Fundamentals of direct limit Lie theory

Helge Glöckner, March 4, 2004

Abstract. We show that every countable direct system of finite-dimensional real or complex Lie groups has a direct limit in the category of Lie groups modelled on locally convex spaces. This enables us to push all basic constructions of finite-dimensional Lie theory to the case of direct limit groups. In particular, we obtain an analogue of Lie's third theorem: Every countable-dimensional locally finite real or complex Lie algebra arises as the Lie algebra of some regular Lie group (a suitable direct limit group).

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Introduction

In this paper, we develop the foundations of Lie theory for countable direct limits of finite-dimensional Lie groups. For the purposes of this introduction, consider an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of finite-dimensional real Lie groups, such that the inclusion maps are smooth homomorphisms. Then $G := \bigcup_{n\in\mathbb{N}} G_n$ is a group in a natural way, and it becomes a topological group when equipped with the final topology with respect to the inclusion maps $G_n \to G$ ([20], [32]). A simple example is $\mathrm{GL}_\infty(\mathbb{R})$, the group of invertible matrices of countable size, differing from the unit matrix at only finitely many places. Here $\mathrm{GL}_\infty(\mathbb{R}) = \bigcup_n \mathrm{GL}_n(\mathbb{R})$, where $\mathrm{GL}_1(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R}) \subseteq \cdots$ identifying $A \in \mathrm{GL}_n(\mathbb{R})$ with $\mathrm{diag}(A,1) \in \mathrm{GL}_{n+1}(\mathbb{R})$. Our goal is to make $G = \bigcup_n G_n$ a (usually infinite-dimensional) Lie group, and to discuss the fundamental constructions of Lie theory for such groups.

Existing methods. Provided certain technical conditions are satisfied (ensuring in particular that $\exp_G := \varinjlim \exp_{G_n} : \varinjlim L(G_n) \to \varinjlim G_n = G$ is a local homeomorphism at 0), the map $\exp_G$ restricts to a chart making G a Lie group (see [27], [28] and [29, Appendix]). This method applies, in particular, to $\mathrm{GL}_\infty(\mathbb{R})$ and other direct limits of linear Lie groups. It produces Lie groups which are not only smooth, but real analytic in the sense of convenient differential calculus [10, Rem. 6.5]. It is also known that every Lie subalgebra of $\mathfrak{gl}_\infty(\mathbb{R}) := \varinjlim \mathfrak{gl}_n(\mathbb{R})$ integrates to a subgroup of $\mathrm{GL}_\infty(\mathbb{R})$ [23, Thm. 47.9]; this provides an alternative construction of the Lie group structure on various direct limit groups. But neither of these methods is general enough to tackle arbitrary direct limits of Lie groups. In particular, examples show that $\exp_G$ need not be injective on any 0-neighbourhood [10, Example 5.5]. Therefore a general construction of a Lie group structure on $G = \bigcup_n G_n$ cannot make use of $\exp_G$.

A general construction principle. In [10], a smooth Lie group structure on $G = \bigcup_n G_n$ was constructed in the case where all inclusion maps are embeddings (for "strict" direct systems). Strict direct limits of Lie groups are discussed there as special cases of direct limits of direct sequences $M_1 \subseteq M_2 \subseteq \cdots$ of finite-dimensional smooth manifolds and embeddings onto closed submanifolds. To make $M := \bigcup_n M_n$ a smooth manifold, one starts with a chart ϕ_{n_0} of some M_{n_0} and then uses tubular neighbourhoods to extend ϕ_n already constructed (possibly restricted to a smaller open set) to a chart ϕ_{n+1} of M_{n+1}. Then $\varinjlim \phi_n$ is a chart for M. In the present article, we generalize this construction principle in two ways. First, we are able to remove the strictness condition. This facilitates to make $\bigcup_n M_n$ a smoothly paracompact, smooth manifold, for any ascending sequence of paracompact, finite-dimensional smooth manifolds and injective immersions (Theorem 3.1, Proposition 3.6). Second, we generalize the method from the case of smooth manifolds over $\mathbb{R}$ to the case of real- and complex analytic manifolds (Theorem 3.1, Proposition 3.8). This enables us to turn $G := \bigcup_n G_n$ into a real analytic Lie group in the sense of convenient differential calculus, resp., a complex Lie group, for any ascending sequence of finite-dimensional real or complex Lie groups (Theorem 4.3).[1] Each direct limit group G is regular in the convenient sense (the argument from [23, Thm. 47.8] carries over). Moreover, G is a regular Lie group in Milnor's sense (Theorem 8.1): this is much harder to prove.

Lie theory for direct limit groups. Despite the fact that $\exp_G$ need not be well-behaved, all of the basic constructions of finite-dimensional Lie theory can be pushed to the case of direct limit groups $G = \bigcup_n G_n$. Thus, subgroups and Hausdorff quotient groups are Lie groups (Propositions 7.2 and 7.5), a universal complexification $G_{\mathbb{C}}$ exists (Proposition 7.13), subalgebras of $L(G)$ integrate to analytic subgroups (Proposition 7.11), and Lie algebra homomorphisms integrate to group homomorphisms in the expected way (Proposition 7.10). Furthermore (Theorem 5.1), every locally finite real or complex Lie algebra of countable dimension is enlargible, i.e., it arises as the Lie algebra of some Lie group (a suitable direct limit group). Such Lie algebras have been studied by Yu. Bahturin, A. A. Baranov, I. Dimitrov, K.-H. Neeb, I. Penkov, H. Strade, N. Stumme, A. E. Zalesskii, and others. If $H \subseteq G$ is a closed subgroup, then H is a conveniently real analytic ($c^\omega_{\mathbb{R}}$-) submanifold of G. Furthermore, the homogeneous space G/H can be given a $c^\omega_{\mathbb{R}}$-manifold structure making $\pi : G \to G/H$ a $c^\omega_{\mathbb{R}}$-principal bundle (Proposition 7.5). Similar results are available for complex Lie groups. We remark that special cases of complexifications and homogeneous spaces of direct limit groups have already been used in [29] and [35], in the context of a Bott-Borel-Weil theorem, resp., direct limits of principal series representations. Universal complexifications of "linear" direct limit groups $G \subseteq \mathrm{GL}_\infty(\mathbb{R})$ have been discussed in [8], in the framework of BCH-Lie groups. For some special examples of direct limit manifolds of relevance for algebraic topology, see [23, §47].

Variants. Although our main results concern the real and complex cases, some of the constructions apply just as well to Lie groups over local fields (i.e., totally disconnected,

[1] More generally, we can create direct limit Lie groups for arbitrary countable direct systems of finite-dimensional real or complex Lie groups. The bonding maps need not be injective.

locally compact, non-discrete topological fields, such as the p-adic numbers), and are formulated accordingly. Readers mainly interested in the real and complex cases are invited to read "$\mathbb{K}$" as $\mathbb{R}$ or $\mathbb{C}$, ignore the definition of smooth maps over general topological fields, and assume that all Lie groups are modelled on real or complex locally convex spaces.

1 Basic definitions and facts

We are working in two settings of differential calculus in parallel: 1. The Convenient Differential Calculus of Frölicher, Kriegl and Michor. 2. Keller's C^∞_c-theory (going back to Michal and Bastiani), as used, e.g., in [25], [26], [8], [9], and generalized to a general differential calculus over topological fields in [2]. For the basic notions of infinite-dimensional Lie theory ($L(G)$, $\exp_G$, logarithmic derivative, product integral), see [23] and [26].

1.1 Convenient differential calculus. Our source for Convenient Differential Calculus is [23], and we presume familiarity with the basic ideas. The smooth maps and manifolds from convenient calculus will be called $c^\infty_{\mathbb{R}}$*-maps* and $c^\infty_{\mathbb{R}}$*-manifolds* here. Maps and manifolds which are holomorphic in the convenient sense will be called $c^\infty_{\mathbb{C}}$ or $c^\omega_{\mathbb{C}}$. Real analytic maps and manifolds in the convenient sense will be called $c^\omega_{\mathbb{R}}$. Likewise for Lie groups. The regular $c^\infty_{\mathbb{R}}$-Lie groups from convenient calculus (see [23, Defn. 38.4]) will be called $c^\infty_{\mathbb{R}}$*-regular*; we call a $c^\infty_{\mathbb{C}}$-Lie group $c^\infty_{\mathbb{C}}$*-regular* or $c^\omega_{\mathbb{C}}$*-regular* if its underlying $c^\infty_{\mathbb{R}}$-Lie group is $c^\infty_{\mathbb{R}}$-regular. A $c^\omega_{\mathbb{R}}$-Lie group G will be called $c^\omega_{\mathbb{R}}$*-regular* if it is $c^\infty_{\mathbb{R}}$-regular and the right product integral $\mathrm{Evol}^r_G(\gamma)\colon \mathbb{R} \to G$ of each real analytic curve $\gamma\colon \mathbb{R} \to L(G)$ is real analytic. The definitions of $c^\omega_{\mathbb{R}}$-regularity and $c^\omega_{\mathbb{C}}$-regularity ensure the following:

Lemma 1.2 *Given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, suppose that G and H are $c^\omega_{\mathbb{K}}$-Lie groups, where G is simply connected and H is $c^\omega_{\mathbb{K}}$-regular. Then, for every bounded $\mathbb{K}$-Lie algebra homomorphism $\alpha\colon L(G) \to L(H)$, there exists a unique $c^\omega_{\mathbb{K}}$-homomorphism $\beta\colon G \to H$ such that $L(\beta) = \alpha$.*

Proof. By [23, Thm. 40.3], there exists a unique $c^\infty_{\mathbb{R}}$-homomorphism $\beta\colon G \to H$ such that $L(\beta) = \alpha$. If $\mathbb{K} = \mathbb{R}$ and $\gamma\colon \mathbb{R} \to G$ is a real analytic curve, then $\beta \circ \gamma\colon \mathbb{R} \to H$ is a smooth curve with right logarithmic derivative $\delta^r(\beta \circ \gamma) = L(\beta) \circ \delta^r\gamma = \alpha \circ \delta^r\gamma$. Here $\alpha \circ \delta^r\gamma$ is real analytic, whence its right product integral $\beta \circ \gamma$ is real analytic, by $c^\omega_{\mathbb{R}}$-regularity. Hence β is $c^\omega_{\mathbb{R}}$. If $\mathbb{K} = \mathbb{C}$, then β is a $c^\infty_{\mathbb{R}}$-homomorphism such that $T_x(\beta)$ is $\mathbb{C}$-linear for each $x \in G$, as $T_1(\beta) = \alpha$ is $\mathbb{C}$-linear. Hence β is $c^\omega_{\mathbb{C}}$ by [23, Thm. 7.19 (8)]. □

1.3 Keller's C^∞_c-theory and analytic maps. Let E and F be locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $U \subseteq E$ be open and $f\colon U \to F$ be a map. If $\mathbb{K} = \mathbb{R}$ and $r \in \mathbb{N}_0 \cup \{\infty\}$, then f is called $C^r_{\mathbb{R}}$ if it is continuous and, for all $k \in \mathbb{N}_0$ such that $k \leq r$, the iterated directional derivatives $d^kf(x, y_1, \ldots, y_k) := D_{y_1} \cdots D_{y_k} f(x)$ exist for all $x \in U$ and $y_1, \ldots, y_k \in E$, and define a continuous map $d^kf\colon U \times E^k \to F$. The $C^\infty_{\mathbb{R}}$-maps are also called *smooth*. If $\mathbb{K} = \mathbb{C}$, we call f a $C^\infty_{\mathbb{C}}$*-map*, $C^\omega_{\mathbb{C}}$, or *complex analytic*, if it is continuous and given locally by a pointwise convergent series of continuous homogeneous polynomials [3, Defn. 5.6]. If $\mathbb{K} = \mathbb{R}$, we call f *real analytic* or $C^\omega_{\mathbb{R}}$ if it extends to a complex analytic map between open subsets of the complexifications of E and F. See [25], [26], or [9] for further information (also concerning the corresponding smooth and $\mathbb{K}$-analytic Lie groups and manifolds).

1.4 General differential calculus. Let E and F be (Hausdorff) topological vector spaces over a non-discrete topological field $\mathbb{K}$, $U \subseteq E$ be open, and $f : U \to F$ a map. According to [2], f is called $C^1_{\mathbb{K}}$ if it is continuous and there exists a (necessarily unique) continuous map $f^{[1]} \colon U^{[1]} \to F$ on $U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{K} \colon x + ty \in U\}$ such that $f^{[1]}(x, y, t) = \frac{1}{t}(f(x + ty) - f(x))$ for all $(x, y, t) \in U^{[1]}$ such that $t \neq 0$. Inductively, f is called $C^{k+1}_{\mathbb{K}}$ if it is $C^1_{\mathbb{K}}$ and $f^{[1]}$ is $C^k_{\mathbb{K}}$; it is $C^\infty_{\mathbb{K}}$ if it is $C^k_{\mathbb{K}}$ for all k. As shown in [2], compositions of $C^k_{\mathbb{K}}$-maps are $C^k_{\mathbb{K}}$, and being $C^k_{\mathbb{K}}$ is a local property. For maps between open subsets of locally convex spaces, the present definitions of $C^k_{\mathbb{R}}$-maps and $C^\infty_{\mathbb{C}}$-maps are equivalent to those from **1.3** ([2], Prop. 7.4 and 7.7). Analytic maps between open subsets of Banach spaces over a complete valued field $\mathbb{K}$ (as used in [4] or [31]) are $C^\infty_{\mathbb{K}}$ [2, Prop. 7.20]. For further information, also concerning $C^\infty_{\mathbb{K}}$-manifolds and Lie groups modelled on topological $\mathbb{K}$-vector spaces, we refer to [2], [11], [12], and [13].

1.5 Direct limits. A *direct system* in a category $\mathbb{A}$ is a pair $\mathcal{S} = ((X_i)_{i\in I}, (\phi_{i,j})_{i\geq j})$, where $(I, \leq)$ is a directed set, each X_i an object of $\mathbb{A}$, and each $\phi_{i,j} \colon X_j \to X_i$ a morphism ("bonding map") such that $\phi_{i,i} = \mathrm{id}_{X_i}$ and $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ if $i \geq j \geq k$. A *cone over* $\mathcal{S}$ is a pair $(X, (\phi_i)_{i\in I})$, where $X \in \operatorname{ob} \mathbb{A}$ and $\phi_i \colon X_i \to X$ is a morphism for $i \in I$ such that $\phi_i \circ \phi_{i,j} = \phi_j$ if $i \geq j$. A cone $(X, (\phi_i)_{i\in I})$ is a *direct limit cone* over $\mathcal{S}$ in the category $\mathbb{A}$ if, for every cone $(Y, (\psi_i)_{i\in I})$ over $\mathcal{S}$, there exists a unique morphism $\psi \colon X \to Y$ such that $\psi \circ \phi_i = \psi_i$ for each i. We then write $(X, (\phi_i)_{i\in I}) = \varinjlim \mathcal{S}$. If the bonding maps and "limit maps" ϕ_i are understood, we simply call X the *direct limit* of $\mathcal{S}$ and write $X = \varinjlim X_i$. If also $\mathcal{T} = ((Y_i)_{i\in I}, (\psi_{i,j})_{i\leq j})$ is a direct system over I and $(Y, (\psi_i)_{i\in I})$ a cone over $\mathcal{T}$, we call a family $(\eta_i)_{i\in I}$ of morphisms $\eta_i \colon X_i \to Y_i$ *compatible* if $\eta_i \circ \phi_{i,j} = \psi_{i,j} \circ \eta_j$ for $i \geq j$. Then $(Y, (\psi_i \circ \eta_i)_{i\in I})$ is a cone over $\mathcal{S}$; write $\varinjlim \eta_i := \eta$ for the morphism $\eta \colon X \to Y$ such that $\eta \circ \phi_i = \psi_i \circ \eta_i$. If there is a compatible family $(\eta_i)_{i\in I}$ with each η_i an isomorphism, $\mathcal{S}$ and $\mathcal{T}$ are called *equivalent*. Then $\mathcal{S}$ has a direct limit if and only if so does $\mathcal{T}$; in this case, $\varinjlim \eta_i$ is an isomorphism. Every countable direct set has a *cofinal subsequence*, whence countable direct systems can be replaced by *direct sequences*, viz. $I = (\mathbb{N}, \leq)$.

1.6 Direct limits of sets, topological spaces, and groups. If $\mathcal{S} = ((X_i)_{i\in I}, (\phi_{i,j})_{i\geq j})$ is a direct system of sets, write $(j, x) \sim (k, y)$ if there exists $i \geq j, k$ such that $\phi_{i,j}(x) = \phi_{i,k}(y)$; then $X := \big(\coprod_{i\in I} X_i\big)/\sim$, together with the maps $\phi_i : X_i \to X$, $\phi_i(x) := [(i, x)]$, is the direct limit of $\mathcal{S}$ in the category of sets. Here $X = \bigcup_{i\in I} \phi_i(X_i)$. If each $\phi_{i,j}$ is injective, then so is each ϕ_i, whence $\mathcal{S}$ is equivalent to the direct system of the subsets $\phi_i(X_i) \subseteq X$, together with the inclusion maps. This facilitates to replace injective direct systems by direct systems in which all bonding maps are inclusion maps. If $\mathcal{S} := ((X_i)_{i\in I}, (\phi_{i,j}))$ is a direct system of topological spaces and continuous maps, then the direct limit $(X, (\phi_i)_{i\in I})$ of the underlying sets becomes the direct limit in the category of topological spaces and continuous maps if we equip X with the *DL-topology*, the final topology with respect to the family $(\phi_i)_{i\in I}$. Thus $U \subseteq X$ is open if and only if $\phi^{-1}(U)$ is open in X_i, for each i. If $\mathcal{S}$ is *strict* in the sense that each $\phi_{i,j}$ is a topological embedding, then also each ϕ_i is a topological embedding [28, La. A.5]. If $((G_i)_{i\in I}, (\phi_{i,j})_{i\geq j})$ is a direct system of groups and homomorphisms, then the direct limit $(G, (\phi_i)_{i\in I})$ of the underlying sets becomes the direct limit in the category of groups and homomorphisms when equipped with the unique group structure making each ϕ_i a homomorphism; the group inversion and multiplication on G are $\varinjlim \kappa_i$ and $\varinjlim \mu_i$, in terms of those on the G_i's.

For further information concerning direct limits of topological groups and topological spaces, see [10], [18], [20], and [32].

Lemma 1.7 *Let $X_1 \subseteq X_2 \subseteq \cdots$ be an ascending sequence of topological spaces such that the inclusion maps are continuous; equip $X := \bigcup_{n\in\mathbb{N}} X_n$ with the final topology with respect to the inclusion maps $\lambda_n : X_n \to X$ (the DL-topology). Then the following holds:*

(a) *If each X_n is T_1, then so is X.*

(b) *If $U_n \subseteq X_n$ is open and $U_1 \subseteq U_2 \subseteq \cdots$, then $U := \bigcup_n U_n$ is open in X and the DL-topology on $U = \varinjlim U_n$ coincides with the topology induced by X.*

(c) *If each X_n is locally compact, then X is Hausdorff.*

(d) *If each X_n is T_1 and $K \subseteq X$ is compact, then $K \subseteq X_n$ for some n.*

Proof. (a) Let $x \in X$. Then $\lambda_n^{-1}(\{x\})$ is either $\{x\}$ or empty, hence closed in the T_1-space X_n. Hence $\{x\}$ is closed in X.

(b) and (c): This is proved in [18, Prop. 4.1 (ii)] and [10, La. 3.1] for strict direct sequences, but the strictness is not used in the proofs.

(d) If not, for each n we find $x_n \in K \setminus X_n$. Then $D := \{x_n : n \in \mathbb{N}\} \subseteq K$ is closed in X (and thus compact), as $D \cap X_n$ is finite and thus closed, for each n. On the other hand, $D = \varinjlim (D \cap X_n)$ for the topology induced by X, as D is closed in X. Now $D \cap X_n$ being discrete, this entails D is discrete and hence finite (being also compact). Contradiction. □

1.8 Let E be a countable-dimensional vector space over a non-discrete, locally compact topological field $\mathbb{K}$ (e.g., $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then the finest vector topology on E is locally convex and coincides with the so-called *finite topology*, the final topology with respect to the inclusion maps $F \to E$, where F ranges through the set of finite-dimensional vector subspaces of E (and F is equipped with its canonical Hausdorff vector topology). Thus, the finite topology on E is the DL-topology on $E = \varinjlim F$. See [10] and the references therein for these standard facts. The space $\mathbb{K}^\infty := \mathbb{K}^{(\mathbb{N})} = \varinjlim \mathbb{K}^n$ of finite sequences will always be equipped with the finite topology. We shall frequently identify $\mathbb{K}^n$ with the subspace $\mathbb{K}^n \times \{0\}$ of $\mathbb{K}^\infty$, and $\mathbb{K}^m$ with $\mathbb{K}^m \times \{0\} \subseteq \mathbb{K}^n$ if $n \geq m$.

