Band-dominated operators on l^p-spaces: Fredholm indices and finite sections

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Abstract

We derive an index formula for band-dominated operators on $l^p(\mathbb{Z})$ when $1 < p < \infty$ in terms of local indices of their limit operators. This formula is applied to verify the stability of the finite section method for invertible band-dominated operators with slowly oscillating coefficients. Hilbert space versions of these results (partially under further restrictions) were obtained in [9] and [6], respectively.

1 Introduction

Let $l^\infty(\mathbb{Z}^N)$ denote the Banach space of all bounded functions $x : \mathbb{Z}^N \to \mathbb{C}$ provided with the supremum norm. If $x \in l^\infty(\mathbb{Z}^N)$ and $n \in \mathbb{Z}^N$, we will usually write x_n instead of $x(n)$.

For $k \in \mathbb{Z}^N$, the shift operator V_k on $l^\infty(\mathbb{Z}^N)$ is defined by $(V_k x)_n := x_{n-k}$. Further, each function $a \in l^\infty(\mathbb{Z}^N)$ induces an operator on $l^\infty(\mathbb{Z}^N)$ via $(ax)_n := a_n x_n$. We denote this operator by aI and call it the operator of multiplication by a. Finally, each finite sum $\sum a_k V_k$ with $a_k \in l^\infty(\mathbb{Z}^N)$ is called a *band operator* on $l^\infty(\mathbb{Z}^N)$, and the a_k are called the *coefficients* of that operator.

For $1 \le p < \infty$, let $l^p(\mathbb{Z}^N)$ refer to the (non-closed) subspace of $l^\infty(\mathbb{Z}^N)$ which consists of all functions x with

$$\|x\|_p^p := \sum_{n \in \mathbb{Z}^N} |x_n|^p < \infty.$$

Clearly, band operators leave each of these subspaces invariant. If A is a band operator on $l^\infty(\mathbb{Z}^N)$, then we call its restriction to $l^p(\mathbb{Z}^N)$ a *band operator* on $l^p(\mathbb{Z}^N)$, and we denote this restriction be the same letter A again. Clearly, an operator A on $l^p(\mathbb{Z}^N)$ with $1 \le p < \infty$ is a band operator if, and only if, its matrix representation (a_{ij}) with respect to the standard basis of $l^p(\mathbb{Z})$ has a band structure, i.e. there is an integer k such that $a_{ij} = 0$ whenever $|i - j| > k$.

The closure of the set of all band operators on $l^p(\mathbb{Z}^N)$ in the operator norm is a closed subalgebra of the algebra $L(l^p(\mathbb{Z}^N))$ of all bounded linear operators on $l^p(\mathbb{Z}^N)$ which we denote by $\mathcal{A}_p(\mathbb{Z}^N)$. The elements of $\mathcal{A}_p(\mathbb{Z}^N)$ are called *band-dominated operators* on $l^p(\mathbb{Z}^N)$.

Sometimes we will also have to work on l^p-spaces over the non-negative integers $\mathbb{Z}_+$ and over the negative integers $\mathbb{Z}_-$ and to consider band and band-dominated operators over them. These operators as well as the corresponding algebras $\mathcal{A}_p(\mathbb{Z}_+)$ and $\mathcal{A}_p(\mathbb{Z}_-)$ are defined in complete analogy with the case of operators on $l^p(\mathbb{Z})$. We will often identify $l^p(\mathbb{Z}_+)$ and $l^p(\mathbb{Z}_-)$ with closed subspaces of $l^p(\mathbb{Z})$ in the obvious way, and we denote by P and Q the canonical projection from $l^p(\mathbb{Z})$ onto $l^p(\mathbb{Z}_+)$ and $l^p(\mathbb{Z}_-)$, respectively. Thus, $P + Q = I$.

