Extensions of Algebraic Groups

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Introduction

Let G be a connected complex algebraic group and A an abelian connected algebraic group, together with an algebraic action of G on A via group automorphisms. The aim of this note is to study the set of isomorphism classes $\mathrm{Ext}_{\mathrm{alg}}(G, A)$ of extensions of G by A in the algebraic group category. The following is our main result (cf. Theorem 1.8).

0.1 Theorem. *For G and A as above, there exists an exact sequence of abelian groups:*

$$0 \to \mathrm{Hom}(\pi_1([G,G]), A) \to \mathrm{Ext}_{\mathrm{alg}}(G, A) \xrightarrow{\pi} H^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u) \to 0\,,$$

where A_u is the unipotent radical of A, G_{red} is a Levi subgroup of G, $\mathfrak{g}_{\mathrm{red}}, \mathfrak{g}, \mathfrak{a}_u$ are the Lie algebras of G_{red}, G, A_u respectively, and $H^*(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u)$ is the Lie algebra cohomology of the pair $(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}})$ with coefficients in the $\mathfrak{g}$-module $\mathfrak{a}_u$.

Our next main result is the following analogue of the Van-Est Theorem for the algebraic group cohomology (cf. Theorem 2.2).

0.2 Theorem. *Let G be a connected algebraic group and let $\mathfrak{a}$ be a finite-dimensional algebraic G-module. Then, for any $p \geq 0$,*

$$H^p_{\mathrm{alg}}(G, \mathfrak{a}) \simeq H^p(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}).$$

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By an algebraic group G we mean an affine algebraic group over the field of complex numbers $\mathbb{C}$ and the varieties are considered over $\mathbb{C}$. The Lie algebra of G is denoted by $L(G)$.

1 Extensions of Algebraic Groups

1.1 Definition. Let G be an algebraic group and A an abelian algebraic group, together with an algebraic action of G on A via group automorphisms, i.e., a morphism of varieties $\rho : G \times A \to A$ such that the induced map $G \to \operatorname{Aut} A$ is a group homomorphism. Such an A is called an *algebraic group with G-action.*

By $\operatorname{Ext}_{\mathrm{alg}}(G, A)$ we mean the set of isomorphism classes of extensions of G by A in the algebraic group category, i.e., quotient morphisms $q : \widehat{G} \to G$ with kernel isomorphic to A as an algebraic group with G-action. We obtain on $\operatorname{Ext}_{\mathrm{alg}}(G, A)$ the structure of an abelian group by assigning to two extensions $q_i : \widehat{G}_i \to G$ of G by A the fiber product extension $\widehat{G}_1 \times_G \widehat{G}_2$ of G by $A \times A$ and then applying the group morphism $m_A : A \times A \to A$ fiberwise to obtain an A-extension of G (this is the Baer sum of two extensions). Then $\operatorname{Ext}_{\mathrm{alg}}$ assigns to a pair of an algebraic group G and an abelian algebraic group A with G-action, an abelian group, and this assignment is contravariant in G (via pulling back the action of G and the extension) and if G is fixed, $\operatorname{Ext}_{\mathrm{alg}}(G, \cdot)$ is a covariant functor from the category of abelian algebraic groups with G-actions to the category of abelian groups. Here we assign to a G-equivariant morphism $\gamma : A_1 \to A_2$ of abelian algebraic groups and an extension $q : \widehat{G} \to G$ of G by A_1 the extension

$$\gamma_* \widehat{G} := (A_2 \rtimes \widehat{G})/\Gamma(\gamma) \to G, \quad [(a, g)] \mapsto q(g),$$

where $\Gamma(\gamma)$ is the graph of γ in $A_2 \times A_1$ and the semidirect product refers to the action of $\widehat{G}$ on A_2 obtained by pulling back the action of G on A_2 to $\widehat{G}$. In view of the equivariance of γ, its graph is a normal algebraic subgroup of $A_2 \rtimes \widehat{G}$, so that we can form the quotient $\gamma_* \widehat{G}$.

We define a map

$$D : \operatorname{Ext}_{\mathrm{alg}}(G, A) \to \operatorname{Ext}(L(G), L(A))$$

by assigning to an extension

$$1 \to A \xrightarrow{i} \widehat{G} \xrightarrow{q} G \to 1$$

of algebraic groups the corresponding extension

$$0 \to L(A) \xrightarrow{di} L(\widehat{G}) \xrightarrow{dq} L(G) \to 0$$

of Lie algebras. Since i is injective, di is injective. Similarly, dq is surjective. Moreover, $\dim G = \dim L(G)$ and hence the above sequence of Lie algebras is indeed exact.