Lemma 1.9 *Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or a local field, and E be a $\mathbb{K}$-vector space of countable dimension, equipped with the finite topology. Let $E_1 \subseteq E_2 \subseteq \cdots$ be an ascending sequence of vector subspaces of E such that $\bigcup_{n\in\mathbb{N}} E_n = E$, and $U_n \subseteq E_n$ be open subsets such that $U_1 \subseteq U_2 \subseteq \ldots$. Let $f : U \to F$ be a map into a topological $\mathbb{K}$-vector space F on the open subset $U := \bigcup_{n\in\mathbb{N}} U_n$ of E. Then the following holds:*

(a) *Given $r \in \mathbb{N}_0 \cup \{\infty\}$, f is $C^r_{\mathbb{K}}$ if and only if $f_n := f|_{U_n} \colon U_n \to F$ is $C^r_{\mathbb{K}}$ for each n.*

(b) *If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and F is locally convex and Mackey complete, then f is $C^\infty_{\mathbb{K}}$ if and only if it is $c^\infty_{\mathbb{K}}$. Furthermore, f is $c^\omega_{\mathbb{K}}$ if and only if $f|_{U_n}$ is $c^\omega_{\mathbb{K}}$ for each $n \in \mathbb{N}$.*

Proof. (a)[2] We may assume $r < \infty$. Lemma 1.7 (b) settles the case $r = 0$. If $r \geq 1$, note that $U_1^{[1]} \subseteq U_2^{[1]} \subseteq \cdots$ and $U^{[1]} = \bigcup_n U_n^{[1]}$. The product topology on $E \times E \times \mathbb{K}$ is the finite topology (cf. [10, Prop. 3.3]) and hence induces on $U^{[1]}$ the topology making it the direct limit topological space $U^{[1]} = \varinjlim U_n^{[1]}$ (Lemma 1.7 (b)). By induction, the cone $(F, (f_n^{[1]})_{n \in \mathbb{N}})$ of $C^{r-1}_{\mathbb{K}}$-maps induces a $C^{r-1}_{\mathbb{K}}$-map $g \colon U^{[1]} = \varinjlim U_n^{[1]} \to F$, determined by $g|_{U_n^{[1]}} = f_n^{[1]}$. As g is continuous and extends the difference quotient map, f is $C^1_{\mathbb{K}}$ with $f^{[1]} = g$. Now f being $C^1_{\mathbb{K}}$ with $f^{[1]} = g$ of class $C^{r-1}_{\mathbb{K}}$, the map f is $C^r_{\mathbb{K}}$.

(b) If f is $C^\infty_{\mathbb{K}}$, then it is $c^\infty_{\mathbb{K}}$. If f is $c^\infty_{\mathbb{K}}$, then $f|_{U_n}$ is $c^\infty_{\mathbb{K}}$ for each n and thus $C^\infty_{\mathbb{K}}$, as $\dim_{\mathbb{K}}(E_n) < \infty$. Hence f is $C^\infty_{\mathbb{K}}$, by (a). Given a real analytic curve $\gamma \colon \mathbb{R} \to U$ and $t_0 \in \mathbb{R}$, pick an open relatively compact neighbourhood $J \subseteq \mathbb{R}$ of t_0. Then $\gamma(J) \subseteq U_n$ for some n by Lemma 1.7 (d), and thus $\gamma|_J$ is $c^\omega_{\mathbb{R}}$ if so is $f|_{U_n}$. The remainder is now obvious. □

A map $f \colon \mathbb{R}^\infty \to \mathbb{R}$ which is $C^\omega_{\mathbb{R}}$ on each $\mathbb{R}^n$ need not be $C^\omega_{\mathbb{R}}$ [23, Ex. 10.8]. For this reason, we have to work with the weaker concept of $c^\omega_{\mathbb{R}}$-maps.

2 Extension of charts

In this section, we explain how a chart of a submanifold $M_1 \subseteq M_2$ (or its restriction to a slightly smaller open set) can be extended to a chart of M_2.

Lemma 2.1 *Let M_1 and M_2 be finite-dimensional smooth (resp., real analytic) manifolds over $\mathbb{R}$, of dimensions m_1 and m_2, respectively. Assume that $M_1 \subseteq M_2$ and assume that the inclusion map $\lambda \colon M_1 \to M_2$ is a smooth (resp., real analytic) immersion. Let $\phi_1 \colon U_1 \to V_1$ be a chart of M_1, where U_1 is open in $\mathbb{R}^{m_1}$ and V_1 is an open, relatively compact, contractible subset of M_1. Then there exists a chart $\phi_2 \colon U_2 \to V_2$ of M_2 such that $U_2 \cap (\mathbb{R}^{m_1} \times \{0\}) = U_1 \times \{0\}$, $\phi_2(x, 0) = \phi_1(x)$ for all $x \in U_1$, and such that $V_2 \subseteq M_2$ is relatively compact and contractible.*

Proof. Because $C := \overline{V_1} \subseteq M_1$ is compact, the map $\lambda|_C$ is a topological embedding. Now V_1 being open in C, we deduce that $V_1 = \lambda|_C(V_1)$ is open in $\lambda(C)$, whence there exists an open subset $W \subseteq M_2$ such that $W \cap \lambda(C) = V_1$. Since $\lambda(C)$ is closed in M_2, the preceding formula shows that V_1 is closed in W. After shrinking W, we may assume that W is σ-compact, and relatively compact in M_2. Then V_1 is a closed submanifold of the σ-compact, relatively compact, open submanifold W of M_2. *Smooth case*: By [24,

[2] For $\mathbb{K} = \mathbb{R}$ and locally convex F, see also [10], lines preceding La. 4.1. This implies the claim for $\mathbb{K} = \mathbb{C}$, $r = \infty$, F locally convex because then f is $C^\infty_{\mathbb{R}}$ with $df(x, \bullet)$ complex linear for each x (because $df(x, \bullet)|_{E_n} = df_n(x, \bullet)$), whence f is complex analytic by [9, La. 2.5].

Thm. IV.5.1], V_1 admits a smooth tubular neighbourhood in W, i.e., there exists a $C^\infty_{\mathbb{R}}$-diffeomorphism $\psi : V_2 \to P$ from some open neighbourhood V_2 of V_1 in W onto some open neighbourhood P of the zero-section of some smooth vector bundle $\pi : E \to V_1$ over V_1, such that $\psi|_{V_1} = \mathrm{id}_{V_1}$ (identifying V_1 with the zero-section of E). *Real analytic case*: Being σ-compact, W is $C^\omega_{\mathbb{R}}$-diffeomorphic to a closed real analytic submanifold of $\mathbb{R}^k$ for some $k \in \mathbb{N}_0$ (see [16, Thm. 3]), whence W admits a real analytic Riemannian metric g. Using the real analytic Riemannian metric, the classical construction of tubular neighbourhoods provides a real analytic tubular neighbourhood $\psi\colon V_1 \supseteq V_2 \to P \subseteq E$.

In either case, after shrinking V_2 and P, we may assume that P is balanced, i.e., $[-1, 1]P \subseteq P$ (using the scalar multiplication in the fibres of E). Being a vector bundle over a contractible, σ-compact base manifold, E is trivial. This is well-known in the smooth case [21, Cor. 4.2.5]. For the real analytic case, note that E is associated to a real analytic $\mathrm{GL}(F)$-principal bundle over the σ-compact, contractible $C^\omega_{\mathbb{R}}$-manifold V_1, where $F := \mathbb{R}^{m_2-m_1}$ is the fibre of E. This principal bundle is trivial by [33, Teorema 5] (combined with [21, Cor. 4.2.5]), and hence so is E. (Compare also [1] and [17]).

By the preceding, we find an isomorphism of smooth (resp., real analytic) vector bundles $\theta : E \to V_1 \times \mathbb{R}^{m_2-m_1}$. Then $\kappa : \phi_1^{-1} \times \mathrm{id} : V_1 \times \mathbb{R}^{m_2-m_1} \to U_1 \times \mathbb{R}^{m_2-m_1} \subseteq \mathbb{R}^{m_2}$ is a $C^\infty_{\mathbb{R}}$- (resp., $C^\omega_{\mathbb{R}}$-) diffeomorphism, and $U_2 := \kappa(\theta(P))$ is an open subset of $\mathbb{R}^{m_2}$ such that $U_2 \cap (\mathbb{R}^{m_1} \times \{0\}) = U_1$. Then $\phi_2 := (\kappa \circ \theta \circ \psi)^{-1}|^{V_2}_{U_2} : U_2 \to V_2$ is a $C^\infty_{\mathbb{R}}$- (resp., $C^\omega_{\mathbb{R}}$-) diffeomorphism from U_2 onto the open subset V_2 of M_2, such that $\phi_2(x, 0) = \phi_1(x)$ for all $x \in U_1$. Since $V_2 \subseteq W$, the set V_2 is relatively compact in M_2. To see that V_2 is contractible, we only need to show that so is P, as V_2 and P are homeomorphic. Let $H\colon [0, 1] \times V_1 \to V_1$ be a homotopy from id_{V_1} to a constant map. The map $[0, 1] \times P \to P$, $(t, x) \mapsto (1 - t)x$ (which uses scalar multiplication in the fibres) is a homotopy from id_P to $\pi|_P$. The map $[0, 1] \times P \to P$, $(t, x) \mapsto H(t, \pi(x))$ is a homotopy from $\pi|_P$ to a constant map. Thus id_P is homotopic to a constant map and thus P is contractible. □

Definition 2.2 Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or a local field, and $|.|$ be an absolute value on $\mathbb{K}$ defining its topology. Given $n \in \mathbb{N}_0$ and $r > 0$, we let

$$\Delta^n_r := \{(x_1, \ldots, x_n) \in \mathbb{K}^n \colon\ |x_j| < r \text{ for all } j = 1, \ldots, n\}$$

be the n-dimensional polydisk of radius r around 0. If we wish to emphasize the ground field, we also write $\Delta^n_r(\mathbb{K})$ for Δ^n_r.

If $\mathbb{K}$ is a local field, we define $C^\infty_{\mathbb{K}}$-immersions (and $C^\infty_{\mathbb{K}}$-submersions) between finite-dimensional $C^\infty_{\mathbb{K}}$-manifolds analogous to the $\mathbb{K}$-analytic case [31]. Because an Inverse Function Theorem holds for $C^\infty_{\mathbb{K}}$-maps [11], $C^\infty_{\mathbb{K}}$-immersions and submersions have the usual properties.

Lemma 2.3 (Extension Lemma) *Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or a local field. Let M be a finite-dimensional $C^\infty_{\mathbb{K}}$-manifold (or a finite-dimensional real analytic manifold), of dimension $m \in \mathbb{N}_0$, and $\phi : \Delta^n_r \to M$ be a $C^\infty_{\mathbb{K}}$ (resp., real analytic) injective immersion, where $n \in \{0, 1, \ldots, m\}$ and $r > 0$. Then, for every $s \in \,]0, r[$, there exists a $C^\infty_{\mathbb{K}}$-diffeomorphism* (*resp., a real analytic diffeomorphism*) $\psi\colon \Delta^m_s \to V$ *onto an open subset* V *of* M *such that* $\psi(x,0)=\phi(x)$ *for all* $x\in\Delta^n_s$. *If* $\mathbb{K}$ *is a local field, the conclusion remains valid for* $s=r$. *The subset* $V\subseteq M$ *can be chosen relatively compact.*

Proof. Let $s\in\,]0,r[$ and $t\in\,]s,r[$.

The case of smooth or analytic manifolds over $\mathbb{K}=\mathbb{R}$. We equip $M_1:=\phi(\Delta^n_r)$ with the smooth (resp., real analytic) manifold structure making $\phi|^{M_1}\colon \Delta^n_r\to M_1$ a diffeomorphism. Then the inclusion map $\lambda\colon M_1\to M$ is an immersion, $V_1:=\phi(\Delta^n_t)$ is a relatively compact, contractible, σ-compact open subset of M_1, and $\phi_1:=\phi|^{V_1}_{\Delta^n_t}\colon \Delta^n_t\to V_1$ is a chart for M_1. By Lemma 2.1, there exists a $C^\infty_{\mathbb{R}}$- (resp., $C^\omega_{\mathbb{R}}$-) diffeomorphism $\phi_2\colon U_2\to V_2$ from an open subset U_2 of $\mathbb{R}^m$ onto an open subset V_2 of M such that $U_2\cap(\mathbb{R}^n\times\{0\})=\Delta^n_t\times\{0\}$ and $\phi_2(x,0)=\phi_1(x)=\phi(x)$ for all $x\in\Delta^n_t$. Now $\overline{\Delta^n_s}\subseteq\mathbb{R}^n$ being compact, we find $\varepsilon>0$ such that $\overline{\Delta^n_s}\times\Delta^{m-n}_\varepsilon\subseteq U_2$. Then

$$\psi\colon \Delta^m_s\to M,\quad \psi(x,y):=\phi_2(x,\tfrac{\varepsilon}{s}y)\qquad \text{for } x\in\Delta^n_s,\ y\in\Delta^{m-n}_s$$

is a mapping with the required properties.

The case $\mathbb{K}=\mathbb{C}$. The map $\phi|_{\Delta^n_t}$ is an embedding of complex manifolds, and hence so is $f:\Delta^n_1\to M$, $f(x):=\phi(tx)$. By [30, Prop. 1], there exists a holomorphic embedding $F:\Delta^m_{s/t}\times\Delta^{n-m}_1\to M$ such that $F(x,0)=f(x)$ for all $x\in\Delta^n_{s/t}$. Then $\psi:\Delta^m_s\to M$, $\psi(x,y):=F(\frac{1}{t}x,\frac{1}{s}y)$ (where $x\in\Delta^n_s$, $y\in\Delta^{m-n}_s$) is a holomorphic embedding with the desired properties.

Relative compactness of V. By the real or complex case already discussed, there exists an extension $\widetilde{\psi}\colon\Delta^m_t\to\widetilde{V}$ of $\phi|_{\Delta^n_t}$. Then $V:=\widetilde{\psi}(\Delta^m_s)$ is a relatively compact open subset of M, and $\psi:=\widetilde{\psi}|^V_{\Delta^m_s}$ has the desired properties.

The case where $\mathbb{K}$ *is a local field.* In this case, Δ^n_r is compact, whence ϕ is a $C^\infty_{\mathbb{K}}$-diffeomorphism from Δ^n_r onto the compact $C^\infty_{\mathbb{K}}$-submanifold $M_1:=\operatorname{im}\phi$ of M. The proof of [10, La. 8.1] (tackling the $\mathbb{K}$-analytic case) carries over verbatim to the case of $C^\infty_{\mathbb{K}}$-manifolds; we therefore find a $C^\infty_{\mathbb{K}}$-diffeomorphism $\theta\colon\Delta^n_r\times\mathbb{O}^{m-n}\to M$ such that $\theta(x,0)=\phi(x)$, where $\mathbb{O}$ is the maximal compact subring of $\mathbb{K}$. Pick $a\in\mathbb{K}^\times$ such that $a\Delta^{m-n}_r\subseteq\mathbb{O}^{m-n}$; then $\psi\colon\Delta^m_r\to M$, $\psi(x,y):=\theta(x,ay)$ for $x\in\Delta^n_r$, $y\in\Delta^{m-n}_r$ (resp., its restriction to Δ^m_s) is the required chart for M. □

3 Direct limits of finite-dimensional manifolds

Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or a local field. Throughout this section, we let $\mathcal{S}:=((M_i)_{i\in I},(\lambda_{i,j})_{i\geq j})$ be a direct system of finite-dimensional $C^\infty_{\mathbb{K}}$-manifolds M_i and injective $C^\infty_{\mathbb{K}}$-immersions $\lambda_{i,j}\colon M_j\to M_i$. We let $(M,(\lambda_i)_{i\in I})$ be the direct limit of $\mathcal{S}$ in the category of topological spaces, and abbreviate $s:=\sup\{\dim_{\mathbb{K}}(M_i)\colon i\in I\}\in\mathbb{N}_0\cup\{\infty\}$. Our goal is to make M a manifold, and study its basic properties.

Theorem 3.1 *There exists a uniquely determined $C^\infty_{\mathbb{K}}$-manifold structure on M, modelled on the complete, locally convex topological $\mathbb{K}$-vector space $\mathbb{K}^s$, which makes $\lambda_i \colon M_i \to M$ a $C^\infty_{\mathbb{K}}$-map, for each $i \in I$, and such that $(M, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of $C^\infty_{\mathbb{K}}$-manifolds modelled on topological $\mathbb{K}$-vector spaces (and $C^\infty_{\mathbb{K}}$-maps). For each $i \in I$ and $x \in M_i$, the differential $T_x(\lambda_i) : T_x(M_i) \to T_{\lambda_i(x)}(M)$ is injective. For each $r \in \mathbb{N}_0$, the $C^r_{\mathbb{K}}$-manifold underlying M satisfies $(M, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of $C^r_{\mathbb{K}}$-manifolds modelled on topological $\mathbb{K}$-vector spaces.*

Proof. After passing to a cofinal subsequence of an equivalent direct system (cf. **1.6**), we may assume without loss of generality that $I = \mathbb{N}$, $M_1 \subseteq M_2 \subseteq \cdots$, and that the immersion $\lambda_{n,m}$ is the inclusion map for all $n, m \in \mathbb{N}$ such that $n \geq m$. We let $M := \bigcup_{n\in\mathbb{N}} M_n$, equipped with the final topology with respect to the inclusion maps $\lambda_n \colon M_n \to M$; then $(M, (\lambda_n)_{n\in\mathbb{N}}) = \varinjlim ((M_n), (\lambda_{n,m}))$ in the category of topological spaces. We abbreviate $d_n := \dim_{\mathbb{K}}(M_n)$ and $c_n := d_{n+1} - d_n$.

Let $\mathcal{A}$ be the set of all maps $\phi : P_\phi \to Q_\phi \subseteq M$ such that $P_\phi = \bigcup_{n\in\mathbb{N}} P_n \subseteq \mathbb{K}^s$, $Q_\phi = \bigcup_{n\in\mathbb{N}} Q_n$, and $\phi = \varinjlim \phi_n$ for some sequence $(\phi_n)_{n\in\mathbb{N}}$ of charts $\phi_n \colon P_n \to Q_n \subseteq M_n$, where each P_n is an open (possibly empty) subset of $\mathbb{K}^{d_n}$, Q_n open in M_n, and $Q_m \subseteq Q_n$ and $\phi_n|_{Q_m} = \phi_m$ whenever $n \geq m$. Here Lemma 1.7 (b) allows us to interpret the open subsets $P_\phi \subseteq \mathbb{K}^s$ and $Q_\phi \subseteq M$ as the direct limits $\varinjlim Q_n$ and $\varinjlim P_n$ in the category of topological spaces, whence ϕ is continuous. Because each ϕ_n is injective, also ϕ is injective, and furthermore ϕ is surjective, by definition of Q_ϕ. If $V \subseteq P_\phi$ is open, then $V \cap P_n$ is open in P_n, whence $S_n := \phi_n(V \cap P_n)$ is open in Q_n. Because $S_1 \subseteq S_2 \subseteq \cdots$, the union $\phi(V) = \bigcup_{n\in\mathbb{N}} S_n$ is open in Q_ϕ (Lemma 1.7 (b)). Thus ϕ is an open map. We have shown that ϕ is a homeomorphism.

We claim that $\mathcal{A}$ is a $C^\infty_{\mathbb{K}}$-atlas for M. We first show that $\bigcup_{\phi\in\mathcal{A}} Q_\phi = M$. To this end, let $x \in M$. Then there exists $\ell \in \mathbb{N}_0$ such that $x \in M_\ell$. Define $r_n := 1 + 2^{-n}$ for $n \in \mathbb{N}$. We let $\phi_n : P_n \to Q_n$ be the chart of M_n with $P_n := Q_n := \emptyset$, for all $n < \ell$. We pick a chart $\psi_\ell \colon \Delta^{d_\ell}_{r_\ell}(\mathbb{K}) \to W_\ell \subseteq M_\ell$ of M_ℓ around x, such that $\psi_\ell(0) = x$. Inductively, the Extension Lemma 2.3 provides charts $\psi_n \colon \Delta^{d_n}_{r_n} \to W_n \subseteq M_n$ for $n \in \{\ell+1, \ell+2, \dots\}$ such that $\psi_n|_{\Delta^{d_{n-1}}_{r_n}} = \psi_{n-1}|_{\Delta^{d_{n-1}}_{r_n}}$ (identifying $\mathbb{K}^{d_{n-1}}$ with $\mathbb{K}^{d_{n-1}} \times \{0\} \subseteq \mathbb{K}^{d_n}$). Define $P_n := \Delta^{d_n}_1$, $Q_n := \psi_n(P_n)$, and $\phi_n := \psi_n|^{Q_n}_{P_n} : P_n \to Q_n$ for $n \geq \ell$. Then $P_\phi := \bigcup_{n\in\mathbb{N}} P_n$ is open in $\mathbb{K}^s$, $Q_\phi := \bigcup_{n\in\mathbb{N}} Q_n$ is open in M, and $\phi := \varinjlim \phi_n \colon P_\phi \to Q_\phi$ is an element of $\mathcal{A}$, with $x \in Q_\phi$, as desired.

Compatibility of the charts. Assume that $\phi := \varinjlim \phi_n : P_\phi \to Q_\phi$ and $\psi := \varinjlim \psi_n : P_\psi \to Q_\psi$ are elements of $\mathcal{A}$, where $\phi_n : P_n \to Q_n$ and $\psi_n : A_n \to B_n$. Suppose that $x \in \phi^{-1}(Q_\psi)$. Then $\phi(x) \in Q_\phi \cap Q_\psi$, entailing that there exists $\ell \in \mathbb{N}$ such that $\phi(x) \in Q_\ell \cap B_\ell$. Then $x \in P_n \cap \phi_n^{-1}(B_n) =: X_n$ for all $n \geq \ell$. Since X_n is open in $\mathbb{K}^{d_n}$ and $X_\ell \subseteq X_{\ell+1} \subseteq \cdots$, the union $X := \bigcup_{n\geq\ell} X_n$ is open in $\mathbb{K}^s$. Furthermore, the coordinate changes $\tau_n := \psi_n^{-1}|_{Q_n\cap B_n} \circ \phi_n|_{X_n} : X_n \to \psi_n^{-1}(Q_n) =: Y_n$ are $C^\infty_{\mathbb{K}}$-diffeomorphisms, for all $n \geq \ell$. By Lemma 1.9 (a), the map $\psi^{-1}|^Y_{\phi(X)} \circ \phi|_X = \varinjlim_{n\geq\ell} \tau_n \colon X \to \bigcup_{n\geq\ell} Y_n =: Y$ is $C^\infty_{\mathbb{K}}$, entailing that the bijection $\tau := \psi^{-1}|_{Q_\phi\cap Q_\psi} \circ \phi|_{\phi^{-1}(Q_\psi)} \colon \phi^{-1}(Q_\psi) \to \psi^{-1}(Q_\phi)$ is $C^\infty_{\mathbb{K}}$ on some open neighbourhood of x. As x was arbitrary, τ is $C^\infty_{\mathbb{K}}$ and the same reasoning shows that so is τ^{-1}. Thus $\mathcal{A}$ is an atlas making M a $C^\infty_{\mathbb{K}}$-manifold modelled on $\mathbb{K}^s$.

