On two-dimensional immersions of prescribed mean curvature in $\mathbb{R}^n$

Matthias Bergner, Steffen Fröhlich

Abstract

We consider two-dimensional immersions of disc-type in $\mathbb{R}^n$. We focus well known classical concepts and study the nonlinear elliptic systems of such mappings. Using an Osserman-type condition we give a priori-estimates of the principle curvatures for certain graphs in $\mathbb{R}^4$ with prescribed mean curvature.

Introduction

In 1917, S. Bernstein proved the following theorem:

If the twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ represents a graph of a complete minimal surface in $\mathbb{R}^3$, then f is linear, i.e., the minimal surface is a plane.

This result indicates the difference between linear and non-linear partial differential equations. Over the years much work has been devoted in finding new proofs and generalizations.

To exemplify, let us indicate a method which goes back to works of E. Heinz and F. Sauvigny, and which applies in the paper at hand: Using isothermal parameters, a minimal surface X and its unit normal mapping N satisfy the nonlinear elliptic systems

$$\triangle X = 0, \quad \triangle N = -2KWN.$$

Here, K means the Gaussian curvature and W the surface area element of the immersion X. If one can establish certain bounds for the size of X and N, the gradient estimates in [3] ensure upper bounds for the first and second derivatives of X in terms of given a priori-data. A Harnack-type inequality for pseudo-holomorphic functions would give lower bounds for W. All together, we would estimate the principle curvatures of the immersion, namely in such a way that if the surface growths to become complete, the principle curvatures vanish identically.

This method generalizes to various varitional problems if the appearing differential systems satisfy structure conditions of the form

$$|\triangle Z| \le \text{const}\, |\nabla Z|^2 .$$

For example, the mean curvature system $\triangle X = 2HWN$ and the above system for N satisfy this assumption. Some problems arise in higher codimension.

In 1964, Osserman [5] proved that a complete two-dimensional minimal surface in $\mathbb{R}^n$ is a plane if all of its normal vectors make a certain positive angle with a fixed axis in space. The assumption that *all normals make an acute angle with some fixed direction* makes the difference to the Bernstein theorem in $\mathbb{R}^3$ where it is sufficient to assume that the normal of the immersion omits a certain neighbourhood of some point of the unit sphere.

Osserman's method is based essentially on results of complex analysis due to the fact that the components of a minimal surface are harmonic functions.

Although it is rather difficult to realize Osserman's condition, it is an important fact that any condition of this type is necessary for curvature estimates.

In this paper we want to examine how to apply the methods developed in [3] and [6] to establish certain a priori-estimates for the principle curvatures of graphs with prescribed mean curvature. In the first part, we concentrate on some geometric basics of two-dimensional immersion in $\mathbb{R}^n$ which are motivated from [2], and we will derive nonlinear elliptic differential systems to describe the surfaces analytically. The second part contains the mentioned estimates.

1. Surfaces in $\mathbb{R}^n$

The essential principles of the following are motivated from [1], [2], [5], and [7].

Let

$$B := \{(u,v) \in \mathbb{R}^2 \ : \ u^2+v^2<1\}$$

denote the open unit disc, $\overline{B} \subset \mathbb{R}^2$ its topological closure. For integer $n \geq 3$ we consider immersions

$$\begin{aligned} &X = X(u,v) \in C^{3+\alpha}(B,\mathbb{R}^n) \cap C^0(\overline{B},\mathbb{R}^n), \quad \alpha \in (0,1), \\ &X(u,v) = (x^1(u,v), x^2(u,v), \ldots, x^n(u,v)), \end{aligned}$$

with the property

$$\operatorname{rank} \partial X(u,v) \equiv \operatorname{rank} \begin{pmatrix} x_u^1(u,v) & x_v^1(u,v) \\ \vdots & \vdots \\ x_u^n(u,v) & x_v^n(u,v) \end{pmatrix} = 2 \quad \text{for all } (u,v) \in B.$$

This means that $X_u = X_u(u,v)$ and $X_v = X_v(u,v)$ are linearly independent in every point $(u,v) \in B$ and, therefore, span the two-dimensional tangent plane

$$\mathcal{T}_X(w) := \operatorname{Span}\{X_u(w), X_v(w)\} \quad \text{at } w \in B.$$

There exists a normal space $\mathcal{N}_X(w)$ spanned by $n-2$ linearly independent vectors $N_1, N_2, \ldots, N_{n-2}$ which can be assumed to satisfy the orthonormality relations

$$N_\Sigma \cdot N_\Theta^t = \delta_{\Sigma\Theta} := \begin{cases} 1 & \text{if } \Sigma = \Theta \\ 0 & \text{if } \Sigma \neq \Theta \end{cases} \qquad \text{for all } \Sigma, \Theta \in \{1,2,\ldots,n-2\}.$$

Here, by the upper t we denote the transposed vector.

The choice of the normal vectors $N_\Sigma(w)$, $\Sigma = 1, \ldots, n-2$, is not unique.

Example 1*:* The unit vectors

$$\begin{aligned} N_1 &:= \frac{1}{\sqrt{1+|\nabla\varphi_1|^2}}\,(-\varphi_{1,x}, -\varphi_{1,y}, 1, 0, \ldots, 0), \\ N_2 &:= \frac{1}{\sqrt{1+|\nabla\varphi_2|^2}}\,(-\varphi_{2,x}, -\varphi_{2,y}, 0, 1, \ldots, 0), \ \ldots \end{aligned}$$

are normal to the graph $X(x,y) = (x, y, \varphi_1(x,y), \ldots, \varphi_n(x,y))$.

Assumption: *There exists an orthonormal basis $N_1(w), N_2(w), \ldots, N_{n-2}(w)$ of the normal space $\mathcal{N}_X(w)$ such that $N_\Sigma \in C^{2+\alpha}(B, S^{n-1})$.*

In the following we will use such regular bases.

1.1 The first fundamental form

We consider surfaces $X = X(u,v)$ immersed into the Euclidean space $\mathbb{R}^n$. Inserting $X = X(u,v)$ into the element $ds^2 := (dx^1)^2 + \ldots + (dx^n)^2$ yields

$$ds^2 = h_{11}\,du^2 + 2h_{12}\,dudv + h_{22}\,dv^2 = h_{ij}\,du^i du^j$$

(note the summation convention in the last term) with the first fundamental form

$$I := (h_{ij})_{i,j=1,2} \equiv (X_{u^i} \cdot X_{u^j}^t)_{i,j=1,2} \subset \mathbb{R}^{2\times 2}\,.$$

1.2 The mean curvature

We introduce the second fundamental form $I\!I_\Sigma = (L_{\Sigma,ij})_{i,j=1,2}$ w.r.t. the unit normal N_Σ by

$$L_{\Sigma,ij} := -X_{u^i} \cdot N_{\Sigma,u^j}^t = X_{u^iu^j} \cdot N_\Sigma^t\,, \quad i,j = 1,2.$$

Consider the form

$$(L_{\Sigma,i}^k)_{i,k=1,2} := (L_{\Sigma,ij}h^{jk})_{i,k=1,2} \in \mathbb{R}^{2\times 2}$$

with the coefficients h^{ij} of the inverse of the first fundamental form, i.e.

$$h_{ij}h^{jk} = \delta_i^k := \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}\,, \quad i,k = 1,2.$$

Definition 1: The *mean curvature* H_Σ *in direction* N_Σ is defined as

$$H_\Sigma := \frac{1}{2}\,\text{trace}\,(L_{\Sigma,i}^k)_{i,k=1,2}\,.$$

1.3 The principle curvatures

Beside the mean curvature

$$H_\Sigma = \frac{1}{2}\,\text{trace}\,(L_{\Sigma,i}^k)_{i,k=1,2} = \frac{L_{\Sigma,11}h_{11} - 2L_{\Sigma,12}h_{12} + L_{\Sigma,22}h_{22}}{2(h_{11}h_{22} - h_{12}^2)}\,,$$

we introduce the *Gaussian curvature w.r.t. to* N_Σ as

$$K_\Sigma := \det\,(L_{\Sigma,i}^k)_{i,k=1,2} = \frac{L_{\Sigma,11}L_{\Sigma,22} - L_{\Sigma,12}^2}{h_{11}h_{22} - h_{12}^2}\,.$$