It is clear from the definition of D that it is a homomorphism of abelian groups. If $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{a}$ the Lie algebra of A, then the group $\operatorname{Ext}(\mathfrak{g}, \mathfrak{a})$ is isomorphic to the second Lie algebra cohomology space $H^2(\mathfrak{g}, \mathfrak{a})$ of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $\mathfrak{a}$ (with respect to the derived action) ([CE]). Therefore the description of the group $\operatorname{Ext}_{\mathrm{alg}}(G, A)$ depends on a good description of kernel and cokernel of D which will be obtained below in terms of an exact sequence involving D.

In the following G is always assumed to be connected. The following lemma reduces the extension theory for connected algebraic groups A with G-actions to the two cases of a torus A_s and the case of a unipotent group A_u.

1.2 Lemma. *Let G be connected and A be a connected algebraic group with G-action. Further, let $A = A_u A_s$ denote the decomposition of A into its unipotent and reductive factors. Then $A \cong A_u \times A_s$ as a G-module, where G acts trivially on A_s and G acts on A_u as a G-stable subgroup of A. Thus, we have*

$$\mathrm{Ext}_{\mathrm{alg}}(G, A) \cong \mathrm{Ext}_{\mathrm{alg}}(G, A_u) \oplus \mathrm{Ext}_{\mathrm{alg}}(G, A_s). \tag{1}$$

Proof. Decompose

$$A = A_u A_s, \tag{2}$$

where A_s is the set of semisimple elements of A and A_u is the set of unipotent elements of A. Then A_s and A_u are closed subgroups of A and (2) is a direct product decomposition (see [H, Theorem 15.5]). The action of G on A clearly keeps A_s and A_u stable separately. Also, G acts trivially on A_s since $\mathrm{Aut}(A_s)$ is discrete and G is connected (by assumption). Thus the action of G on A decomposes as the product of actions on A_s and A_u with the trivial action on A_s. Hence the isomorphism (1) follows from the functoriality of $\mathrm{Ext}_{\mathrm{alg}}(G, \cdot)$. $\square$

If $G = G_u \rtimes G_{\mathrm{red}}$ is a Levi decomposition of G, then G_u being simply-connected,

$$\pi_1(G) \cong \pi_1(G_{\mathrm{red}}),$$

where G_u is the unipotent radical of G, G_{red} is a Levi subgroup of G and π_1 denotes the fundamental group. The connected reductive group G_{red} is a product of its connected center $Z := Z(G_{\mathrm{red}})_0$ and its commutator group $G'_{\mathrm{red}} := [G_{\mathrm{red}}, G_{\mathrm{red}}]$ which is a connected semisimple group. Thus, G'_{red} has an algebraic universal covering group $\tilde{G}'_{\mathrm{red}}$, with the finite abelian group $\pi_1(G'_{\mathrm{red}})$ as its fiber. We write $\tilde{G}_{\mathrm{red}} := Z \times \tilde{G}'_{\mathrm{red}}$ which is an algebraic covering group of G_{red}; denote its kernel by Π_G and observe that

$$\tilde{G} := G_u \rtimes \tilde{G}_{\mathrm{red}}$$

is a covering of G with Π_G as its fiber. We write $q_G : \tilde{G} \to G$ for the corresponding covering map.

1.3 Lemma. *If G and A are tori, then* $\mathrm{Ext}_{\mathrm{alg}}(G, A) = 0$.

Proof. Let $q : \widehat{G} \to G$ be an extension of the torus G by A. Then, as is well known, $\widehat{G}$ is again a torus (cf. [B, §11.5]). Since any character of a subtorus of a torus extends to a character of the whole groups ([B, §8.2]), the identity $I_A : A \to A$ extends to a morphism $f : \widehat{G} \to A$. Now $\ker f$ yields a splitting of the above extension. $\square$

The following proposition deals with the case $A = A_s$.

1.4 Proposition. *If $A = A_s$, then $D = 0$ and we obtain an exact sequence*

$$\mathrm{Hom}(\tilde{G}, A_s) \xrightarrow{res} \mathrm{Hom}(\Pi_G, A_s) \xrightarrow{\Phi} \mathrm{Ext}_{\mathrm{alg}}(G, A_s),$$

where Φ assigns to any $\gamma \in \mathrm{Hom}(\Pi_G, A_s)$ the extension $\gamma_* \tilde{G}$. The kernel of Φ consists of those homomorphisms vanishing on the fundamental group $\pi_1(G'_{\mathrm{red}})$ of G'_{red} and Φ factors through an isomorphism

$$\Phi' : \mathrm{Hom}(\pi_1(G'_{\mathrm{red}}), A_s) \simeq \mathrm{Ext}_{\mathrm{alg}}(G, A_s).$$