The first goal of the present paper is to derive an index formula for band-dominated operators on $l^p(\mathbb{Z})$ in terms of their limit operators. The prototype of results of this kind has been achieved in [10] in the Hilbert space case $p = 2$ by means of K-theoretic arguments. The result expresses the index of a band-dominated operator A as a sum of two local indices of two limit operators of A. Here we will prove the corresponding result for $p \neq 2$. In particular, we will see that, if A is a band operator which is Fredholm on one of the spaces $l^p(\mathbb{Z})$, then it is Fredholm on each of these spaces, and its index is independent of p.

Recall in this connection that a bounded linear operator A on a Banach space X is said to be *Fredholm* if its kernel $\ker A := \{x \in X : Ax = 0\}$ and its cokernel $\operatorname{coker} A := X/\operatorname{im} A$ are finite-dimensional linear spaces, and that in this case the integer

$$\operatorname{ind} A := \dim \ker A - \dim \operatorname{coker} A$$

is called the Fredholm index of A.

The second concern of this paper is the stability of the so-called finite section method for band-dominated operators with slowly oscillating coefficients. By definition, a function $a \in l^\infty(\mathbb{Z})$ is *slowly oscillating* if

$$\lim_{m \to -\infty} (a_{m+1} - a_m) = 0 \quad \text{and} \quad \lim_{m \to +\infty} (a_{m+1} - a_m) = 0,$$

and we call a function $a \in l^\infty(\mathbb{Z}_+)$ slowly oscillating if it satisfies the second of these conditions. Further, a band-dominated operator is said to be an *operator with slowly oscillating coefficients* if it is the norm limit of a sequence of band operators with slowly oscillating coefficients. For a more detailed treatise of band-dominated operators with slowly oscillating (even operator-valued) coefficients see [8] as well as Section 2.4 in [12].

For $n \in \mathbb{N}$, consider the projection operators

$$R_n : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (R_n x)_m := \begin{cases} x_m & \text{if } |m| \le n \\ 0 & \text{if } |m| > n \end{cases}$$

and

$$P_n : l^p(\mathbb{Z}_+) \to l^p(\mathbb{Z}_+), \quad P_n = P R_n P.$$

The *finite section method* consists in replacing the operator equation $Au = f$ with $A \in L(l^p(\mathbb{Z}))$ by the sequence of the linear systems

$$R_n A R_n u_n = R_n f, \quad n \in \mathbb{N}. \tag{1}$$

Here, the $R_n A R_n$ are viewed of as operators on the Banach space Im R_n, provided with the norm induced by the norm on $L(l^p(\mathbb{Z}))$. The finite section method is called *stable* if the matrices $R_n A R_n$ are invertible for sufficiently large n and if the norms of their inverses are uniformly bounded. In this case one also says that the sequence $(R_n A R_n)$ is stable. If the finite section method for A is stable, then there is an $n_0 \in \mathbb{N}$ such that the equations (1) are uniquely solvable for each right hand side $f \in l^p(\mathbb{Z})$ and for each $n \geq n_0$, and their solutions u_n converge to a solution of the equation $Au = f$ in the norm of $l^p(\mathbb{Z})$. Analogously, the finite section method $(P_n A P_n)$ for an operator $A \in L(l^p(\mathbb{Z}_+))$ is defined.

A simple necessary condition for the stability of the finite section method of the operator A is the invertibility of A. In [6] it is shown that the invertibility of A is also sufficient for the stability of the finite section method if A is a band operator with slowly oscillating coefficients on $l^p(\mathbb{Z}_+)$. Here we will extent these result to arbitrary band-dominated operators with slowly oscillating coefficients on $l^p(\mathbb{Z}_+)$ with $1 < p < \infty$.

