The Structure of Abelian Pro-Lie Groups

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Abstract. A *pro-Lie group* is a projective limit of a projective system of finite dimensional Lie groups. A *prodiscrete* group is a complete abelian topological group in which the open normal subgroups form a basis of the filter of identity neighborhoods. It is shown here that an abelian pro-Lie group is a product of (in general infinitely many) copies of the additive topological group of reals and of an abelian pro-Lie group of a special type; this last factor has a compact connected component, and a characteristic closed subgroup which is a union of all compact subgroups; the factor group modulo this subgroup is pro-discrete and free of nonsingleton compact subgroups. Accordingly, a connected abelian pro-Lie group is a product of a family of copies of the reals and a compact connected abelian group. A topological group is called *compactly generated* if it is algebraically generated by a compact subset, and a group is called *almost connected* if the factor group modulo its identity component is compact. It is further shown that a compactly generated abelian pro-Lie group has a characteristic almost connected locally compact subgroup which is a product of a finite number of copies of the reals and a compact abelian group such that the factor group modulo this characteristic subgroup is a compactly generated prodiscrete group without nontrivial compact subgroups.

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Introduction

The category of locally compact abelian topological groups and continuous group homomorphisms fails to have products and so is not a closed subcategory of the category of all abelian topological groups and continuous group homomorphisms. The smallest closed subcategory containing all locally compact abelian groups is the category of abelian pro-Lie groups whose structure theory is the object of this paper. For a topological abelian group G we let $\mathcal{N}(G)$ denote the collection of all closed subgroups N such that G/N is a finite dimensional Lie group which, being abelian, is isomorphic to the product of a finite dimensional vector group, a finite dimensional torus and an arbitrary discrete abelian group. We shall write all abelian groups additively. The group G is called a *pro-Lie group* if G is complete and if every identity neighborhood contains a member of $\mathcal{N}(G)$. The set $\mathcal{N}(G)$ is a filter basis and thus the quotient groups G/N form a projective system, and G is its projective limit. The converse saying that a projective limit of an arbitrary projective system (see e.g. [8], Chapter 1 or [11]) of abelian Lie groups is a pro-Lie group is a hard fact that was proved in [9] (for not necessarily abelian groups). By contrast with the category of locally compact abelian groups, the category $\mathbb{ABproLIEGR}$ of abelian pro-Lie groups is closed in the category of all abelian topological groups and continuous group morphisms under the formation of arbitrary limits and the passing to closed subgroups (see [9]). Each locally compact abelian group is a pro-Lie group and thus arbitrary products and of locally compact abelian groups and arbitrary closed subgroups of such products are abelian pro-Lie groups. Therefore, the category $\mathbb{AB}\text{pro}\mathbb{LIEGR}$ is rather large. In the first section below we shall record some individual examples so that the reader can form a first impression of the type of topological abelian groups we face. This addresses in particular those readers whose intuition is trained by studying locally compact abelian groups which are the daily bread of classical harmonic analysis and whose structure is known and is described for instance in [6], Chapters 7 and 8.

One of the most prominent examples of abelian pro-Lie groups is the additive topological group of the dual vector space of an arbitrary real vector space, where the dual $E' = \text{Hom}_{\mathbb{R}}(E, \mathbb{R})$ of a real vector space is given the topology of pointwise convergence; this topology is also called the weak-$*$-topology. It makes the dual into a complete topological vector space which is called a *weakly complete topological vector space*. Therefore its additive topological group is called a *weakly complete vector group*, and whenever it occurs as a subgroup of a topological group, it is called a *weakly complete vector subgroup*. Weakly complete vector groups have a perfect duality theory as was discussed in [6]. The character group and the vector spaces dual of a topological vector space are naturally isomorphic. The compact open topology on the dual of a weakly complete vector space is the finest locally convex topology on the vector space dual. Since the character group of a weakly complete vector group is a vector space and thus is a direct sum of copies of $\mathbb{R}$, every weakly complete vector group is isomorphic to a vector group of the form $\mathbb{R}^J$ for some set J. Thus a weakly complete vector group is locally compact if and only if it is finite dimensional. The transfinite topological dimension of $\mathbb{R}^J$ as discussed in [7] is $\text{card}\, J$. Therefore

the topological dimension of a weakly complete vector group agrees with the linear dimension of its dual.

Pontryagin duality establishes a dual equivalence between the categories of compact abelian groups and the category of (discrete) abelian groups. Therefore an abelian pro-Lie group is a direct product of a weakly complete vector group and compact abelian group if and only if its dual is a direct sum of a real vector space given its finest locally convex topology, and some discrete abelian group.

The main result of this paper is that every abelian pro-Lie group G is isomorphic as a topological group to the direct product $W \times H$ of a weakly complete vector subgroup W of G and an abelian pro-Lie group H whose identity component H_0 is compact and is the intersection of open subgroups. In particular, a connected abelian pro-Lie group is the direct product of a vector group and a compact connected group. For connected pro-Lie groups it turns out that one is reduced to the theory of weakly complete vector spaces on the one hand and to the more familiar locally compact situation on the other hand. From this result we derive that a necessary and sufficient condition for a connected abelian pro-Lie group to be locally compact is that is algebraically generated by a compact set.

These results say for instance that, while the category $\mathbb{AB}\text{pro}\mathbb{LIEGR}$ contains all products of locally compact abelian groups, the connected objects in it are just products of connected locally compact abelian groups. Thus, while $\mathbb{AB}\mathrm{pro}\mathbb{LIEGR}$ is large, it is just large enough on the level of connected groups.