Proof. Consider an extension

$$1 \to A_s \to \widehat{G} \to G \to 1.$$

Since A_s is a central torus in $\widehat{G}$, the unipotent radical $\widehat{G}_u$ of $\widehat{G}$ maps isomorphically on G_u. Also

$$1 \to A_s \to \widehat{G}_{\mathrm{red}} \to G_{\mathrm{red}} \to 1$$

is an extension whose restriction to Z splits by the preceding lemma. On the other hand the commutator group of $\widehat{G}_{\mathrm{red}}$ has the same Lie algebra as G'_{red}, hence is a quotient of $\tilde{G}'_{\mathrm{red}}$. Thus $\widehat{G}_{\mathrm{red}}$ is a quotient of $A_s \times Z \times \tilde{G}'_{\mathrm{red}}$, which implies that $\widehat{G}$ is a quotient of $A_s \times \tilde{G}$. Hence $\widehat{G}$ is obtained from $A_s \times \tilde{G}$ via taking its quotient by the graph of a homomorphism $\Pi_G \to A_s$. Conversely, any such extension $\widehat{G}$ of G is obtained this way. This proves that Φ is surjective. In particular, the pullback $q_G^* \widehat{G}$ of $\widehat{G}$ to $\tilde{G}$ always splits.

We next show that $\ker \Phi$ coincides with the image of the restriction map from $\mathrm{Hom}(\tilde{G}, A_s)$ to $\mathrm{Hom}(\Pi_G, A_s)$. Assume that the extension $\widehat{G}_\gamma = \gamma_* \tilde{G}$ defined by $\gamma \in \mathrm{Hom}(\Pi_G, A_s)$ splits. Let $\sigma : G \to \widehat{G}_\gamma$ be a splitting morphism. Pulling σ back via q_G, we obtain a splitting morphism

$$\tilde{\sigma} : \tilde{G} \to q_G^* \widehat{G}_\gamma \cong A_s \times \tilde{G}.$$

Thus, there exists a morphism $\delta : \tilde{G} \to A_s$ of algebraic groups such that σ satisfies $\sigma(q_G(g)) = \beta(\delta(g), g)$ for all $g \in \tilde{G}$, where $\beta : A_s \times \tilde{G} \to \widehat{G}_\gamma = (A_s \times \tilde{G})/\Gamma(\gamma)$ is the standard quotient map. For $g \in \Pi_G = \ker q_G$ we have $\beta(\delta(g), g) = 1$, and therefore $\delta(g) = \gamma(g)$ for all $g \in \Pi_G$. This shows that δ is an extension of γ to $\tilde{G}$. Conversely, if γ extends to $\tilde{G}$, $\widehat{G}_\gamma$ is a trivial extension of G.

That $D = 0$ follows from the fact that $\widehat{G}$ and $q_G^* \widehat{G}$ have the same Lie algebras, which is a split extension of $\mathfrak{g}$ by $\mathfrak{a}_s$.

We recall that $\tilde{G} = G_u \rtimes (Z \times \tilde{G}'_{\mathrm{red}})$. If a homomorphism $\gamma : \Pi_G \to A_s$ extends to $\tilde{G}$, then it must vanish on the subgroup $\pi_1(G'_{\mathrm{red}})$ of Π_G since, $\tilde{G}'_{\mathrm{red}}$ being a semisimple group, there are no nonconstant homomorphisms from $\tilde{G}'_{\mathrm{red}} \to A_s$. Conversely, if a homomorphism $\gamma : \Pi_G \to A_s$ vanishes on $\pi_1(G'_{\mathrm{red}})$, then γ defines a homomorphism

$$Z \cap G'_{\mathrm{red}} \cong \Pi_G / \pi_1(G'_{\mathrm{red}}) \to A_s.$$

But A_s being a torus, this extends to a morphism $f : Z \to A_s$ ([B, §8.2]) which in turn can be pulled back via $Z \cong \tilde{G}/(G_u \rtimes \tilde{G}'_{\mathrm{red}})$ to a morphism $\tilde{f} : \tilde{G} \to A_s$

extending γ. This proves that the image of $\mathrm{Hom}(\tilde{G}, A_s)$ under the restriction map in $\mathrm{Hom}(\Pi_G, A_s)$ is the annihilator of $\pi_1(G'_{\mathrm{red}})$, so that

$$\Phi : \mathrm{Hom}(\Pi_G, A_s) \to \mathrm{Ext}_{\mathrm{alg}}(G, A_s)$$

factors through an isomorphism

$$\Phi' : \mathrm{Hom}(\pi_1(G'_{\mathrm{red}}), A_s) \simeq \mathrm{Ext}_{\mathrm{alg}}(G, A_s).$$

$\square$

1.5 Remark. A unipotent group A_u over $\mathbb{C}$ has no non-trivial finite subgroups, so that

$$\mathrm{Hom}(\pi_1(G'_{\mathrm{red}}), A_s) \cong \mathrm{Hom}(\pi_1(G'_{\mathrm{red}}), A).$$

Now we turn to the study of extensions by unipotent groups. In contrast to the situation for tori, we shall see that these extensions are faithfully represented by the corresponding Lie algebra extensions.