Theorem 1.1 *Let $1 < p < \infty$, and let $A \in L(l^p(\mathbb{Z}_+))$ be a band-dominated operator with slowly oscillating coefficients. Then the finite section method $(P_n A P_n)$ is stable if and only if the operator A is invertible.*

This result is well known in case of band-dominated operators on $l^p(\mathbb{Z}_+)$ with constant coefficients, which are called Toeplitz operators. Recall in this connection that a function $a \in L^1(\mathbb{T})$ with Fourier coefficient sequence $(a_n)_{n \in \mathbb{Z}}$ is called an l^p*-multiplier* if

$$\sup \|L(a)x\|_p / \|x\|_p < \infty$$

where the supremum is taken over all non-zero and finitely supported sequences $x : \mathbb{Z} \to \mathbb{C}$ and where the *Laurent operator* $L(a)$ acts via

$$(L(a)x)_n := \sum_{k \in \mathbb{Z}} a_{n-k} x_k.$$

If a is an l^p-multiplier, then the operator $L(a)$ extends to a bounded linear operator on $l^p(\mathbb{Z})$ which we denote by $L(a)$ again. The compression $PL(a)P$ of the Laurent operator $L(a)$ onto $l^p(\mathbb{Z}_+)$ is the *Toeplitz operator* $T(a)$. Overviews on Toeplitz and Laurent operators with continuous generating functions, including their Fredholmness, invertibility, and stability of the finite section method, can be found in [2, 3, 4], for example.

We will also prove a version of Theorem 1.1 which holds for band-dominated operators with slowly oscillating coefficients on $l^p(\mathbb{Z})$ in which case the invertibility of the operator A is no longer sufficient for the stability of the finite section method $(R_n A R_n)$. Rather, an additional condition stated in terms of the Fredholm index of $PAP + Q$ will be needed.

2 Fredholm indices of band-dominated operators

In spite of the fact that some of the following considerations hold for $p = \infty$ and $p = 1$ as well, we let for simplicity $1 < p < \infty$ in what follows. In [10], Theorem 8 (see also [12], Theorem 2.5.7), it is shown that, if A is a band operator which is Fredholm on one of the spaces $l^p(\mathbb{Z}^N)$, then it is Fredholm on each of these spaces. The following lemma states that also the Fredholm index of A is independent of the choice of p.

Lemma 2.1 *If A is a band operator which is Fredholm on one of the spaces $l^p(\mathbb{Z}^N)$ with $1 < p < \infty$, then it is Fredholm on each of these spaces, and its Fredholm index is independent of p.*

Proof. Let $\alpha = (1, 0, 0, \ldots, 0) \in \mathbb{Z}^N$, and let Π denote the projection on $l^\infty(\mathbb{Z}^N)$ given by

$$(\Pi x)_n := \begin{cases} x_n & \text{if } n \in \mathbb{Z}_+\alpha \\ 0 & \text{if } n \notin \mathbb{Z}_+\alpha. \end{cases}$$

Then

$$U := \Pi V_\alpha \Pi + (I - \Pi) \quad \text{and} \quad U^{-1} := \Pi V_\alpha^{-1} \Pi + (I - \Pi)$$

are Fredholm operators with indices -1 and 1, respectively, on each of the spaces $l^p(\mathbb{Z}^N)$. Thus, for each $k \in \mathbb{Z}$, the operator

$$U_k := \begin{cases} U^k & \text{if } k \geq 0 \\ (U^{-1})^{-k} & \text{if } k < 0 \end{cases}$$

is a Fredholm operator of index $-k$ on each of the spaces $l^p(\mathbb{Z}^N)$. Notice also that the U_k are band operators.

Let now A be Fredholm on one of the spaces $l^p(\mathbb{Z}^N)$. Then, as already mentioned, A is Fredholm on $l^2(\mathbb{Z}^N)$. Let $r := \operatorname{ind}_2 A$, where we agree upon denoting the Fredholm index of the operator A on $l^p(\mathbb{Z}^N)$ by $\operatorname{ind}_p A$. Hence, $U_r A \in L(l^2(\mathbb{Z}^N))$ is a Fredholm band operator with $\operatorname{ind}_2(U_r A) = 0$. Thus, there is a compact operator $K \in L(l^2(\mathbb{Z}^N))$ such that the operator $U_r A + K$ is invertible on $l^2(\mathbb{Z}^N)$ (Theorem 6.2 in Chapter 4 of [5]).