The *principle curvatures* $\kappa_{\Sigma,1}$, $\kappa_{\Sigma,2}$ w.r.t. N_Σ are the eigenvalues of the form $(L_{\Sigma,i}^k)_{i,k=1,2}$, that is

$$H_\Sigma = \frac{\kappa_{\Sigma,1} + \kappa_{\Sigma,2}}{2}\,, \quad K_\Sigma = \kappa_{\Sigma,1}\kappa_{\Sigma,2}\,.$$

1.4 Minimal surfaces

Minimal surfaces can be introduced analytically via the first Beltrami operator

$$\overline{\nabla}_{ds^2}(X, N_\Sigma) := h^{ij}X_{u^i} \cdot N_{\Sigma,u^j}^t = -h^{ij}X_{u^iu^j} \cdot N_\Sigma^t = -h^{ij}L_{\Sigma,ij} = -2H_\Sigma$$

w.r.t. to the metric element ds^2.

Definition 2: Let $\{N\}$ be an orthonormal basis of the normal space. The immersion $X = X(u,v)$ is called a minimal surface iff

$$\overline{\nabla}_{ds^2}(X, N_\Sigma) = 0 \quad \text{in } B \quad \text{for all } N_\Sigma \in \{N\}.$$

In fact, it holds $\overline{\nabla}_{ds^2}(X, Z) = 0$ for all $Z \in \mathcal{N}_X$ with $|Z| = 1$. Namely, let $Z = \sum_{\Sigma=1}^{n-2} \lambda_\Sigma N_\Sigma$ with coefficients $\lambda_\Sigma \in \mathbb{R}$, $\Sigma = 1, 2, \ldots, n-2$, then we calculate

$$\overline{\nabla}_{ds^2}(X, Z) = h^{ij} X_{u^i} Z_{u^j}^t = -h^{ij} X_{u^i u^j} \cdot Z^t = -\sum_{\Sigma=1}^{n-2} \lambda_\Sigma (h^{ij} X_{u^i u^j} \cdot N_\Sigma^t) = 0.$$

Now, we proceed as follows: Introducing conformal parameters $(u,v) \in B$, there hold the conformality relations

$$h_{11} = W = h_{22}\,, \quad h_{12} = 0 \quad \text{in } B$$

with the surface area element W. The, from Beltrami's equation for a minimal surface we get

$$\triangle X \cdot N_\Sigma^t = 0 \quad \text{in } B \quad \text{for all } \Sigma = 1, 2, \ldots, n-2.$$

Further, differentiation of the conformality relations $|X_u|^2 - |X_v|^2 = 0$ and $X_u \cdot X_v^t = 0$ yields

$$\begin{array}{ll} X_u \cdot X_{uu}^t - X_v \cdot X_{uv}^t = 0, & X_u \cdot X_{vv}^t + X_v \cdot X_{uv}^t = 0, \\ X_u \cdot X_{uv}^t - X_v \cdot X_{vv}^t = 0, & X_u \cdot X_{uv}^t + X_v \cdot X_{uu}^t = 0. \end{array}$$

Thus, there also hold

$$\triangle X \cdot X_u^t = 0 \quad \text{and} \quad \triangle X \cdot X_v^t = 0 \quad \text{in } B.$$

Proposition 1: *A minimal surface $X = X(u,v)$, given in conformal parameters $(u,v) \in B$, satisfies the elliptic system*

$$\triangle X = 0 \quad \textit{in } B.$$

Therefore, its components $x^i = x^i(u,v)$, $i = 1, 2, \ldots, n$, are harmonic functions in B.

1.5 The differential equations

Let $\{N\}$ be an orthonormal basis of the normal space. We rewrite the derivatives N_{Σ,u^i}, where $N_\Sigma \in \{N\}$ for $\Sigma = 1, 2, \ldots, n-2$, and $X_{u^i u^j}$ for and $i, j = 1, 2$ in terms of the moving frame $X_u, X_v, N_1, \ldots, N_{n-2}$. The following is discussed in more detail in [2].

1.5.1 The Weingarten equations

Due to $N_{\Sigma,u^i} \cdot N_\Sigma^t = 0$, $i = 1, 2$, we make the ansatz

$$N_{\Sigma,u^i} = a_{\Sigma,i}^j X_{u^j} + b_{\Sigma,i}^\Theta N_\Theta\,, \quad i = 1, 2.$$

We determine the coefficients $a_{\Sigma,i}^j$ and $b_{\Sigma,i}^\Theta$ as follows: From

$$-L_{\Sigma,ik} = a_{\Sigma,i}^j X_{u^k} \cdot X_{u^j}^t = a_{\Sigma,i}^j h_{jk}$$

we conclude

$$N_{\Sigma,u^i} = -L_{\Sigma,ij} h^{jk} X_{u^k} + b_{\Sigma,i}^\Theta N_\Theta\,.$$

Now, let us define the *torsion coefficients*

$$\sigma_{\Sigma,i}^\Theta := \begin{cases} N_{\Sigma,u^i} \cdot N_\Theta^t & \text{if } \Sigma \neq \Theta \\ 0 & \text{if } \Sigma = \Theta \end{cases}.$$

Proposition 2: *Let $\{N\}$ be an orthonormal basis of the normal space. The Weingarten equations are*

$$N_{\Sigma,u^i} = -L_{\Sigma,ij}h^{jk}X^t_{u^k} + \sigma^{\Theta}_{\Sigma,i}N_{\Theta}\,, \quad i = 1,2,\ \Sigma = 1,2,\ldots,n-2.$$

1.5.2 The Gauss equations

We complete the set of differential equations by

Proposition 3: *Let $\{N\}$ be an orthonormal basis of the normal space. The Gauss equations are*

$$X_{u^iu^j} = \Gamma^k_{ij}X_{u^k} + \sum_{\Sigma=1}^{n-2} L_{\Sigma,ij}N_{\Sigma}$$

with the Christoffel symbols

$$\Gamma^k_{ij} := \frac{1}{2}\,h^{kl}(h_{jl,i} + h_{li,j} - h_{ij,l}), \quad h_{ij,k} := h_{ij,u^k}\,.$$

For the proof we make the ansatz

$$X_{u^iu^j} = a^k_{ij}X_{u^k} + \sum_{\Sigma=1}^{n-2} b_{\Sigma,ij}N_{\Sigma}$$

with unknown coefficients a^k_{ij} and $b_{\Sigma,ij}$. Multiplication with N_Σ gives

$$L_{\Sigma,ij} = X_{u^iu^j}\cdot N^t_{\Sigma} = b_{\Sigma,ij}\,, \quad i,j = 1,2.$$

Furthermore, we calculate

$$X_{u^iu^j}\cdot X^t_{u^l} = a^k_{ij}X_{u^k}\cdot X^t_{u^l} = a^k_{ij}h_{kl} =: \alpha_{ilj}\,.$$

Note the symmetry $\alpha_{ilj} = \alpha_{jli}$. It holds

$$\alpha_{ilj} = (X_{u^i}\cdot X^t_{u^l})_{u^j} - X_{u^lu^j}\cdot X^t_{u^i}\,,$$

and, therefore,

$$\alpha_{ilj} = h_{il,j} - \alpha_{lij} \quad \text{or} \quad h_{il,j} = \alpha_{ilj} + \alpha_{lij}\,.$$

From the above symmetry we get

$$h_{jl,i} + h_{li,j} - h_{ij,l} = \alpha_{jli} + \alpha_{lji} + \alpha_{lij} + \alpha_{ilj} - \alpha_{ijl} - \alpha_{jil} = 2\alpha_{jli}\,.$$