Each λ_n is smooth. To see this, assume that $n \in \mathbb{N}$ and $x \in M_n$. As just shown, there exists a chart $\phi\colon P_\phi \to Q_\phi$ in $\mathcal{A}$, say $\phi = \varinjlim \phi_k$ with charts $\phi_k\colon P_k \to Q_k \subseteq M_k$ for $k \in \mathbb{N}$, such that $x \in P_n$. Then $\phi^{-1} \circ \lambda_n \circ \phi_n = \overrightarrow{\phi}^{-1} \circ \phi_n : \mathbb{K}^{d_n} \supseteq P_n \to P \subseteq \mathbb{K}^s$ is the inclusion map and hence smooth, and its differential at x is injective. Hence λ_n is smooth on the open neighbourhood Q_n of x, and $T_x(\lambda_n)$ is injective. As x was arbitrary, λ_n is smooth.

Direct limit property and uniqueness. Fix $r \in \mathbb{N}_0 \cup \{\infty\}$. Assume that Y is a $C^r_{\mathbb{K}}$-manifold modelled on a topological $\mathbb{K}$-vector space E and $f_n : M_n \to Y$ a $C^r_{\mathbb{K}}$-map for each $n \in \mathbb{N}$ such that $(Y, (f_n)_{n\in\mathbb{N}})$ is a cone over $\mathcal{S}$; thus $f_n|_{M_m} = f_m$ if $n \geq m$. Then there is a uniquely determined map $f\colon M \to Y$ such that $f|_{M_n} = f_n$ for all $n \in \mathbb{N}$. Since $M = \varinjlim M_n$ as a topological space, f is continuous. If $x \in M$, we find a chart $\phi\colon P_\phi \to Q_\phi$ of M around x in the atlas $\mathcal{A}$, where $\phi = \varinjlim \phi_n$ for charts $\phi_n : P_n \to Q_n \subseteq M_n$. Let $\psi : V \to W \subseteq Y$ be a chart for Y, where $V \subseteq E$ is open. Then $U := (f \circ \phi)^{-1}(W)$ is an open subset of $P_\phi \subseteq \mathbb{K}^s$, and $U_n := U \cap P_n$ is open in $P_n \subseteq \mathbb{K}^{d_n}$ for each n. Consider $g := \psi^{-1} \circ (f \circ \phi)|_U^W : U \to V$. Then $g|_{U_n} = \psi^{-1} \circ (f_n \circ \phi_n)|_{U_n}^W : U_n \to V$ is $C^r_{\mathbb{K}}$ for each $n \in \mathbb{N}$. Hence g is $C^r_{\mathbb{K}}$ by Lemma 1.9 (a), whence so is f on the open neighbourhood Q_ϕ of x and hence on all of M, as x was arbitrary. Thus $(M, (\lambda_n)_{n\in\mathbb{N}}) = \varinjlim \mathcal{S}$ in the category of $C^r_{\mathbb{K}}$-manifolds, for all $r \in \mathbb{N}_0 \cup \{\infty\}$. The uniqueness of a $C^\infty_{\mathbb{K}}$-manifold structure on M with the described properties follows from the universal property of direct limits. □

Convention. Throughout the remainder of this section, M will be equipped with the $C^\infty_{\mathbb{K}}$-manifold structure just defined. In the proofs, we shall always reduce to the case where $I = \mathbb{N}$ and $M_1 \subseteq M_2 \subseteq \ldots$ (by the above argument), without further mention.

Proposition 3.2 *If $\mathbb{F} \subseteq \mathbb{K}$ is a non-discrete, closed subfield, then $(M, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ also in the category of $C^\infty_{\mathbb{F}}$-manifolds. (E.g., $\mathbb{K} = \mathbb{C}$, $\mathbb{F} = \mathbb{R}$).*

Proof. Let $\mathcal{A}$ be the $C^\infty_{\mathbb{K}}$-atlas of M described in the proof of Theorem 3.1. Given a non-discrete closed subfield $\mathbb{F} \subseteq \mathbb{K}$, let $\mathcal{A}_{\mathbb{F}}$ be the corresponding atlas obtained when considering each M_i merely as a $C^\infty_{\mathbb{F}}$-manifold over $\mathbb{F}$. Then $\mathcal{A} \subseteq \mathcal{A}_{\mathbb{F}}$, entailing that $(M, (\lambda_n)_{n\in N}) = \varinjlim \mathcal{S}$ also in the category of $C^\infty_{\mathbb{F}}$-manifolds. □

Proposition 3.3 *Assume that $U_i \subseteq M_i$ is open and $\lambda_{i,j}(U_j) \subseteq U_i$ whenever $i \geq j$. Then $U := \bigcup_{i\in I} U_i$ is open in M. For the $C^\infty_{\mathbb{K}}$-manifold structure induced by M on its open subset U, we have $(U, (\lambda_i|_{U_i}^U)_{i\in I}) = \varinjlim((U_i)_{i\in I}, (\lambda_{i,j}|_{U_j}^{U_i})_{i\geq j})$ in the category of $C^\infty_{\mathbb{K}}$-manifolds.*

Proof. Given open subsets $U_n \subseteq M_n$ such that $M_1 \subseteq M_2 \subseteq \cdots$, their union $U := \bigcup_{n\in\mathbb{N}} U_n$ is open in M and the topology induced by M on U makes U the direct limit $\varinjlim U_n$ (Lemma 1.7 (b)). We define an atlas $\mathcal{A}_U$ for U turning U into the direct limit of the $C^\infty_{\mathbb{K}}$-manifolds U_n, analogous to the definition of $\mathcal{A}$ in the proof of (a). Then $\mathcal{A}_U \subseteq \mathcal{A}$, whence $(U, \mathcal{A}_U)$ coincides with U, considered as an open submanifold of M. □

Proposition 3.4 *Assume that $f\colon X \to M$ is a $C^r_{\mathbb{K}}$-map, where $r \in \mathbb{N}_0 \cup \{\infty\}$ and X is a $C^r_{\mathbb{K}}$-manifold modelled on a metrizable topological $\mathbb{K}$-vector space E (or a metrizable, locally path-connected topological space, if $r = 0$). Then every $x \in X$ has an open neighbourhood S such that $f(S) \subseteq \lambda_i(M_i)$ for some $i \in \mathbb{N}$ and such that $\lambda_i^{-1} \circ f|_S^{\lambda_i(M_i)}\colon S \to M_i$ is $C^r_{\mathbb{K}}$.*

Proof. Let $x \in X$. The assertion being local, in the case of manifolds we may assume that X is an open subset of E. Choose a metric d on X defining its topology, and $k \in \mathbb{N}$ such that $f(x) \in M_k$. Let $\phi = \varinjlim \phi_n\colon P \to Q$ be a chart of M around $f(x)$, where ϕ_n is a chart of M_n for all $n \geq k$, of the form $\phi_n\colon \Delta_1^{d_n} \to Q_n \subseteq M_n$ (see proof of Theorem 3.1). If $f^{-1}(Q_n)$ is not a neighbourhood of x for any $n \geq k$, we find $x_n \in f^{-1}(Q) \setminus f^{-1}(Q_n)$ such that $d(x_n, x) < 2^{-n}$. Thus $x_n \to x$, entailing that $C := \{f(x_n)\colon n \in \mathbb{N}\} \cup \{f(x)\}$ is a compact subset of Q such that $C \not\subseteq Q_n$ for any $n \geq k$. Since $Q = \varinjlim Q_n$, this contradicts Lemma 1.7 (d). Hence there exists $n \geq k$ such that $f^{-1}(Q_n)$ is a neighbourhood of x. Let $S := (f^{-1}(Q_n))^0$ be its interior. Then $S \to \mathbb{K}^s$, $y \mapsto \phi^{-1}(f(y)) = \phi_n^{-1}(f(y))$ is a $C^r_{\mathbb{K}}$-map taking its values in the closed vector subspace $\mathbb{K}^{d_n}$ of $\mathbb{K}^s$, whence also its co-restriction $\phi_n^{-1} \circ f|_S^{Q_n}\colon S \to \Delta_1^{d_n}$ is $C^r_{\mathbb{K}}$ [2, La. 10.1]. As ϕ_n is a chart, this means that $f|_S^{M_n}$ is $C^r_{\mathbb{K}}$. □

Proposition 3.5 *If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $x \in M$, where $x = \lambda_j(y)$ say, then the connected component C of $x \in M$ in M is $\bigcup_{i \in I} \lambda_i(C_i) \cong \varinjlim_{i \geq j} C_i$, where C_i is the connected component of $\lambda_{i,j}(y)$ in M_i.*

Proof. Given $x \in M_n$, we let C and C_m be its connected component in M and M_m, respectively, for $m \geq n$. Then $\bigcup_{m \geq n} C_m \subseteq C$. If $z \in C$, then we find a continuous curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = z$. Since $[0,1]$ is compact, using Proposition 3.4 we find $m \geq n$ such that $\gamma([0,1]) \subseteq M_m$, and such that $\gamma|^{M_m}\colon [0,1] \to M_m$ is continuous. Thus $z \in C_m$. Hence indeed $C = \bigcup_{m \geq n} C_m$. □

Proposition 3.6 *If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and M_i is paracompact for each $i \in I$, then the $C^\infty_{\mathbb{K}}$-manifold M also is a $c^\infty_{\mathbb{K}}$-manifold, and $(M, (\lambda_i)_{i \in I}) = \varinjlim \mathcal{S}$ in the category of $c^\infty_{\mathbb{K}}$-manifolds. Furthermore, M is smoothly paracompact as a $C^\infty_{\mathbb{R}}$-manifold: For every open cover of M, there exists a $C^\infty_{\mathbb{R}}$-partition of unity subordinate to the cover.*

Proof. Assume that $\mathbb{K} = \mathbb{R}$. In order that M be smoothly paracompact, we only need to show that every connected component C of M is smoothly paracompact. Pick $c \in C$. We may assume that $c \in M_1$ after passing to a cofinal subsystem; we let C_n the connected component of c in M_n for each $n \in \mathbb{N}$. Then $\bigcup_{n \in \mathbb{N}} C_n$ is the connected component of c in M (see Proposition 3.5) and hence coincides with C; furthermore, $C = \varinjlim C_n$, by Proposition 3.3. After replacing M with C and M_n with C_n for each n, we may assume that each M_n is a connected, paracompact finite-dimensional $C^\infty_{\mathbb{R}}$-manifold and hence σ-compact. This entails that $M = \bigcup_{n \in \mathbb{N}} M_n$ is σ-compact and therefore Lindelöf. Hence, by [23, Thm. 16.10], M will be smoothly paracompact if we can show that M is *smoothly* *regular* in the sense that, for every $x \in M$ and open neighbourhood Ω of x in M, there exists a smooth function ("bump function") $f\colon M \to \mathbb{R}$ such that $f(x) \neq 0$ and $f|_{M\setminus\Omega} = 0$.

If each $\lambda_{n,m}$ is a topological embedding onto a closed submanifold, then M is a regular topological space (see [18, Prop. 4.3 (ii)]), whence smooth regularity passes from the modelling space[3] to M (cf. [10, proof of Thm. 6.4]). In the fully general case to be investigated here, we do not know *a priori* that M is regular, whence we have to prove smooth regularity of M by hand. Essentially, we need to go once more through our construction of charts and build up bump functions step by step. Let $x \in M$ and Ω be an open neighbourhood of x in M. Passing to a co-final subsequence, we may assume that $x \in M_1$.

Let $r_n := 1 + 2^{-n}$ for $n \in \mathbb{N}$ and $\Delta^{d_n}_{r_n} := \Delta^{d_n}_{r_n}(\mathbb{R})$. Pick a chart $\psi_1\colon \Delta^{d_1}_{r_1} \to W_1 \subseteq M_1$ of M_1 around x, such that $\psi_1(0) = x$ and such that W_1 is relatively compact in $M_1 \cap \Omega$. Define $Q_1 := \psi_1(\Delta^{d_1}_1)$. We choose compact subsets $K_{1,j}$ of M_1 such that $W_1 \subseteq K^0_{1,1} \subseteq K_{1,1} \subseteq K^0_{1,2} \subseteq K_{1,2} \subseteq K^0_{1,3} \subseteq \cdots$ and $M_1 = \bigcup_{j\in\mathbb{N}} K_{1,j}$. There exists a smooth function $h_1\colon \Delta^{d_1}_{r_1} \to \mathbb{R}$ such that $\operatorname{supp}(h_1) \subseteq \Delta^{d_1}_1$ and $h_1(0) = 1$. Define $f_1\colon M_1 \to \mathbb{R}$, $f_1(y) := 0$ if $y \notin W_1$, $f_1(y) := h_1(\psi_1^{-1}(y))$ if $y \in W_1$. Then f_1 is smooth, $\operatorname{supp}(f_1) \subseteq Q_1$, and $f_1(x) = 1$.

The Extension Lemma 2.3 provides a chart $\psi_2 : \Delta^{d_2}_{r_2} \to W_2 \subseteq M_2$ onto an open, relatively compact subset W_2 of $M_2 \cap \Omega$ such that $\psi_2|_{\Delta^{d_1}_{r_2}} = \psi_1|_{\Delta^{d_1}_{r_2}}$. We choose compact subsets $K_{2,j}$ of M_2 such that $K_{1,j} \subseteq K^0_{2,j}$ and $W_2 \subseteq K^0_{2,1} \subseteq K_{2,1} \subseteq K^0_{2,2} \subseteq K_{2,2} \subseteq \cdots$ and $M_2 = \bigcup_{j\in\mathbb{N}} K_{2,j}$. Then $K_{1,1} \setminus Q_1$ is a compact subset of M_1 and hence also of M_2. Therefore $A := \psi_2^{-1}(K_{1,1} \setminus Q_1)$ is closed in $\Delta^{d_2}_{r_2}$, and it does not meet the compact subset $\operatorname{supp}(h_1) \subseteq \Delta^{d_1}_1 \subseteq \Delta^{d_2}_{r_2}$ (which is mapped into Q_1 by ψ_2). Hence, there exists $\varepsilon \in \,]0, 1[$ such that $A \cap (\operatorname{supp}(h_1) \times \Delta^{d_2-d_1}_{\varepsilon}) = \emptyset$. We let $\xi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\xi(0) = 1$ and $\operatorname{supp}(\xi) \subseteq \,]-\varepsilon^2, \varepsilon^2[$. Then

$$h_2\colon \Delta^{d_2}_{r_2} \to \mathbb{R}, \qquad h_2(y, z) := h_1(y) \cdot \xi((\|z\|_2)^2) \quad \text{for } y \in \Delta^{d_1}_{r_2},\ z \in \Delta^{d_2-d_1}_{r_2}$$

(where $\|.\|_2$ is the euclidean norm on $\mathbb{R}^{d_2-d_1}$) is a smooth map such that $\operatorname{supp}(h_2) \subseteq \Delta^{d_2}_1$. Then $f_2(y) := 0$ if $y \notin W_2$, $f_2(y) := h_2(\psi_2^{-1}(y))$ for $y \in W_2$ defines a smooth function $f_2\colon M_2 \to \mathbb{R}$. We have $\operatorname{supp}(f_2) \subseteq Q_2 := \psi_2(\Delta^{d_2}_1)$, and $f_2|_{K_{1,1}} = f_1|_{K_{1,1}}$, because $f_2|_{Q_1} = f_1|_{Q_1}$ by definition and also $f_2|_{K_{1,1}\setminus Q_1} = 0 = f_1|_{K_{1,1}\setminus Q_1}$.

Proceeding in this way, we find charts $\psi_n\colon \Delta^{d_n}_{r_n} \to W_n \subseteq M_n$ with relatively compact image $W_n \subseteq \Omega$, compact subsets $K_{n,j}$ of M_n with union M_n such that $W_n \subseteq K_{n,1}$, $K_{n,j} \subseteq K^0_{n,j+1}$ and $K_{n-1,j} \subseteq K^0_{n,j}$ for all $n, j \in \mathbb{N}$, $n \geq 2$; and smooth maps $f_n\colon M_n \to \mathbb{R}$ such that $\operatorname{supp}(f_n) \subseteq Q_n := \psi_n(\Delta^{d_n}_1)$ and $f_{n+1}|_{K_{n,n}} = f_n|_{K_{n,n}}$ for all $n \in \mathbb{N}$, whence

$$f_m|_{K_{n,n}} = f_n|_{K_{n,n}} \qquad \text{for all } n, m \in \mathbb{N} \text{ such that } m \geq n. \tag{1}$$

Let U_n be the interior $K^0_{n,n}$ of $K_{n,n}$ in M_n. Then $U_1 \subseteq U_2 \subseteq \cdots$ and $M = \bigcup_{n\in\mathbb{N}} U_n$, whence $M = \varinjlim U_n$ as a smooth manifold by Proposition 3.3. By (1), the smooth maps $f_n|_{U_n}$ form a cone and hence induce a smooth map $f : M \to \mathbb{R}$, such that $f|_{U_n} = f_n|_{U_n}$

[3]See [23, Thm. 16.10] or [10], proof of Thm. 6.4 for the smooth regularity of $\mathbb{R}^\infty$.

for each $n \in \mathbb{N}$. Then $f(x) = f_1(x) = 1$. If $y \in M \setminus \Omega$, we find $n \in \mathbb{N}$ such that $y \in U_n$. Then $f(y) = f_n(y) = 0$ because $\operatorname{supp}(f_n) \subseteq Q_n \subseteq W_n \subseteq \Omega$. Hence f is a bump function around x carried by Ω, as desired. Thus M is smoothly paracompact.

Direct limit properties when $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Since M (resp., its underlying real manifold) is smoothly regular, M is *smoothly Hausdorff*, i.e., $C^\infty_{\mathbb{R}}(M, \mathbb{R})$ separates points on M. Let $\mathcal{A}$ be an $C^\infty_{\mathbb{K}}$-atlas for M. Being a smoothly Hausdorff $C^\infty_{\mathbb{K}}$-manifold modelled on a Mackey complete locally convex space, $(M, \mathcal{A})$ can be made a $c^\infty_{\mathbb{K}}$-manifold $(c^\infty(M), \mathcal{A})$ by replacing its topology with the final topology with respect to the given charts, when the topology on the modelling space has been replaced with its c^∞-topology. Since $c^\infty(\mathbb{K}^s) = \mathbb{K}^s$, the topology on M remains unchanged, and thus $c^\infty(M) = M$. In view of Lemma 1.9 (b), the desired direct limit properties can be checked as in the proof of Theorem 3.1. □

Proposition 3.7 *Assume that also* $\mathcal{T} := ((N_i)_{i\in I}, (\mu_{i,j})_{i\geq j})$ *is a direct system of finite-dimensional* $C^\infty_{\mathbb{K}}$*-manifolds and injective* $C^\infty_{\mathbb{K}}$*-immersions, over the same index set. Then also* $\mathcal{P} := ((M_i \times N_i)_{i\in I}, (\lambda_{i,j} \times \mu_{i,j})_{i\geq j})$ *is such a direct system. Let* $(N, (\mu_i)) = \varinjlim \mathcal{T}$*. The* $C^\infty_{\mathbb{K}}$*-maps* $\eta_i := \lambda_i \times \mu_i \colon M_i \times N_i \to M \times N$ *define a cone* $(M \times N, (\eta_i)_{i\in I})$ *over* $\mathcal{P}$*, which induces a* $C^\infty_{\mathbb{K}}$*-diffeomorphism* $\eta \colon \varinjlim (M_i \times N_i) \to \big(\varinjlim M_i\big) \times \big(\varinjlim N_i\big)$.

Proof. Let $e_n := \dim_{\mathbb{K}}(N_n)$ and $t := \sup\{e_n : n \in \mathbb{N}\}$. The natural map $\zeta : \mathbb{K}^{s+t} = \varinjlim \mathbb{K}^{d_n+e_n} \to \mathbb{K}^s \times \mathbb{K}^t$ analogous to η is an isomorphism of topological vector spaces ([10, Prop. 3.3]; [20, Thm. 4.1]). Let $\mathcal{A}$ be the atlas for $M = \bigcup_{n\in\mathbb{N}} M_n$ from the proof of Theorem 3.1; let $\mathcal{B}$ and $\mathcal{C}$ be analogous atlases for $N = \bigcup_{n\in\mathbb{N}} N_n$ and $P := \bigcup_{n\in\mathbb{N}}(M_n \times N_n)$. Then $\mathcal{D} := \{\phi \times \psi \colon \phi \in \mathcal{A}, \psi \in \mathcal{B}\}$ is a $C^\infty_{\mathbb{K}}$-atlas making $M \times N$ the direct product of M and N in the category of $C^\infty_{\mathbb{K}}$-manifolds. Since $\{(\phi \times \psi) \circ \zeta|_{\zeta^{-1}(P_\phi \times P_\psi)} \colon \phi \in \mathcal{A}, \psi \in \mathcal{B}\} \subseteq \mathcal{C}$, the map $\eta = \mathrm{id} \colon P \to M \times N$ is a $C^\infty_{\mathbb{K}}$-diffeomorphism. □

Proposition 3.8 *If* $\mathbb{K} = \mathbb{R}$*, each* M_i *is a finite-dimensional, real analytic manifold and each* $\lambda_{i,j}$ *an injective, real analytic immersion, then there exists a* $c^\omega_{\mathbb{R}}$*-manifold structure on* M *such that* $M = \varinjlim \mathcal{S}$ *in the category of* $c^\omega_{\mathbb{R}}$*-manifolds (and* $c^\omega_{\mathbb{R}}$*-maps), and which is compatible with the above* $C^\infty_{\mathbb{R}}$*-manifold structure on* M*. Analogues of Propositions* 3.3, 3.4 *and* 3.7 *hold for the* $c^\omega_{\mathbb{R}}$*-manifold structures.*

Proof. Using the $C^\omega_{\mathbb{R}}$-case of the Extension Lemma 2.3, the construction described in the proof of Theorem 3.1 provides a subatlas $\mathcal{B}$ of the $C^\infty_{\mathbb{R}}$-atlas $\mathcal{A}$ consisting of charts $\phi = \varinjlim \phi_n$ where each ϕ_n is a $C^\omega_{\mathbb{R}}$-diffeomorphism. Using Lemma 1.9 (b), the above arguments show that the chart changes for charts in $\mathcal{B}$ are $c^\omega_{\mathbb{R}}$, whence indeed M has a compatible $c^\omega_{\mathbb{R}}$-manifold structure. Similarly, Lemma 1.9 (b) entails the validity of the $c^\omega_{\mathbb{R}}$-analogues of Proposition 3.3 and 3.7. The proof of Proposition 3.4 carries over directly. □

Remark 3.9 For the singular homology groups of M over a commutative ring R, we have $(H_k(M), (H_k(\lambda_i))_{i\in I}) = \varinjlim (H_k(M_i), (H_k(\lambda_{i,j})_{i\geq j}))$ for all $k \in \mathbb{N}_0$, as a consequence of Proposition 3.4 (and Proposition 3.5). Likewise, given $\ell \in I$ and $x \in M_\ell$, for the homotopy groups we have $\pi_k(M, \lambda_\ell(x)) = \varinjlim \pi_k(M_i, \lambda_{i,\ell}(x))$.