Since the R_n converge to I *-strongly as $n \to \infty$, and since K is compact, the operators $U_r A + R_n K R_n$ converge to $U_r A + K$ in the norm of $L(l^2(\mathbb{Z}^N))$. By Neumann series, there is an n_0 such that the operator $A' := U_r A + R_{n_0} K R_{n_0}$ is invertible on $l^2(\mathbb{Z}^N)$. Being a band operator, the operator A' belongs to the Wiener algebra $\mathcal{W}(\mathbb{Z}^N)$. This set is defined as the closure of the set of all band operators

$C := \sum a_\alpha V_\alpha$ in the norm $\|C\|_{\mathcal{W}} := \sum \|a_\alpha\|_\infty$. It is well known that $\mathcal{W}(\mathbb{Z}^N)$ is indeed a Banach algebra under the norm $\|.\|_{\mathcal{W}}$ and that the elements of the Wiener algebra act as bounded linear operators on each of the spaces $l^p(\mathbb{Z}^N)$. Moreover, the Wiener algebra is inverse closed in $L(l^p(\mathbb{Z}^N))$, which means that, whenever an operator $A \in \mathcal{W}(\mathbb{Z}^N)$ is invertible in $L(l^p(\mathbb{Z}^N))$, then its inverse belongs to the Wiener algebra again (see [7] and Theorem 2.5.2 in [12]). This explains the existence of an operator $B \in \mathcal{W}(\mathbb{Z}^N)$ such that $BA' = I$ or, equivalently,

$$BU_r A = I - BR_{n_0} K R_{n_0}. \tag{2}$$

This identity holds on each of the spaces $l^p(\mathbb{Z}^N)$ since all occurring operators (i.e. the operators B, U_r, A and $R_{n_0} K R_{n_0}$) belong to the Wiener algebra. Considering (2) as an identity between operators on $l^p(\mathbb{Z}^N)$ and computing the indices yields

$$0 - r + \mathrm{ind}_p\, A = \mathrm{ind}_p\, B + \mathrm{ind}_p\, U_r + \mathrm{ind}_p\, A = \mathrm{ind}_p\, (I - BR_{n_0} K R_{n_0}) = 1.$$

Thus, $\mathrm{ind}_p\, A = r$ independently of p. ■

From here on, we specify N to be 1. In [9], a formula for the Fredholm index of a band-dominated operator A on $l^2(\mathbb{Z})$ is derived, which expresses the index of A in terms of local indices of the limit operators of A. To restate this result, we first recall the notion of a limit operator and the Fredholm criterion for band-dominated operators derived in [10].

Let $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}$ which tend to infinity. An operator A_h is called a *limit operator* of $A \in L(l^p(\mathbb{Z}))$ with respect to the sequence $h \in \mathcal{H}$ if $V_{-h(n)} A V_{h(n)}$ tends *-strongly to A_h as $n \to \infty$. Clearly, every operator A can have at most one limit operator with respect to a given sequence $h \in \mathcal{H}$, which justifies this notation. The set $\sigma_{op}(A)$ of all limit operators of a given operator A is the *operator spectrum* of A. Clearly, the operator spectrum splits into

$$\sigma_{op}(A) = \sigma_+(A) \cup \sigma_-(A)$$

where $\sigma_+(A)$ and $\sigma_-(A)$ stand for the sets of all limit operators of A which correspond to sequences tending to $+\infty$ and to $-\infty$, respectively. It is also clear that every limit operator of a compact operator is 0 and that every limit operator of a Fredholm operator is invertible. It is a basic result of [10] that the operator spectrum of a *band-dominated operator* is rich enough in order to guarantee the reverse implications.

Theorem 2.2 *Let A be a band-dominated operator on $l^p(\mathbb{Z})$. Then*

(a) every sequence $h \in \mathcal{H}$ possesses a subsequence g such that the limit operator A_g exists.

(b) the operator A is compact if and only if $\sigma_{op}(A) = \{0\}$.

(c) the operator A is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded.