The main result entails at once that every abelian pro-Lie group is homotopy equivalent to an abelian pro-Lie group whose identity component is compact. In particular, every connected abelian pro-Lie group is homotopy equivalent to a compact abelian group and thus has the homotopy and cohomology of a compact abelian group which can be completely calculated from its character group (see for instance [6], pages 418 and 430).

1. Examples of Abelian Pro-Lie Groups

We shall often refer to a theorem which is proved in [8] 3.35, or in [9], Corollary 4.9:

Theorem CL. (The Closed Subgroup Theorem) *Every closed subgroup of a pro-Lie group is a pro-Lie group.*

This is a result which we keep in mind as well when we look at the elementary examples.

We begin by offering some orientation on the class of abelian pro-Lie groups by presenting a list of examples. Let us firstly recall (see for instance [17]) the following basic types of locally compact nondiscrete fields:

(a) The field $\mathbb{R}$ of real numbers.

(b) The field $\mathbb{Q}_p$ of p-adic rationals for some prime p.

(c) The field $\mathrm{GF}(p)[[X]]$ of Laurent series in one variable with the exponent valuation over the field with p elements.

All other nondiscrete locally compact fields are finite extensions of these; in cases (a) and (b) the characteristic is 0 and in case (c) it is finite. Of course, every field F with the discrete topology is a locally compact field.

Let $\mathbb{Z}(p^\infty) = \left(\bigcup_{n=1}^{\infty} \frac{1}{n}\cdot\mathbb{Z}\right)/\mathbb{Z}$ denote the Prüfer group of all elements of p-power order in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We consider on the groups $\mathbb{Z}$ of integers, $\mathbb{Q}$ of rationals, and the Prüfer group $\mathbb{Z}(p^\infty)$ only their discrete topologies.

Example 1.1. Let J be an arbitrary infinite set. The following examples are abelian pro-Lie groups.

(i) All locally compact abelian groups.

(ii) All products of locally compact abelian groups, specifically:

(iii) the groups $\mathbb{R}^J$;

(iv) the groups $(\mathbb{Q}_p)^J$;

(v) the groups $\mathbb{Q}^J$;

(vi) the groups $\mathbb{Z}^J$;

(vii) the groups $\mathbb{Z}(p^\infty)^J$. □

An infinite product of noncompact locally compact groups is not locally compact, so none of the groups in (iii)–(vi) is locally compact, but if J is countable, they are Polish (that is, completely metrizable and second countable). A countable product of discrete infinite countable sets in the product topology is homeomorphic to the space $\mathbb{R}\setminus\mathbb{Q}$ of the irrational numbers in the topology induced by $\mathbb{R}$ (see [2], Chap. IX, §6, Exercise 7). So $\mathbb{Q}^{\mathbb{N}}$, $\mathbb{Z}^{\mathbb{N}}$, and $\mathbb{Z}(p^\infty)^{\mathbb{N}}$ are abelian prodiscrete groups on the Polish space of irrational numbers. Elementary examples such as pro-Lie group topologies on the space of irrationals in its natural topology illustrate once more the fact that the category of abelian pro-Lie groups is considerably larger than that of locally compact groups. The groups in (iii), (iv) and (v) are divisible and torsion free, and the groups in (vii) are divisible and have a dense torsion group.

There is a less obvious but very instructive example which we present separately. If J is any set and $j\in J$, then $\delta_j\colon J\to\mathbb{R}$ is defined by

$$\delta_j(x)=\begin{cases}1 & \text{if } x=j,\\ 0 & \text{otherwise.}\end{cases}$$

For any subset $S\subseteq\mathbb{R}^J$ let $\langle S\rangle$ denote the subgroup algebraically generated by S as is usual.

For an arbitrary set J and an abelian group A we use the notation $A^{(J)}$ to denote the subgroup of the power A^J consisting of all J-tuples $(a_j)_{j\in J}$ for which $a_j=0$ with at most finitely many exceptions. We note that a weakly complete topological vector space of topological dimension $2^{\aleph_0}$ is isomorphic to $\mathbb{R}^{\mathbb{R}}$ as a topological vector space. For any topological abelian group G the topological space $\mathrm{Hom}(\mathbb{R},G)$ of continuous group homomorphisms $X\colon\mathbb{R}\to G$ endowed with the compact open topology becomes a topological vector space, denoted $\mathfrak{L}(G)$, when given the pointwise addition and the scalar multiplication defined by $(r{\cdot}X)(s)=X(sr)$. (See for instance [6], p. 334 ff.)

Proposition 1.2. *The free abelian group $\mathbb{Z}^{(\mathbb{N})}$ has a nondiscrete topology making it into a prodiscrete abelian group F in such a fashion that the following conditions are satisfied:*

(i) *There is an injection $j\colon F\to W$ mapping F isomorphically (algebraically and topologically) onto a closed subgroup of the weakly complete vector group W of topological dimension $2^{\aleph_0}$ such that W/F is an incomplete group whose completion is a compact connected and locally connected group, and that the $\mathbb{R}$–linear span $\mathrm{span}_{\mathbb{R}}(\mathrm{F})$ is dense in W.*

(ii) *The subset $B\stackrel{\mathrm{def}}{=}\{\delta_n : n\in\mathbb{N}\}$ satisfies $F=\langle B\rangle$ and $j(B)$ is unbounded in W.*