4 Direct limits of finite-dimensional Lie groups

Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or a local field, and $\mathcal{S} := ((G_i)_{i\in I}, (\lambda_{i,j})_{i\geq j})$ be a countable direct system of finite-dimensional $C^\infty_\mathbb{K}$-Lie groups G_i and $C^\infty_\mathbb{K}$-homomorphisms $\lambda_{i,j}\colon G_j \to G_i$; if $\mathrm{char}(\mathbb{K}) > 0$, we assume that each $\lambda_{i,j}$ is an injective immersion. In this section, we construct a direct limit Lie group for $\mathcal{S}$, and discuss some of its properties.

Remark 4.1 (a) As in the classical real and complex cases, also every $C^\infty_\mathbb{K}$-Lie group over a local field $\mathbb{K}$ of characteristic 0 admits a $C^\infty_\mathbb{K}$-compatible analytic Lie group structure, and every $C^\infty_\mathbb{K}$-homomorphism between such groups is $\mathbb{K}$-analytic [12].

(b) Note that the squaring map $\sigma\colon \mathbb{F}_2[[X]]^\times \to \mathbb{F}_2[[X]]^\times$, $\sigma(x) := x^2$ is an analytic (and hence smooth) homomorphism which is injective (since $\mathbb{F}[[X]]^\times$ is isomorphic to the power $(\mathbb{Z}_2)^\mathbb{N}$ of the group of 2-adic integers as a group by [34, Chap. II-4, Prop. 10], and thus torsion-free) but *not* an immersion because $f'(x) = 2\,\mathrm{id} = 0$ for all $x \in \mathbb{F}_2[[X]]^\times$ (where $\mathbb{F}_2[[X]]$ denotes the field of formal Laurent series over the field $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$). This explains that an extra hypothesis is needed in positive characteristic.[4]

Associated injective quotient system. If $\mathbb{K} = \mathbb{R}$ (or if $\mathrm{char}(\mathbb{K}) > 0$, in which case we obtain trivial subgroups), we let $N_j := \overline{\bigcup_{i\geq j} \ker \lambda_{i,j}}$ for $j \in I$. If $\mathbb{K}$ is a local field of characteristic 0, we let $N_j = \bigcup_{i\geq j} \ker \lambda_{i,j}$ for $j \in I$ and note that N_j is locally closed and hence closed in G_j, because G_j has an open compact subgroup U which satisfies an ascending chain condition on closed subgroups. If $\mathbb{K} = \mathbb{C}$, we let $N_j \subseteq G_j$ be the intersection of all closed complex Lie subgroups S of G_j such that $\bigcup_{i\geq j} \ker \lambda_{i,j} \subseteq S$. Then N_j is a complex Lie subgroup of G_j (as G_j satisfies a descending chain condition on closed, connected subgroups), and thus N_j is the smallest closed, complex Lie subgroup of G_j containing $\bigcup_{i\geq j} \ker \lambda_{i,j}$. By minimality, N_j is invariant under inner automorphisms and hence a normal subgroup of G_j.

Then, in either case, there is a uniquely determined $\mathbb{K}$-Lie group structure on $\overline{G}_j := G_j/N_j$ which makes the canonical quotient homomorphism $q_j : G_j \to \overline{G}_j$ a submersion. Each $\lambda_{i,j}$ factors to a $C^\infty_\mathbb{K}$-homomorphism $\overline{\lambda}_{i,j}\colon \overline{G}_j \to \overline{G}_i$, uniquely determined by $\overline{\lambda}_{i,j} \circ q_j = q_i \circ \lambda_{i,j}$. Then $\overline{\mathcal{S}} = ((\overline{G}_i)_{i\in I}, (\overline{\lambda}_{i,j})_{i\geq j})$ is a direct system of finite-dimensional $C^\infty_\mathbb{K}$-Lie groups and *injective* $C^\infty_\mathbb{K}$-homomorphisms $\overline{\lambda}_{i,j}\colon \overline{G}_j \to \overline{G}_i$; it is called the *injective quotient system associated with* $\mathcal{S}$ (cf. [27]). Each $\overline{\lambda}_{i,j}$ is an injective immersion of class $C^\infty_\mathbb{K}$.

Remark 4.2 The example $\mathbb{C}^\times \xrightarrow{\sigma} \mathbb{C}^\times \xrightarrow{\sigma} \cdots$ with the squaring map $\sigma : \mathbb{C}^\times \to \mathbb{C}^\times$, $\sigma(z) := z^2$ shows that $\overline{\bigcup_{i\geq j} \ker \lambda_{i,j}}$ need not be a *complex* Lie subgroup of G_j.

[4] Slightly more generally, to establish Theorem 4.3 for $\mathrm{char}(\mathbb{K}) > 0$, it would be enough to assume that that G_j/N_j admits a $C^\infty_\mathbb{K}$-Lie group structure for each $j \in I$ which makes the quotient map $G_j \to G_j/N_j$ a submersion, where $N_j := \overline{\bigcup_{i\geq j} \ker \lambda_{i,j}}$, and that the induced homomorphisms $G_j/N_j \to G_i/N_i$ be immersions, for all $i \geq j$.

Theorem 4.3 *For $\mathcal{S}$ and $\overline{\mathcal{S}}$ as before, the following holds:*

(a) *A direct limit $(G, (\overline{\lambda}_i)_{i\in I})$ for $\overline{\mathcal{S}}$ exists in the category of $C^\infty_{\mathbb{K}}$-Lie groups modelled on topological $\mathbb{K}$-vector spaces; G is modelled on the locally convex topological $\mathbb{K}$-vector space $\mathbb{K}^s$, where $s := \sup\{\dim_{\mathbb{K}} \overline{G}_i : i \in I\} \in \mathbb{N}_0 \cup \{\infty\}$. Forgetting the $\mathbb{K}$-Lie group structure, we have $(G, (\overline{\lambda}_i)_{i\in I}) = \varinjlim \overline{\mathcal{S}}$ also in the categories of sets, abstract groups, topological spaces, topological groups, the category of $C^\infty_{\mathbb{K}}$-manifolds modelled on topological $\mathbb{K}$-vector spaces, and the category of $C^\infty_{\mathbb{F}}$-Lie groups modelled on topological $\mathbb{F}$-vector spaces, for every non-discrete closed subfield $\mathbb{F}$ of $\mathbb{K}$. Furthermore, $L(\overline{\lambda}_i)\colon L(\overline{G}_i) \to L(G)$ is injective for each $i \in I$, and*

$$\big(L(G), (L(\overline{\lambda}_i))_{i\in I}\big) \;=\; \varinjlim \big((L(\overline{G}_i))_{i\in I}, (L(\overline{\lambda}_{i,j}))_{i\geq j}\big) \tag{2}$$

in the category of locally convex $\mathbb{K}$-vector spaces (and in the categories of sets, $\mathbb{K}$-Lie algebras, topological spaces, topological $\mathbb{K}$-Lie algebras, topological $\mathbb{K}$-vector spaces, and $C^\infty_{\mathbb{K}}$-manifolds; also in the category of $c^\omega_{\mathbb{K}}$-manifolds, if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).

(b) *Set $\lambda_i := \overline{\lambda}_i \circ q_i$ for $i \in I$. If $\mathbb{K} \neq \mathbb{C}$ or if $\mathbb{K} = \mathbb{C}$ and $N_j = \overline{\bigcup_{i\geq j} \ker \lambda_{i,j}}$ for each $j \in I$, then $(G, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of $C^\infty_{\mathbb{K}}$-Lie groups modelled on topological $\mathbb{K}$-vector spaces, and also in the categories of smooth manifolds modelled on topological $\mathbb{K}$-vector spaces, Hausdorff topological spaces, and (Hausdorff) topological groups.*

(c) *If $\mathbb{K} = \mathbb{C}$, then $(G, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of complex Lie groups modelled on complex locally convex spaces.*

(d) *If $\mathbb{K} = \mathbb{R}$, then G is a $c^\infty_{\mathbb{R}}$-regular $c^\infty_{\mathbb{R}}$-Lie group which is smoothly paracompact. Furthermore, $(G, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of $c^\infty_{\mathbb{R}}$-Lie groups, and in the category of $c^\infty_{\mathbb{R}}$-manifolds.*

(e) *If $\mathbb{K} = \mathbb{C}$, then G is a $c^\infty_{\mathbb{C}}$-regular, $c^\infty_{\mathbb{C}}$-Lie group such that $(G, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of $c^\infty_{\mathbb{C}}$-Lie groups and $(G, (\lambda_i)_{i\in I}) = \varinjlim \overline{\mathcal{S}}$ in the category of $c^\infty_{\mathbb{C}}$-manifolds.*

(f) *If $\mathbb{K} = \mathbb{R}$, then there exists a $c^\omega_{\mathbb{R}}$-regular, $c^\omega_{\mathbb{R}}$-Lie group structure on G, compatible with the $C^\infty_{\mathbb{R}}$-Lie group structure from (a), such that $(G, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of $c^\omega_{\mathbb{R}}$-Lie groups. For the underlying $c^\omega_{\mathbb{R}}$- manifold, we have $(G, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of such manifolds.*

Proof. (a) Let $(G, (\overline{\lambda}_i)_{i\in I})$ be a direct limit for $\overline{\mathcal{S}}$ in the category of topological groups; then $(G, (\overline{\lambda}_i)_{i\in I}) = \varinjlim \overline{\mathcal{S}}$ also in the categories of sets, groups, and topological spaces [32, Thm. 2.7]. Thus Theorem 3.1 provides a $C^\infty_{\mathbb{K}}$-manifold structure on G making $(G, (\overline{\lambda}_i)_{i\in I})$ a direct limit of $\overline{\mathcal{S}}$ in the category of $C^\infty_{\mathbb{K}}$-manifolds modelled on topological $\mathbb{K}$-vector spaces, and also in the category of $C^\infty_{\mathbb{F}}$-manifolds, for every non-discrete closed subfield $\mathbb{F} \subseteq \mathbb{K}$ (Proposition 3.2). Let $\theta : G \to G$, $g \mapsto g^{-1}$ and $\theta_i : \overline{G}_i \to \overline{G}_i$ be the inversion maps; let $\mu : G \times G$ and $\mu_i : \overline{G}_i \times \overline{G}_i \to \overline{G}_i$. Then $\theta := \varinjlim \theta_i$ is $C^\infty_{\mathbb{K}}$, as $G = \varinjlim \overline{G}_i$ in the category of $C^\infty_{\mathbb{K}}$-manifolds. Likewise, $\mu = \varinjlim \mu_i$ is $C^\infty_{\mathbb{K}}$ on $\varinjlim (\overline{G}_i \times \overline{G}_i)$ and hence on $G \times G$, in view of Proposition 3.7. Hence G is a $C^\infty_{\mathbb{K}}$-Lie group. Every cone $(H, (f_i)_{i\in I})$ of $C^\infty_{\mathbb{K}}$-homomorphisms $f_i : \overline{G}_i \to H$ to a $C^\infty_{\mathbb{K}}$-Lie group H uniquely determines a map $f\colon G \to H$ such that $f \circ \overline{\lambda}_i = f_i$ for all i; then f is a homomorphism since $G = \varinjlim \overline{G}_i$ as a group, and f is $C^\infty_{\mathbb{K}}$ since $G = \varinjlim \overline{G}_i$ as a $C^\infty_{\mathbb{K}}$-manifold. Thus $G = \varinjlim \overline{G}_i$ as a $C^\infty_{\mathbb{K}}$-Lie group (and, likewise, as a $C^\infty_{\mathbb{F}}$-Lie group).

Determination of the Lie algebra of $L(G)$. Since $L(\overline{\lambda}_{i,j})$ is injective for all $i \geq j$, $\mathcal{T} := ((L(\overline{G}_i))_{i\in I}, (L(\overline{\lambda}_{i,j}))_{i\geq j})$ is a countable, strict direct system of Lie algebras. We recall from [10] or [20] that $\mathcal{T}$ has a direct limit $(\mathfrak{g}, (\phi_i)_{i\in I})$ in the category of topological $\mathbb{K}$-Lie algebras; here $\mathfrak{g}$ carries the finite topology (see **1.8**), each ϕ_i is injective, and $(\mathfrak{g}, (\phi_i)_{i\in I}) = \varinjlim \mathcal{T}$ also holds in the categories of sets, $\mathbb{K}$-Lie algebras, topological spaces, topological $\mathbb{K}$-vector spaces, and locally convex topological $\mathbb{K}$-vector spaces. By Lemma 1.9 (a), furthermore $(\mathfrak{g}, (\phi_i)_{i\in I}) = \varinjlim \mathcal{T}$ in the category of $C^\infty_{\mathbb{K}}$-manifolds and $C^\infty_{\mathbb{K}}$-maps (and also in the category of $c^\omega_{\mathbb{K}}$-manifolds and $c^\omega_{\mathbb{K}}$-maps by Lemma 1.9 (b), if $\mathbb{K} = \mathbb{R}$ of $\mathbb{C}$). The mappings $L(\overline{\lambda}_i) : L(\overline{G}_i) \to L(G)$ form a cone over $\mathcal{T}$ and hence induce a continuous Lie algebra homomorphism $\Lambda\colon \mathfrak{g} = \varinjlim L(\overline{G}_i) \to L(G)$, determined by $\Lambda \circ \phi_i = L(\overline{\lambda}_i)$. To see that Λ is surjective, let a geometric tangent vector $v \in L(G) = T_1 G$ be given, say $v = [\gamma]$ where $\gamma : U \to G$ is a smooth map on an open 0-neighbourhood $U \subseteq \mathbb{K}$, such that $\gamma(0) = 1$. By Proposition 3.4, after shrinking U we may assume that $\gamma(U) \subseteq \overline{\lambda}_i(\overline{G}_i)$ for some $n \in \mathbb{N}$, and that $\gamma = \overline{\lambda}_i \circ \eta$ for some smooth map $\eta\colon U \to \overline{G}_i$. Then $\Lambda(\phi_i([\eta])) = L(\overline{\lambda}_i)([\eta]) = [\overline{\lambda}_i \circ \eta] = [\gamma] = v$, as desired. Because $\mathfrak{g} = \bigcup_{i\in I} \operatorname{im} \phi_i$ and $\Lambda \circ \phi_i = L(\overline{\lambda}_i)$, *injectivity* of Λ follows from the injectivity of the maps $L(\overline{\lambda}_i) = T_1(\overline{\lambda}_i)$ established in Theorem 3.1. By the preceding, Λ is an isomorphism of Lie algebras; as both $\mathfrak{g}$ and $L(G) \cong \mathbb{K}^s$ are equipped with the finite topology, Λ also is an isomorphism of topological vector spaces. Hence $L(G) \cong \mathfrak{g} = \varinjlim L(\overline{G}_i)$ naturally. The desired direct limit properties carry over from $\mathfrak{g}$ to $L(G)$.

(b) and (c): Assume that H is a $C^\infty_{\mathbb{K}}$-Lie group modelled on a topological $\mathbb{K}$-vector space and $(f_i)_{i\in I}$ a family of $C^\infty_{\mathbb{K}}$-homomorphisms $f_i : G_i \to H$ such that $(H, (f_i)_{i\in I})$ is a cone over $\mathcal{S}$. Given $j \in I$, for any $i \geq j$ we then have $f_j = f_i \circ \lambda_{i,j}$ and thus $\ker\, \lambda_{i,j} \subseteq \ker\, f_j$, entailing that $\overline{\bigcup_{i\geq j} \ker\, \lambda_{i,j}} \subseteq \ker\, f_j$. In the situation of (b), this means that $N_j \subseteq \ker\, f_j$. In the situation of (c), we assume that H is modelled on a complex locally convex space. Then $\ker\, f_j$ is a complex Lie subgroup of G_j (Lemma 4.4), which contains $\bigcup_{i\geq j} \ker\, \lambda_{i,j}$; hence also $N_j \subseteq \ker\, f_j$ in (c). In any case, we deduce that $f_j = \overline{f}_j \circ q_j$ for a homomorphism $\overline{f}_j\colon \overline{G}_j \to H$, which is $C^\infty_{\mathbb{K}}$ because q_j admits smooth local sections. Then $((\overline{f}_i)_{i\in I}, H)$ is a cone over $\overline{\mathcal{S}}$ and hence induces a unique $C^\infty_{\mathbb{K}}$-homomorphism $f\colon G \to H$ such that $f \circ \overline{\lambda}_i = \overline{f}_i$, since $(G, (\overline{\lambda}_i)_{i\in I}) = \varinjlim \overline{\mathcal{S}}$ in the category of $C^\infty_{\mathbb{K}}$-Lie groups. Then $f \circ \lambda_i = f \circ \overline{\lambda}_i \circ q_i = \overline{f}_i \circ q_i = f_i$ for each $i \in I$, and clearly f is determined by this property. Thus $(G, (\lambda_i)_{i\in I}) = \varinjlim \mathcal{S}$ in the category of $C^\infty_{\mathbb{K}}$-Lie groups. In the situation of (b), the universal property of direct limit in the other categories of interest can be proved by trivial adaptations of the argument just given.

(d) To establish the first assertion, we may assume that $I = \mathbb{N}$, and after replacing $\mathcal{S}$ by a system equivalent to $\overline{\mathcal{S}}$ we may assume that $G_1 \subseteq G_2 \subseteq \cdots$, each $\lambda_{n,m}$ being the respective inclusion map. Then $L(G) = \bigcup_{n\in\mathbb{N}} L(G_n)$. If $\gamma : \mathbb{R} \to L(G)$ is a smooth curve, then for each $k \in \mathbb{Z}$, there exists $n_k \in \mathbb{N}$ such that the relatively compact set $\gamma(]k-1, k+2[)$ is contained in $L(G_{n_k})$. The finite-dimensional Lie group G_{n_k} being $c^\infty_{\mathbb{R}}$-regular, we find a smooth curve $\eta_k :]k-1, k+2[\to G_{n_k}$ such that $\eta_k(k) = 1$ and $\delta^r(\eta_k) = \gamma_k$. We define $\eta(t) := \eta_k(t)\eta_{k-1}(k)\cdots\eta_1(2)\eta_0(1)$ for $t \in [k, k+1]$ with $k \geq 0$, and $\eta(t) := \eta_k(t)\eta_k(k+1)^{-1}\cdots\eta_{-2}(-1)^{-1}\eta_{-1}(0)^{-1}$ for $t \in [k, k+1]$ with $k < 0$. Then $\eta\colon \mathbb{R} \to G$ is a smooth curve such that $\eta(0) = 1$ and $\delta^r(\eta) = \gamma$. Thus every $\gamma \in C^\infty(\mathbb{R}, L(G))$ has a right product integral $\mathrm{Evol}^r_G(\gamma) := \eta \in C^\infty(\mathbb{R}, G)$. We define

$$\mathrm{evol}^r_G\colon C^\infty(\mathbb{R}, L(G)) \to G, \qquad \mathrm{evol}^r_G(\gamma) := \mathrm{Evol}^r_G(\gamma)(1)\,.$$

To see that evol^r_G is $c^\infty_{\mathbb{R}}$, let $\sigma : \mathbb{R} \to C^\infty(\mathbb{R}, L(G))$ be a smooth curve. Given $t_0 \in \mathbb{R}$, let $U \subseteq \mathbb{R}$ be a relatively compact neighbourhood of t_0. We show that $\mathrm{evol}^r_G \circ \sigma : \mathbb{R} \to G$ is smooth on U. The evaluation map $C^\infty(\mathbb{R}, L(G)) \times \mathbb{R} \to L(G)$, $(\gamma, t) \mapsto \gamma(t)$ being continuous (cf. Thm. 3.4.3 and Prop. 2.6.11 in [7]), $\sigma(U)([-1, 2])$ is a compact subset of $L(G)$ and hence contained in $L(G_n)$ for some $n \in \mathbb{N}$, by Lemma 1.7 (d). We now consider

$$\tau\colon U \to C^\infty(]-1, 2[, L(G_n)), \quad \tau(t) := \sigma(t)|^{L(G_n)}_{]-1,2[}\,.$$

Then τ is smooth, because the restriction map $C^\infty(\mathbb{R}, L(G)) \to C^\infty(]-1, 2[, L(G))$ is continuous linear, and $C^\infty(]-1, 2[, L(G_n))$ is a closed vector subspace of $C^\infty(]-1, 2[, L(G))$. The group G_n being regular, $\mathrm{evol}^r_{G_n} : C^\infty(]-1, 2[, L(G_n)) \to G_n$ is smooth. Since $\mathrm{evol}^r_G \circ \sigma|_U = \mathrm{evol}^r_{G_n} \circ \tau$ apparently, we deduce that $\mathrm{evol}^r_G \circ \sigma|_U$ is smooth. Thus evol^r_G is $c^\infty_{\mathbb{R}}$. The desired direct limit properties can be proved as in (a) and (b), based on Proposition 3.6.

(e) As a consequence of Proposition 3.6, the $C^\infty_{\mathbb{C}}$-Lie group G also is a $c^\infty_{\mathbb{C}}$-Lie group. It is $c^\infty_{\mathbb{C}}$-regular because its underlying $c^\infty_{\mathbb{R}}$-Lie group is $c^\infty_{\mathbb{R}}$-regular by (d). The desired direct limit property can be proved as in (a) and (b).