It follows, that

$$\alpha_{ilj} = a^k_{ij}h_{kl} = \frac{1}{2}\,(h_{jl,i} + h_{li,j} - h_{ij,l}),$$

and, finally,

$$a^k_{ij} = \frac{1}{2}\,h^{kl}(h_{jl,i} + h_{li,j} - h_{ij,l}) = \Gamma^k_{ij}\,.$$

1.6 The first variation of the area functional

Minimal surfaces are critical points of the area functional:
Let $\alpha,\beta,\varphi^\Sigma\in C_0^\infty(B,\mathbb{R})$, $\Sigma=1,2,\ldots,n-2$, and $\varepsilon\in(-\varepsilon_0,+\varepsilon_0)$. We consider the variation

$$\widetilde{X}(u,v):=X(u,v)+\varepsilon\alpha(u,v)X_u(u,v)+\varepsilon\beta(u,v)X_v(u,v)+\varepsilon\sum_{\Sigma=1}^{n-2}\varphi^\Sigma(u,v)N_\Sigma(u,v),\quad (u,v)\in B.$$

Taking the Gauss equations into account we calculate

$$\begin{aligned}
\widetilde{X}_u &= X_u+\varepsilon(\alpha_u+\alpha\Gamma_{11}^1+\beta\Gamma_{12}^1-W^{-1}\varphi^\Sigma L_{\Sigma,11})X_u+\varepsilon(\beta_u+\alpha\Gamma_{11}^2+\beta\Gamma_{12}^2-W^{-1}\varphi^\Sigma L_{\Sigma,12})X_v\\
&\quad+\varepsilon\sum_{\Sigma=1}^{n-2}\left(\alpha L_{\Sigma,11}+\beta L_{\Sigma,12}+\varphi_u^\Sigma+\varphi^\Theta\sigma_{\Theta,1}^\Sigma\right)N_\Sigma\,,\\
\widetilde{X}_v &= X_v+\varepsilon(\alpha_v+\alpha\Gamma_{12}^1+\beta\Gamma_{22}^1-W^{-1}\varphi^\Sigma L_{\Sigma,12})X_u+\varepsilon(\beta_v+\alpha\Gamma_{12}^2+\beta\Gamma_{22}^2-W^{-1}\varphi^\Sigma L_{\Sigma,22})X_v\\
&\quad+\varepsilon\sum_{\Sigma=1}^{n-2}\left(\alpha L_{\Sigma,12}+\beta L_{\Sigma,22}+\varphi_v^\Sigma+\varphi^\Theta\sigma_{\Theta,2}^\Sigma\right)N_\Sigma\,.
\end{aligned}$$

We use conformal parameters to get

$$\begin{aligned}
\widetilde{X}_u^2 &= W+\varepsilon(2\alpha_uW+\alpha W_u+\beta W_v-2\varphi^\Sigma L_{\Sigma,11}),\\
\widetilde{X}_v^2 &= W+\varepsilon(2\beta_vW+\alpha W_u+\beta W_v-2\varphi^\Sigma L_{\Sigma,22}),\\
\widetilde{X}_u\cdot\widetilde{X}_v^t &= \varepsilon(\alpha_v+\beta_u)W-2\varepsilon\varphi^\Sigma L_{\Sigma,12}\,.
\end{aligned}$$

Thus, the following variational formula hold true:

$$\begin{aligned}
\delta h_{11} &= 2\alpha_uW+\alpha W_u+\beta W_v-2\varphi^\Sigma L_{\Sigma,11},\\
\delta h_{12} &= (\alpha_v+\beta_u)W-2\varphi^\Sigma L_{\Sigma,12}\,,\\
\delta h_{22} &= 2\beta_vW+\alpha W_u+\beta W_v-2\varphi^\Sigma L_{\Sigma,22}\,.
\end{aligned}$$

From the identity

$$2W\delta W=h_{22}\delta h_{11}-2h_{12}\delta h_{12}+h_{11}\delta h_{22}$$

we infer, using conformal parameters,

$$\delta W=\frac{1}{2}\,\delta h_{11}+\frac{1}{2}\,\delta h_{22}=\operatorname{div}(\alpha W,\beta W)-2\varphi^\Sigma H_\Sigma W.$$

Consequently, the first variation of the area functional

$$\mathcal{A}[X]:=\iint\limits_B\sqrt{h_{11}h_{22}-h_{12}^2}\,dudv$$

takes the form

$$\delta\mathcal{A}[X]=\iint\limits_B\operatorname{div}(\alpha W,\beta W)\,dudv-2\iint\limits_B\varphi^\Sigma H_\Sigma W\,dudv=-2\iint\limits_B\varphi^\Sigma H_\Sigma W\,dudv$$

because α,β vanish on the boundary ∂B. The equation is independent from the tangential variation. For a critical point $X_0=X_0(u,v)$ it holds $\delta\mathcal{A}[X_0]=0$ for all possible test functions $\varphi^\Sigma\in C_0^\infty(B,\mathbb{R})$. Therefore, the mean curvature in all directions must be zero, and a critical point is a minimal surface.

1.7 An elliptic system for X

Let $(u,v)\in B$ be conformal parameters. Then, there hold

$$\Gamma_{11}^1+\Gamma_{22}^1=0,\quad \Gamma_{11}^2+\Gamma_{22}^2=0.$$

Thus, the Gauss equations from Proposition 3 yield

$$\triangle X=(\Gamma_{11}^1+\Gamma_{22}^1)X_u+(\Gamma_{11}^2+\Gamma_{22}^2)X_v+\sum_{\Sigma=1}^{n-2}(L_{\Sigma,11}+L_{\Sigma,22})N_\Sigma=2\sum_{\Sigma=1}^{n-2}H_\Sigma WN_\Sigma\,.$$

This system is called the *mean curvature system.*
Let $|H_1|+|H_2|+\ldots+|H_{n-2}|\le h_0$ with a real constant $h_0\in(0,+\infty)$. We estimate

$$|\triangle X|\le 2h_0|X_u||X_v|\le h_0|\nabla X|^2$$

due to $W=\sqrt{h_{11}h_{22}}=|X_u||X_v|$. We point out the quadratic growth of the gradient on the right hand side. Elliptic systems with this structure are the matter of investigations in [3].

1.8 The Codazzi-Mainardi equations

The following integrability conditions enable us to investigate the structure of the normal system. Again, let $\{N\}$ be an orthonormal basis of the normal space. From the Gauss equations we have

$$\begin{aligned} X_{u^iu} &= \Gamma_{i1}^1X_u+\Gamma_{i1}^2X_v+\sum_{\Sigma=1}^{n-2}L_{\Sigma,i1}N_\Sigma\,,\\ X_{u^iv} &= \Gamma_{i2}^1X_u+\Gamma_{i2}^2X_v+\sum_{\Sigma=1}^{n-2}L_{\Sigma,i2}N_\Sigma \end{aligned}$$

for $i=1,2$. We evaluate the integrability conditions

$$X_{u^iuv}-X_{u^ivu}=0\quad\text{for } i=1,2.$$

For this, we calculate

$$\begin{aligned} X_{u^iuv} &= \Gamma_{i1,v}^1X_u+\Gamma_{i1,v}^2X_v+\Gamma_{i1}^1X_{uv}+\Gamma_{i1}^2X_{vv}+\sum_{\Sigma=1}^{n-2}L_{\Sigma,i1,v}N_\Sigma+\sum_{\Sigma=1}^{n-2}L_{\Sigma,i1}N_{\Sigma,v}\\ &= (\Gamma_{i1,v}^1+\Gamma_{i1}^1\Gamma_{12}^1+\Gamma_{i1}^2\Gamma_{22}^1)X_u+(\Gamma_{i1,v}^2+\Gamma_{i1}^1\Gamma_{12}^2+\Gamma_{i1}^2\Gamma_{22}^2)X_v\\ &\quad+\sum_{\Sigma=1}^{n-2}\left\{L_{\Sigma,i1,v}+\Gamma_{i1}^1L_{\Sigma,12}+\Gamma_{i1}^2L_{\Sigma,22}+\sum_{\Theta=1}^{n-2}L_{\Theta,i1}\sigma_{\Theta,2}^\Sigma\right\}N_\Sigma-\sum_{\Sigma=1}^{n-2}L_{\Sigma,i1}L_{\Sigma,2l}h^{lm}X_{u^m}^t\,, \end{aligned}$$