Next observe that, if A is a band-dominated operator on $l^p(\mathbb{Z})$, then each of the operators PAQ and QAP is compact (they are of finite rank if A is a band operator). Hence, the operators $A - (PAP + Q)(P + QAQ)$ and $A - (P + QAQ)(PAP + Q)$ are compact for every band-dominated operator A, and this shows that a band-dominated operator A is Fredholm if and only if both operators $PAP + Q$ and $P + QAQ$ are Fredholm. In this case, we call

$$\mathrm{ind}_p^+ A := \mathrm{ind}_p\,(PAP + Q) \quad \text{and} \quad \mathrm{ind}_p^- A := \mathrm{ind}_p\,(P + QAQ)$$

the *plus-index* and the *minus-index* of A. One can think of the plus- and the minus-index of A as local indices at $+\infty$ and $-\infty$. Evidently,

$$\mathrm{ind}_p A = \mathrm{ind}_p^+ A + \mathrm{ind}_p^- A \tag{3}$$

for every Fredholm band-dominated operator $A \in L(l^p(\mathbb{Z}))$.

In [9] (see also Section 2.7 of [12]), a surprisingly simple formula for the index of band-dominated operators is derived in the Hilbert space case $p = 2$. We will show now that verbatim the same result holds in case $p \neq 2$.

Theorem 2.3 *Let $1 < p < \infty$, and let A be a Fredholm band-dominated operator on $l^p(\mathbb{Z})$. Then*
(a) for all $B_\pm \in \sigma_\pm(A)$,

$$\mathrm{ind}_p^\pm B_\pm = \mathrm{ind}_p^\pm A.$$

(b) all operators in $\sigma_+(A)$ have the same plus-index, and all operators in $\sigma_-(A)$ have the same minus-index.
(c) for arbitrarily chosen operators $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,

$$\mathrm{ind}_p A = \mathrm{ind}_p^+ B_+ + \mathrm{ind}_p^- B_-. \tag{4}$$

Proof. Due to (3), it is clearly sufficient to prove assertion (a), and we will give this proof for the plus-index, for example. Choose a sequence (A_n) of band operators such that

$$\|A - A_n\|_{L(l^p(\mathbb{Z}))} \to 0, \tag{5}$$

Then, in particular,

$$\|(PAP + Q) - (PA_nP + Q)\|_{L(l^p(\mathbb{Z}))} \to 0.$$

Since $PAP + Q$ is Fredholm whenever A is Fredholm, this shows that $PA_nP + Q$ is a Fredholm operator for all sufficiently large n, say for $n \geq n_0$, and that

$$\mathrm{ind}_p^+ A = \mathrm{ind}_p(PAP + Q) = \mathrm{ind}_p(PA_nP + Q) = \mathrm{ind}_p^+ A_n \quad \text{for } n \geq n_0. \tag{6}$$

Let now $B_+ \in \sigma_+(A)$ be a limit operator of $A \in L(l^p(\mathbb{Z}))$. By a Cantor diagonal argument, one finds a sequence $h \in \mathcal{H}$ which tends to $+\infty$ such that the limit operators $(A_n)_h$ exist for each $n \in \mathbb{N}$ and that

$$\|B_+ - (A_n)_h\|_{L(l^p(\mathbb{Z}))} \to 0 \tag{7}$$

(see also Proposition 1 (d) in [10] and Proposition 1.2.2 (e) in [12]). Now we make use of the following simple observation (Proposition 29 in [10] and Proposition 2.5.6 in [12]): If C is a band operator, and if the limit operator A_h of A with respect to $h \in \mathcal{H}$ exists on one of the spaces $l^p(\mathbb{Z})$, then A_h is a limit operator of A on each of these spaces. Thus, $(A_n)_h$ is also a limit operator of A_n if the operators are considered as acting on $l^2(\mathbb{Z})$, and $(A_n)_h \in \sigma_+(A_n)$. The l^2-version of Theorem 2.3 (a) established in [9] yields

$$\operatorname{ind}_2^+ A_n = \operatorname{ind}_2^+ (A_n)_h = \operatorname{ind}_2(P(A_n)_h P + Q) \quad \text{for all } n.$$