(f) Using the $c^\omega_{\mathbb{R}}$-analogue of Proposition 3.7 (see Proposition 3.8), we see as in the proof of (a) that the $C^\infty_{\mathbb{R}}$-compatible $c^\omega_{\mathbb{R}}$-manifold structure on G from Proposition 3.8 turns G into a $c^\omega_{\mathbb{R}}$-Lie group. By (d), the latter is $c^\infty_{\mathbb{R}}$-regular. To see that it is $c^\omega_{\mathbb{R}}$-regular, let $\gamma : \mathbb{R} \to L(G)$ be a real analytic curve and $\eta := \mathrm{Evol}^r_G(\gamma)$ be its right product integral. Using Proposition 3.4 and its real analytic analogue (Proposition 3.8), for each $k \in \mathbb{N}$ we find $n \in \mathbb{N}$ such that $\gamma([-k, k]) \subseteq L(G_n)$, $\eta([-k, k]) \subseteq G_n$, and such that $\sigma := \gamma|^{L(G_n)}_{]-k,k[}$ is real analytic and $\tau := \eta|^{G_n}_{]-k,k[}$ smooth. The finite-dimensional Lie group G_n being $c^\omega_{\mathbb{R}}$-regular, the product integral τ of the real analytic curve σ has to be real analytic. Hence $\eta|_{]-k,k[}$ is real analytic for each $k \in \mathbb{N}$ and thus η is real analytic. Hence G is $c^\omega_{\mathbb{R}}$-regular. The direct limit property can be established as in (b). □

We needed to assume local convexity in Theorem 4.3 (c) because the proof of the following simple lemma depends on local convexity.

Lemma 4.4 *Let $\phi\colon G \to H$ be a $C^\infty_{\mathbb{C}}$-homomorphism from a finite-dimensional complex Lie group to a complex Lie group modelled on a locally convex complex topological vector space. Then $K := \ker\phi$ is a complex Lie subgroup of G. The same conclusion holds if H is a $c^\omega_{\mathbb{C}}$-Lie group and ϕ a $c^\omega_{\mathbb{C}}$-homomorphism.*

Proof. Being a closed subgroup of G, K is a real Lie subgroup, with Lie algebra

$$\mathfrak{k} = \{X \in L(G)\colon \exp_G(\mathbb{R}X) \subseteq K\} = \{X \in L(G)\colon \phi(\exp_G(\mathbb{R}X)) = \{1\}\,\}\,.$$

Given $X \in \mathfrak{k}$, the map $f\colon \mathbb{C} \to H$, $f(z) := \phi(\exp_G(zX))$ is complex analytic and $f|_{\mathbb{R}} = 1$, whence $f = 1$ by the Identity Theorem. Hence $\mathbb{C}X \subseteq \mathfrak{k}$, whence $\mathfrak{k}$ is a complex Lie subalgebra of $L(G)$. Therefore K is a complex Lie subgroup [4, Ch. III, §4.2, Cor. 2]. □

Remark 4.5 (a) In the situation of Remark 4.2, the direct system $((\mathbb{C}^\times)_{n\in\mathbb{N}}, (\sigma)_{n\geq m})$ has the direct limit $\mathbb{R}$ in the category of real Lie groups, whereas its direct limit in the category of complex Lie groups is the trivial group. Hence the conclusions of Theorem 4.3 (a) become false in general if we replace the injective quotient system $\overline{\mathcal{S}}$ by a non-injective system $\mathcal{S}$. Also note that $\varinjlim L(\mathbb{C}^\times) = \mathbb{C} \neq \{0\} = L(\{1\}) = L\big(\varinjlim \mathbb{C}^\times\big)$ here.

(b) If each $\lambda_{i,j}$ is injective, then the direct systems $\mathcal{S}$ and $\overline{\mathcal{S}}$ are equivalent, whence Theorem 4.3 (a) remains valid when $\overline{\mathcal{S}}$ is replaced with $\mathcal{S}$ and all bars are omitted.

Proposition 4.6 *Assume that $((G_i)_{i\in I}, (\lambda_{i,j})_{i\geq j})$ is a countable direct system of finite-dimensional Lie groups over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and injective $C^\infty_{\mathbb{K}}$-homomorphisms, with direct limit $(G, (\lambda_i)_{i\in I})$. Then the following holds:*

(a) *$\exp_G = \varinjlim \exp_{G_i} : L(G) = \varinjlim L(G_i) \to \varinjlim G_i = G$ is the exponential map of G, where $(L(G), (L(\lambda_i))_{i\in I}) = \varinjlim \big((L(G_i)), (L(\lambda_{i,j}))\big)$. The map $\exp_G$ is $c^\omega_{\mathbb{K}}$.*

(b) *The Trotter Product Formula $\exp_G(x+y) = \lim_{n\to\infty} \big(\exp_G(\frac{1}{n}x)\exp_G(\frac{1}{n}y)\big)^n$ holds and $\exp_G([x,y]) = \lim_{n\to\infty} \big(\exp_G(\frac{1}{n}x)\exp_G(\frac{1}{n}y)\exp_G(-\frac{1}{n}x)\exp_G(-\frac{1}{n}y)\big)^{n^2}$ (the Commutator Formula), for all $x, y \in L(G)$.*

(c) *Let $(H, (\mu_i)) = \varinjlim \mathcal{T}$ for a direct system $\mathcal{T} = ((H_i), (\mu_{i,j}))$ of finite-dimensional $\mathbb{K}$-Lie groups and injective $C^\infty_{\mathbb{K}}$-homomorphisms, and assume that $f_i : G_i \to H_i$ are $C^\infty_{\mathbb{K}}$-homomorphisms. Then $L\Big(\varinjlim f_i\Big) = \varinjlim L(f_i)$. Furthermore, every continuous homomorphism $G \to H$ is $c^\omega_{\mathbb{R}}$.*

Proof. (a) By Theorem 4.3 (a), $L(G) = \varinjlim L(G_i)$ as a $c^\omega_{\mathbb{K}}$-manifold. The family $(\exp_{G_i})_{i\in I}$ of $c^\omega_{\mathbb{K}}$-maps being compatible with the direct systems by naturality of exp, there is a unique $c^\omega_{\mathbb{K}}$-map $\exp_G := \varinjlim \exp_{G_i}$ such that $\exp_G \circ L(\lambda_i) = \lambda_i \circ \exp_{G_i}$ for each i. Given $y \in L(G)$, there are $j \in I$ and $x \in L(G_j)$ such that $y = L(\lambda_j)(x)$. Then $\xi : \mathbb{R} \to G$, $\xi(t) := \exp_G(tx) = \lambda_j(\exp_{G_j}(ty))$ is a smooth homomorphism such that $\xi'(0) =$

$L(\lambda_j)(\exp'_{G_j}(0).y) = L(\lambda_j)(y) = x$. Hence G admits an exponential map (in the sense of [23, Defn. 36.8]), and it is given by $\exp_G$ from above and hence $c^\omega_{\mathbb{K}}$.

(b) Using (a), the assertions directly follow from the finite-dimensional case.

(c) By Theorem 4.3 (a), $\alpha := \varinjlim L(f_i)$ is a continuous $\mathbb{K}$-Lie algebra homomorphism. Abbreviate $f := \varinjlim f_i$. From $\exp_H \circ \alpha = \big(\varinjlim \exp_{G_i}\big) \circ \big(\varinjlim L(f_i)\big) = \varinjlim \big(\exp_{H_i} \circ L(f_i)\big) = \varinjlim \big(f_i \circ \exp_{G_i}\big) = f \circ \exp_G$ we deduce that $\alpha = T_0(\exp_H \circ \alpha) = T_0(f \circ \exp_G) = L(f)$.

Now suppose that $h \colon G \to H$ is a continuous homomorphism. We may assume that $I = \mathbb{N}$ and $G_1 \subseteq G_2 \subseteq \ldots$, $H_1 \subseteq H_2 \subseteq \cdots$. After replacing G by its identity component G_0, we may assume that each G_n is connected. Using Proposition 3.4, we find $m(n) \in \mathbb{N}$ such that $h(G_n) \subseteq H_{m(n)}$, and such that $h_n := h|_{G_n}^{H_{m(n)}}$ is continuous and hence $C^\omega_{\mathbb{R}}$. We may assume that $m(1) < m(2) < \cdots$; after passing to a cofinal subsequence of the H_n's, without loss of generality $m(n) = n$ for each n. Thus $h = \varinjlim h_n$ is $c^\omega_{\mathbb{R}}$, by Theorem 4.3 (f). □

Remark 4.7 (a) The exponential map of a direct limit group need not be well-behaved, as the example $G = \mathbb{C}^\infty \rtimes_\alpha \mathbb{R} = \varinjlim (\mathbb{C}^n \rtimes \mathbb{R})$ with $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathbb{C}^\infty)$, $\alpha(t)((z_k)_{k\in\mathbb{N}}) := (e^{ikt} z_k)_{k\in\mathbb{N}}$ shows. Here $\exp_G$ fails to be injective on any 0-neighbourhood, and the exponential image $\mathrm{im}(\exp_G)$ is not an identity neighbourhood in G [10, Example 5.5].

(b) As a consequence of (a), also the exponential map $\exp_H$ of the complex analytic Lie group $H := \mathbb{C}^\infty \rtimes_\beta \mathbb{C} = \varinjlim (\mathbb{C}^n \rtimes \mathbb{C})$ with $\beta(z)((z_k)_{k\in\mathbb{N}}) := (e^{ikz} z_k)_{k\in\mathbb{N}}$ is not injective on any 0-neighbourhood. This settles an open problem by J. Milnor [25, p. 31] in the negative, who asked whether every complex analytic Lie group is "of Campbell-Hausdorff type."

5 Integration of locally finite Lie algebras

A Lie algebra is *locally finite* if every finite subset generates a finite-dimensional subalgebra.

Theorem 5.1 *Let $\mathfrak{g}$ be a countable-dimensional locally finite Lie algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then there exists a $c^\omega_{\mathbb{K}}$-regular, $c^\omega_{\mathbb{K}}$-Lie group G, which also is a regular $C^\infty_{\mathbb{K}}$-Lie group in Milnor's sense, such that $L(G) \cong \mathfrak{g}$, equipped with the finite topology.*

Proof. As $\mathfrak{g}$ is locally finite and $\dim_{\mathbb{K}}(\mathfrak{g}) \leq \aleph_0$, there is an ascending sequence $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots$ of finite-dimensional subalgebras of $\mathfrak{g}$, with union $\mathfrak{g}$. For each $n \in \mathbb{N}$, let G_n be a simply connected $\mathbb{K}$-Lie group with Lie algebra $L(G_n) \cong \mathfrak{g}_n$; fix an isomorphism $\kappa_n \colon L(G_n) \to \mathfrak{g}_n$. If $m \geq n$, then the Lie algebra homomorphism $\kappa_{m,n} := \kappa_m^{-1} \circ \kappa_n : L(G_n) \to L(G_m)$ (corresponding to the inclusion map $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_m$) induces a $C^\omega_{\mathbb{K}}$-homomorphism $\phi_{m,n} \colon G_n \to G_m$ such that $L(\phi_{m,n}) = \kappa_{m,n}$. Since $L(\phi_{k,m} \circ \phi_{m,n}) = L(\phi_{k,m}) \circ L(\phi_{m,n}) = \kappa_{k,m} \circ \kappa_{m,n} = \kappa_{k,n} = L(\phi_{k,n})$, we have $\phi_{k,m} \circ \phi_{m,n} = \phi_{k,n}$ for all $k \geq m \geq n$, whence $((G_n)_{n\in\mathbb{N}}, (\phi_{m,n})_{m\geq n})$ is a direct system of $C^\omega_{\mathbb{K}}$-Lie groups. Now take $G := \varinjlim G_n$ in the category of $c^\omega_{\mathbb{K}}$-Lie groups. We shall presently show that, for each n, the normal subgroup $K_n := \bigcup_{m\geq n} \ker \phi_{m,n}$ of G_n is discrete. Hence, by Theorem 4.3, G is a $c^\omega_{\mathbb{K}}$-regular $c^\omega_{\mathbb{K}}$-Lie group, $G = \varinjlim G_n/K_n$, and $L(G) = \varinjlim L(G_n/K_n) = \varinjlim L(G_n) \cong \varinjlim \mathfrak{g}_n = \mathfrak{g}$. For Milnor regularity, see Theorem 8.1.

Each K_n is discrete: We show that the closure $N_n := \overline{K_n} \subseteq G_n$ is discrete. The homomorphism $\phi_{m,n}$ has discrete kernel for all $m, n \in \mathbb{N}$ with $m \geq n$, because $L(\phi_{m,n}) = \kappa_{m,n}$ is injective. Now $\ker \phi_{m,n}$ being a discrete normal subgroup of the connected group G_n, it is central. This entails that $N_n \subseteq Z(G_n)$, for each n, whence $(N_n)_0 \subseteq Z(G_n)_0$ is a vector group being a connected closed subgroup of a vector group (Lemma 5.2). It is obvious from the definitions that $\phi_{m,n}(K_n) \subseteq K_m$ for all $m \geq n$, whence $\phi_{m,n}(N_n) \subseteq N_m$ and $\phi_{m,n}((N_n)_0) \subseteq (N_m)_0$. Being a continuous homomorphism between vector groups, $\psi_{m,n} := \phi_{m,n}|_{(N_n)_0}^{(N_m)_0}$ is real linear. Hence, being a real linear map with discrete kernel, $\psi_{m,n}$ is injective. Thus $(N_n)_0 = \overline{\bigcup_{m \geq n} \ker \psi_{m,n}} = \{1\}$, whence N_n is discrete. □

Here, we used the following fact:

Lemma 5.2 *Let G be a simply connected, finite-dimensional real Lie group. Then $Z(G)_0$ is a vector group, i.e., $Z(G)_0 \cong \mathbb{R}^m$ for some $m \in \mathbb{N}_0$.*

Proof. By Lévi's Theorem, $L(G) = \mathfrak{r} \rtimes \mathfrak{s}$ internally, where $\mathfrak{r}$ is the radical of $L(G)$ and $\mathfrak{s}$ a Lévi complement ([31], Part I, Ch. VI, Thm. 4.1 or [4], Ch. I, §6.8, Thm. 5). Let R and S be the analytic subgroups of G corresponding to $\mathfrak{r}$ and $\mathfrak{s}$, respectively. Then R and S are simply connected, R is a closed normal subgroup of G, S a closed subgroup, and $G = R \rtimes S$ internally [19, Kor. III.3.16]. Now consider the identity component $Z(G)_0$ of the centre $Z(G)$ of G. Let $\pi : G \to S$ be the projection onto S, with kernel R. Then $\pi(Z(G)_0) \subseteq Z(S)_0 = \{1\}$, entailing that $Z(G)_0 \subseteq R$. Thus $Z(G)_0$ is an analytic subgroup of the simply connected solvable Lie group R, whence $Z(G)_0$ is simply connected [19, Satz III.3.31]. Being a simply connected abelian Lie group, $Z(G)_0$ is a vector group. □

6 Extension of sections in principal bundles

We prove a preparatory result concerning sections in principal bundles, which will be used later to discuss closed subgroups and homogeneous spaces of direct limit groups.

Lemma 6.1 *Given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $\pi_1 : E_1 \to M_1$ be a $C^\infty_\mathbb{K}$-map between finite-dimensional $C^\infty_\mathbb{K}$-manifolds and $\pi_2 : E_2 \to M_2$ be a finite-dimensional G-principal bundle of class $C^\infty_\mathbb{K}$ whose structure group G is a finite-dimensional $C^\infty_\mathbb{K}$-Lie group. Let $m_1 := \dim_\mathbb{K}(M_1)$ and $m_2 := \dim_\mathbb{K}(M_2)$. Assume that $\lambda : M_1 \to M_2$ is an injective $C^\infty_\mathbb{K}$-immersion and $\Lambda : E_1 \to E_2$ a $C^\infty_\mathbb{K}$-map such that $\pi_2 \circ \Lambda = \lambda \circ \pi_1$. Assume that $\phi_1 : \Delta^{m_1}_r(\mathbb{K}) \to U_1 \subseteq M_1$ a chart for M_1, where $r > 0$, and $\sigma_1 : U_1 \to E_1$ a $C^\infty_\mathbb{K}$-section of π_1. Then, for every $s \in \,]0, r[$, there exists a chart $\phi_2 : \Delta^{m_2}_s \to U_2 \subseteq M_2$ and a $C^\infty_\mathbb{K}$-section $\sigma_2 : U_2 \to E_2$ of π_2 such that $\phi_2(x, 0) = \lambda(\phi_1(x))$ for all $x \in \Delta^{m_1}_s$ and $\sigma_2 \circ \lambda|_W = \Lambda \circ \sigma_1|_W$, where $W := \phi_1(\Delta^{m_1}_s)$. If $\mathbb{K} = \mathbb{R}$ and all of E_1, M_1, π_1, the principal bundle π_2, λ, Λ, ϕ_1 and σ_1 are real analytic, then also ϕ_2 and σ_2 can be chosen as real analytic maps.*

Proof. Since $\lambda \circ \phi_1$ is an injective immersion, there is a chart $\phi_2 : \Delta^{m_2}_s \to U_2 \subseteq M_2$ such that $\phi_2(x, 0) = \lambda(\phi_1(x))$ for all $x \in \Delta^{m_1}_s$ (Lemma 2.3).

The $C^\infty_{\mathbb{K}}$-case. If $\mathbb{K}=\mathbb{R}$, then $E_2|_{U_2}$ is trivial as a G-principal bundle of class $C^\infty_{\mathbb{R}}$, since $U_2 \cong \Delta_s^{m_2}$ is paracompact and contractible (this is a well-known fact, which can be proved exactly as [21, Cor. 4.2.5]). If $\mathbb{K}=\mathbb{C}$, then $E_2|_{U_2}$ is trivial as a G-principal bundle of class $C^\infty_{\mathbb{C}}$, since $U_2 \cong \Delta_s^{m_2}(\mathbb{C})$ is a contractible Stein manifold [15, Satz 6].

Real analytic case. Since $U_2 \cong \Delta_s^{m_2}$ is σ-compact and contractible, $E_2|_{U_2}$ is trivial as a topological G-principal bundle and therefore also as a real analytic G-principal bundle, by injectivity of ϑ in [33, Teorema 5].

By the preceding, in either case, we find a $C^\infty_{\mathbb{K}}$ (resp., real analytic) trivialization $\theta\colon E_2|_{U_2} \to U_2 \times G$. Let $\theta_2 : E_2|_{U_2} \to G$ be the second coordinate function of θ. Define $\sigma_2 := \zeta \circ \phi_2^{-1}\colon U_2 \to E_2$, where $\zeta\colon \Delta_s^{m_2} \to E_2$ is defined via

$$\zeta(x,y) \;:=\; \theta^{-1}\big(\phi_2(x,y),\; \theta_2((\Lambda\circ\sigma_1\circ\phi_1)(x))\big) \quad \text{for } x\in\Delta_s^{m_1},\ y\in\Delta_s^{m_2-m_1}.$$

Then $\sigma_2\colon U_2 \to E_2$ is a $C^\infty_{\mathbb{K}}$-section (resp., $C^\omega_{\mathbb{R}}$-section) with the required properties. □

7 Fundamentals of Lie theory for direct limit groups

In this section, we develop the basics of Lie theory for direct limit groups. Throughout the following, $G_1 \subseteq G_2 \subseteq \cdots$ is an ascending sequence of finite-dimensional Lie groups over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, such that the inclusion maps $\lambda_{n,m}\colon G_m \to G_n$ are $C^\omega_{\mathbb{K}}$-homomorphisms, and $G := \bigcup_{n\in\mathbb{N}} G_n$, equipped with the $c^\omega_{\mathbb{K}}$-Lie group structure such that $G = \varinjlim G_n$ in the category of $c^\omega_{\mathbb{K}}$-Lie groups (and $C^\infty_{\mathbb{K}}$-Lie groups).

7.1 A map $f\colon M \to N$ between $c^\omega_{\mathbb{K}}$-manifolds is called $c^\omega_{\mathbb{K}}$-*final* if a map $g\colon N \to Z$ into a $c^\omega_{\mathbb{K}}$-manifold is $c^\omega_{\mathbb{K}}$ if and only if $g\circ f$ is $c^\omega_{\mathbb{K}}$. The map f is $c^\omega_{\mathbb{K}}$-*initial* if a map $g\colon Z \to M$ from a $c^\omega_{\mathbb{K}}$-manifold Z to M is $c^\omega_{\mathbb{K}}$ if and only if $f\circ g$ is $c^\omega_{\mathbb{K}}$. Obvious adaptations are used to define $c^\infty_{\mathbb{K}}$-final, $C^r_{\mathbb{K}}$-final, $c^\infty_{\mathbb{K}}$-initial, and $C^r_{\mathbb{K}}$-initial maps, where $r\in\mathbb{N}_0\cup\{\infty\}$. A subset $M\subseteq N$ of a $c^\omega_{\mathbb{K}}$-manifold M is called a (split) submanifold if there exists a (complemented) closed vector subspace F of the modelling space E of N such that, for every $x\in M$, there exists a chart $\phi\colon U\to V\subseteq N$ of N around x such that $\phi(U\cap F) = M\cap V$. Then M, with the induced topology, can be made a $c^\omega_{\mathbb{K}}$-manifold modelled on F, in an apparent way.