(iii) *If $K\subseteq F$ is any compact subset, then there is a finite subset $M\subseteq\mathbb{N}$ such that $k\in M\subseteq\mathbb{Z}^{(\mathbb{N})}$ implies that the support $\mathrm{supp}(k)=\{m\in\mathbb{N} : k(n)\neq 0\}$ is contained in M. In particular, every compact subset of F is contained in a finite rank subgroup of F.*

Proof. (i) This example is based on an example Poguntke and the authors described in [10]. Let G be the character group of the discrete group $\mathbb{Z}^{\mathbb{N}}$. Then G is a compact, connected and locally connected but not arcwise connected group. Let G_a denote the arc component of the identity element in G. It was proved in [10] that the corestriction of the exponential function $\exp'_G\colon\mathfrak{L}(G)\to G_a$ was a quotient map. Let $W = \mathfrak{L}(G) \cong \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \cong \mathbb{R}^{2^{\aleph_0}}$ and $F = \ker \exp_G \subseteq \mathfrak{L}(G) = W$. The exponential function of (locally) compact abelian groups is extensively discussed in [6], p. 344ff; in [6], p. 355, the kernel of the exponential function is denoted by $\mathfrak{K}(G)$. Now F as a closed subgroup of a pro-Lie group is a pro-Lie group by the Closed Subgroup Theorem for Pro-Lie Groups **CL**. It is totally disconnected (see [6], p. 355, Theorem 7.66(ii)), and so by Lemma [8] 3.31, or by [9], 4.6, F is a prodiscrete group. By [6], p. 332, Proposition 7.35(v)(d), the linear span $\mathrm{span}_{\mathbb{R}}(\mathrm{F}) = \mathrm{span}_{\mathbb{R}}(\mathfrak{K})(G)$ is dense in $\mathfrak{L}(G) = W$. Since G_a does not contain any copy of a cofinite dimensional closed vector subspace of $\mathfrak{L}(G)$, the function $\exp'_G \colon \mathfrak{L}(G) \to G_a$ cannot induce a local isomorphism and thus its kernel $F = \mathfrak{K}(G)$ is not discrete.

Furthermore, from [6], p. 355, Theorem 7.66(ii) we observe that

$$F \cong \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}),$$

algebraically, and from [4], p. 53 Corollary 15 and p. 60, Corollary 24, we get that $\Phi \colon \mathbb{Z}^{(\mathbb{N})} \to \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$, $\Phi\big((p_m)_{m\in\mathbb{N}}\big)\big((z_m)_{m\in\mathbb{N}}\big) = \sum_{m\in\mathbb{N}} p_m z_m$ is an isomorphism of abelian groups and that, accordingly, F is a free group algebraically generated by $\Phi(B)$.

(ii) We now prove that $\Phi(B) \subseteq \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \subseteq \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) = W$ is unbounded. Note that $\Phi(\delta_n) \colon \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ is simply the evaluation $\mathrm{ev}_n \colon \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$, $ev_n(f) = f(n)$. Since $\mathbb{Z}^{\mathbb{N}}$ is considered with the discrete topology, the topology on $W \cong \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ is that of pointwise convergence, that is, the topology induced from $\mathbb{Z}^{\mathbb{Z}^{\mathbb{N}}}$. Let $s \colon \mathbb{N} \to \mathbb{Z}$ be an arbitrary element of $\mathbb{Z}^{\mathbb{N}}$. Then $\mathrm{pr}_s \colon \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \to \mathbb{Z}$ is given by $\mathrm{pr}_s(\varphi) = \varphi(s)$ and thus $\mathrm{pr}_s\big(\Phi(B)\big) = \{\mathrm{ev}_n(s) : n \in \mathbb{N}\} = \{s(n) : n \in \mathbb{N}\} = \mathrm{im}\, s$. Therefore the projection of B into the s-component of $\mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \subseteq \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \cong W$ is bounded if and only if s is bounded. Since there are unbounded elements $s \in \mathbb{Z}^{\mathbb{N}}$, the set $j(B)$ is unbounded in W.

(iii) Let $K \subseteq \mathbb{Z}^{(\mathbb{N})}$ be a compact subset and suppose that it fails to satisfy the claim. Then there is a sequence of elements $k_n \in K$ such that $M_n = \max \mathrm{supp}\, k_n$ is a strictly increasing sequence. Now we define a function $s \colon \mathbb{N} \to \mathbb{Z}$ recursively as follows: Set $s(1) = 0$ and $s(m) = 0$ for $m \notin \{M_n : n \in \mathbb{N}\}$. Assume that $s(m)$ has been defined for $1 \le m \le M_n$ in such a way that $\sigma_m = \sum_{j=1}^{M_m} k_m(j)s(j) \ge m$. Now solve the inequality

$$n + 1 \le \sigma_{n+1} \stackrel{\mathrm{def}}{=} \sum_{j=1}^{M_n} k_{n+1}(j)s(j) + k_{n+1}(M_{n+1})s(M_{n+1})$$

for $s(M_{n+1})$. The s-projection $\mathrm{pr}_s\big(\Phi(K)\big) = \{\sum_{n\in\mathbb{N}} k(n)s(n) : k \in K\} \supseteq \{\sigma_n = \sum_{m\in\mathbb{N}} k_n(m)s(m) : m \in J\}$ contains arbitrarily large elements $\sigma_n \ge n$ and thus cannot be bounded, in contradiction to the compactness of K. ◻

We realize that (ii) is implied by (iii); we have preferred to give separate proofs for better elucidation of this remarkable example.