as well as

$$\begin{aligned} X_{u^ivu} &= (\Gamma_{i2,u}^1+\Gamma_{i2}^1\Gamma_{11}^1+\Gamma_{i2}^2\Gamma_{12}^1)X_u+(\Gamma_{i2,u}^2+\Gamma_{i2}^1\Gamma_{11}^2+\Gamma_{i2}^2\Gamma_{12}^2)X_v\\ &\quad+\sum_{\Sigma=1}^{n-2}\left\{L_{\Sigma,i2,u}+\Gamma_{i2}^1L_{\Sigma,11}+\Gamma_{i2}^2L_{\Sigma,12}+\sum_{\Theta=1}^{n-2}L_{\Theta,i2}\sigma_{\Theta,1}^\Sigma\right\}N_\Sigma-\sum_{\Sigma=1}^{n-2}L_{\Sigma,i2}L_{\Sigma,1l}h^{lm}X_{u^m}^t\,. \end{aligned}$$

Evaluating the normal directions gives

Proposition 4: *The Codazzi-Mainardi equations are*

$$L_{\Sigma,i1,v} + \Gamma_{i1}^1 L_{\Sigma,12} + \Gamma_{i1}^2 L_{\Sigma,22} + \sum_{\Theta=1}^{n-2} L_{\Theta,i1}\sigma_{\Theta,2}^{\Sigma} = L_{\Sigma,i2,u} + \Gamma_{i2}^1 L_{\Sigma,11} + \Gamma_{i2}^2 L_{\Sigma,12} + \sum_{\Theta=1}^{n-2} L_{\Theta,i2}\sigma_{\Theta,1}^{\Sigma}$$

for $i = 1,2$ *and* $\Sigma = 1,2,\ldots,n-2$.

Explicitely, we write

$$\begin{aligned} L_{\Sigma,11,v} + \Gamma_{11}^m L_{\Sigma,m2} + \sum_{\Theta=1}^{n-2} L_{\Theta,11}\sigma_{\Theta,2}^{\Sigma} &= L_{\Sigma,12,u} + \Gamma_{12}^m L_{\Sigma,m1} + \sum_{\Theta=1}^{n-2} L_{\Theta,12}\sigma_{\Theta,1}^{\Sigma}\,, \\ L_{\Sigma,21,v} + \Gamma_{21}^m L_{\Sigma,m2} + \sum_{\Theta=1}^{n-2} L_{\Theta,21}\sigma_{\Theta,2}^{\Sigma} &= L_{\Sigma,22,u} + \Gamma_{22}^m L_{\Sigma,m1} + \sum_{\Theta=1}^{n-2} L_{\Theta,22}\sigma_{\Theta,1}^{\Sigma} \end{aligned}$$

for $\Sigma = 1,2,\ldots,n-2$.

1.9 An elliptic system for N_Σ

Using conformal parameters, from the Weingarten equations we have

$$\begin{aligned} N_{\Sigma,u} &= -\frac{L_{\Sigma,11}}{W}\,X_u - \frac{L_{\Sigma,12}}{W}\,X_v + \sigma_{\Sigma,1}^{\Theta} N_\Theta\,, \\ N_{\Sigma,v} &= -\frac{L_{\Sigma,12}}{W}\,X_u - \frac{L_{\Sigma,22}}{W}\,X_v + \sigma_{\Sigma,2}^{\Theta} N_\Theta\,. \end{aligned}$$

Differentiation yields

$$\begin{aligned} N_{\Sigma,uu} &= -\left(\frac{L_{\Sigma,11}}{W}\right)_u X_u - \left(\frac{L_{\Sigma,12}}{W}\right)_u X_v - \frac{L_{\Sigma,11}}{W}\,X_{uu} - \frac{L_{\Sigma,12}}{W}\,X_{uv} \\ &\quad + \sigma_{\Sigma,1,u}^{\Theta} N_\Theta + \sigma_{\Sigma,1}^{\Theta} N_{\Theta,u}\,, \\ N_{\Sigma,vv} &= -\left(\frac{L_{\Sigma,12}}{W}\right)_v X_u - \left(\frac{L_{\Sigma,22}}{W}\right)_v X_v - \frac{L_{\Sigma,12}}{W}\,X_{uv} - \frac{L_{\Sigma,22}}{W}\,X_{vv} \\ &\quad + \sigma_{\Sigma,2,v}^{\Theta} N_\Theta + \sigma_{\Sigma,2}^{\Theta} N_{\Theta,v}\,. \end{aligned}$$

These terms can be evaluated using the Codazzi-Mainardi equations to get

$$\begin{aligned} \triangle N_\Sigma &= -\left\{2H_u + \frac{1}{W}\sum_{\Theta=1}^{n-2}\big(\sigma_{\Theta,1}^{\Sigma}L_{\Theta,22} - \sigma_{\Theta,2}^{\Sigma}L_{\Theta,12} + \sigma_{\Sigma,1}^{\Theta}L_{\Theta,11} + \sigma_{\Sigma,2}^{\Theta}L_{\Theta,12}\big)\right\} X_u \\ &\quad - \left\{2H_v + \frac{1}{W}\sum_{\Theta=1}^{n-2}\big(\sigma_{\Theta,2}^{\Sigma}L_{\Theta,11} - \sigma_{\Theta,1}^{\Sigma}L_{\Theta,12} + \sigma_{\Sigma,1}^{\Theta}L_{\Theta,12} + \sigma_{\Sigma,2}^{\Theta}L_{\Theta,22}\big)\right\} X_v \\ &\quad - \sum_{\Sigma=1}^{n-2}\frac{L_{\Sigma,11}^2 + 2L_{\Sigma,12}^2 + L_{\Sigma,22}^2}{W}\,N_\Sigma + (\sigma_{\Sigma,1}^{\Theta}\sigma_{\Theta,1}^{\Lambda} + \sigma_{\Sigma,2}^{\Theta}\sigma_{\Theta,2}^{\Lambda})N_\Lambda + (\sigma_{\Sigma,1,u}^{\Theta} + \sigma_{\Sigma,2,v}^{\Theta})N_\Theta\,. \end{aligned}$$

In contrast to the mean curvature equation, this nonlinear system is not in diagonal form. Quadratic growth in $|\nabla X|$ and $|\nabla N|$ on the right hand side would presuppose a priori-data on the derivatives of the torsion.

2. A priori-estimates for graphs in $\mathbb{R}^4$

In this second part we consider immersions

$$X = X(u,v) \in C^{2+\alpha}(B,\mathbb{R}^4) \cap C^0(\overline{B},\mathbb{R}^4), \quad \alpha \in (0,1),$$
$$X(u,v) = (x^1(u,v),\dots,x^4(u,v)).$$

Let a function

$$\overline{\mathcal{H}}(X) : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

be given. We define a scalar function $\mathcal{H}(X,Z) : \mathbb{R}^4 \times S^3 \to \mathbb{R}$ by

$$\mathcal{H}(X,Z) := \overline{\mathcal{H}}(X) \cdot Z^t \quad \text{for } X \in \mathbb{R}^4\,,\ Z \in S^3\,.$$

We now consider surfaces of prescribed mean curvature $\mathcal{H} = \mathcal{H}(X,Z)$ in the following sense.