Proposition 7.2 (Subgroups are Lie groups) *Every subgroup $H\subseteq G$ admits a $c^\omega_{\mathbb{K}}$-Lie group structure with Lie algebra $L(H) = \{v\in L(G)\colon \exp_G(\mathbb{K}v)\subseteq H\} =: \mathfrak{h}$ which makes the inclusion map $\lambda\colon H\to G$ a $C^\infty_{\mathbb{K}}$-homomorphism and both a $c^\omega_{\mathbb{K}}$-initial and a $c^\infty_{\mathbb{K}}$-initial map, and such that $L(\lambda)\colon L(H)\to L(G)$ is an embedding of topological $\mathbb{K}$-vector spaces. Furthermore, $H = \varinjlim H_n$ in the category of $c^\omega_{\mathbb{K}}$-Lie groups, where $H_n := H\cap G_n$ is equipped with the finite-dimensional $\mathbb{K}$-Lie group structure induced by G_n.*

Proof. We equip H_n with the finite-dimensional $C^\omega_{\mathbb{K}}$-Lie group structure induced by G_n, which makes the inclusion map $\lambda_n : H_n \to G_n$ an immersion and a $C^\omega_{\mathbb{K}}$-initial and $C^\infty_{\mathbb{K}}$-initial map inside the category of finite-dimensional $C^\omega_{\mathbb{K}}$- and $C^\infty_{\mathbb{K}}$-manifolds, respectively (see [4], Ch. III, §4.5, Defn. 3 and Prop. 9). Then the inclusion maps $\mu_{n,m}\colon H_m \to H_n$ are $C^\omega_{\mathbb{K}}$-immersions for $n \geq m$; we give $H = \bigcup_{n\in\mathbb{N}} H_n$ the $c^\omega_{\mathbb{K}}$-Lie group structure such that $(H, (\mu_n)_{n\in\mathbb{N}}) = \varinjlim ((H_n)_{n\in\mathbb{N}}, (\mu_{n,m}))$ in the category of $c^\omega_{\mathbb{K}}$-Lie groups, where $\mu_n\colon H_n \to H$ is the inclusion map (see Theorem 4.3). Then $\lambda = \varinjlim \lambda_n\colon H \to G$ is $c^\omega_{\mathbb{K}}$. We have $L(H) = \bigcup_{n\in\mathbb{N}} L(H_n)$ (with obvious identifications) and $L(G) = \bigcup_{n\in\mathbb{N}} L(G_n)$ by Theorem 4.3 (a), and each of $L(\lambda_n) : L(H_n) \to L(G_n)$ as well as $L(\lambda) : L(H) \to L(G)$ is the respective inclusion map. Thus $L(\lambda)$ is injective. Being an injective linear map between locally convex spaces equipped with their finest locally convex topologies, $L(\lambda)$ is a topological embedding (cf. [22, Prop. 7.25 (ii)]). Clearly $L(H) \subseteq \mathfrak{h}$. If $v \in \mathfrak{h}$, then $v \in L(G_n)$ for some n and thus $\exp_{G_n}(\mathbb{K}v) \subseteq G_n \cap H = H_n$, whence $v \in L(H_n) \subseteq L(H)$. Thus $L(H) = \mathfrak{h}$. Now assume that M is a $c^\infty_{\mathbb{K}}$-manifold and $f\colon M \to H$ a map such that $\lambda \circ f\colon M \to G$ is $c^\infty_{\mathbb{K}}$. Then, for every smooth map $\gamma\colon \mathbb{K} \supseteq U \to M$ on an open 0-neighbourhood $U \subseteq \mathbb{K}$, the composition $f \circ \gamma$ maps some 0-neighbourhood $V \subseteq U$ into some G_n and $(f \circ \gamma)|_V^{G_n}$ is $C^\infty_{\mathbb{K}}$, by Proposition 3.4. Since $f(\gamma(V)) \subseteq G_n \cap H = H_n$ and H_n is $C^\infty_{\mathbb{K}}$-initial for maps from finite-dimensional $C^\infty_{\mathbb{K}}$-manifolds, we deduce that $(f \circ \gamma)|_V^{H_n}$ is $C^\infty_{\mathbb{K}}$, whence $(f \circ \gamma)|_V$ is $c^\infty_{\mathbb{K}}$. This entails that f is $c^\infty_{\mathbb{K}}$. Thus $\lambda\colon H \to G$ is $c^\infty_{\mathbb{K}}$-initial. Similarly, λ is $c^\omega_{\mathbb{K}}$-initial. □

Lemma 7.3 *If $\mathbb{K} = \mathbb{C}$ in the situation of Proposition 7.2, define $\mathfrak{h}$ as before and $\mathfrak{s} := \{v \in L(G) : \exp_G(\mathbb{R}v) \subseteq H\}$. Let S_n be H_n, equipped with the $C^\omega_{\mathbb{R}}$-Lie group structure induced by the $C^\omega_{\mathbb{R}}$-Lie group $(G_n)_{\mathbb{R}}$ underlying G_n, and define $S := \varinjlim S_n$. Thus $S = H$ as a set and an abstract group, and* $\mathrm{id}\colon H_{\mathbb{R}} \to S$ *is $c^\omega_{\mathbb{R}}$. Then $\mathfrak{h} = \mathfrak{s}$ (as a set or real Lie algebra) if and only if $(H_n)_{\mathbb{R}} = S_n$ (as a real Lie group) for each $n \in \mathbb{N}$, if and only if $H_{\mathbb{R}} = S$ (as a $c^\omega_{\mathbb{R}}$-Lie group).*

Proof. If $\mathfrak{h} = \mathfrak{s}$, then for every $n \in \mathbb{N}$ we have $L(S_n) + iL(S_n) \subseteq L(H_m)$ for some $m \geq n$. Let $v \in L(S_n)$. Then $\exp_{G_n}(\mathbb{C}v) = \exp_{G_m}(\mathbb{C}v) \subseteq H_m \cap G_n = H_n$, entailing that $v \in L(H_n)$. Thus $L(S_n) \subseteq L(H_n)$ and hence $L(S_n) = L(H_n)$, whence $S_n = (H_n)_{\mathbb{R}}$.

If $S_n = (H_n)_{\mathbb{R}}$ for each $n \in \mathbb{N}$, then $(\varinjlim H_n)_{\mathbb{R}} = \varinjlim (H_n)_{\mathbb{R}} = \varinjlim S_n$, by Theorem 4.3.

Now suppose that $H_{\mathbb{R}} = S$. We have $\mathfrak{h} \subseteq \mathfrak{s}$ by definition. If $v \in \mathfrak{s}$, then $\exp_G(\mathbb{R}v) \subseteq H$ and $\xi : \mathbb{R} \to S$, $\xi(t) = \exp_G(tv) = \exp_S(tv)$ is a $c^\omega_{\mathbb{R}}$-homomorphism. Since $S = H_{\mathbb{R}} = \varinjlim (H_n)_{\mathbb{R}}$ (see Theorem 4.3), Proposition 3.4 entails that $\mathrm{im}(\xi) \subseteq (H_n)_{\mathbb{R}}$ for some $n \in \mathbb{N}$ and that $\xi|^{(H_n)_{\mathbb{R}}}$ is $C^\omega_{\mathbb{R}}$. Hence $\xi = \exp_{H_n}(\bullet w)$ for some $w \in L(H_n)$, where $w = v$ clearly and thus $\exp_G(\mathbb{C}v) = \exp_{H_n}(\mathbb{C}v) \subseteq H_n \subseteq H$, whence $v \in \mathfrak{h}$. Therefore $\mathfrak{s} = \mathfrak{h}$. □

7.4 We now specialize to the case where H is a *closed* subgroup of G; if $\mathbb{K} = \mathbb{C}$, we assume that $\mathfrak{s} := \{v \in L(G)\colon \exp_G(\mathbb{R}v) \subseteq H\}$ is a complex Lie subalgebra of $L(G)$. Then the finite-dimensional $C^\omega_{\mathbb{K}}$-Lie group structure induced by G_n on its closed subgroup $H_n := G_n \cap H$ makes H_n a closed $C^\omega_{\mathbb{K}}$-submanifold of G_n (this is obvious in the real case, and follows for $\mathbb{K} = \mathbb{C}$ using Lemma 7.3). For each $n \in \mathbb{N}$, we give G_n/H_n the finite-dimensional $C^\omega_{\mathbb{K}}$-manifold structure making the canonical quotient map $q_n : G_n \to G_n/H_n$ a submersion. Let $q_{n,m} : G_m/H_m \to G_n/H_n$ be the uniquely determined $C^\omega_{\mathbb{K}}$-maps such that $q_{n,m} \circ q_m =$

$q_n \circ \lambda_{n,m}$. Then $\mathcal{T} := ((G_n/H_n)_{n\in\mathbb{N}}, (q_{n,m})_{n\geq m})$ is a direct system of paracompact, finite-dimensional $C^\omega_\mathbb{K}$-manifolds and injective $C^\omega_\mathbb{K}$-immersions, whence $(M, (\psi_n)_{n\in\mathbb{N}}) := \varinjlim \mathcal{T}$ exists in the category of $c^\omega_\mathbb{K}$-manifolds (Proposition 3.8). We have $(M, (\psi_n)_{n\in\mathbb{N}}) = \varinjlim \mathcal{T}$ also in the categories of $C^\infty_\mathbb{K}$-manifolds, and the category of sets. Consider the quotient map $q : G \to G/H$ and the inclusion maps $\lambda_n : G_n \to G$. For each $n \in \mathbb{N}$, the map $q \circ \lambda_n$ factors to an injective map $\mu_n : G_n/H_n \to G/H$, determined by $\mu_n \circ q_n = q \circ \lambda_n$. Then $(G/H, (\mu_n)_{n\in\mathbb{N}})$ is a cone over $\mathcal{T}$, and hence induces a map $\mu : M \to G/H$. It is easy to see that μ is a bijection; we give G/H the $c^\omega_\mathbb{K}$-manifold structure making μ a $c^\omega_\mathbb{K}$-diffeomorphism; thus $G/H \cong \varinjlim G_n/H_n$. Then also $(G/H, (\mu_n)_{n\in\mathbb{N}}) = \varinjlim \mathcal{T}$ in the category of $c^\omega_\mathbb{K}$-manifolds. Since $q = \varinjlim q_n$, the map q is $c^\omega_\mathbb{K}$.

Proposition 7.5 (Closed subgroups, quotient groups, homogeneous spaces) *Let H be a closed subgroup of G; if $\mathbb{K} = \mathbb{C}$, assume that $\{v \in L(G) \colon \exp_G(\mathbb{R}v) \subseteq H\}$ is a complex Lie subalgebra of G. Equip G/H with the $c^\omega_\mathbb{K}$-manifold structure described in* **7.4***; thus $G/H \cong \varinjlim G_n/H_n$ as a $c^\omega_\mathbb{K}$-manifold. Then the following holds:*

(a) *$q \colon G \to G/H$ admits local $c^\omega_\mathbb{K}$-sections, i.e., $q \colon G \to G/H$ is an H-principal bundle of class $c^\omega_\mathbb{K}$. Therefore q is $c^\omega_\mathbb{K}$-final, $c^\infty_\mathbb{K}$-final and $C^r_\mathbb{K}$-final, for each $r \in \mathbb{N}_0 \cup \{\infty\}$.*

(b) *H is a closed, split $c^\omega_\mathbb{K}$-submanifold of G. The $c^\omega_\mathbb{K}$-submanifold structure on H and the $c^\omega_\mathbb{K}$-manifold structure underlying the $c^\omega_\mathbb{K}$-Lie group structure induced by G on H (as described in Proposition 7.2) coincide. This manifold structure makes the inclusion map $H \to G$ a $c^\omega_\mathbb{K}$-initial, $c^\infty_\mathbb{K}$-initial, and $C^r_\mathbb{K}$-initial map, for each $r \in \mathbb{N}_0 \cup \{\infty\}$. If $L(H) = \{0\}$, then H is discrete in the topology induced by G.*

(c) *If H is furthermore a normal subgroup of G, then the $c^\omega_\mathbb{K}$-manifold structure on G/H makes the quotient group G/H a $c^\omega_\mathbb{K}$-regular $c^\omega_\mathbb{K}$-Lie group such that $G/H = \varinjlim G_n/H_n$ in the category of $c^\omega_\mathbb{K}$-Lie groups.*

Proof. (a) Let $x \in G/H$; then there exists $k \in \mathbb{N}$ and $y \in G_k$ such that $x = q(y)$. Define $z := q_k(y)$. Define $r_n := 1 + 2^{-n}$ for $n \in \mathbb{N}$, and $d_n := \dim_\mathbb{K}(G_n/H_n)$. There exists a $C^\omega_\mathbb{K}$-section $\tau \colon V \to G_k$ of q_k on some open neighbourhood V of z in G_k/H_k, and a chart $\phi_k \colon \Delta^{d_k}_{r_k} \to U_k \subseteq G_k/H_k$ such that $U_k \subseteq V$; we define $\sigma_k := \tau|_{U_k}$. Inductively, Lemma 6.1 provides charts $\phi_n : \Delta^{d_n}_{r_n} \to U_n \subseteq G_n/H_n$ and $C^\omega_\mathbb{K}$-sections $\sigma_n : U_n \to G_n$ such that $q_{n,m} \circ \phi_m|_{\Delta^{d_m}_{r_n}} = \phi_n|_{\Delta^{d_m}_{r_n}}$ for all $n \geq m \geq k$ and $\sigma_n(q_{n,m}(w)) = \sigma_m(w)$ for all $w \in \phi_m(\Delta^{d_m}_{r_n})$. Define $W_n := \phi_n(\Delta^{d_n}_1)$ for $n \in \mathbb{N}$, $n \geq k$. Then $W := \bigcup_{n\geq k} \mu_n(W_n)$ is an open subset of G/H, and $(W, (\mu_n|^W_{W_n})_{n\geq k}) = \varinjlim \mathcal{W}$ as a $c^\omega_\mathbb{K}$-manifold, where $\mathcal{W} := ((W_n)_{n\geq k}, (q_{n,m}|^{W_n}_{W_m}))$. Now $\sigma := \varinjlim (\sigma|_{W_n}) \colon W = \varinjlim W_n \to \varinjlim G_n = G$ is a $c^\omega_\mathbb{K}$-map, and it is a section for q because $q \circ \sigma = \varinjlim (q_n \circ \sigma_n|_{W_n}) = \varinjlim j_n = j$, where $j_n \colon W_n \to G_n/H_n$ and $j \colon W \to G/H$ are the inclusion maps. The remainder is obvious.

(b) For the $c^\omega_\mathbb{K}$-Lie group structure induced by G on H, we have $H = \varinjlim H_n$ by Proposition 7.2, and this then also holds for the underlying $c^\omega_\mathbb{K}$ and $c^\infty_\mathbb{K}$-manifold structures (Theorem 4.3 (d)–(f)). Hence, by the proof of Theorem 3.1, there exists a chart of H around 1 of the form $\phi = \varinjlim \phi_n : P \to Q \subseteq H$, where, for each n, the map $\phi_n : P_n \to Q_n \subseteq H_n$ is a chart of H_n around 1, defined on an open subset $P_n \subseteq \mathbb{K}^{h_n}$ (where $h_n := \dim_{\mathbb{K}}(H_n)$), $P := \bigcup_{n\in\mathbb{N}} P_n \subseteq \mathbb{K}^t$ (where $t := \sup\{h_n : n \in \mathbb{N}\} \in \mathbb{N}_0 \cup \{\infty\}$), and $Q := \bigcup_{n\in\mathbb{N}} Q_n \subseteq H$. By the proof of Part (a) of the present proposition, there exist charts $\psi_n \colon \Delta_1^{d_n} \to W_n$ onto open neighbourhoods $W_n \subseteq G_n/H_n$ of $q_n(1)$ (where $d_n := \dim_{\mathbb{K}}(G_n/H_n)$) and $C^\omega_{\mathbb{K}}$-sections $\sigma_n \colon W_n \to G_n$ of q_n, such that $q_{n,m}(W_m) \subseteq W_n$, $q_{n,m} \circ \psi_m = \psi_n|_{\Delta_1^{d_m}}$, and $\sigma_n \circ q_{n,m}|_{W_m} = \sigma_m$ for all $m, n \in \mathbb{N}$ such that $n \geq m$. Define $V_n := \operatorname{im}(\sigma_n) Q_n \subseteq G_n$ and

$$\theta_n \colon \Delta_1^{d_n+h_n} \to V_n, \quad \theta_n(x,y) := \sigma_n(\psi_n(x))\phi_n(y) \quad \text{for } x \in \Delta_1^{d_n},\ y \in \Delta_1^{h_n}.$$

Since σ_n is a section of q_n, the map θ_n is easily seen to be injective. Using the inverse function theorem, one verifies that V_n is open in G_n and that θ_n is a $C^\omega_{\mathbb{K}}$-diffeomorphism onto V_n. Then $V := \bigcup_{n\in\mathbb{N}} V_n$ is open in G, and $\theta := \varinjlim \theta_n : \varinjlim \Delta_1^{d_n+h_n} \to V$ is a $c^\omega_{\mathbb{K}}$-diffeomorphism. Let $\zeta : \mathbb{K}^{s+t} = \varinjlim \mathbb{K}^{d_n+h_n} \to \mathbb{K}^s \times \mathbb{K}^t$ be the natural isomorphism of topological vector spaces (cf. Proposition 3.7), and $\Omega := \zeta(\bigcup_n \Delta_1^{d_n+h_n}) \subseteq \mathbb{K}^s \times \mathbb{K}^t$. Then $\kappa := \theta \circ \zeta^{-1}|_\Omega \colon \Omega \to V$ is a $C^\infty_{\mathbb{K}}$-diffeomorphism. By construction of θ, we have $V \cap H = Q$ and $\kappa^{-1}(V \cap H) = \Omega \cap (\{0\} \times \mathbb{K}^t)$, where $\{0\} \times \mathbb{K}^t$ is a closed, complemented vector subspace of $\mathbb{K}^s \times \mathbb{K}^t$. Hence H is a split $c^\omega_{\mathbb{K}}$-submanifold of G. As the restriction of κ to a submanifold chart of H is the given chart ϕ of H, the submanifold structure and the above manifold structure on H coincide. If $L(H) = 0$, then the topology on the Lie group H is discrete and hence so is the topology on H as a submanifold of G, the induced topology.

(c) By construction, $(G/H, (\mu_n)) = \varinjlim ((G_n/H_n), (q_{n,m}))$ as a $c^\omega_{\mathbb{K}}$-manifold. Since each $q_{n,m}$ also is a homomorphism, Theorem 4.3 shows that there is a group structure on the $c^\omega_{\mathbb{K}}$-manifold G/H making it a Lie group, and such that each μ_n becomes a homomorphism. This requirement entails that $q \colon G \to G/H$ is a homomorphism, whence the group structure on G/H is the one of the quotient group. For the second assertion, see Theorem 4.3. □

Proposition 7.5 (a) entails that the surjection q is an open map. Hence q is a topological quotient map, and the manifold G/H carries the quotient topology.

Example 7.6 If G_n closed in G_{n+1} for each n and $K_n \subseteq G_n$ a maximal compact subgroup such that $K_1 \subseteq K_2 \subseteq \cdots$, then $K := \bigcup_{n\in\mathbb{N}} K_n$ is a closed subgroup of $G = \bigcup_{n\in\mathbb{N}} G_n$. In fact, $K_m \cap G_n = K_n$ for $m \geq n$ by maximality, whence $K \cap G_n = K_n$ is closed in G_n.

Proposition 7.7 *If $f : G \to A$ is $C^\infty_{\mathbb{K}}$- (resp., $c^\infty_{\mathbb{K}}$-) homomorphism from $G = \bigcup_{n\in\mathbb{N}} G_n$ into a $C^\infty_{\mathbb{K}}$-Lie group modelled on a locally convex space (resp., a $c^\infty_{\mathbb{K}}$- Lie group), then $H := \ker(f)$ satisfies the hypotheses of Proposition 7.5, and $L(H) = \ker L(f)$.*

Proof. In the complex case, $H \cap G_n = \ker(f|_{G_n})$ is a complex Lie subgroup of G_n by Lemma 4.4, whence the specific hypothesis of Proposition 7.5 is satisfied, by Lemma 7.3. If $w \in \ker L(f)$, then $\xi \colon \mathbb{K} \to H$, $\xi(t) := f(\exp_G(tw))$ is a $C^\infty_{\mathbb{R}}$- (resp., $c^\infty_{\mathbb{R}}$-) homomorphism such that $L(\xi) = L(f)(w) = 0$ and thus $\xi = 1$ ([26, La. 7.1], [23, La. 36.7]). Hence $\exp_G(\mathbb{K}w) \subseteq H$ and therefore $w \in L(H) = \{v \in L(G) : \exp_G(\mathbb{K}v) \subseteq H\}$. The inclusion $L(H) \subseteq \ker L(f)$ is trivial. □

Proposition 7.8 (Canonical Factorization) *Let $f : G \to A$ be a $c^\infty_{\mathbb{K}}$-homomorphism between direct limit groups, where G is connected, $G = \bigcup_{n\in\mathbb{N}} G_n$, and $A = \bigcup_{n\in\mathbb{N}} A_n$. Equip $G/\ker(f)$ with the $c^\omega_{\mathbb{K}}$-Lie group structure from Proposition 7.5 (c), and $\operatorname{im}(f)$ with the $c^\omega_{\mathbb{K}}$-Lie group structure induced by A (as in Proposition 7.2). Let $\overline{f}\colon G/\ker(f) \to \operatorname{im}(f)$ be the bijective homomorphism induced by f. Then $\overline{f}$ is a $c^\omega_{\mathbb{K}}$-diffeomorphism.*

Proof. In view of Proposition 3.5, we may assume that each G_n and A_n is connected. Note that $\overline{f}$ is $c^\omega_{\mathbb{K}}$ because the inclusion map $\operatorname{im}(f) \to A$ is $c^\omega_{\mathbb{K}}$-initial and the quotient map $G \to G/\ker(f)$ is $c^\omega_{\mathbb{K}}$-final. Replacing G with $G/\ker(f)$ and A with $\operatorname{im}(f)$, we may therefore assume that f is bijective, and have to show that f^{-1} is $c^\omega_{\mathbb{K}}$. Then $L(f)$ is injective, by Proposition 7.7.