1.6 Lemma. *The canonical restriction map*

$$H^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u) \longrightarrow H^2(\mathfrak{g}, \mathfrak{a}_u)$$

is injective.

Proof. Let $\omega \in Z^2(\mathfrak{g}, \mathfrak{a}_u)$ be a Lie algebra cocycle representing an element of $H^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u)$ and suppose that the class $[\omega] \in H^2(\mathfrak{g}, \mathfrak{a}_u)$ vanishes, so that the extension

$$\widehat{\mathfrak{g}} := \mathfrak{a}_u \oplus_\omega \mathfrak{g} \to \mathfrak{g}, \quad (a, x) \mapsto x$$

with the bracket $[(a, x), (a', x')] = (x.a' - x'.a + \omega(x, x'), [x, x'])$ splits. We have to find a $\mathfrak{g}_{\mathrm{red}}$-module map $f : \mathfrak{g} \to \mathfrak{a}_u$ vanishing on $\mathfrak{g}_{\mathrm{red}}$ with

$$\omega(x, x') = (d_{\mathfrak{g}} f)(x, x') := x.f(x') - x'.f(x) - f([x, x']), \quad x, x' \in \mathfrak{g}.$$

Since the space $C^1(\mathfrak{g}, \mathfrak{a}_u)$ of linear maps $\mathfrak{g} \to \mathfrak{a}_u$ is a semisimple $\mathfrak{g}_{\mathrm{red}}$-module ($\mathfrak{a}_u$ being a G-module, in particular, a G_{red}-module), we have

$$C^1(\mathfrak{g}, \mathfrak{a}_u) = C^1(\mathfrak{g}, \mathfrak{a}_u)^{\mathfrak{g}_{\mathrm{red}}} \oplus \mathfrak{g}_{\mathrm{red}}.C^1(\mathfrak{g}, \mathfrak{a}_u)$$

and similarly for the space $Z^2(\mathfrak{g}, \mathfrak{a}_u)$ of 2-cocycles. As the Lie algebra differential $d_{\mathfrak{g}} : C^1(\mathfrak{g}, \mathfrak{a}_u) \to Z^2(\mathfrak{g}, \mathfrak{a}_u)$ is a $\mathfrak{g}_{\mathrm{red}}$-module map, each $\mathfrak{g}_{\mathrm{red}}$-invariant coboundary is the image of a $\mathfrak{g}_{\mathrm{red}}$-invariant cochain in $C^1(\mathfrak{g}, \mathfrak{a}_u)$. We conclude, in particular, that $\omega = d_{\mathfrak{g}} h$ for some $\mathfrak{g}_{\mathrm{red}}$-module map $h : \mathfrak{g} \to \mathfrak{a}_u$. For $x \in \mathfrak{g}_{\mathrm{red}}$ and $x' \in \mathfrak{g}$ it follows that

$$\begin{aligned} 0 &= \omega(x, x') = x.h(x') - x'.h(x) - h([x, x']) \\ &= h([x, x']) - x'.h(x) - h([x, x']) = -x'.h(x), \end{aligned}$$

showing that $h(\mathfrak{g}_{\mathrm{red}}) \subseteq \mathfrak{a}_u^{\mathfrak{g}}$, which in turn leads to $[\mathfrak{g}_{\mathrm{red}}, \mathfrak{g}_{\mathrm{red}}] \subseteq \ker h$. As $\mathfrak{z}(\mathfrak{g}_{\mathrm{red}}) \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$, the map $h|_{\mathfrak{z}(\mathfrak{g}_{\mathrm{red}})}$ extends to a linear map $f : \mathfrak{g} \to \mathfrak{a}_u^{\mathfrak{g}}$ vanishing on $[\mathfrak{g}, \mathfrak{g}]$. Moreover, since f vanishes on $[\mathfrak{g}, \mathfrak{g}]$, f is clearly a $\mathfrak{g}$-module map, in particular, a $\mathfrak{g}_{\mathrm{red}}$-module map. Then $d_{\mathfrak{g}} f = 0$, so that $d_{\mathfrak{g}}(h - f) = \omega$, and $h - f$ vanishes on $\mathfrak{g}_{\mathrm{red}}$. ☐

1.7 Proposition. *For $A = A_u$ the map $D : \mathrm{Ext}_{\mathrm{alg}}(G, A_u) \to H^2(\mathfrak{g}, \mathfrak{a}_u)$ induces a bijection*

$$D : \mathrm{Ext}_{\mathrm{alg}}(G, A_u) \to H^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u).$$