The group F cannot be metrizable, because as a complete abelian group its underlying space would be a Baire space and thus as a countable topological group would have to be discrete. It is therefore noteworthy that there are pro-Lie groups whose underlying space is not a Baire space. As a countable group, it is the countable union of compact sets. A topological space said to be *σ-compact*, if it is a countable union of compact subspaces. Thus F is trivially σ-compact.

We observe in conclusion of our brief discussion of examples of abelian pro-Lie groups that we have seen pro-discrete abelian groups on the space $\mathbb{R}\setminus\mathbb{Q}$ of irrationals and on a countable nondiscrete space.

2. Weil's Lemma

In the domain of of locally compact groups, Weil's Lemma says

Let g be an element of a locally compact group and $\langle g\rangle$ the subgroup generated by it. Then one (and only one) of the two following cases occurs
(i) $n\mapsto g^n:\mathbb{Z}\to\langle g\rangle$ *is an isomorphism of topological groups.*
(ii) $\overline{\langle g\rangle}$ *is compact.*
(See for instance [6], p. 342, Proposition 7.43.)

Theorem 2.1. (Weil's Lemma for pro-Lie groups) *Let E be either $\mathbb{R}$ or $\mathbb{Z}$ and $X\colon E\to G$ a morphism of topological groups into a pro-Lie group G. Then one and only one of the two following cases occurs*
(i) $r\mapsto X(r):E\to X(E)$ *is an isomorphism of topological groups.*
(ii) $\overline{X(E)}$ *is compact.*

Proof. The closed subgroup $A\overset{\text{def}}{=}\overline{X(E)}$ is an abelian pro-Lie group by the Closed Subgroup Theorem for Pro-Lie Groups **CL**. Thus we may assume that G is abelian and the image of X is dense. Now let $N\in\mathcal{N}(G)$ and $p_N\colon G\to G/N$ the quotient map. Then G/N is an abelian Lie group for a morphism $p_N\circ X\colon E\to G/N$ with dense image. By Weil's Lemma for locally compact groups, either $p_N\circ X$ is an isomorphism of topological groups or else G/N is compact. If $M\supseteq N$ in $\mathcal{N}(G)$ and and $p_N\circ X$ is an isomorphism, then $p_M\circ X$ and the bonding map $G/M\to G/N$ are isomorphisms as well. Thus there are two mutually exclusive cases:
(A) $(\forall N\in\mathcal{N}(G))\,p_N\circ X$ is an isomorphism of topological groups and all bonding maps $G/M\to G/N$ are isomorphisms.
(B) There is a cofinal subset $\mathcal{N}_c(G)\subseteq\mathcal{N}(G)$ such that G/N is compact.

In Case (A), the limit $G\cong\lim_{N\in\mathcal{N}(G)}G/N$ is isomorphic to E.

In Case (B) let $N\in\mathcal{N}(G)$ and let $\uparrow N=\{P\in\mathcal{N}(G):P\subseteq N\}$. Then for all $P\in\mathcal{N}(G)$ the quotient G/P is compact. By the Cofinality Lemma [8] 1.21(ii) we know $G\cong\lim_{P\in\uparrow N}G/P$. Since all G/P are compact for $P\in\uparrow N$, the group G is compact in this case, and the Theorem is proved. □

Definition 2.2. (i) Let G be a topological group. Then $\mathrm{comp}(\mathfrak{L})(G)$ denotes the set of all $X\in\mathfrak{L}(G)$ such that $\overline{\exp_G\mathbb{R}\cdot X}$ is compact. A one-parameter subgroup

$X \in \mathrm{comp}(\mathfrak{L})(G)$ is called a *relatively compact one-parameter subgroup.* Furthermore, $\mathrm{comp}(G)$ denotes the set $\{x \in G : \overline{\langle x\rangle} \text{ is compact}\}$. An element $x \in \mathrm{comp}(G)$ is called a *relatively compact element* of G.

(ii) A topological abelian group G is said to be *elementwise compact* if $G = \mathrm{comp}(G)$. It is said to be *compactfree* if $\mathrm{comp}(G) = \{0\}$. □

In any topological abelian group G, the set $\mathrm{comp}(G)$ of relatively compact elements is a subgroup, since $g, h \in \mathrm{comp}(G)$ implies that gh is contained in the compact subgroup $\overline{\langle g\rangle\langle h\rangle}$.

A discrete abelian group is elementwise compact if and only if it is a torsion group; it is compactfree if and only if it is torsionfree. If $f: G \to H$ is a morphism of abelian pro-Lie groups, then $f\big(\mathrm{comp}(G)\big) \subseteq \mathrm{comp}\,\big(f(G)\big) \subseteq \mathrm{comp}(H)$; accordingly $f\big(\mathrm{comp}(G)\big) \subseteq \mathrm{comp}(H)$. The quotient morphism $f: \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ shows that in general $f\big(\mathrm{comp}(G)\big) \neq \mathrm{comp}(H)$ even for quotient morphisms.

