all } v \in V \text{ and } s \in I. \tag{29}$$

Pick $r \in \mathbb{K}^\times$ such that $ry \in V$. As $(y_\alpha, t_\alpha) \to (y,0)$, there exists β such that $f'(x).(y_\alpha - y) \in W$, $v_\alpha := ry_\alpha \in V$, $s_\alpha := r^{-1}t_\alpha \in I$, and $|\theta(s_\alpha)|/|s_\alpha| \leq |r|$ for all $\alpha \geq \beta$. For any such α, (29) applied to $x + t_\alpha y_\alpha = x + s_\alpha v_\alpha$ shows that

$$\widetilde{f}_x(y_\alpha, t_\alpha) - \widetilde{f}_x(y,0) \in f'(x).y_\alpha - f'(x).y + \frac{\theta(s_\alpha)}{t_\alpha}W \subseteq W + \frac{\theta(s_\alpha)}{rs_\alpha}W \subseteq W + W \subseteq W_1\,.$$

Thus indeed $\widetilde{f}_x(y_\alpha, t_\alpha) \to \widetilde{f}_x(y,0)$. □

Proof of 1.18. We define $\widetilde{f}_x : \widetilde{U}_x \to F$ and $\widetilde{g}_{f(x)} : \widetilde{V}_{f(x)} \to H$ as in **1.16** and abbreviate $h := g \circ f \colon U \to H$. For any $y \in E$ and $t \in \mathbb{K}^\times$ such that $x + ty \in U$, we calculate

$$\frac{h(x+ty) - h(x)}{t} = \frac{g\Big(f(x) + t\,\frac{f(x+ty)-f(x)}{t}\Big) - g(f(x))}{t} = \widetilde{g}_{f(x)}\big(\widetilde{f}_x(y,t), t\big) = \widetilde{h}_x(y,t)$$

where $\widetilde{h}_x : \widetilde{U}_x \to H$, $\widetilde{h}_x(y,t) := \widetilde{g}_{f(x)}\big(\widetilde{f}_x(y,t),t\big)$ is continuous, and the map $\widetilde{h}_x(\bullet, 0) = g'(f(x)) \circ f'(x)$ is continuous linear. Thus $h = g \circ f$ is feebly differentiable at x. □

B Hölder continuity at 1 entails Hölder continuity

So far, we only considered Hölder continuity *at a point.* We now discuss *globally* Hölder continuous maps. Basic facts are provided and a proof for Proposition 3.9 is given.

Definition B.1 Let E and F be polynormed vector spaces over a valued field $\mathbb{K}$, and $U \subseteq E$ be open. A map $f : U \to F$ is called *Hölder continuous of degree* α (or H_α, for short) if, for every $x_0 \in U$ and continuous seminorm q on F, there exists a continuous seminorm p on E and $\delta > 0$ such that $B^p_\delta(x_0) \subseteq U$ and $\|f(y) - f(x)\|_q \leq \|y - x\|^\alpha_p$, for all $x, y \in B^p_\delta(x_0)$. If f is H_1, we also say that f is *Lipschitz continuous*.

Lemma B.2 *For maps between open subsets of polynormed $\mathbb{K}$-vector spaces, we have:*

(a) *If $f : E \supseteq U \to F$ is H_α, then f is H_α at each $x \in U$.*

(b) *If f is H_α then f is continuous.*

(c) *If $\alpha \geq \beta$ and f is H_α, then f is H_β.*

(d) *If f and g are composable maps such that f is H_α and g is H_β, then $g \circ f$ is $H_{\alpha \cdot \beta}$.*

(e) *Any C^1-map is Lipschitz continuous.*

Proof. (a), (b) and (c) are obvious; (d) can be proved as Lemma 1.9 (d).

(e) We use the first order Taylor expansion $f(x + ty) - f(x) = tdf(x, y) + tR_1(x, y, t)$ of the C^1-map $f : E \supseteq U \to F$. Here

$$R_1(x, y, 1) = tR_1(x, t^{-1}y, t) \quad \text{for } t \in \mathbb{K}^\times \text{ and } (x, y) \in U \times E \text{ such that } x + y \in U.$$

Fix $x_0 \in U$. Let q be a continuous seminorm on F. Pick $a \in \mathbb{K}^\times$ such that $|a| < 1$. Since $df(x_0, 0) = 0$, using the continuity of df we find a continuous seminorm r on E such that $B^r_1(x_0) \subseteq U$ and $\|df(x, y)\|_q \leq |a|$ for all $x \in B^r_1(x_0)$ and $y \in B^r_1(0)$, whence $\|df(x, y)\|_q \leq \|y\|_r$ for all $x \in B^r_1(x_0)$ and $y \in E$. Since $R_1(x_0, 0, 0) = 0$, we find a continuous seminorm p on E and $\rho \in]0, 1]$ such that $B^p_{2\rho}(x_0) \subseteq U$ and $\|R_1(x, y, t)\|_q \leq 1$ for all $x \in B^p_\rho(x_0)$, $y \in B^p_\rho(0)$ and $t \in B_\rho(0) \subseteq \mathbb{K}$; we may assume that $p \geq r$. Define $\delta := \frac{1}{2}\rho^2|a|$. Given $x, y \in B^p_\delta(x_0)$, set $z := y - x$. If $\|z\|_p > 0$, there is $k \in \mathbb{Z}$ such that $|a|^{k+1} \leq \rho^{-1}\|z\|_p < |a|^k$. Then $\|a^{-k}z\|_p < \rho$ and $|a^k| \leq |a|^{-1}\rho^{-1}\|z\|_p < \rho$, whence $\|R_1(x, z, 1)\|_q = |a^k|\, \|R_1(x, a^{-k}z, a^k)\|_q \leq |a^k| \leq |a|^{-1}\rho^{-1}\|z\|_p$ and thus $\|f(x+z)-f(x)\|_q \leq \|df(x, z)\|_q + \|R_1(x, z, 1)\|_q \leq (1 + |a|^{-1}\rho^{-1})\|z\|_p$. Hence