Since $(A_n)_h$ is a band operator again, we conclude via Lemma 2.1 that

$$\operatorname{ind}_p^+ A_n = \operatorname{ind}_p^+ (A_n)_h = \operatorname{ind}_p(P(A_n)_h P + Q) \quad \text{for all } n. \tag{8}$$

Further, by (7) we have

$$\|(PB_+P + Q) - (P(A_n)_h P + Q)\|_{L(l^p(\mathbb{Z}))} \to 0. \tag{9}$$

Since B_+ is invertible (it is the limit operator of a Fredholm operator), the operator $PB_+P + Q$ is Fredholm. Thus, (9) implies that the operators $P(A_n)_h P + Q$ are Fredholm and that

$$\operatorname{ind}_p^+ (A_n)_h = \operatorname{ind}_p (P(A_n)_h P + Q) = \operatorname{ind}_p (PB_+P + Q) = \operatorname{ind}_p^+ B_+ \tag{10}$$

for sufficiently large n (say $n \geq n_0$ again). From (6), (8) and (10) we get the assertion. ∎

In particular we rediscover the well known fact that the index of a Toeplitz operator $T(a)$ with symbol a in the Wiener algebra of functions on the unit circle does not depend on the space $l^p(\mathbb{Z}_+)$ on which the operator $T(a)$ is considered. Let us also mention that Theorem 2.3 can be proved in the same vein for band-dominated operators acting on the closed subspace $c_0(\mathbb{Z})$ of $l^\infty(\mathbb{Z})$ which consists of all sequences vanishing at infinity.

3 Finite sections of band-dominated operators

In this section, where we let $1 < p < \infty$ again, we will first prove Theorem 1.1, and then we are going to generalize this theorem to operators on $l^p(\mathbb{Z})$. As in [6], our approach is based on the fact that the stability of the finite section method of a band-dominated operator A is equivalent to the Fredholmness of an associated band-dominated operator (see also [11] and Section 6.1 in [12]. Indeed, the finite section method $(R_n A R_n)$ for the operator $A \in L(l^p(\mathbb{Z}))$ is stable if and only if the block diagonal operator

$$\mathbf{A} := \operatorname{diag}(R_0 A R_0,\ R_1 A R_1,\ R_2 A R_2,\ \ldots),$$

considered as acting on $l^p(\mathbb{Z}_+)$, is a Fredholm operator. If A is a band operator, then $\mathbf{A}$ is a band operator, too, and Theorem 2.2 applies to study its Fredholmness. Basically, one has to compute the limit operators of $\mathbf{A}$, which leads to the following.

Theorem 3.1 ([11], Theorem 3) *Let $A \in L(l^p(\mathbb{Z}))$ be a band-dominated operator. Then the finite section method (R_nAR_n) is stable if and only if the operator A, all operators*

$$QA_hQ + P \quad \textit{with} \quad A_h \in \sigma_+(A)$$

and all operators

$$PA_hP + Q \quad \textit{with} \quad A_h \in \sigma_-(A)$$

are invertible on $l^p(\mathbb{Z})$, and if the norms of their inverses are uniformly bounded. The condition of the uniform boundedness of the inverses is redundant if A is a band operator.

Specifying this result to the case of a band-dominated operator A on $l^p(\mathbb{Z}_+)$ we get the following, where J refers to the unitary operator

$$l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (Jx)_m := x_{-m-1},$$

and where we define $\sigma_+(A)$ as $\sigma_+(PAP + Q)$.

Theorem 3.2 *Let $A \in L(l^p(\mathbb{Z}_+))$ be a band-dominated operator. Then the finite section method (P_nAP_n) is stable if and only if the operator A and all operators*

$$JQA_hQJ \quad \textit{with} \quad A_h \in \sigma_+(A)$$

are invertible on $l^p(\mathbb{Z}_+)$, and if the norms of their inverses are uniformly bounded. The condition of the uniform boundedness of the inverses is redundant if A is a band operator.