Proof. In view of the preceding lemma, we may identify $H^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u)$ with a subspace of $H^2(\mathfrak{g}, \mathfrak{a}_u)$. First we claim that $\mathrm{im}(D)$ is contained in this subspace. For any extension

$$1 \to A_u \to \widehat{G} \to G \to 1, \tag{3}$$

we choose a Levi subgroup $\widehat{G}_{\mathrm{red}} \subset \widehat{G}$ mapping to G_{red} under the above map $\widehat{G} \to G$. Then

$$\widehat{G}_{\mathrm{red}} \cap A_u = \{1\}.$$

Moreover, $\widehat{G}_{\mathrm{red}} \to G_{\mathrm{red}}$ is surjective and hence an isomorphism. This shows that the extension (3) restricted to G_{red} is trivial and that $\widehat{\mathfrak{g}}_u$ contains a $\widehat{\mathfrak{g}}_{\mathrm{red}}$-invariant complement to $\mathfrak{a}_u$. Therefore $\widehat{\mathfrak{g}}$ can be described by a cocycle $\omega \in Z^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u)$, in particular, ω vanishes on $\mathfrak{g} \times \mathfrak{g}_{\mathrm{red}}$. This shows that $\mathrm{Im}\ D \subset H^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u)$.

If the image of the extension (3) under D vanishes, then the extension $\mathfrak{a}_u \hookrightarrow \widehat{\mathfrak{g}}_u \twoheadrightarrow \mathfrak{g}_u$ splits, which implies that the corresponding extension of unipotent groups $A_u \hookrightarrow \widehat{G}_u \twoheadrightarrow G_u$ splits. Moreover, the splitting map can be chosen to be G_{red}-equivariant, since ω is G_{red}-invariant. This means that we have a morphism $G_u \rtimes G_{\mathrm{red}} \to \widehat{G} \cong \widehat{G}_u \rtimes G_{\mathrm{red}}$ splitting the extension (3). This proves that D is injective.

To see that D is surjective, let $\omega \in Z^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u)$. Let $q : \widehat{\mathfrak{g}} := \mathfrak{a}_u \oplus_\omega \mathfrak{g} \to \mathfrak{g}$ denote the corresponding Lie algebra extension. Since $\mathfrak{a}_u$ is a nilpotent module of $\mathfrak{g}_u$, the subalgebra $\widehat{\mathfrak{g}}_u := \mathfrak{a}_u \oplus_\omega \mathfrak{g}_u$ of $\widehat{\mathfrak{g}}$ is nilpotent, hence corresponds to a unipotent algebraic group $\widehat{G}_u$ which is an extension of G_u by A_u. Further, the G_{red}-invariance of the decomposition $\widehat{\mathfrak{g}} = \mathfrak{a}_u \oplus \mathfrak{g}$ implies that G_{red} acts algebraically on $\widehat{\mathfrak{g}}_u$ and hence on $\widehat{G}_u$, so that we can form the semidirect product $\widehat{G} := \widehat{G}_u \rtimes G_{\mathrm{red}}$ which is an extension of G by A_u mapped by D onto $\widehat{\mathfrak{g}}$. ☐

1.8 Theorem. *For a connected algebraic group G and a connected abelian algebraic group A with G-action, there exists an exact sequence of abelian groups:*

$$0 \to \mathrm{Hom}(\pi_1([G, G]), A) \to \mathrm{Ext}_{\mathrm{alg}}(G, A) \xrightarrow{\pi} H^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}_u) \to 0\,,$$

where $\mathfrak{a} = L(A)$, G_{red} is a Levi subgroup of G, $\mathfrak{g}_{\mathrm{red}} = L(G_{\mathrm{red}})$, $\mathfrak{g} = L(G)$ and $\mathfrak{a}_u = L(A_u)$.

(Observe that, by the following proof, the fundamental group $\pi_1([G, G])$ is a finite group.)

Proof. In view of the Levi decomposition of the commutator $[G,G] = [G,G]_u \rtimes G'_{\mathrm{red}}$, we have $\pi_1([G,G]) = \pi_1(G'_{\mathrm{red}})$. Now we only have to use Lemma 1.2 to combine the preceding results Propositions 1.4 and 1.7 on extensions by A_s and A_u to complete the proof. $\square$

2 Analogue of Van-Est Theorem for algebraic group cohomology

2.1 Definition. Let G be an algebraic group and A an abelian algebraic group with G-action. For any $n \geq 0$, let $C^n_{\mathrm{alg}}(G,A)$ be the abelian group consisting of all the variety morphisms $f : G^n \to A$ under the pointwise addition. Define the differential

$$\delta : C^n_{\mathrm{alg}}(G,A) \to C^{n+1}_{\mathrm{alg}}(G,A) \qquad \text{by}$$
$$(\delta f)(g_0, \cdots, g_n) = g_0 \cdot f(g_1, \cdots, g_n) + (-1)^{n+1} f(g_0, \cdots, g_{n-1}) + \sum_{i=0}^{n-1} (-1)^{i+1} f(g_0, g_1, \cdots, g_i g_{i+1}, \cdots, g_n).$$