Theorem 2.3. *Let G be an abelian pro-Lie group. Then*
(i) $\mathrm{comp}(G)$ *is a closed subgroup of G and therefore is also a pro-Lie group.*
(ii) $\mathrm{comp}(G) \cong \lim_{N\in\mathcal{N}(G)} \mathrm{comp}(G/N)$.
(iii) *For $N \in \mathcal{N}(G)$, let $C_N \subseteq G$ be the closed subgroup of G containing* $\mathrm{comp}(G)$ *for which* $C_N/\mathrm{comp}(G) = \mathrm{comp}(G/N)$. *Then* $\mathrm{comp}(G) = \bigcap_{N\in\mathcal{N}(G)} C_N$.
(iv) *The factor group $G/\mathrm{comp}(G)$ is compactfree.*

Proof. (i) Since $G \cong \lim_{N\in\mathcal{N}(G)} G/N \subseteq \prod_{N\in\mathcal{N}(G)} G/N$ we may assume that G is a closed subgroup of a product $P = \prod_{j\in J} L_j$ of abelian Lie groups L_j. Any abelian Lie group is of the form $L = \mathbb{R}^p \times \mathbb{T}^q \times D$ for a discrete group D (see for instance [6], p. 349, Corollary 7.58(iii)). Then $\mathrm{comp}(L) = \{0\} \times \mathbb{T}^q \times \mathrm{tor}\, D$ for the torsion subgroup $\mathrm{tor}\, D$ of D. Hence $\mathrm{comp}(L)$ is closed. Thus $\overline{\mathrm{pr}_j(\mathrm{comp}(G))} \subseteq \mathrm{pr}_j\left(\overline{\mathrm{comp}(G)}\right) \subseteq \mathrm{comp}(L_j)$ as $\mathrm{comp}(L_j)$ is closed. Now let $g = (g_j)_{j\in J} \in \overline{G} \subseteq P$. Then $g_j \in \mathrm{comp}(L_j)$ and thus $\overline{\langle g_j\rangle}$ is compact and $g \in K \stackrel{\text{def}}{=} \prod_{j\in J} \overline{\langle g_j\rangle}$. As a product of compact groups, K is compact and thus $\overline{\langle g\rangle} \subseteq K$ is compact and so $g \in \mathrm{comp}(G)$. Hence $\mathrm{comp}(G)$ is closed and thus is a pro-Lie group by the Closed Subgroup Theorem for Pro-Lie Groups **CL**.

(ii) The assignment $G \mapsto \mathrm{comp}(G)$ defines a functor from the category of abelian pro-Lie groups to itself. If G is an elementwise compact abelian pro-Lie group and H is any abelian pro-Lie group, then any morphism of topological groups $f: G \to H$ factors through $\mathrm{comp}(H)$, that is, is of the form $f = \mathrm{incl}_{\mathrm{comp}(H),H} \circ f'$ with a unique morphism $f': H \to \mathrm{comp}(H)$. That is, comp is a right adjoint to the inclusion functor of the full subcategory of elementwise compact groups in the category $\mathbb{AB}\mathrm{pro}\mathbb{LIEGR}$ of abelian pro-Lie groups. See for instance [6], p. 718, Proposition A3.36. Therefore it preserves limits (see for instance [6], p. 723, Theorem A3.52). Since comp preserves limits, it preserves, in particular, projective limits.

(iii) follows from the Closed Subgroup Theorem for Projective Limits [8] 1.34(v) or [9], 2.2(iv). [Here is the proof: Since the family of $\{C_N : N \in \mathcal{N}\}$ is filtered, the system of natural quotient maps $gN \mapsto gM : K_N/N \to K_M/M$ for $N \subseteq M$ in $\mathcal{N}$ is a projective system and the projective limit $L \stackrel{\rm def}{=} \lim_{N\in\mathcal{N}} C_N/N$ is well defined. We have a natural map $\delta\colon \mathrm{comp}(G) \to L$, $\delta(h) = (hN)_{N\in\mathcal{N}}$. Just as was in the proof of Theorem 2.2(i) in [9] for the family $\{H_N : N \in \mathcal{N}\}$, we can define an inverse $\sigma\colon L \to \mathrm{comp}(G)$, $\sigma\big((g_N N)_{N\in\mathcal{N}}\big) = \lim_{N\in\mathcal{N}} g_N$, since $(g_N)_{N\in\mathcal{N}}$ is a Cauchy net. The fact that σ is a morphism of topological groups and inverts δ is shown in a fashion that is completely analogous to that which we applied in [9], 2.2(i).]

(iv) Let $g\,\mathrm{comp}(G)$ be nonzero in $G/\mathrm{comp}(G)$ that is, $g \notin \mathrm{comp}(G)$. Then by (ii) there is an $N \in \mathcal{N}(G)$ such that $g \notin C_N$. Then $gN \notin \mathrm{comp}(G/N)$ and then $(gN)\,\mathrm{comp}(G/N)$ is not relatively compact in $(G/N)/\mathrm{comp}(G/N)$. Since the quotient morphism $G \to G/N$ maps $\mathrm{comp}(G)$ into $\mathrm{comp}(G/N)$, there is an induced quotient morphism $F\colon G/\mathrm{comp}(G) \to (G/N)/\mathrm{comp}(G/N)$ given by $F\big(g\,\mathrm{comp}(G)\big) = (gN)\,\mathrm{comp}(G/N)$. Since this element is not relatively compact, the element $g\,\mathrm{comp}(G)$ cannot be relatively compact in $G/\mathrm{comp}(G)$. Thus $G/\mathrm{comp}(G)$ is compactfree. ☐

The Examples 1.1(iv) and (vii) are elementwise compact abelian pro-Lie groups G (that is, they satisfy $G = \mathrm{comp}(G)$), but they are not locally compact. The Examples 1.1(ii) and (vi), and the Example in Proposition 1.2 are compactfree abelian prodiscrete (pro-Lie) groups.

Definition 2.4. Let G be a topological group.