Definition 3: A surface $X = X(u,v)$ is called a (conformally parametrized) surface of prescribed mean curvature $\mathcal{H} : \mathbb{R}^4 \times S^3 \to \mathbb{R}$ if the immersion satisfies the system

$$\triangle X = 2\mathcal{H}(X,N_1)WN_1 + 2\mathcal{H}(X,N_2)WN_2\,,$$
$$|X_u|^2 = W = |X_v|^2\,, \quad X_u \cdot X_v^t = 0 \qquad \text{in } B$$

for an orthonormal basis N_1, N_2 of the normal space.

Remarks:

1. By the definition of $\mathcal{H}$, the above system is satisfied for all orthonormal bases of the normal space if it is satisfied for at least one orthonormal basis.
2. Using conformal parameters the geometric mean curvature $H_\Sigma(X) \equiv H(X,N_\Sigma)$ of X in direction of the unit normal N_Σ can be calculated by

$$H_\Sigma(X) = \frac{(h_{11}X_{uu} - 2h_{12}X_{uv} + h_{22}X_{vv}) \cdot N_\Sigma^t}{2(h_{11}h_{22} - h_{12}^2)} = \frac{\triangle X \cdot N_\Sigma^t}{2W}\,.$$

   Thus, by section 1.7, the geometric mean curvature $H_\Sigma(X)$ is equal to $\mathcal{H}(X,N_\Sigma)$ if the the normal basis is continuously differentiable.

Our main result is the following.

Theorem 1:

Assumptions: *Let $X \in C^{2+\alpha}(B,\mathbb{R}^4) \cap C^0(\overline{B},\mathbb{R}^4)$, $\alpha \in (0,1)$, be a surface of prescribed mean curvature $\mathcal{H}(X,Z) \in C^0(\mathbb{R}^4 \times S^3,\mathbb{R})$ with the following properties:*

(A1) The immersion $X = X(u,v)$ is a conformal parametrization of a graph of the form

$$X(x,y) = (x,y,\varphi(x,y),\psi(x,y)), \quad (x,y) \in \overline{\Omega},$$

with an open, bounded and simply-connected domain $\Omega \subset \mathbb{R}^2$, and $\varphi,\psi \in C^{2+\alpha}(\overline{\Omega},\mathbb{R})$.

(A2) The mean curvature $\mathcal{H} = \mathcal{H}(X,Z)$ satisfies

$$|\mathcal{H}(X,Z)| \le \frac{h_0}{2} \quad \textit{for all } X \in \mathbb{R}^4\,,\ Z \in S^3\,,$$

as well as

$$|\mathcal{H}(X_1,Z_1) - \mathcal{H}(X_2,Z_2)| \le h_1|X_1 - X_2|^\alpha + h_2|Z_1 - Z_2| \quad \textit{for all } X_1,X_2 \in \mathbb{R}^4\,,\ Z_1,Z_2 \in S^3$$

with constants $h_0,h_1,h_2 \in [0,+\infty)$.

(A3) The surface represents a geodesic disc of radius $r > 0$ and with center $X(0,0) = (0,0,0,0)$.

(A4) With a real constant $d_0 > 0$, the Dirichlet energy can be estimated by

$$\mathcal{D}[X] := \iint\limits_B |\nabla X(u,v)|^2\, dudv \le d_0 r^2\,.$$

(A5) At every point $w \in B$, each normal vector of the immersion makes an angle of at least $\omega > 0$ with the x_1-axis.

Statement: *Then, for orthonormal vectors N_1 and N_2 spanning the normal space at the point $X(0,0)$, there exists a constant $\Theta = \Theta(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, \alpha) \in (0,+\infty)$ such that holds*

$$\kappa_{\Sigma,1}(0,0)^2 + \kappa_{\Sigma,2}(0,0)^2 \le \frac{1}{r^2}\Big\{(h_0 r)^2 + \Theta\Big\}$$

for the principle curvature $\kappa_{\Sigma,1}$ and $\kappa_{\Sigma,2}$ in direction N_Σ.

Proof: The proof follows the lines of [6], § 2. It holds

$$\begin{aligned}\kappa_{\Sigma,1}(0,0)^2 + \kappa_{\Sigma,2}(0,0)^2 &= 4H_\Sigma(0,0)^2 - 2K_\Sigma(0,0) \ \le\ h_0^2 + 2|K_\Sigma(0,0)| \\ &= \frac{1}{r^2}\Big\{(h_0 r)^2 + r^2|K_\Sigma(0,0)|\Big\}\end{aligned}$$

for $\Sigma = 1,2$. The desired estimate follows from

$$K_\Sigma(0,0) = \frac{(X_{uu}\cdot N_\Sigma)(X_{vv}\cdot N_\Sigma) - (X_{uv}\cdot N_\Sigma)^2}{W^2}\Big|_{(0,0)}$$

w.r.t. a normal basis $\{N_1, N_2\}$. Thus, our proof consists of two steps: First, we have to find a lower bound for the surface area element, and, then, an upper bound for the second derivatives of the immersion.

1. In the first part we will prove the estimate

$$\frac{W(w)}{r^2} \ge C_1 \quad \text{for } w \in B_{\frac{1}{2}}(0,0)$$

with a constant $C_1 = C_1(h_0 r, d_0, \sin\omega) > 0$.

1.1 We fix a normal basis of the surface. Note that the vectors $e_3 := (0,0,1,0)$ and $e_4 := (0,0,0,1)$ are not in any tangent plane of the surface due to the fact that

$$\frac{1}{\sqrt{1+|\nabla\varphi|^2}}\,(-\varphi_x, -\varphi_y, 1, 0), \qquad \frac{1}{\sqrt{1+|\nabla\psi|^2}}\,(-\psi_x, -\psi_y, 0, 1)$$

are normal to the surface (in the corresponding surface point). We see that the inner product of the first vector by e_3 and the inner product of the second vector by e_4 do not vanish. Therefore, the projections

$$N_1^* := e_3 - \frac{e_3\cdot X_u^t}{|X_u|^2}\,X_u - \frac{e_3\cdot X_v^t}{|X_v|^2}\,X_v\,, \quad N_2^* := e_4 - \frac{e_4\cdot X_u^t}{|X_u|^2}\,X_u - \frac{e_4\cdot X_v^t}{|X_v|^2}\,X_v$$

are a basis of the normal space and can be transformed into an orthonormal basis $\{N_1, N_2\}$ of the normal space.

1.2 In the first part of the proof we will work with the above orthonormal basis $\{N_1, N_2\}$. Using the differential equation (see Remark 1 above)

$$\triangle X = 2\mathcal{H}(X, N_1)WN_1 + 2\mathcal{H}(X, N_2)WN_2 \quad \text{in } B,$$

Assumption (A1) yields the estimate

$$|\triangle X(u,v)| \le 2h_0|\nabla X(u,v)|^2 \quad \text{in } B.$$

The special structure of this differential inequality - the quadratic growth in the gradient on the right hand side - allows us to apply the methods of [3].

We cite two important consequences of our assumptions.

1.3 From Assumption (A3) we conclude:

Let $\Gamma(B)$ denote the set of all continuous and piecewise differentiable curves $\gamma : [0,1] \to \overline{B}$, such that $\gamma(0) = (0,0)$ and $\gamma(1) \in \partial B$. Then, it holds

$$\inf_{\gamma\in\Gamma(B)} \int_0^1 \left|\frac{d}{dt} X \circ \gamma(t)\right| dt \ge r.$$

For a proof we refer the reader to [6].

1.4 Assumption (A5) enables us to interpolate the surface area element $W = W(u,v)$ in terms of $|\nabla x^1|^2$: Namely, it holds

$$|\nabla x^1|^2 \ge W \sin^2\omega \quad \text{in } B.$$

The proof can be extracted from [5], Lemma 1.1 (note that this result of Osserman makes only use of the conformal representation of the surface).