Proof of Theorem 1.1. Let $A \in L(l^p(\mathbb{Z}_+))$ be a band-dominated operator with slowly oscillating coefficients. If the finite section method (P_nAP_n) is stable, then A is invertible. (This fact holds for arbitrary bounded linear operators A.) Let, conversely, A be an invertible operator. We identify A with the operator $PAP + Q$ acting on $l^p(\mathbb{Z})$. Clearly, this operator is invertible, too. Hence, all limit operators of $PAP+Q$ are invertible by Theorem 2.2 (c). It is easy to check that the part $\sigma_-(PAP+Q)$ of the operator spectrum of $PAP+Q$ consists of the identity operator only. Let A_h be a limit operator in $\sigma_+(PAP + Q)$. Since the coefficients of A (hence, the coefficients of $PAP + Q$) are slowly oscillating, the operator A_h is shift invariant (Proposition 30 in [10]). From Proposition 2.4 in [2] we conclude that there is an l^p-multiplier a_h such that $A_h = L(a_h)$. Further, an elementary calculation shows that $JQA_hQJ = PJL(a_h)JP$ is just the Toeplitz operator $T(\widetilde{a_h})$ where $\tilde{a}(t) := a(1/t)$ for a function a on the unit circle.

Since the operator A is invertible, the plus-index of $PAP + Q$ is zero. By Theorem 2.3, the plus- and minus-indices of each limit operator of $PAP + Q$ are zero, too. In particular, the index of $QA_hQ + P$ (which is the minus-index of A_h) is zero. This implies that the index of $JQA_hQJ = T(\widetilde{a_h})$ is zero, whence the invertibility of $T(\widetilde{a_h})$ via Coburn's theorem (Theorems 2.38 (b) and 2.40 in [2]).

Via Theorem 3.2, this settles the assertion of Theorem 1.1 in case A is a band operator. For a general band-dominated operator A, we still have to show the uniform boundedness of the norms of the inverses of the operators $QA_hQ + P$ where the A_h run through $\sigma_+(A)$.

For, we show that the limit operators of A can be labeled by the points of a compact metric space, X, in such a way that the mapping which associates a limit operator A^x of A with each point $x \in X$ becomes norm continuous. The same idea has been used in [10] in order to show that the uniform boundedness of the inverses of the limit operators in Theorem 2.2 is redundant in case A is a band-dominated operator with slowly oscillating coefficient (see also Section 2.4.9 in [12] and [1] for another occurrence of the same idea) .

Let (A_n) be a sequence of band operators with slowly oscillating coefficients which tends to A in the norm of $L(l^p(\mathbb{Z}_+))$. By SO_A we denote the smallest closed and symmetric subalgebra of $l^\infty(\mathbb{Z}^+)$ which contains all coefficients of all operators A_n as well as all sequences tending to zero. Then SO_A is a commutative and separable (since generated by a countable number of functions) C^*-algebra. We denote its maximal ideal space by $M(SO_A)$ and write $M^{+\infty}(SO_A)$ for its fiber over $+\infty$, i.e. for the collection of all non-trivial multiplicative functionals $f : SO_A \to \mathbb{C}$ which vanish over $c_0(\mathbb{Z}_+)$. Since SO_A is a separable algebra, its maximal ideal space is metrizable, and so is the fiber $X := M^{+\infty}(SO_A)$.

In [10] it is shown that, if $B = \sum b_k PV_kP$ is a band operator with coefficients b_k in SO_A, and if $h \in \mathcal{H}$ is a sequence which tends to $+\infty$ and for which the limit operator B_h exists, then there is a unique point $x \in M^{+\infty}(SO_A)$ such that

$$B_h = \sum b_k(x)V_k =: B^x. \tag{11}$$

Conversely, given $x \in M^{+\infty}(SO_A)$, one finds a sequence $h \in \mathcal{H}$ such that the limit operator of B with respect to h exists and that (11) holds (the metrizability of $M^{+\infty}(SO_A)$ is used in order to approximate x by a *sequence* of points in $\mathbb{Z}_+$ from which one can choose h as a subsequence). Evidently, the function

$$M^{+\infty}(SO_A) \to L(l^p(\mathbb{Z})), \quad x \mapsto \sum b_k(x)V_k$$

is continuous.