$$\|f(y) - f(x)\|_q \leq (1 + |a|^{-1}\rho^{-1})\|y - x\|_p\,. \tag{30}$$

If $\|z\|_p = 0$, given $\varepsilon > 0$ pick $t \in \mathbb{K}^\times$ such that $|t| < \min\{\rho, \varepsilon\}$. Then $\|df(x, z)\|_q = 0$ and $\|R_1(x, z, 1)\|_q = |t|\, \|R_1(x, t^{-1}z, t)\|_q \leq |t| \leq \varepsilon$, whence $\|R_1(x, z, 1)\|_q = 0$ (as ε was arbitrary). Thus (30) also holds if $\|z\|_p = 0$. □

Definition B.3 Let $f : M \to N$ be a map between $C^1_\mathbb{K}$-manifolds modelled on polynormed $\mathbb{K}$-vector spaces, and $\alpha \in]0, 1]$. We say that f is *Hölder continuous of degree* α (or briefly: f is H_α), if f is continuous and, for each $x_0 \in M$, there exist a chart $\phi : U_1 \to U$ of M around x_0 and a chart $\psi : V_1 \to V$ of N around $f(x_0)$, such that $\phi(f^{-1}(V_1) \cap U_1) \to V$, $y \mapsto \psi(f(\phi^{-1}(y)))$ is H_α. (This then holds for any choice of ϕ and ψ, by Lemma B.2).

Proof of Proposition 3.9. Let f be H_α at 1. If we can show that $f|_U$ is H_α for an open identity neighbourhood $U \subseteq G$, then $f|_{xU} = \lambda^H_{f(x)} \circ f|_U \circ \lambda^G_{x^{-1}}|^U_{xU}$ (with left translation maps as indicated) will be H_α by Lemma B.2 (d) and (e), for each $x \in G$, whence f will be H_α.

We choose charts $\phi\colon P_1 \to P \subseteq L(G)$ and $\psi\colon Q_1 \to Q \subseteq L(H)$ around 1 of G and H, respectively, such that $\phi(1) = 0$, $\psi(1) = 0$ and $f(P_1) \subseteq Q_1$. There are symmetric identity neighbourhoods $X_1, U_1 \subseteq G$ and $Y_1, V_1 \subseteq H$ such that $X_1X_1 \subseteq P_1$, $U_1U_1 \subseteq X_1$, $Y_1Y_1 \subseteq Q_1$, $V_1V_1 \subseteq Y_1$, $f(X_1) \subseteq Y_1$, and $f(U_1) \subseteq V_1$; set $X := \phi(X_1)$, $U := \phi(U_1)$, $Y := \psi(Y_1)$, and $V := \psi(V_1)$. We write $\mu\colon X \times X \to P$ and $\nu\colon Y \times Y \to Q$ (or "$*$") for the local multiplications obtained from the respective group multiplication, and define $g := \psi \circ f|_P \circ \phi^{-1}|_P\colon P \to Q$. Let q be a continuous seminorm on $L(H)$, and $x_0 \in U$. As ν is H_1, there is a continuous seminorm p on $L(H)$ such that $B^p_1(g(x_0)) \times B^p_1(0) \subseteq V \times V$ and

$$\|\nu(u,v) - \nu(u',v')\|_q \leq \max\{\|u'-u\|_p, \|v'-v\|_p\} \quad \text{for all } u, u' \in B^p_1(x_0),\ v, v' \in B^p_1(0).$$

Now g being H_α at 0, there exists a continuous seminorm r on $L(G)$ such that $B^r_1(0) \subseteq X$ and $\|g(x)\|_p \leq \|x\|^\alpha_r$ for all $x \in B^r_1(0)$. The map $h\colon U \times U \to X$, $h(u,v) := u^{-1} * v$ being Lipschitz continuous, there is a continuous seminorm $s \geq r$ on $L(G)$ such that $B^s_1(x_0) \subseteq U$, $h(B^s_1(x_0) \times B^s_1(x_0)) \subseteq B^r_1(0)$, and

$$\|h(u,v) - h(u',v')\|_r \leq \max\{\|u-u'\|_s, \|v-v'\|_s\} \quad \text{for all } u, u', v, v' \in B^s_1(x_0).$$

For any $x, y \in B^s_1(x_0) \subseteq U$, we obtain

$$\begin{aligned}
\|g(y) - g(x)\|_q &= \|g(x) * g(x^{-1} * y) - g(x)\|_q = \|\nu(g(x), g(x^{-1}*y)) - \nu(g(x), 0)\|_q \\
&\leq \max\{\|g(x) - g(x)\|_p, \|g(x^{-1}*y)\|_p\} = \|g(x^{-1}*y)\|_p \\
&\leq \|x^{-1}*y\|^\alpha_r = \|h(x,y) - h(x,x)\|^\alpha_r \leq \|y - x\|^\alpha_s.
\end{aligned}$$

Hence $g|_U$ is H_α indeed. □

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