Now, we estimate the surface area element. For this, we define several two-dimensional auxiliary functions and apply Heinz' results for elliptic systems in the plane (cp. [3]).

1.5 Let

$$F^*(u,v) := (x^1(u,v), x^2(u,v)) \,:\, \overline{B} \longrightarrow \mathbb{R}^2$$

denote the plane mapping w.r.t. $X = X(u,v)$. Then, there hold

(i) $|\triangle F^*(w)| \le \dfrac{4h_0}{\sin^2\omega}\,|\nabla F^*(w)|^2$

(ii) $|\nabla X(w)|^2 \le \dfrac{2}{\sin^2\omega}\,|\nabla F^*(w)|^2$

for all $w \in B$.

For the proof we note that the second inequality follows from $W + W = |\nabla X|^2$ and from the Osserman inequality in 1.4. The differential inequality $|\triangle X| \le 2h_0|\nabla X|^2 = 4h_0 W$ (see 1.2) yields

$$|\triangle F^*| \le |\triangle X| \le 4h_0 W \le \frac{4h_0}{\sin^2\omega}\,|\nabla x^1|^2 \le \frac{4h_0}{\sin^2\omega}\,|\nabla F^*|^2\,.$$

1.6 Let $w_0 \in B$ and $\nu \in (0,1)$ be given such that $B_{2\nu}(w_0) := \{w \in B \,:\, |w - w_0| < 2\nu\} \subset B$. We consider the mapping

$$Y(w) := \frac{1}{r}\,\{X(w_0 + 2\nu w) - X(w_0)\}, \quad w \in \overline{B},$$

as well as the corresponding plane mapping

$$F(w) := (y^1(w), y^2(w)) \; : \; \overline{B} \longrightarrow \mathbb{R}^2 \, .$$

The immersion $Y = Y(u,v)$ satisfies

$$|Y_u(w)|^2 = \frac{4\nu^2}{r^2}\, W(w_0 + 2\nu w) = |Y_v(w)|^2 \, , \quad Y_u(w) \cdot Y_v(w)^t = 0 \quad \text{in } B$$

and

$$|\triangle Y(w)| \le 2(h_0 r)|\nabla Y(w)|^2 \quad \text{in } B.$$

1.7 From inequality 1.5 and 1.6 we infer

$$\begin{aligned} |\triangle F(w)| \; &\le \; |\triangle Y(w)| \; \le \; 2(h_0 r)|\nabla Y(w)|^2 \\ &= \frac{16\nu^2(h_0 r)}{r^2}\, W(w_0 + 2\nu w) \; = \; \frac{8\nu^2(h_0 r)}{r^2}\, |\nabla X(w_0 + 2\nu w)|^2 \\ &\le \frac{16\nu^2(h_0 r)}{r^2 \sin^2 \omega}\, |\nabla F^*(w_0 + 2\nu w)|^2 \; = \; \frac{16\nu^2(h_0 r)}{r^2 \sin^2 \omega}\, \frac{r^2}{4\nu^2}\, |\nabla F(w)|^2 \\ &\le \frac{4(h_0 r)}{\sin^2 \omega}\, |\nabla F(w)|^2 \end{aligned}$$

for all $w \in B$. Furthermore, by the assumption that the immersion represents a graph of the form $(x, y, \varphi(x,y), \psi(x,y))$, $(x,y) \in \overline{\Omega}$, we conclude that the plane mapping $F = F(u,v)$ is one-to-one and has positive Jacobian $J_F(w) > 0$ for all $w \in B$. Additionally, Assumption (A4) gives

$$\mathcal{D}[F] \le \mathcal{D}[Y] \le \frac{1}{r^2}\, \mathcal{D}[X] \le d_0$$

for the Dirichlet integrals. We now apply [3], Theorem 6, page 254, which gives the following inner gradient estimate: There is a constant $c_1 = c_1(h_0 r, d_0, \sin\omega) \in (0, +\infty)$ such that

$$|\nabla F(u,v)| \le c_1(h_0 r, d_0, \sin\omega) \quad \text{for all } (u,v) \in B_{\frac{1}{2}}(0,0).$$

1.8 From 1.7 we infer

$$\frac{4\nu^2}{r^2}\, W(w_0 + 2\nu w) \le \frac{1}{\sin^2 \alpha}\, |\nabla F(w)|^2 \quad \text{for } w \in B.$$

Thus, we arrive at the following estimate

$$\frac{1}{r^2}\, W(w) \le \frac{1}{4\nu^2 \sin^2 \omega}\, c_1(h_0 r, d_0, \sin\omega) =: c_2(h_0 r, d_0, \sin\omega, \nu)$$

for all $w \in B_\nu(w_0)$, and for all $w_0 \in B$ such that $B_{2\nu}(w_0) \subset B$. For $w_0 \in B_{\frac{1}{2}}(0,0)$ and $\nu \le \frac{1}{4}$ we get

$$\frac{1}{r^2}\, W(w) \le c_2(h_0 r, d_0, \sin\omega, \nu) \quad \text{for all } w \in B_{\frac{1}{2}}(0,0).$$

This estimate will be used in the second part of the proof.

1.9 The properties $J_F(w) > 0$ in B and $\mathcal{D}[F] \le d_0$, as well as the structure of the differential inequality for $|\triangle F(w)|$ (see 1.7) make [3], Lemma 17, page 255 applicable: There exists a constant $c_3 = c_3(h_0 r, d_0, \sin\omega) \in (0, +\infty)$ such that

$$|\nabla F(w)|^2 \le c_3(h_0 r, d_0, \sin\omega)|\nabla F(0,0)|^{\frac{2}{5}} \quad \text{for all } w \in B_{\frac{1}{2}}(0,0).$$

It follows, that (cp. 1.6)

$$\begin{aligned}\frac{4\nu^2}{r^2} W(w_0 + 2\nu w) &\le \frac{1}{\sin^2\omega}|\nabla F(w)|^2 \le \frac{c_3(h_0 r, d_0, \sin\omega)}{\sin^2\omega}|\nabla F(0,0)|^{\frac{2}{5}} \\ &\le \frac{c_3(h_0 r, d_0, \sin\omega)}{\sin^2\omega}|\nabla Y(0,0)|^{\frac{2}{5}} = \frac{c_3(h_0 r, d_0, \sin\omega)}{\sin^2\omega}\left[\frac{8\nu^2}{r^2} W(w_0)\right]^{\frac{1}{5}}.\end{aligned}$$

Rearranging yields the important inequality of Harnack type

$$\left[\frac{W(w_0)}{r^2}\right]^{\frac{1}{5}} \ge \frac{8^{\frac{17}{15}}\nu^{\frac{8}{5}}\sin^2\omega}{c_3(h_0 r, d_0, \sin\omega)}\,\frac{W(w_0 + 2\nu w)}{r^2} \quad \text{for all } w \in B_{\frac{1}{2}}(0,0),$$

or, which is equivalent,

$$c_4(h_0 r, d_0, \sin\omega, \nu)\left[\frac{W(w)}{r^2}\right]^5 \le \frac{W(w_0)}{r^2} \quad \text{for all } w \in B_\nu(w_0)$$

with the constant

$$c_4(h_0 r, d_0, \sin\omega, \nu) := \frac{8^{\frac{7}{15}}\nu^8 \sin^{10}\omega}{c_3(h_0 r, d_0, \sin\omega)^5} \in (0, +\infty).$$

1.10 Assumption (A4) also ensures that we can estimate the surface area element in at least one point: There exists a $w^* \in B_{1-\nu_0}(0,0)$, $\nu_0 := \min(e^{-4\pi d_0}, \frac{1}{2})$ such that

$$\frac{W(w^*)}{r^2} \ge \frac{1}{4(1 - e^{-4\pi d_0})} =: c_5(d_0) > 0\,.$$

The constant ν_0 arises from an application of the Courant-Lebesgue lemma.