Then, as is well known (and easy to see),

$$\delta^2 = 0. \tag{4}$$

The *algebraic group cohomology* $H^*_{\mathrm{alg}}(G,A)$ *of* G *with coefficients in* A is defined as the cohomology of the complex

$$0 \to C^0_{\mathrm{alg}}(G,A) \xrightarrow{\delta} C^1_{\mathrm{alg}}(G,A) \xrightarrow{\delta} \cdots .$$

We have the following analogue of the Van-Est Theorem [V] for the algebraic group cohomology.

2.2 Theorem. *Let G be a connected algebraic group and let $\mathfrak{a}$ be a finite-dimensional algebraic G-module. Then, for any $p \geq 0$,*

$$H^p_{\mathrm{alg}}(G, \mathfrak{a}) \simeq H^p(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}),$$

where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}_{\mathrm{red}}$ is the Lie algebra of a Levi subgroup G_{red} of G as in Section 1.

Proof. Consider the homogeneous affine variety $X := G/G_{\mathrm{red}}$ and let $\Omega^q(X,\mathfrak{a})$ denote the complex vector space of algebraic de Rham forms on X with values in the vector space $\mathfrak{a}$. Since X is a G-variety under the left multiplication of G and $\mathfrak{a}$ is a G-module, Ω^q has a natural locally-finite algebraic G-module structure. Define a double cochain complex $A = \bigoplus_{p,q \geq 0} A^{p,q}$, where

$$A^{p,q} := C^p_{\mathrm{alg}}(G, \Omega^q(X, \mathfrak{a}))$$

and $C^p_{\mathrm{alg}}(G, \Omega^q(X, \mathfrak{a}))$ consists of all the maps $f : G^p \to \Omega^q(X, \mathfrak{a})$ such that $\operatorname{im} f \subset M_f$, for some finite-dimensional G-stable subspace $M_f \subset \Omega^q(X, \mathfrak{a})$ and, moreover, the map $f : G^p \to M_f$ is algebraic. Let $\delta : A^{p,q} \to A^{p+1,q}$ be the group cohomology differential as in Section 2.1 and let $d : A^{p,q} \to A^{p,q+1}$ be induced from the standard de Rham differential $\Omega^q(X, \mathfrak{a}) \to \Omega^{q+1}(X, \mathfrak{a})$, which is a G-module map. It is easy to see that $d\delta - \delta d = 0$ and, of course, $d^2 = \delta^2 = 0$. Thus, (A, δ, d) is a double cochain complex. This gives rise to two spectral sequences both converging to the cohomology of the associated single complex $(C, \delta + d)$ with their E_1-terms given as follows:

$$(5) \qquad {}'E_1^{p,q} = H^q_d(A^{p,*}), \qquad \text{and}$$

$$(6) \qquad {}''E_1^{p,q} = H^q_\delta(A^{*,p}).$$

We now determine ${}'E_1$ and ${}''E_1$ more explicitly in our case.

Since X is a contractible variety, by the algebraic de Rham theorem [GH, Chap. 3, §5], the algebraic deRham cohomology

$$H^q_{\mathrm{dR}}(X, \mathfrak{a}) \begin{cases} \simeq \mathfrak{a}, & \text{if } q = 0 \\ = 0, & \text{otherwise.} \end{cases}$$

Thus,

$${}'E_1^{p,q} \begin{cases} \simeq C^p_{\mathrm{alg}}(G, \mathfrak{a}), & \text{if } q = 0 \\ = 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$(7) \qquad {}'E_2^{p,q} = H^p_\delta(H^q_d(A)) = \begin{cases} H^p_{\mathrm{alg}}(G, \mathfrak{a}), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the spectral sequence ${}'E_*$ collapses at ${}'E_2$. From this we see that there is a canonical isomorphism

$$(8) \qquad H^p_{\mathrm{alg}}(G, \mathfrak{a}) \simeq H^p(C, \delta + d).$$

We next determine ${}''E_1$ and ${}''E_2$. But first we need the following two lemmas.

2.3 Lemma. *For any $p \geq 0$,*

$$H^q_{\mathrm{alg}}(G, \Omega^p(X, \mathfrak{a})) = \begin{cases} \Omega^p(X, \mathfrak{a})^G, & \textit{if } q = 0 \\ 0, & \textit{otherwise,} \end{cases}$$

where $\Omega^p(X, \mathfrak{a})^G$ denotes the subspace of G-invariants in $\Omega^p(X, \mathfrak{a})$.