(i) G is said to be *almost connected* if G/G_0 is compact.

(ii) G is said to be *compactly generated* if there is a compact subset K of G such that $G = \langle K\rangle$.

(iii) G is said to be *compactly topologically generated* if there is a compact subset K of G such that $G = \overline{\langle K\rangle}$. ☐

Lemma 2.5. *Each quotient group of an almost connected topological group is almost connected.*

Proof. Let G be a topological group such that G/G_0 is compact and let N be a closed normal subgroup of G. Then $\overline{G_0N}/N \subseteq (G/N)_0$, and $G/\overline{G_0N}$ is compact as a continuous image of G/G_0. So $(G/N)/(G/N)_0$ is a homomorphic image of the compact group $G/\overline{G_0N}$ and is, therefore, compact. ☐

Proposition 2.6. *Assume that G is an abelian pro-Lie group satisfying at least one of the two conditions*

(i) *G is almost connected.*

(ii) *G is compactly topologically generated.*

Then $\mathrm{comp}(G)$ *is compact and therefore is the unique largest compact subgroup of G.*

Proof. We may assume that $G = \overline{\mathrm{comp}(G)}$ and must show that G is compact. Then by Theorem 2.3(ii) we have $G = \lim_{N\in\mathcal{N}(()G)} \mathrm{comp}(G/N)$; it therefore suffices to verify that $\mathrm{comp}(G/N)$ is compact for all N. Let $N \in \mathcal{N}(G)$. The Lie group G/N is isomorphic to $L = \mathbb{R}^p \times \mathbb{T}^q \times D$ for a discrete group D (see the proof of 2.3(i)), and thus $\mathrm{comp}(L) = \{0\} \times \mathbb{T}^q \times \mathrm{tor}(D)$ for the torsion group $\mathrm{tor}(D)$ of D. Thus we have to verify that $\mathrm{tor}(D)$ is finite.

In case (i), $G/N \cong L$ is an almost connected Lie group by Lemma 2.5 and thus D is itself finite.

In case (ii), $G/N \cong L$ has a dense subgroup generated by a compact set, and this is then true for the discrete factor D. Then D is finitely generated and thus is isomorphic to the direct sum of a finite group and a finitely generated free group (see for instance [6], p. 623, Theorem A1.11). Thus $\mathrm{tor}(D)$ is finite in this case as well. □

Lemma 2.7. *Assume that G is an abelian pro-Lie group. Then $\mathfrak{L}\big(\mathrm{comp}(G)\big) = \mathrm{comp}(\mathfrak{L})(G)\big)$, and there is a closed vector subspace W such that*

$$(X,Y) \mapsto X+Y \colon W \times \mathrm{comp}(\mathfrak{L})(G)\big) \to W \oplus \mathrm{comp}(\mathfrak{L})(G) = \mathfrak{L}(G)$$

is an isomorphism of weakly complete vector spaces.

Proof. By Definition 2.2, a one-parameter subgroup $X\colon \mathbb{R} \to G$ is in $\mathrm{comp}(\mathfrak{L})(G)$ iff $\overline{X(\mathbb{R})}$ is compact iff $X(\mathbb{R}) \in \mathrm{comp}(G)$. Thus $\mathrm{comp}(\mathfrak{L})(G) = \mathfrak{L}\big(\mathrm{comp}(G)\big)$. Since $\mathrm{comp}(G)$ is a closed subgroup of G by Theorem 2.3(i), $\mathfrak{L}\big(\mathrm{comp}(G)\big)$ is a closed vector subspace of $\mathfrak{L}(G)$. Then it has an algebraic and topological vector space complement by [6], p. 325, Theorem 7.30(iv). □

3. Vector group splitting theorems

Recall from Proposition 2.6 that $\mathrm{comp}(G)$ for an abelian pro-Lie group G is compact if G is almost connected or compactly topologically generated.

Lemma 3.1. *Let G be an abelian pro-Lie group, and assume that* $\mathrm{comp}(G)$ *is compact. Then the following conclusions hold.*

(i) *There is a weakly complete vector group W and a compact abelian group C which is a product of circle groups for which G may be considered as a closed subgroup of $W \times C$ such that $G \cap (\{0\} \times C) = \mathrm{comp}(G)$.*

(ii) *$G/\mathrm{comp}(G)$ is embedded as a closed subgroup into the weakly complete vector group W.*

Proof. (i) For each $N \in \mathcal{N}(G)$ we have an embedding $i_N\colon G \to W(N) \times C(N)$ for a finite dimensional vector group $W(N)$ and a finite dimensional torus $C(N)$. Hence

$$G \cong \lim_{N\in\mathcal{N}(G)} G/N \subseteq \prod_{N\in\mathcal{N}(G)} G/N \xrightarrow{\prod_{N\in\mathcal{N}(G)} i_N} \prod_{N\in\mathcal{N}(G)} W(N)\times C(N) = W\times C$$

for a weakly complete vector group $W=\prod_{N\in\mathcal{N}(G)} W(N)$ and a compact group $C\cong\prod_{N\in\mathcal{N}(G)} C(N)$. Since i is an embedding we may write $G\subseteq W\times C$ and assume that G is closed. Then $\mathrm{comp}(G)\subseteq\mathrm{comp}(W\times C)=\{0\}\times C$. But conversely, every element in $\{0\}\times C$ being relatively compact, we have $G\cap(\{0\}\times C)\subseteq\mathrm{comp}\,G$.