1.11 We now show an estimate of the surface area element:
We set $\nu := \frac{1}{2}\nu_0 \in (0, \frac{1}{4}]$ and choose an integer $n = n(\nu) \in \mathbb{N}$ such that $1 - 2\nu \le \frac{n}{2}\nu \le 1 - \nu$. For an arbitrary $w_0 \in B_{1-\nu_0}(0,0)$ we define the following points

$$w_j := \frac{j}{n}w^* + \frac{n-j}{n}w_0 \quad \text{for } j = 0, \ldots, n\,.$$

Together with 1.10 we have

$$|w_j| \le \frac{j}{n}|w^*| + \frac{n-j}{n}|w_0| < 1 - \nu_0$$

and therefore

$$B_{2\nu}(w_j) = B_{\nu_0}(w_j) \subset B\,.$$

Furthermore, we have

$$|w_{j+1} - w_j| = |\frac{1}{n}w^* - \frac{1}{n}w_0| \le \frac{1}{n}|w^* - w_0| \le \frac{2(1-\nu_0)}{n} \le \nu.$$

This implies

$$w_{j+1} \in B_\nu(w_j) \quad \text{for } j = 0, \ldots, n-1\,.$$

We apply the Harnack inequality from 1.9 and obtain

$$\frac{W(w_0)}{r^2} \geq c_4 \left[\frac{W(w_1)}{r^2}\right]^5 \geq c_4^{1+5} \left[\frac{W(w_2)}{r^2}\right]^{5^2} \geq \ldots \geq c_4^{1+5+5^2+\ldots+5^{n-1}} \left[\frac{W(w_n)}{r^2}\right]^{5^n}\,.$$

Recalling $w_n = w^*$, 1.10 gives

$$\frac{W(w_0)}{r^2} \geq c_4^{1+5+5^2+\ldots+5^{n-1}} c_5(d_0)^{5^n} =: C_1(h_0 r, d_0, \sin\omega) > 0$$

for all $w_0 \in B_{1-\nu_0}(0,0)$. From $\nu_0 \leq \frac{1}{2}$ we now conclude

$$\frac{W(w)}{r^2} \geq C_1(h_0 r, d_0, \sin\omega) \quad \text{for all } w \in B_{\frac{1}{2}}(0,0)\,.$$

This ends the first part of the proof.

2. In the second part of the proof we have to estimate the second derivatives of X using the differential equation

$$\triangle X = 2\mathcal{H}(X, N_1) W N_1 + 2\mathcal{H}(X, N_2) W N_2\,.$$

To do so, we have to give Hölder estimates of the right hand side of this equation. Especially, we have to construct an orthonormal basis $\{N_1, N_2\}$ of the normal space at each point, whose Hölder norm can be estimated.

2.1 We define the auxiliary function

$$Z(u,v) = \frac{1}{r}\{X(u,v) - X(0,0)\} = \frac{1}{r} X(u,v), \quad (u,v) \in \overline{B}\,.$$

Denoting by W_Z the surface area element of Z, we have

$$|Z_u|^2 = W_Z = |Z_v|^2\,, \quad Z_u \cdot Z_v^t = 0 \quad \text{in } B.$$

It holds $r^2 W_Z = W_X$ with $W_X := |X_u|^2 = |X_v|^2$. We calculate

$$\begin{aligned} \triangle Z &= \frac{2}{r}\mathcal{H}(X, N_1) W_X N_1 + \frac{2}{r}\mathcal{H}(X, N_2) W_X N_2 \\ &= 2r\mathcal{H}(rZ, N_1) W_Z N_1 + 2r\mathcal{H}(rZ, N_2) W_Z N_2\,. \end{aligned}$$

2.2 Due to 1.8 we have the estimate

$$|\triangle Z(w)| \leq 4(rh_0) c_2(h_0 r, d_0, \sin\omega) \quad \text{for all } w \in B_{\frac{1}{2}}(0,0).$$

Furthermore, we get

$$|Z(u,v)| = |Z(u,v) - Z(0,0)| \leq \frac{1}{2}|\nabla Z(\widetilde{w})| \leq \sqrt{c_2(h_0 r, d_0, \sin\omega)} \quad \text{in } B_{\frac{1}{2}}(0,0).$$

Now, by potential theoretic estimates, there exists a constant $c_6(h_0 r, d_0, \sin\omega, \alpha)$ such that the Hölder estimate

$$|Z_{u^i}(w_1) - Z_{u^i}(w_2)| \leq c_6(h_0 r, d_0, \sin\omega, \alpha)|w_1 - w_2|^\alpha \quad \text{for } w_1, w_2 \in B_{\frac{1}{4}}(0,0), \quad i = 1,2$$

holds true. Therefore, for the surface area element it holds

$$|W_Z(w_1) - W_Z(w_2)| \leq c_7(h_0 r, d_0, \sin\omega, \alpha)|w_1 - w_2|^\alpha \quad \text{for all } w_1, w_2 \in B_{\frac{1}{4}}(0,0)$$

with the constant $c_7 := 4\sqrt{c_2}\, c_6$.

2.3 Using the mean value theorem we have the Lipschitz estimate

$$|Z(w_1)-Z(w_2)| \le \sqrt{c_2(h_0 r, d_0, \sin\alpha)}\,|w_1-w_2| \quad \text{for } w_1, w_2 \in B_{\frac{1}{2}}(0,0).$$

In a certain neighborhood of the origin, we now construct an orthonormal basis $\{N_1, N_2\}$ of the normal space whose Hölder norm can be estimated.

First we choose vectors $\overline{N}_1, \overline{N}_2 \in \mathbb{R}^4$ such that

$$\overline{N}_i \cdot Z_{u^j}(0,0)^t = 0, \quad \overline{N}_i \cdot \overline{N}_j^t = \delta_{ij}\,,$$

i.e. $\{\overline{N}_1, \overline{N}_2\}$ form a orthonormal basis of the normal space at the point $Z(0,0)$. Due to the Gram-Schmidt orthonormalization, for $k=1,2$ we define vectors

$$N_k^*(w) := \overline{N}_k - \frac{\overline{N}_k \cdot Z_u(w)}{|Z_u(w)|^2}\, Z_u(w) - \frac{\overline{N}_k \cdot Z_v(w)}{|Z_v(w)|^2}\, Z_v(w) \quad \text{in } B\,.$$

These vectors both belong to the normal space at $Z(w)$ but they may not be linearly independent. At first, we determine a $\nu_1 = \nu_1(h_0 r, d_0, \sin\omega, \alpha) > 0$ such that

$$|N_k^*(w)|^2 = 1 - \frac{[\overline{N}_k \cdot Z_u(w)]^2}{W_Z(w)} - \frac{[\overline{N}_k \cdot Z_v(w)]^2}{W_Z(w)} \ge \frac{1}{2} \quad \text{in } B_{\nu_1}(0,0).$$

This is possible because, first, we calculate

$$\begin{aligned} |\overline{N}_k \cdot Z_{u^\ell}(w)|^2 \;&=\; |\overline{N}_k \cdot \{Z_{u^\ell}(w) - Z_{u^\ell}(0,0)\}|^2 \;\le\; |Z_{u^\ell}(w) - Z_{u^\ell}(0,0)|^2 \\ &\le\; c_6(h_0 r, d_0, \sin\omega, \alpha)^2 |w|^{2\alpha} \end{aligned}$$

by 2.2, and next, by 1.11, we have the lower bound

$$W_Z(w) \ge C_1(h_0 r, d_0, \sin\omega) \quad \text{in } B_{\frac{1}{2}}(0,0).$$