Proof. The assertion for $q = 0$ follows from the general properties of group cohomology. So we need to consider the case $q > 0$ now.

Since $L := G_{\text{red}}$ is reductive, any algebraic L-module M is completely reducible. Let

$$\pi^M : M \to M^L$$

be the unique L-module projection onto the space of L-module invariants M^L of M. Taking M to be the ring of regular functions $\mathbb{C}[L]$ on L under the left regular representation, i.e., under the action

$$(k \cdot f)(k') = f(k^{-1}k^1), \quad \text{for } f \in \mathbb{C}[L], k, k' \in L,$$

we get the L-module projection $\pi = \pi^{\mathbb{C}[L]} : \mathbb{C}[L] \to \mathbb{C}$. Thus, for any complex vextor space V, we get the projection $\pi \otimes I_V : \mathbb{C}[L] \otimes V \to V$, which we abbreviate simply by π, where I_V is the identity map of V. We define a 'homotopy operator' H, for any $q \geq 0$,

$$H : C^{q+1}_{\text{alg}}(G, \Omega^p(X, \mathfrak{a})) \to C^q_{\text{alg}}(G, \Omega^p(X, \mathfrak{a}))$$

by

$$\Big((Hf)(g_1, \cdots, g_q)\Big)_{g_0 L} = \pi\big(\Theta^f_{(g_0, \cdots, g_q)}\big),$$

for $f \in C^{q+1}_{\text{alg}}(G, \Omega^p(X, \mathfrak{a}))$ and $g_0, \cdots, g_q \in G$, where $\Theta^f_{(g_0, \cdots, g_q)} : L \to \Omega^p(X, \mathfrak{a})_{g_0 L}$ is defined by

$$\Theta^f_{(g_0, \cdots, g_q)}(k) = \Big((g_0 k) \cdot f(k^{-1} g_0^{-1}, g_1, g_2, \cdots, g_q)\Big)_{g_0 L},$$

for $k \in L$. (Here $\Omega^p(X, \mathfrak{a})_{g_0 L}$ denotes the fiber at $g_0 L$ of the vector bundle of p-forms in X with values in $\mathfrak{a}$ and, for a form ω, $\omega_{g_0 L}$ denotes the value of the form ω at $g_0 L$.) It is easy to see that on $C^q_{\text{alg}}(G, \Omega^p(X, \mathfrak{a}))$, for any $q \geq 1$,

$$H\delta + \delta H = I. \tag{9}$$

To prove this, take any $f \in C^q_{\text{alg}}(G, \Omega^p(X, \mathfrak{a}))$ and $g_0, \cdots, g_q \in G$. Then,

$$\begin{aligned}
\Big((H\delta f)(g_1, \cdots, g_q)\Big)_{g_0 L} &= \pi\big(\Theta^{\delta f}_{(g_0, \cdots, g_q)}\big) \\
&= \big(f(g_1, \cdots, g_q)\big)_{g_0 L} \\
&\quad + (-1)^{q+1} \pi\Big(\big((g_0 k) \cdot f(k^{-1} g_0^{-1}, g_1, \cdots, g_{q-1})\big)_{g_0 L}\Big) \\
&\quad + \sum_{i=1}^{q-1} (-1)^{i+1} \pi\Big(\big((g_0 k) \cdot f(k^{-1} g_0^{-1}, g_1, \cdots, g_i g_{i+1}, \cdots, g_q)\big)_{g_0 L}\Big) \\
&\quad - \pi\Big(\big((g_0 k) \cdot f(k^{-1} g_0^{-1} g_1, g_2, \cdots, g_q)\big)_{g_0 L}\Big),
\end{aligned} \tag{10}$$

where $\big((g_0k)\cdot f(k^{-1}g_0^{-1},g_1,\cdots,g_{q-1})\big)_{g_0L}$ means the function from L to $\Omega^p(X,\mathfrak{a})_{g_0L}$ defined as $k\mapsto \big((g_0k)\cdot f(k^{-1}g_0^{-1},g_1,\cdots,g_{q-1})\big)_{g_0L}$. Similarly,

$$
\begin{aligned}
\big((\delta Hf)(g_1,\cdots,g_q)\big)_{g_0L} &= \Big(g_1\cdot\big((Hf)(g_2,\cdots,g_q)\big)\Big)_{g_0L}\\
&+(-1)^q\big((Hf)(g_1,\cdots,g_{q-1})\big)_{g_0L}\\
&+\sum_{i=1}^{q-1}(-1)^i\big((Hf)(g_1,\cdots,g_ig_{i+1},\cdots,g_q)\big)_{g_0L}\\
&=\Big(g_1\cdot\big((Hf)(g_2,\cdots,g_q)\big)\Big)_{g_0L}\\
&+(-1)^q\pi\Big(\big((g_0k)\cdot f(k^{-1}g_0^{-1},g_1,\cdots,g_{q-1})\big)_{g_0L}\Big)\\
&+\sum_{i=1}^{q-1}(-1)^i\pi\Big(\big((g_0k)\cdot f(k^{-1}g_0^{-1},g_1,\cdots,g_ig_{i+1},\cdots,g_q)\big)_{g_0L}\Big).
\end{aligned}
\tag{11}
$$