(ii) By (i) above we may assume $G\subseteq W\times C$ for a weakly complete vector group W and torus C and $\{0\}\times C$ is the maximal compact subgroup of $W\times C$. Hence $\mathrm{comp}(G)\subseteq\{0\}\times C$, and since C is compact $\overline{\mathrm{comp}(G)}\subseteq\{0\}\times C$. Let $p\colon P\to W$ be the projection of $W\times C$ onto W with kernel $\{0\}\times C$. Then p is a proper and hence closed morphism of topological groups; therefore $p|G\colon G\to p(G)$ is a quotient morphism onto a closed subgroup of W. Since $\ker(p|G)=\mathrm{comp}(G)$ we have $G/\mathrm{comp}(G)\cong p(G)$ and the Lemma follows. □

Lemma 3.2. *Let G be an abelian pro-Lie group such that* $\mathrm{comp}(G)=\{1\}$. *Then* $\exp_G\colon\mathfrak{L}(G)\to G_0$ *is an isomorphism of topological groups.*

Proof. It is no loss of generality to assume that $G=G_0$, and we shall do that from now on. If $X\in\ker\exp_G$, that is $X(1)=0$, then $\exp_G\mathbb{R}.X=X(\mathbb{R})$ is a homomorphic image of $\mathbb{R}/\mathbb{Z}$ and is therefore compact. Hence $X(\mathbb{R})\subseteq\mathrm{comp}(G)=\{1\}$ and thus $X=0$. So $\exp_G$ is injective.

Let $N\in\mathcal{N}(G)$. Since G is connected, $\mathrm{comp}(G)$ is compact, and thus Lemma 3.1 applies. and we may and will now assume that G is a closed subgroup of a weakly complete vector group V. We let $i\colon G\to V$ denote the inclusion map and identify V with $\mathfrak{L}(V)$ and $\exp_V$ with id_V. There is a commutative diagram

$$\begin{array}{ccc}
\mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(i)} & V\\
{\scriptstyle\exp_G}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{id}_V}\\
G & \xrightarrow[i]{} & V.
\end{array}$$

By the Correspondence Theorem of Subalgebras and Subgroups of [8] 7.22(v), which in the present case is easily verified directly, the morphism $\mathfrak{L}(i)$ implements an isomorphism of $\mathfrak{L}(G)$ onto a closed vector subspace of V. This implies that the corestriction $\exp_G'\colon\mathfrak{L}(G)\to\exp_G\mathfrak{L}(G)$ is an isomorphism of topological groups, and thus $\exp_G\mathfrak{L}(G)$ is a closed vector subspace of V via i. By [8], Corollary 4.22, or by [9], Corollary 6.8(i), $\exp\mathfrak{L}(G)$ is dense in $G=G_0$. Since $\exp_G\mathfrak{L}(G)$ is closed, we have $G=\exp_G\mathfrak{L}(G)$, and $\exp_G\colon\mathfrak{L}(G)\to G$ is an isomorphism. □

Lemma 3.3. (Vector Group Splitting Lemma for Connected Abelian Pro-Lie Groups) *Let G be a connected abelian pro-Lie group. Then there is a closed subgroup V of G such that* $\exp_G|\mathfrak{L}(V)\colon\mathfrak{L}(V)\to V$ *is an isomorphism of topological* *groups and that*

$$(X, g) \mapsto (\exp_G X) + g : \mathfrak{L}(V) \times \operatorname{comp}(G) \to V \oplus \operatorname{comp}(G) = G$$

is an isomorphism of topological groups.

In particular, every connected abelian pro-Lie group is isomorphic to $\mathbb{R}^J \times C$ *for some set* J *and some compact connected abelian group* C. (Compare [12].)

Proof. By Theorem 2.3(i), $\operatorname{comp}(G)$ is a closed subgroup. Let $q: G \to H \stackrel{\text{def}}{=} G/\operatorname{comp}(G)$ be the quotient morphism. By the Strict Exactness Theorem for $\mathfrak{L}$ of [8] 4.20, or by [9], Theorem 6.7, we have a commutative diagram of strict exact sequences

$$\begin{array}{ccccccc}
0 \to & \operatorname{comp}(\mathfrak{L})(G) & \xrightarrow{\text{incl}} & \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(q)} & \mathfrak{L}(H) & \to 0 \\
& {\scriptstyle \exp_G \mid \operatorname{comp}(\mathfrak{L})(G)}\downarrow & & \downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp_H} & \\
0 \to & \operatorname{comp}(G) & \xrightarrow[\text{incl}]{} & G & \xrightarrow[q]{} & H & \to 0.
\end{array}$$

By Lemma 2.7, the morphism $\mathfrak{L}(q)$ splits, that is, there is a morphism of weakly complete vector spaces $\sigma: \mathfrak{L}(H) \to \mathfrak{L}(G)$ such that $\mathfrak{L}(q) \circ \sigma = \mathrm{id}_{\mathfrak{L}(H)}$. By 2.3(iv) we have $\operatorname{comp}(H) = \{1\}$. Now we use the fact that $\operatorname{comp}(G)$ is compact by Proposition 2.6 to conclude that H is complete and thus is a pro-Lie group, see [8], Theorem 4.28(iii), or [16], p. 206, Theorem 11.18 and p. 242, Lemma 13.13. Since G is connected, H is connected. Then Lemma 3.2 implies that $\exp_H: \mathfrak{L}(H) \to H$ is an isomorphism. We define $s: H \to G$ by $s = \exp_G \circ \sigma \circ \exp_H^{-1}$. Then $q \circ s = q \circ \exp_G \circ \sigma \circ \exp_H^{-1} = \exp_H \circ \mathfrak{L}(q) \circ \sigma \circ \exp_H^{-1} = \exp_H \circ \exp_H^{-1} = \mathrm{id}_H$. Thus q splits. Now let $\mu: V \times \operatorname{comp}(G) \to G$ be defined by $\mu(v, g) = v + g$ and $\nu: G \to V \times G$ by $\nu(g) = \big(g - s(q(g)), s(q(g))\big)$. Then μ and ν are inverses of each other, and this completes the proof of the Lemma. □

Thus connected abelian pro-Lie groups are already reduced to weakly complete vector groups (Example 1.1(iii)) and compact connected groups.