2.4 We remark that the vectors $N_k^*(w)$, $k=1,2$, are Hölder continuous in $B_{\nu_1}(0,0)$ and the Hölder estimate

$$|N_k^*(w_1) - N_k^*(w_2)| \le c_8(h_0 r, d_0, \sin\omega, \alpha)|w_1 - w_2|^\alpha\,, \qquad w_1, w_2 \in B_{\nu_1}(0,0),$$

holds true with a constant $c_8(h_0 r, d_0, \sin\omega, \alpha)$ coming from the Hölder estimate for Z_{u_j} and the lower bound of W_Z. Now, for $k=1,2$ we define

$$\widetilde{N}_k(w) := \frac{N_k^*(w)}{|N_k^*(w)|} \quad \text{in } B_{\nu_1}(0,0).$$

These vectors are well defined because of $|N_k^*(w)|^2 \ge \frac{1}{2}$ in $B_{\nu_1}(0,0)$, but they are not orthogonal. Note that

$$N_1^* \cdot N_2^* = -\frac{(\overline{N}_1 \cdot Z_u^t)(\overline{N}_2 \cdot Z_u^t)}{W_Z} - \frac{(\overline{N}_1 \cdot Z_v^t)(\overline{N}_2 \cdot Z_v^t)}{W_Z}\,,$$

and, therefore, due to 2.3

$$|\widetilde{N}_1 \cdot \widetilde{N}_2| = \frac{|N_1^* \cdot N_2^*|}{|N_1^*||N_2^*|} \le \frac{2}{C_1}\left(|\overline{N}_1 \cdot Z_u^t||\overline{N}_2 \cdot Z_u^t| + |\overline{N}_1 \cdot Z_v^t||\overline{N}_2 \cdot Z_v^t|\right) \le \frac{4c_6^2}{C_1}\,|w|^{2\alpha}\,.$$

Thus we can find a $\nu_2 = \nu_2(h_0 r, d_0, \sin\omega, \alpha)$ with $0 < \nu_2 \le \nu_1$ such that

$$|\widetilde{N}_1 \cdot \widetilde{N}_2| \le \frac{1}{2} \quad \text{in } B_{\nu_2}(0,0)\,.$$

We now define the vectors

$$N_1(w) := \widetilde{N}_1(w), \quad N_2(w) := \frac{\widetilde{N}_2(w) - (N_1 \cdot \widetilde{N}_2)N_1}{1 - (N_1 \cdot \widetilde{N}_2)^2} \quad \text{in } B_{\nu_2}(0,0)\,.$$

N_2 is well defined because for its denominator we have

$$1 - (N_1 \cdot \widetilde{N}_2)^2 = 1 - (\widetilde{N}_1 \cdot \widetilde{N}_2)^2 \ge \frac{3}{4}\,.$$

The vectors $N_1(w), N_2(w)$ are an orthonormal basis of the normal space at each point $Z(w)$ for $w \in B_{\nu_2}(0,0)$. Furthermore, the Hölder estimate

$$|N_k(w_1) - N_k(w_2)| \le c_9(h_0 r, d_0, \sin\omega, \alpha)|w_1 - w_2|^\alpha \quad \text{for } w_1, w_2 \in B_{\nu_2}(0,0)$$

holds true with a constant $c_9(h_0 r, d_0, \sin\omega, \alpha)$ which can be calculated directly using the Hölder estimate for N_k^*.

2.5 We now use the differential equation (recall Remark 1 to Definition 3)

$$\triangle Z = 2r\mathcal{H}(rZ, N_1)W_Z N_1 + 2r\mathcal{H}(rZ, N_2)W_Z N_2 \quad \text{in } B_{\nu_2}(0,0)\,.$$

We already showed the estimate

$$|\triangle Z(w)| \le 4(h_0 r)c_2(h_0 r, d_0, \sin\omega) \quad \text{in } B_{\nu_2}(0,0)\,.$$

Using the assumptions on $\mathcal{H}$ we obtain the Hölder estimate

$$\begin{aligned} &|\mathcal{H}(rZ(w_1), N_k(w_1)) - \mathcal{H}(rZ(w_2), N_k(w_2))| \\ &\quad \le h_1 r^\alpha |Z(w_1) - Z(w_2)|^\alpha + h_2 |N_k(w_1) - N_k(w_2)| \\ &\quad \le h_1 r^\alpha c_2^{\frac{\alpha}{2}} |w_1 - w_2|^\alpha + h_2 c_9 |w_1 - w_2|^\alpha\,. \end{aligned}$$

Thus we can find a constant $c_{10} = c_{10}(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, \alpha)$ such that

$$|\triangle Z(w_1) - \triangle Z(w_2)| \le c_{10}|w_1 - w_2|^\alpha \quad \text{for } w_1, w_2 \in B_{\nu_2}(0,0)\,.$$

Setting $\nu_3 := \frac{1}{2}\nu_2$, the interior Schauder estimates give a constant $C_2 \in (0, +\infty)$ such that

$$|Z_{uu}(w)|, |Z_{uv}(w)|, |Z_{vv}(w)| \le C_2(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, \alpha) \quad \text{in } B_{\nu_3}(0,0).$$

From the beginning of the proof we recall

$$\kappa_{1,\Sigma}(0,0)^2 + \kappa_{2,\Sigma}(0,0)^2 \le \frac{1}{r^2}\left\{(h_0 r)^2 + \frac{|Z_{uu}(0,0)||Z_{vv}(0,0)| + |Z_{uv}(0,0)|}{W_Z(0,0)^2}\right\}.$$

Setting $\Theta(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, \alpha) := \frac{2C_2^2}{C_1^2}$ we arrive at

$$\kappa_{1,\Sigma}(0,0)^2 + \kappa_{2,\Sigma}(0,0)^2 \le \frac{1}{r^2}\Big\{(h_0 r)^2 + \Theta\Big\}.$$

This proves the statement. $\square$

Remarks:

1. Assumption $(A4)$ is basically needed to establish the inequality

$$|\triangle F^*(w)| \leq \frac{4h_0}{\sin^2 \omega} |\nabla F^*(w)|^2$$

for the plane mapping F^*. This is the difference to estimates for surfaces in $\mathbb{R}^3$, where

$$|\triangle F^*(w)| \leq 4h_0 |\nabla F^*(w)|^2$$

holds due to the conformal representation (note $|X_u^3|^2 \leq |X_v^1|^2 + |X_v^2|^2$).

2. The proof does not make use of the special dimension $n = 4$. Thus, it can be carried over to the more general case of dimenion $n \geq 4$.

In the case of minimal surfaces, i.e. $\mathcal{H} \equiv 0$, the proof yields a result of Bernstein type since the a priori constant Θ does not depend on r any more. For $r \to \infty$ we obtain the

Corollary 1: *A complete minimal graph $X(x, y) = (x, y, \varphi(x, y), \psi(x, y))$ which satisfies assumptions (A4) and (A5) is an affine plane.*

Literatur

[1] Blaschke, W.; Leichtweiss, K.: *Elementare Differentialgeometrie.* Grundlehren der mathematischen Wissenschaften **1,** Springer, 1973.

[2] Brauner, H.: *Differentialgeometrie.* Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig, 1981.

[3] Heinz, E.: *On certain nonlinear differential equations and univalent mappings.* Journal d'Analyse Math. **5,** 197-272, 1956.

[4] Lawson, H.B.: *Lectures on minimal submanifolds.* Vol. I, Mathematics Lecture Series **9,** Publish or Perish, Inc., 1980.

[5] Osserman, R.: *Global properties of minimal surfaces in $\mathbb{E}^3$ and $\mathbb{E}^n$.* Ann. Math. **80,** 340-364, 1964.

[6] Sauvigny, F.: *A-priori-Abschätzungen der Hauptkrümmungen für Immersionen vom Mittleren-Krümmungs-Typ mittels Uniformisierung und Sätze vom Bernstein-Typ.* Habilitationsschrift, Göttingen 1988.

[7] Weyl, H.: *Zur Infinitesimalgeometrie: p-dimensionale Fläche im n-dimensionalen Raum.* Math. Z. **12,** 154-160, 1922.