From the definition of the G-action on $\Omega^p(X,\mathfrak{a})$, it is easy to see that

$$
\pi\Big(\big((g_0k)\cdot f(k^{-1}g_0^{-1}g_1,g_2,\cdots,g_q)\big)_{g_0L}\Big)=\Big(g_1\cdot\big((Hf)(g_2,\cdots,g_q)\big)\Big)_{g_0L}.
\tag{12}
$$

Combining (10)-(12), we clearly get (9).

From the above identity (9), we see, of course, that any cocycle in $C^q_{\mathrm{alg}}(G,\Omega^p(X,\mathfrak{a}))$ (for any $q\geq 1$) is a coboundary, proving the lemma. □

2.4 Lemma. *The restriction map $\gamma:\Omega^p(X,\mathfrak{a})^G\to C^p(\mathfrak{g},\mathfrak{g}_{\mathrm{red}},\mathfrak{a})$ (defined below in the proof) is an isomorphism for all $p\geq 0$, where $C^*(\mathfrak{g},\mathfrak{g}_{\mathrm{red}},\mathfrak{a})$ is the standard cochain complex for the Lie algebra pair $(\mathfrak{g},\mathfrak{g}_{\mathrm{red}})$ with coefficient in the $\mathfrak{g}$-module $\mathfrak{a}$. Moreover, γ commutes with differentials. Thus, γ induces an isomorphism in cohomology*

$$H^*(\Omega(X,\mathfrak{a})^G)\xrightarrow{\sim}H^*(\mathfrak{g},\mathfrak{g}_{\mathrm{red}},\mathfrak{a}).$$

Proof. For any $\omega\in\Omega^p(X,\mathfrak{a})^G$, define $\gamma(\omega)$ as the value of ω at eL. Since G acts transitively on X, and ω is G-invariant, γ is injective.

Since any $\omega_o\in C^p(\mathfrak{g},\mathfrak{g}_{\mathrm{red}},\mathfrak{a})$ can be extended (uniquely) to a G-invariant form on X with values in $\mathfrak{a}$, γ is surjective. Further, from the definition of differentials on the two sides, it is easy to see that γ commutes with differentials. □

2.5 Continuation of the proof of Theorem 2.2.

We now determine ${}^{\backslash\backslash}E$. First of all, by (6) of (2.2),

$${}^{\backslash\backslash}E_1^{p,q}=H^q_\delta(A^{*,p})=H^q_{\mathrm{alg}}(G,\Omega^p(X,\mathfrak{a})).$$

Thus, by Lemma 2.3,

$${}^{\backslash\backslash}E_1^{p,0}=H^0_{\mathrm{alg}}(G,\Omega^p(X,\mathfrak{a}))=\Omega^p(X,\mathfrak{a})^G,$$

and

$$ {}^{\backslash\backslash}E_1^{p,q} = 0, \qquad \text{if } q > 0. $$

Moreover, under the above equality, the differential of the spectral sequence $d_1 : {}^{\backslash\backslash}E_1^{p,0} \to {}^{\backslash\backslash}E_1^{p+1,0}$ can be identified with the restriction of the deRham differential

$$ \Omega^p(X, \mathfrak{a})^G \to \Omega^{p+1}(X, \mathfrak{a})^G. $$

Thus, by Lemma 2.4,

$$ {}^{\backslash\backslash}E_2^{p,q} = \begin{cases} H^p(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{13} $$

In particular, the spectral sequence ${}^{\backslash\backslash}E$ as well degenerates at the ${}^{\backslash\backslash}E_2$-term. Moreover, we have a canonical isomorphism

$$ H^p(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}) \simeq H^p(C, \delta + d). \tag{14} $$

Comparing the above isomorphism with the isomorphism (8) of §2.2, we get a canonical isomorphism:

$$ H^p_{\mathrm{alg}}(G, \mathfrak{a}) \simeq H^p(\mathfrak{g}, \mathfrak{g}_{\mathrm{red}}, \mathfrak{a}). $$

This proves Theorem 2.2. □

2.5 Remark. Even though we took the field $\mathbb{C}$ as our base field, all the results of this paper hold (by the same proofs) over any algebraically closed field of char. 0, if we replace the fundamental group π_1 by the algebraic fundamental group.

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