Now we extend this construction to the operator A. Let h be a sequence which tends to $+\infty$ for which the limit operator A_h exists. By a Cantor diagonal argument, there is a subsequence g of h such that all limit operators $(A_n)_g$ exist, too. Let x be the uniquely determined point in $M^{+\infty}(SO_A)$ such that $(A_n)_g = A_n^x$.

Then

$$A_h = \lim_{n\to\infty} (A_n)_g = \lim_{n\to\infty} A_n^x,$$

and we denote this operator by A^x. Being the uniform limit of the continuous functions $x \mapsto A_n^x$, the function

$$M^{+\infty}(SO_A) \to L(l^p(\mathbb{Z})), \quad x \mapsto A^x$$

is continuous. Then, clearly, the function

$$M^{+\infty}(SO_A) \to L(l^p(\mathbb{Z})), \quad x \mapsto QA^xQ + P$$

is continuous, and the values of this function are invertible operators as we have already seen. Since inversion is continuous, the function

$$M^{+\infty}(SO_A) \to L(l^p(\mathbb{Z})), \quad x \mapsto \|(QA^xQ + P)^{-1}\| \tag{12}$$

proves to be continuous, too. Finally, being a continuous function on a compact metric space, the function (12) attains its maximum. Since the values of that function are exactly the norms of the operators $(QA_hQ+P)^{-1}$ with $A_h \in \sigma_+(A)$, we obtain the uniform boundedness of these norms. ■

Now we turn over to the finite section method for band-dominated operators with slowly oscillating coefficients on l^p over $\mathbb{Z}$. In this setting, Theorem 1.1 cannot hold. Actually, it becomes wrong even for band Laurent operators: the finite section method $(R_nL(a)R_n)$ for the *Laurent operator* $L(a)$ is stable if and only if the *Toeplitz operator* $T(a)$ is invertible. Clearly, the invertibility of $T(a)$ implies that of $L(a)$, but if $a(t) := t^{-1}$, then the Laurent operator $L(a)$ is invertible, whereas the Toeplitz operator $T(a)$ has a non-trivial kernel.

Here is the result, which has been proved for band operators on $l^2(\mathbb{Z})$ in [6].

Theorem 3.3 *Let $1 < p < \infty$, and let $A \in L(l^p(\mathbb{Z}))$ be a band-dominated operator with slowly oscillating coefficients. Then the finite section method (R_nAR_n) is stable if and only if the operator A is invertible and if the plus-index of A is zero.*

Proof. The necessity of these conditions can be seen as in [6]. For the readers' convenience we recall the short proof here.

If the finite section method (R_nAR_n) is stable, then A is invertible. Let A_h be a limit operator of A which lies in $\sigma_+(A)$. Then $QA_hQ + P$ is invertible by Theorem 3.1, whence

$$0 = \mathrm{ind}\,(QA_hQ + P) = \mathrm{ind}_-(A_h) = \mathrm{ind}_-(A)$$

by Theorem 2.3. Since A is invertible, this implies that $\mathrm{ind}_+(A) = 0$.

Let, conversely, A be invertible and $\mathrm{ind}_+(A) = 0$, and let A_h be a limit operator in $\sigma_\pm(A)$. Then also $\mathrm{ind}_-(A) = 0$, and we get as above that

$$0 = \mathrm{ind}_\pm(A) = \mathrm{ind}_\pm(A_h),$$

whence

$$\mathrm{ind}\,(PA_hP + Q) = \mathrm{ind}\,(QA_hQ + P) = 0.$$

The shift invariance of the limit operators of A implies that the operators PA_hP+Q and $QA_hQ + P$ can be identified with Toeplitz operators. Since the indices of these operators are zero, Coburn's theorem implies the invertibility of these operators. The uniform boundedness of the inverses of the operators $PA_hP + Q$ and $QA_hQ + P$ can be verified as in the proof of Theorem 1.1 (where one now has to work with both fibers $M^\pm(SO_A)$). ■

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