$L(f)$ is surjective. To see this, let $x \in L(A) = \bigcup_{n\in\mathbb{N}} L(A_n)$; define $\mathfrak{s} := \mathbb{K}x$ and $S := \exp_A(\mathfrak{s})$. If $x \notin \operatorname{im}(L(f))$, then $\mathfrak{h}_n \cap \mathfrak{s} = \{0\}$ for each $n \in \mathbb{N}$, where $\mathfrak{h}_n := L(f)(L(G_n))$. Given n, there exists $m \in \mathbb{N}$ such that $L(A_m) \supseteq \mathfrak{h}_n \cup \mathfrak{s}$. Let H_n and S_n be the analytic subgroups of A_m with Lie algebras $\mathfrak{h}_n$ and $\mathfrak{s}$, respectively. Then $S = S_n$ as a set, and the group $H_n \cap S = H_n \cap S_n$ is countable, because $\mathfrak{h}_n \cap \mathfrak{s} = \{0\}$. Thus $S = \bigcup_{n\in\mathbb{N}}(S \cap H_n)$ is countable. But S is uncountable, contradiction. Therefore $x \in \operatorname{im}(L(f))$.

f^{-1} is $c^\omega_{\mathbb{K}}$. As $A = \varinjlim A_n$, it suffices to show that $f^{-1}|_{A_n}$ is $c^\omega_{\mathbb{K}}$, for each $n \in \mathbb{N}$. Fix n. There exists $m \in \mathbb{N}$ such that $L(f)(L(G_m)) \supseteq L(A_n)$. Let B be the analytic subgroup of G_m with Lie algebra $L(f)^{-1}(L(A_n))$. Then $f|_B^{A_n}$ is a bijective $C^\omega_{\mathbb{K}}$-homomorphism between connected finite-dimensional $\mathbb{K}$-Lie groups and therefore an isomorphism of $C^\omega_{\mathbb{K}}$-Lie groups. Hence $f^{-1}|_{A_n}^B$ is $C^\omega_{\mathbb{K}}$ and hence so is $f^{-1}|_{A_n}$. □

Proposition 7.9 (Universal covering group) *If G_n is connected for each $n \in \mathbb{N}$, let $\pi_n : \widetilde{G}_n \to G_n$ be the universal covering group, and $\widetilde{\lambda}_{n,m} : \widetilde{G}_m \to \widetilde{G}_n$ be the $C^\infty_{\mathbb{K}}$-homomorphism which lifts $\lambda_{n,m} \circ \pi_m$. Then $((\widetilde{G}_n)_{n\in\mathbb{N}}, (\widetilde{\lambda}_{n,m}))$ is a direct system in the category of $C^\infty_{\mathbb{K}}$-Lie groups; let $(\widetilde{G}, (\Lambda_n)_{n\in\mathbb{N}})$ be its direct limit. Then $\widetilde{G}$ is simply connected, and the $C^\infty_{\mathbb{K}}$-homomorphism $\pi := \varinjlim \pi_n \colon \widetilde{G} \to G$ is a universal covering map.*

Proof.[5] As any connected $C^\infty_{\mathbb{K}}$-Lie group, G has a universal covering group $p\colon P \to G$; thus G is a simply connected $C^\infty_{\mathbb{K}}$-Lie group and p a $C^\infty_{\mathbb{K}}$-homomorphism with discrete kernel. Being a regular topological space and locally diffeomorphic to $L(G)$, P is smoothly Hausdorff and hence also is a $c^\infty_{\mathbb{K}}$-Lie group. By [23, Thm. 38.6], P is a $c^\infty_{\mathbb{K}}$-regular Lie group. Let $\lambda_n : G_n \to G$ be the inclusion map. Since P is $c^\infty_{\mathbb{K}}$-regular, $L(\lambda_n)\colon L(\widetilde{G}_n) = L(G_n) \to L(G) = L(P)$ integrates to a $c^\infty_{\mathbb{K}}$-homomorphism $\alpha_n : \widetilde{G}_n \to P$ (Lemma 1.2, [23, Thm. 40.3]). Being a cone, $(P, (\alpha_n))$ induces a $c^\infty_{\mathbb{K}}$-homomorphism $\alpha : \widetilde{G} \to P$, determined by $\alpha \circ \Lambda_n = \alpha_n$. Recall from Theorem 4.3 that $\widetilde{G} = \varinjlim \widetilde{G}_n/D_n$, where $D_n := \ker(\Lambda_n)$ and where the limit map $\mu_n : \widetilde{G}_n/D_n \to \widetilde{G}$ is obtained by factoring Λ_n over $\widetilde{G}_n \to \widetilde{G}_n/D_n$. Because $\pi \circ \Lambda_n = \lambda_n \circ \pi_n$, the subgroup $D_n \subseteq \ker(\pi_n)$ is discrete.

[5] We cannot use Remark 3.9 because the $\widetilde{\lambda}_{n,m}$'s need not be injective.

Hence $L(\widetilde{G}) = \varinjlim L(\widetilde{G}_n/D_n) = \varinjlim L(\widetilde{G}_n) = \varinjlim L(G_n) = L(G)$. It is easily verified that $L(\alpha) = \mathrm{id}_{L(G)}$ with respect to these identifications. Now $\widetilde{G}$ being $c^\infty_{\mathbb{K}}$-regular and P simply connected, $\mathrm{id}\colon L(P) = L(G) \to L(G) = L(\widetilde{G})$ induces a $c^\infty_{\mathbb{K}}$-homomorphism $\beta\colon P \to \widetilde{G}$, determined by $L(\beta) = \mathrm{id}_{L(G)}$. Since $L(\alpha \circ \beta) = \mathrm{id}_{L(G)} = L(\mathrm{id}_P)$, we have $\alpha \circ \beta = \mathrm{id}_P$ by [26, La. 7.1]. Likewise, $\beta \circ \alpha = \mathrm{id}_{\widetilde{G}}$. Thus $\widetilde{G} \cong P$ is the universal covering group. □

Proposition 7.10 (Integration of Lie algebra homomorphisms) *Assume that $G = \bigcup_{n\in\mathbb{N}} G_n$ is simply connected. Then the following holds:*

(a) *Every $\mathbb{K}$-Lie algebra homomorphism $\alpha : L(G) \to L(H)$ into the Lie algebra of a $c^\infty_{\mathbb{K}}$-regular $c^\infty_{\mathbb{K}}$-Lie group H integrates to a $c^\infty_{\mathbb{K}}$-homomorphism $\beta\colon G \to H$ such that $L(\beta) = \alpha$. If $\mathbb{K} = \mathbb{R}$ and H is a $c^\omega_{\mathbb{R}}$-regular $c^\omega_{\mathbb{R}}$-Lie group here, then β is $c^\omega_{\mathbb{R}}$.*

(b) *Every $\mathbb{K}$-Lie algebra homomorphism $\alpha : L(G) \to L(H)$ into the Lie algebra of a $\mathbb{K}$-analytic BCH-Lie group (see [8]) integrates to a $C^\infty_{\mathbb{K}}$- (and $c^\omega_{\mathbb{K}}$-) homomorphism $\beta\colon G \to H$.*

Proof. (a) See [23, Thm. 40.3] and Lemma 1.2.

(b) Let $((\widetilde{G}_n), (\widetilde{\lambda}_{n,m}))$, $(\widetilde{G}, (\Lambda_n))$, $\pi_n\colon \widetilde{G}_n \to G_n$, and $\pi\colon \widetilde{G} \to G$ be as in Proposition 7.9. Because G is simply connected, the covering homomorphism π is an isomorphism. Hence $\widetilde{G} = G$ and $\Lambda_n = \lambda_n \circ \pi_n$ without loss of generality, where $\lambda_n\colon G_n \to G$ is the inclusion map. Now H and $\widetilde{G}_n$ being BCH, the homomorphism $\alpha_n := \alpha \circ L(\Lambda_n)$ integrates to a $C^\omega_{\mathbb{K}}$-homomorphism $\beta_n\colon \widetilde{G}_n \to H$ [8, Prop. 2.8]. Then $(H, (\beta_n))$ is a cone and hence induces a $C^\infty_{\mathbb{K}}$- (and $c^\omega_{\mathbb{K}}$-) homomorphism $\beta\colon G \to H$ such that $\beta \circ \Lambda_n = \beta_n$. Clearly $L(\beta) = \alpha$. □

Proposition 7.11 (Integration of Lie subalgebras) *Given a $\mathbb{K}$-Lie subalgebra $\mathfrak{h}$ of $L(G)$, equip the subgroup $H := \langle \exp_G(\mathfrak{h})\rangle$ with the $c^\omega_{\mathbb{K}}$-Lie group structure described in Proposition 7.2. Then H is connected, and $L(H) = \mathfrak{h}$. Furthermore, $H = \varinjlim H_n$ where $H_n := \langle \exp_{G_n}(\mathfrak{h}_n)\rangle$ is the analytic subgroup of G_n with Lie algebra $\mathfrak{h}_n := \mathfrak{h} \cap L(G_n)$.*

Proof. Consider the inclusion map $f : S \to G$, where $S := \varinjlim H_n = \bigcup_{n\in\mathbb{N}} H_n$. Then $S = H$ as an abstract group. We have $L(S) = \bigcup_{n\in\mathbb{N}} L(H_n) = \mathfrak{h}$, and f is $c^\omega_{\mathbb{K}}$ because each $f|_{H_n}$ is so. By Proposition 7.8, f is a $c^\omega_{\mathbb{K}}$-diffeomorphism onto $\mathrm{im}(f) = H$, equipped with the $c^\omega_{\mathbb{K}}$-Lie group structure induced by G. Thus $S = H$ as $c^\omega_{\mathbb{K}}$-Lie groups. □

Before we can discuss universal complexifications of direct limit groups, we need to re-examine universal complexifications of finite-dimensional Lie groups.

Lemma 7.12 *Let G be a finite-dimensional real Lie group, and $\gamma_G : G \to G_{\mathbb{C}}$ be its universal complexification in the category of finite-dimensional complex Lie groups. Let $\alpha : G \to H$ be a $c^\omega_{\mathbb{R}}$-homomorphism from G to a $c^\omega_{\mathbb{C}}$-Lie group H, respectively, a $c^\omega_{\mathbb{R}}$-homomorphism from G to a $C^\omega_{\mathbb{C}}$-Lie group H modelled on a locally convex space. Then there exists a unique $c^\omega_{\mathbb{C}}$-homomorphism (resp., $C^\omega_{\mathbb{C}}$-homomorphism) $\beta\colon G_{\mathbb{C}} \to H$ such that $\beta \circ \gamma_G = \alpha$. If H is a $c^\omega_{\mathbb{C}}$-regular $c^\omega_{\mathbb{C}}$-Lie group and α a $c^\infty_{\mathbb{R}}$-homomorphism, then the same conclusion holds, and α is $c^\omega_{\mathbb{R}}$.*

Proof. *We assume first that G is connected.* Let $p: \widetilde{G} \to G$ be the universal covering group of G and S be a simply connected complex Lie group with Lie algebra $L(G)_{\mathbb{C}}$. Let $\lambda: L(G) \to L(G)_{\mathbb{C}}$ be the inclusion map and $\kappa: \widetilde{G} \to S$ be the unique $C^{\omega}_{\mathbb{R}}$-homomorphism such that $L(\kappa) = \lambda$. Set $\Pi := \ker(p) \cong \pi_1(G)$ and let $N \subseteq S$ be the smallest closed complex Lie subgroup such that $\kappa(\Pi) \subseteq N$. Let $q: S \to S/N =: G_{\mathbb{C}}$ be the canonical quotient map. Then there exists a $C^{\omega}_{\mathbb{R}}$-homomorphism $\gamma_G: G \to G_{\mathbb{C}}$ such that $\gamma_G \circ p = q \circ \kappa$.

Let $\alpha: G \to H$ be a $c^{\omega}_{\mathbb{R}}$-homomorphism into a $c^{\omega}_{\mathbb{C}}$-Lie group H. Set $\sigma := \alpha \circ p: \widetilde{G} \to H$. Then there exist charts $\phi: L(G) \supseteq U \to \widetilde{G}$ and $\psi: L(H) \supseteq V \to H$ such that $\phi(0) = 1$, $\psi(0) = 1$ and $\sigma(\phi(U)) \subseteq \psi(V)$. After shrinking U, we may assume that $\kappa \circ \phi$ is injective and extends to a chart $\theta: W \to S$ of S, defined on some open neighbourhood W of U in $L(G)_{\mathbb{C}}$. The map $\tau := \psi^{-1} \circ \sigma \circ \phi: U \to V \subseteq L(H)$ is real analytic (see [23, Thm. 10.1]) and therefore extends to a complex analytic map $\tau_{\mathbb{C}}: U_{\mathbb{C}} \to L(H)$, defined on some open neighbourhood $U_{\mathbb{C}}$ of U in $L(G)_{\mathbb{C}} = L(S)$. After shrinking $U_{\mathbb{C}}$, we may assume that $\tau_{\mathbb{C}}(U_{\mathbb{C}}) \subseteq V$ and $U_{\mathbb{C}} \subseteq W$. The sets $\Omega := \{(x,y) \in U \times U: \phi(x)\phi(y) \in \phi(U)\}$ and $\Omega_{\mathbb{C}} := \{\{(x,y) \in U_{\mathbb{C}} \times U_{\mathbb{C}}: \theta(x)\theta(y) \in \theta(U_{\mathbb{C}})\}$ are open 0-neighbourhoods in $U \times U$ and $U_{\mathbb{C}} \times U_{\mathbb{C}}$, respectively. Let $Q \subseteq U_{\mathbb{C}}$ be an open, connected 0-neighbourhood such that $Q \times Q \subseteq \Omega_{\mathbb{C}}$. Then $h: Q \times Q \to H$, $h(x,y) := \psi(\tau_{\mathbb{C}}(x))\psi(\tau_{\mathbb{C}}(y))\psi(\tau_{\mathbb{C}}(\theta^{-1}(\theta(x)\theta(y))))^{-1}$ is a complex analytic map which is identically 1 on $\Omega \cap (Q \times Q)$; hence $h \equiv 1$ by the Identity Theorem [3, Prop. 6.6 II]. As a consequence, the complex analytic map $\zeta: \theta(Q) \to H$, $\zeta(x) := \psi(\tau_{\mathbb{C}}(\theta^{-1}(x)))$ satisfies $\zeta(xy) = \zeta(x)\zeta(y)$ for all $x, y \in \theta(Q)$ such that $xy \in \theta(Q)$. Therefore ζ extends to a homomorphism $\eta: S \to H$, by [22, Cor. A.2.26], which is complex analytic because $\eta|_{\theta(Q)} = \zeta$ is so. Then $\eta \circ \kappa = \sigma = \alpha \circ p$, because $\eta(\kappa(\phi(x))) = \sigma(\phi(x))$ for small x, by definition of η. Thus $\kappa(\Pi) \subseteq \ker(\eta)$, where $\ker(\eta)$ is a closed, complex Lie subgroup of S by Lemma 4.4. Thus $N \subseteq \ker(\eta)$, and thus η factors to a $c^{\omega}_{\mathbb{C}}$-homomorphism $\beta: G_{\mathbb{C}} = S/N \to H$ such that $\beta \circ q = \eta$. From $\beta \circ \gamma_G \circ p = \beta \circ q \circ \kappa = \eta \circ \kappa = \alpha \circ p$ we deduce that $\beta \circ \gamma_G = \alpha$, and clearly β is uniquely determined by this property. By the preceding, $\gamma_G: G \to G_{\mathbb{C}}$ is a universal complexification of the $c^{\omega}_{\mathbb{R}}$-Lie group G in the category of $c^{\omega}_{\mathbb{C}}$-Lie groups; since $G_{\mathbb{C}}$ is finite-dimensional, $\gamma_G: G \to G_{\mathbb{C}}$ also is the universal complexification of G in the category of finite-dimensional complex Lie groups.

If G is not necessarily connected, then the $c^{\omega}_{\mathbb{R}}$-Lie group G_0 has a universal complexification in the category $\mathcal{A}$ of $c^{\omega}_{\mathbb{C}}$-Lie groups, which is finite-dimensional. As in [8, Prop. 5.2], we see that the $c^{\omega}_{\mathbb{R}}$-Lie group G has a universal complexification $\gamma_G: G \to G_{\mathbb{C}}$ in $\mathcal{A}$, and $(G_{\mathbb{C}})_0$ is a universal complexification for G_0 and therefore finite-dimensional. Hence $G_{\mathbb{C}}$ is finite-dimensional, and hence it coincides with the universal complexification of G in the category of finite-dimensional complex Lie groups.

The second assertion (stated in parentheses) can be proved in the same way. To prove the third, let G be connected. Re-using the above notation, Lemma 1.2 provides a $c^{\omega}_{\mathbb{C}}$-homomorphism $\eta: S \to H$ such that $L(\eta)$ is the $\mathbb{C}$-linear extension of $L(\alpha)$. The proof can now be completed as above. As β is $c^{\omega}_{\mathbb{C}}$ and thus $c^{\omega}_{\mathbb{R}}$, the composition $\alpha = \beta \circ \gamma_G$ is $c^{\omega}_{\mathbb{R}}$. $\square$

Proposition 7.13 (Universal complexifications) *Let $\mathcal{S} := ((G_n)_{n\in\mathbb{N}}, (\lambda_{n,m})_{n\geq m})$ be a direct system of finite-dimensional real Lie groups and $C^\omega_\mathbb{R}$-homomorphisms, $(G, (\lambda_n)) := \varinjlim \mathcal{S}$ in the category of $c^\omega_\mathbb{R}$-Lie groups and $(G_\mathbb{C}, (\kappa_n)_{n\in\mathbb{N}}) := \varinjlim \big(((G_n)_\mathbb{C})_{n\in\mathbb{N}}, ((\lambda_{n,m})_\mathbb{C})\big)$ in the category of $c^\omega_\mathbb{C}$-Lie groups, where $\gamma_n\colon G_n \to (G_n)_\mathbb{C}$ is a universal complexification for G_n in the category of finite-dimensional complex Lie groups, and $(\lambda_{n,m})_\mathbb{C}\colon (G_m)_\mathbb{C} \to (G_n)_\mathbb{C}$ the uniquely determined complex analytic homomorphism such that $(\lambda_{n,m})_\mathbb{C} \circ \gamma_m = \gamma_n \circ \lambda_{n,m}$. Let $\gamma_G := \varinjlim \gamma_n\colon G \to G_\mathbb{C}$. Then the following holds:*

(a) *$\gamma_G : G \to G_\mathbb{C}$ is a universal complexification of the $c^\omega_\mathbb{R}$-Lie group G in the category of $c^\omega_\mathbb{C}$-Lie groups in the sense that for every $c^\omega_\mathbb{R}$-homomorphism $\alpha : G \to H$ into a $c^\omega_\mathbb{C}$-Lie group H, there exists a uniquely determined $c^\omega_\mathbb{C}$-homomorphism $\beta\colon G_\mathbb{C} \to H$ such that $\beta \circ \gamma_G = \alpha$.*

(b) *If H is a $c^\omega_\mathbb{C}$-regular $c^\omega_\mathbb{C}$-Lie group, then every $c^\infty_\mathbb{R}$-homomorphism $\alpha\colon G \to H$ is $c^\omega_\mathbb{R}$.*

(c) *If H is a $C^\omega_\mathbb{C}$-Lie group modelled on a locally convex space and $\alpha : G \to H$ a $c^\omega_\mathbb{R}$-homomorphism, then there also exists a unique β as in* (a).

(d) *$\gamma_G|_{G_0}^{(G_\mathbb{C})_0}$ is the universal complexification of G_0, and the map $G/G_0 \to G_\mathbb{C}/(G_\mathbb{C})_0$, $xG_0 \mapsto \gamma_G(x)(G_\mathbb{C})_0$ is a bijection.*

(e) *If G is simply connected, then γ_G has discrete kernel.*

(f) *If γ_G has discrete kernel, then $L(G_\mathbb{C}) = L(G)_\mathbb{C}$, $\operatorname{im}\gamma_G$ is closed in $G_\mathbb{C}$, and $\gamma_G|^{\operatorname{im}\gamma_G}$ is a local $c^\omega_\mathbb{R}$-diffeomorphism onto $\operatorname{im}\gamma_G$, equipped with the $c^\omega_\mathbb{R}$-Lie group structure induced by $(G_\mathbb{C})_\mathbb{R}$.*

Proof. (a)–(c): By Lemma 7.12, for each $n \in \mathbb{N}$ there exists a unique $c^\omega_\mathbb{C}$-homomorphism $\beta_n\colon (G_n)_\mathbb{C} \to H$ such that $\beta_n \circ \gamma_n = \alpha \circ \lambda_n$. Clearly $(H, (\beta_n))$ is a cone, whence there exists a unique $c^\omega_\mathbb{C}$- (resp., $C^\omega_\mathbb{C}$-) homomorphism $\beta\colon G_\mathbb{C} = \varinjlim (G_n)_\mathbb{C} \to H$ such that $\beta \circ \kappa_n = \beta_n$. Then $\beta \circ \gamma_G = \alpha$, and it is easily verified that β is uniquely determined by this property. In case (b), the composition $\alpha = \beta \circ \gamma_G$ is $c^\omega_\mathbb{R}$.

(d) Compare [8, Prop. 5.2].

(e) After replacing $\mathcal{S}$ by the corresponding injective quotient system of quotient groups G_n/N_n (see Section 4) and then by the corresponding direct system of simply connected groups $(G_n/N_n)^\sim$ (cf. Proposition 7.9), we may assume without loss of generality that G_n is simply connected and that $\lambda_n : G_n \to G$ has discrete kernel, for each $n \in \mathbb{N}$. Then also $(G_n)_\mathbb{C}$ is simply connected, γ_n has discrete kernel, and $\ker \kappa_n = \bigcup_{m\geq n} \ker(\lambda_{m,n})_\mathbb{C}$ is discrete (see proof of Theorem 5.1), for each n. As G_n and $(G_n)_\mathbb{C}$ are connected, the discrete subgroups $\ker \gamma_n$ and $\ker \kappa_n$ are countable. Therefore $D_n := \ker(\gamma_G \circ \lambda_n) = \ker(\kappa_n \circ \gamma_n) = \gamma_n^{-1}(\ker \kappa_n)$ is a closed, countable subgroup of G_n, and hence D_n is discrete. Since $\ker \lambda_n \subseteq D_n$, we deduce that $\lambda_n(D_n)$ is discrete in $\overline{G}_n := \operatorname{im}(\lambda_n)$, where the latter group is equipped with the finite-dimensional real Lie group structure $\cong G_n/\ker(\lambda_n)$. Then $G = \bigcup_{n\in\mathbb{N}} \overline{G}_n$, and it is clear from the construction in Section 4 that $G = \varinjlim \overline{G}_n$. The subgroup $H := \ker(\gamma_G)$ is closed in G, and $H_n := H \cap \overline{G}_n = \lambda_n(D_n)$ is discrete, for each $n \in \mathbb{N}$. By Proposition 7.5 (b) and Proposition 7.2, we have $H = \varinjlim H_n$ as a topological space (for the induced topology), whence H is discrete.

(f) Since $\ker(\gamma_G)$ is discrete, $L(\gamma_G)$ is injective (Proposition 7.7), enabling us to identify $L(G)$ with $\operatorname{im} L(\gamma_G)$ as a real locally convex space. Let $(G_{\mathbb{C}})_{\mathrm{op}}$ be $G_{\mathbb{C}}$, equipped with the opposite complex structure; by the universal property of $G_{\mathbb{C}}$, there is a unique $c^\omega_{\mathbb{C}}$-homomorphism $\sigma\colon G_{\mathbb{C}} \to (G_{\mathbb{C}})_{\mathrm{op}}$ such that $\sigma \circ \gamma_G = \gamma_G$. We now consider σ as an antiholomorphic self-map of $G_{\mathbb{C}}$. Thus $L(\sigma)$ is $\mathbb{C}$-antilinear. As in [14, La. IV.2], we see that σ is an involution. We have $L(G) \subseteq L(G_{\mathbb{C}})^\sigma$ for the fixed space of $L(\sigma)$. Since $L(G_{\mathbb{C}}) = L(G) + iL(G)$ by construction of $G_{\mathbb{C}}$, it easily follows that $L(G_{\mathbb{C}}) = L(G) \oplus iL(G) = L(G)_{\mathbb{C}}$ and thus $L(G) = L(G_{\mathbb{C}})^\sigma$. We now give the closed subgroup $(G_{\mathbb{C}})^\sigma := \mathrm{Fix}(\sigma)$ the $c^\omega_{\mathbb{R}}$-Lie group structure induced by $(G_{\mathbb{C}})_{\mathbb{R}}$. Then $\gamma_G(G) \subseteq (G_{\mathbb{C}})^\sigma$, and it is easy to see that $L((G_{\mathbb{C}})^\sigma) := \{v \in L(G_{\mathbb{C}}) : \exp_{G_{\mathbb{C}}}(\mathbb{R}v) \subseteq (G_{\mathbb{C}})^\sigma\} = L(G)$. Thus $C := ((G_{\mathbb{C}})^\sigma)_0 = \langle \exp_{G_{\mathbb{C}}}(L(G))\rangle = \gamma_G(G_0)$, and now Proposition 7.8 entails that $\gamma_G|_{G_0}^C$ is a local $c^\omega_{\mathbb{R}}$-diffeomorphism. To complete the proof, note that $(G_{\mathbb{C}})_0 \cap \gamma_G(G) = \gamma_G(G_0) = C$ by (d), whence $\gamma_G(G)$ is a locally closed subgroup of $G_{\mathbb{C}}$ and hence closed. □

8 Proof of regularity in Milnor's sense

Theorem 8.1 *Every direct limit group $G = \varinjlim G_n$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a regular $C^\infty_{\mathbb{R}}$-Lie group in Milnor's sense. More precisely, for every $k \in \mathbb{N} \cup \{\infty\}$, every $C^k_{\mathbb{R}}$-curve $\gamma : [0,1] \to G$ admits a right product integral $\eta = \mathrm{Evol}^r_G(\gamma) \in C^{k+1}([0,1], G)$ such that $\eta(0) = 1$, and the corresponding right evolution map*

$$\mathrm{evol}^r_G\colon C^k([0,1], L(G)) \to G, \quad \mathrm{evol}^r_G(\gamma) := \mathrm{Evol}^r_G(\gamma)(1)$$

is $C^\infty_{\mathbb{K}}$ and $c^\omega_{\mathbb{K}}$.

Proof. Fix k. The strategy of the proof is as follows. First, we show that product integrals exist and that evol^r_G is continuous. Next, we show that evol^r_G is complex analytic if $\mathbb{K} = \mathbb{C}$. Finally, for $\mathbb{K} = \mathbb{R}$, we deduce smoothness of evol^r_G from the smoothness of $\mathrm{evol}^r_{G_{\mathbb{C}}}$.

Step 1. Since evol^r_G takes its values in the connected component of G, we may assume that G is connected. Using that $\delta^r(p \circ \gamma) = \delta^r\gamma$ for curves in $\widetilde{G}$ (cf. [23, 38.4 (3)]), where $p : \widetilde{G} \to G$ is the universal covering map, we may assume that G is simply connected. Furthermore, we may assume that $G = \bigcup_{n\in\mathbb{N}} G_n$, where $G_1 \subseteq G_2 \subseteq \cdots$ and each G_n is connected. Let $j_n : G_n \to G$ be the inclusion map. We abbreviate $d_n := \dim_{\mathbb{K}}(G_n)$, $s := \sup\{d_n : n \in \mathbb{N}\}$ and let $\phi = \varinjlim \phi_n : P \to Q$ be a chart of G around 1, where $P = \bigcup_{n\in\mathbb{N}} \Delta_2^{d_n}$, $Q := \bigcup_{n\in\mathbb{N}} Q_n$ and $\phi_n : \Delta_2^{d_n} \to Q_n$ is a chart of G_n around 1, such that $\phi_n(0) = 1$. We identify $L(G_n) = T_1(G_n)$ with $\mathbb{K}^{d_n}$ using the chart ϕ_n, and $L(G)$ with $\mathbb{K}^s$ using ϕ; then $L(j_n)\colon \mathbb{K}^{d_n} \to \mathbb{K}^s$ is the inclusion map, for each $n \in \mathbb{N}$.

Step 2: evol^r_G *exists.* To see this, let $\gamma \in C^k([0,1], L(G))$. Then there exists $n \in \mathbb{N}$ such that $\operatorname{im}\gamma \subseteq L(G_n)$. Then $\gamma|^{L(G_n)}$ is C^k. It is a standard fact (based on the local existence and uniqueness of solutions to differential equations) that there exists $\eta \in C^{k+1}([0,1], G_n)$ such that $\delta^r\eta = \gamma|^{L(G_n)}$. Then $\mathrm{Evol}^r_G(\gamma) := j_n \circ \eta$ is C^{k+1} and $\delta^r(j_n \circ \eta) = L(j_n) \circ \gamma|^{L(G_n)} = \gamma$. Thus $\mathrm{evol}^r_G(\gamma)$ exists, and $\mathrm{evol}^r_G \circ C^k([0,1], L(j_n)) = j_n \circ \mathrm{evol}^r_{G_n}$.

Step 3. The inclusion map $C^k([0,1], L(G)) \to C^1([0,1], L(G))$ being continuous linear for each k, it suffices to prove that $\mathrm{evol}^r_G \colon C^1([0,1], L(G)) \to G$ is $C^\infty_{\mathbb{K}}$ and $c^\omega_{\mathbb{K}}$. We may therefore assume that $k = 1$ for the rest of the proof.

Step 4: evol^r_G *is continuous at nice* γ_0*'s.* We show that evol^r_G is continuous at $\gamma_0 \in C^1([0,1], L(G))$, provided that $\operatorname{im}(\gamma_0) \subseteq \mathbb{K}^{d_1} = L(G_1)$ and $\operatorname{im}(\eta_0) \subseteq \phi_1(\Delta^{d_1}_{1/2})$, where $\eta_0 := \mathrm{Evol}^r_G(\gamma_0)$. To this end, let W be an open neighbourhood of $\mathrm{evol}^r_G(\gamma_0) = \eta_0(1)$ in G; abbreviate $\zeta_0 := \phi_1^{-1} \circ \eta_0$. Then $\phi^{-1}(W)$ is an open neighbourhood of $\zeta_0(1)$, whence $\phi^{-1}(W) - \zeta_0(1) \supseteq \Delta^{d_1}_{\varepsilon_1} \oplus \bigoplus_{n \geq 2} \Delta^{d_n - d_{n-1}}_{\varepsilon_n}$ for certain $\varepsilon_n > 0$; we may assume that $1 \geq \varepsilon_1 \geq \varepsilon_2 \geq \cdots$. Define $r_n := 1 - 2^{-n}$ for $n \in \mathbb{N}$. Equip each $\mathbb{K}^{d_n}$ with the supremum norm. There is $R > 0$ such that $\|\gamma_0\|_\infty \leq R$.

For $n \in \mathbb{N}$, consider the map $f_n \colon \mathbb{K}^{d_n} \times \Delta^{d_n}_2 \to \mathbb{K}^{d_n}$, $f_n(y,x) := \frac{d}{ds}\big|_{s=0} \phi_n^{-1}(\phi_n(sy)\phi_n(x))$, which expresses the map $L(G_n) \times G_n \to TG_n$, $(y,x) \mapsto T_1(\rho_x).y$ (with right translation $\rho_x \colon G_n \to G_n$) in local coordinates (forgetting the fibre). Then $\zeta_0'(t) = f_n(\gamma_0(t), \zeta_0(t))$ for all $t \in [0,1]$, because $\delta^r(\eta_0) = \gamma_0$. By compactness of $\overline{\Delta}^{d_n}_1$ and $\overline{\Delta}^{d_n}_{R+n-1}$, there exists $k_n > 0$ such that for the operator norms of the partial differentials we have

$$\|d_2 f_n(v, x, \bullet)\| \leq k_n \quad \text{for all } v \in \Delta^{d_n}_{R+n-1} \text{ and } x \in \Delta^{d_n}_1,$$

and such that for the operator norms of the continuous linear maps $f_n(\bullet, x)$ we have $\|f_n(\bullet, x)\| \leq k_n$ for all $x \in \Delta^{d_n}_1$. Choose $\alpha_n > 0$ so small that

$$\frac{\alpha_n}{k_n}\left(e^{k_n} - 1\right) \;\leq\; 2^{-n-1}\,\varepsilon_n\,. \tag{3}$$

Define $s_n := \min\{\frac{\alpha_n}{k_n}, 1\}$. Suppose that $\gamma : [0,1] \to \Delta^{d_n}_{R+n-1}$ is a C^1-curve for which there exists a C^1-curve $\eta : [0,1] \to \Delta^{d_n}_{1-2^{-n}}$ solving the initial value problem $\eta(0) = 0$, $\eta'(t) = f_n(\gamma(t), \eta(t))$. Then $\|d_2 f_n(\gamma(t), x, \bullet)\| \leq k_n$ for all $t \in [0,1]$ and $x \in \Delta^{d_n}_1$. Let $\overline{\gamma} \colon [0,1] \to \mathbb{K}^{d_n}$ be a C^1-curve such that $\|\overline{\gamma} - \gamma\|_\infty < s_n$. Then $\operatorname{im}(\overline{\gamma}) \subseteq \Delta^{d_n}_{R+n}$, and

$$\|f_n(\overline{\gamma}(t), x) - f_n(\gamma(t), x)\| = \|f_n(\overline{\gamma}(t) - \gamma(t), x)\| \leq \|f_n(\bullet, x)\| \cdot \|\overline{\gamma}(t) - \gamma(t)\| \leq k_n s_n \leq \alpha_n$$

for all $x \in \Delta^{d_n}_1$. Furthermore, $\eta(t) + y \in \Delta^{d_n}_{1-2^{-n-1}} \subseteq \Delta^{d_n}_1$ for all $t \in [0,1]$ and $y \in \mathbb{K}^{d_n}$ such that $\|y\| \leq \frac{\alpha_n}{k_n}(e^{k_n} - 1) \leq 2^{-n-1}\varepsilon_n \leq 2^{-n-1}$. Using [6, (10.5.6)], we therefore find a solution $\xi \colon [0,1] \to \Delta^{d_n}_1$ to the initial value problem $\xi(0) = 0$, $\xi'(t) = f_n(\overline{\gamma}(t), \xi(t))$, such that

$$\|\xi - \eta\|_\infty \;\leq\; \frac{\alpha_n}{k_n}(e^{k_n} - 1) \;\leq\; 2^{-n-1}\varepsilon_n\,. \tag{4}$$

Hence $\operatorname{im}(\xi) \subseteq \Delta^{d_n}_{1-2^{-n-1}}$ in particular.

We now define $\Omega := \Delta^{d_n}_{s_1} \oplus \bigoplus_{n\geq 2} \Delta^{d_n-d_{n-1}}_{s_n}$, considering $\mathbb{K}^s$ as the locally convex direct sum $\mathbb{K}^{d_1} \oplus \bigoplus_{n\geq 2} \mathbb{K}^{d_n-d_{n-1}}$. Then $\gamma_0 + C^1([0,1],\Omega)$ is an open neighbourhood of γ_0 in $C^1([0,1],L(G))$. Let $\gamma \in \gamma_0 + C^1([0,1],\Omega)$. Then $\gamma - \gamma_0 = \sum_{n=1}^{\infty} \gamma_n$, where γ_n is the coordinate function taking its values in $\Delta^{d_1}_{s_1}$, resp., in $\Delta^{d_n-d_{n-1}}_{s_n}$. There exists $\ell \in \mathbb{N}$ such that $\gamma_n = 0$ for all $n \geq \ell$. Considering γ_0, $\gamma_0+\gamma_1$, $\ldots$, $\sum_{n=0}^{\ell} \gamma_n = \gamma$ in turn, from the existence of ζ_0 we inductively deduce by the preceding arguments that there exists a solution $\zeta_n \colon [0,1] \to \Delta^{d_n}_{1-2^{-n-1}}$ to the initial value problem $\zeta_n'(t) = f_n\big(\gamma_0(t)+\cdots+\gamma_n(t),\, \zeta_n(t)\big)$, $\zeta_n(0) = 0$, for $n = 1,\ldots,\ell$, such that $\|\zeta_n - \zeta_{n-1}\|_\infty \leq 2^{-n-1}\varepsilon_n$ (see (4)). Then $\eta := \phi \circ \zeta_\ell$ is the right product integral for γ, and thus $\mathrm{evol}^r_G(\gamma) = \eta(1) \in W$ because

$$\phi^{-1}(\eta(1)) - \phi^{-1}(\eta_0(1)) = \zeta_\ell(1) - \zeta_0(1) = \sum_{n=1}^{\ell} (\zeta_n(1) - \zeta_{n-1}(1)) \in \Delta^{d_1}_{\varepsilon_1} \oplus \bigoplus_{n=2}^{\ell} \Delta^{d_n-d_{n-1}}_{\varepsilon_n} \subseteq \phi^{-1}(W)\,.$$

Hence evol^r_G is indeed continuous at γ_0.

Step 5. evol^r_G *is continuous.* Let $\overline{\gamma} \in C^1([0,1],L(G))$. After passing to a subsequence, we may assume that $\mathrm{im}(\overline{\gamma}) \subseteq \mathbb{K}^{d_1} = L(G_1)$. Let $\overline{\eta} := \mathrm{Evol}^r_G(\overline{\gamma})$. We find a partition $0 = t_0 < t_1 < \ldots < t_\ell = 1$ such that $\overline{\eta}_j([0,1]) \subseteq \phi_1(\Delta^{d_1}_{1/2})$ for each $j \in \{0,\ldots,\ell-1\}$, where $\overline{\eta}_j \colon [0,1] \to G$, $\overline{\eta}_j(t) = \overline{\eta}(t_j + t(t_{j+1}-t_j))\,\overline{\eta}(t_j)^{-1}$. Then $\overline{\eta}_j = \mathrm{Evol}^r_G(\overline{\gamma}_j)$, where $\overline{\gamma}_j \colon [0,1] \to L(G)$, $\overline{\gamma}_j(t) := (t_{j+1}-t_j)\cdot\overline{\gamma}(t_j + t(t_{j+1}-t_j))$ are mappings which satisfy the hypotheses of Step 4. Thus evol^r_G is continuous at $\overline{\gamma}_j$. Since $\gamma \mapsto \gamma_j$ is continuous and $\mathrm{evol}^r_G(\gamma) = \mathrm{evol}^r_G(\gamma_{\ell-1})\cdots\mathrm{evol}^r_G(\gamma_1)\mathrm{evol}^r_G(\gamma_0)$, we deduce that evol^r_G is continuous at $\overline{\gamma}$.[6]

Step 6: evol^r_G *is* $C^\infty_{\mathbb{C}}$ *if* $\mathbb{K} = \mathbb{C}$. It suffices to show that evol^r_G is $C^\infty_{\mathbb{C}}$ on some open neighbourhood of each $\gamma_0 \in C^1([0,1],L(G))$ such that $\gamma_0([0,1]) \subseteq L(G_1)$ and such that $\eta_0 := \mathrm{Evol}^r_G(\gamma_0)$ has image in $\phi_1(\Delta^{d_1}_{1/2})$, by arguments similar to those just employed. Let Ω be as in Step 4, and $U := \gamma_0 + C^1([0,1],\Omega)$. As shown in Step 4, $\eta := \mathrm{Evol}^r_G(\gamma)$ has image in $Q = \mathrm{im}(\phi)$, for each $\gamma \in U$, $\zeta := \phi^{-1}\circ\eta$ satisfies $\zeta(0) = 0$, and $\zeta'(t) = f_n(\gamma(t),\zeta(t))$ for each n such that $\zeta([0,1]) \subseteq \mathbb{C}^{d_n}$. Now suppose that $\gamma \in U$ and $\theta \in C^1([0,1],L(G))$. There exists n (which we fix now) such that γ,θ have image in $\mathbb{C}^{d_n}$. Then $\sigma_z := \gamma + z\theta \in U$ for z in some 0-neighbourhood $V \subseteq \mathbb{C}$, and $\mathrm{im}(\sigma_z) \subseteq \mathbb{C}^{d_n}$ for each $z \in V$. Let $\tau_z := \phi^{-1}\circ\mathrm{Evol}^r_G(\sigma_z)$. Then τ_z solves the initial value problem $\tau_z(0) = 0$, $\tau_z'(t) = f_n(\sigma_z(t),\tau_z(t))$. Consider $f \colon [0,1]\times\Delta^{d_n}_1\times V \to \mathbb{C}^{d_n}$, $f(t,x,z) := f_n(\sigma_z(t),x)$. Then $f(t,x,z) = f_n(\gamma(t),x) + zf_n(\theta(t),x)$, showing that the differentiability requirements of [5, Thm. 3.6.1] are satisfied.[7] Hence $u(t,z) := \tau_z(t)$ is $C^1_{\mathbb{R}}$ in (t,z) on an open neighbourhood of $I\times\{0\}$ in $I\times V$, and the map $h \colon [0,1] \to \mathcal{L}_{\mathbb{R}}(\mathbb{C},\mathbb{C}^{d_n})$, $h(t) := d_2u(t,0,\bullet)$ to the space of $\mathbb{R}$-linear maps $\mathbb{C} \to \mathbb{C}^{d_n}$ is $C^1_{\mathbb{R}}$ and solves the initial value problem

$$h(0) = 0, \quad h'(t) = b(t)\circ h(t) + c(t), \tag{5}$$

where $c(t)(z) = z\cdot f_n(\theta(t),\tau_0(t))$ and $b(t) = d_2f_n(\sigma_0(t),\tau_0(t),\bullet)$. Since $b(t) \in \mathcal{L}_{\mathbb{C}}(\mathbb{C}^{d_n},\mathbb{C}^{d_n})$ actually for each t and $c(t) \in \mathcal{L}_{\mathbb{C}}(\mathbb{C},\mathbb{C}^{d_n})$, we can interpret (5) also as a linear differential

[6] We have even established continuity with respect to the topology of uniform convergence!

[7] To apply the theorem, note that f extends to an open set, because γ and θ extend to open intervals by Borel's theorem.

equation for $\mathcal{L}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}^{d_n})$-valued functions. This implies that $h(t) \in \mathcal{L}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}^{d_n})$ for each t, i.e., $h(t) = d_2u(t, 0, \bullet)$ is complex linear. Hence $\frac{d}{dz}\big|_{z=0}\phi^{-1}(\mathrm{evol}_G^r(\gamma + z\theta)) = \frac{d}{dz}\big|_{z=0}\tau_z(1) = \frac{\partial}{\partial z}\big|_{z=0}u(1, z)$ exists as a complex derivative.

By the preceding, $\psi := \phi^{-1} \circ \mathrm{evol}_G^r|_U : U \to \mathbb{C}^s$ admits complex directional derivatives at each point. Hence ψ is G-analytic in the sense of [3, Defn. 5.5], by [3, Prop. 5.5] and [3, Thm. 3.1]. Being G-analytic and continuous, ψ is complex analytic [3, Thm. 6.1 (i)].

Step 7: evol_G^r *is* $C^\infty_{\mathbb{R}}$ *and* $c^\omega_{\mathbb{R}}$ *if* $\mathbb{K} = \mathbb{R}$. Because G is assumed simply connected, we know that $H := \gamma_G(G)$ is a closed subgroup of $G_{\mathbb{C}}$, that γ_G has discrete kernel, and that γ_G is a local $c^\omega_{\mathbb{R}}$-diffeomorphism onto H, equipped with the real Lie group structure induced by $(G_{\mathbb{C}})_{\mathbb{R}}$ (see Proposition 7.13 (e) and (f)). Since H is $C^\infty_{\mathbb{R}}$-initial in $G_{\mathbb{C}}$ and $c^\omega_{\mathbb{R}}$-initial (Proposition 7.5 (b)), we deduce from the smoothness (and $c^\omega_{\mathbb{R}}$-property) of $\gamma_G \circ \mathrm{evol}_G^r = \mathrm{evol}_{G_{\mathbb{C}}}^r \circ L(\gamma_G)$ that $\gamma_G|^H \circ \mathrm{evol}_G^r$ is $C^\infty_{\mathbb{R}}$ and $c^\omega_{\mathbb{R}}$. As evol_G^r is continuous and $\gamma_G|^H$ a local $C^\infty_{\mathbb{R}}$- (and $c^\omega_{\mathbb{R}}$-) diffeomorphism, this implies that evol_G^r is $C^\infty_{\mathbb{R}}$ and $c^\omega_{\mathbb{R}}$. □

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Helge Glöckner, TU Darmstadt, FB Mathematik AG 5, Schlossgartenstr. 7, 64289 Darmstadt, Germany.
E-Mail: gloeckner@mathematik.tu-darmstadt.de