Definition 3.4. Let G be an abelian pro-Lie group. Let V be any closed subgroup of G such that

(i) V is isomorphic to the additive topological group of a weakly complete vector space,

(ii) $(v, c) \mapsto v + c : V \times \operatorname{comp}(G_0) \to G_0$ is an isomorphism of topological groups.

Then V is called a *vector group complement.* □

A vector group complement is not unique, but all vector group complements are isomorphic to $G_0/\operatorname{comp}(G_0)$.

Remark 3.5. Let V be any vector space complement of an abelian pro-Lie group and let us write $G_0 = V \times C$ with $C = \operatorname{comp}(G_0)$. For any morphism of topological groups $f: V \to C$, the subgroup $\operatorname{graph}(f) = \{\big(v, f(v)\big) : v \in V\}$ is also a vector group complement. All vector group complements are obtained in this way. If f is given, then $\alpha: G_0 \to G_0$, $\alpha(v,c) = \big(v, c+f(v)\big)$ is an automorphism of G_0 mapping $V \times \{0\}$ to graph(f).

Proof. The function $\alpha: G_0 \to G_0$, $\alpha(v,c) = \big(v, c+f(v)\big)$ is an automorphism of topological groups mapping $V \times \{0\}$ onto graph(f).

If $W \subseteq G_0 = V \times C$ is a vector group complement, then since W is a vector group complement there is a projection $\mathrm{pr}_W: G_0 \to W$ with kernel $\{0\} \times C$. Let $\mathrm{pr}_C: G_0 \to C$ denote the projection with kernel $V \times \{0\}$. Define $f: V \to C$ by $\mathrm{pr}_C\big(\mathrm{pr}_W(v,0) - (v,0)\big)$. Then f is a morphism of topological groups; if $w = (v,c) \in W$ then $w = \mathrm{pr}_W(v,0)$ and $w - (v,0) = (0,c)$ whence $c = f(v)$. □

We now work towards removing the hypothesis of connectivity from the Vector Group Splitting Lemma 3.3.

A topological group G for which $\mathcal{N}(G)$ is a filter basis converging to the identity is called a *proto-Lie group.* The group $\widetilde{G} = \lim_{N\in\mathcal{N}(G)} G/N$ is a completion of G and is a pro-Lie group (see [8] 3.26, or verify directly that $g \mapsto (gN)_{N\in\mathcal{N}(G)} : G \to \widetilde{G}$ is a dense embedding into a complete group).

Lemma 3.6. *Let W be a weakly complete vector space and G a proto-Lie group. Assume that $f: W \to G$ is a bijective morphism of abelian groups. Then f is an isomorphism of topological groups.*

Proof. Let $\widetilde{G}$ denote the completion of G. Then $\widetilde{G}$ is a connected abelian pro-Lie group; by the Vector Space Splitting Lemma 3.3, it is therefore of the form $V \oplus C$ algebraically and topologically for a weakly complete vector subgroup V and a compact subgroup C. Let $\mathrm{pr}_V: \widetilde{G} \to V$ denote the projection onto V along C. The function $\varphi \overset{\mathrm{def}}{=} \mathrm{pr}_V \circ f: W \to V$ is a dense morphism of weakly complete vector groups and is therefore a quotient morphism of weakly complete vector spaces by the Duality Theorem of Weakly Complete Vector Spaces (see [6], p. 325, Theorem 7.30, since epics are dual to monics, and the monics in the category of vector spaces are injective and their duals are the quotient morphisms). Thus there are closed vector subspaces V_1 and V_2 of W such that $W = V_1 \oplus V_2$ algebraically and topologically such that $V_2 = \ker\varphi$ and $\varphi \circ i: V_1 \to V$, where $i: V_1 \to W$ is the inclusion, is an isomorphism of weakly complete vector spaces. Now the morphism $\sigma: V \to \widetilde{G}$, $\sigma = f \circ i \circ (\varphi \circ i)^{-1}$ satisfies $\mathrm{pr}_V \circ \sigma(v) = \mathrm{pr}_V \circ f \circ i \circ (\varphi \circ i)^{-1} = \varphi \circ i \circ (\varphi \circ i)^{-1} = \mathrm{id}_V$. This means that $\widetilde{G} = \sigma(V) \oplus C$, $\sigma(V) = f(V_1)$. In order to simplify notation, after replacing V by $\sigma(V)$, if necessary, we may actually assume that $V = f(V_1)$. Then $D \overset{\mathrm{def}}{=} f(V_2) \subseteq C$, and we have $G = V \times D$ for a dense subgroup D of C. We recall that G is a proto-Lie group; then $D \cong G/V$ is a connected proto-Lie group and a dense subgroup of a compact group.

Let $N \in \mathcal{N}(D)$; then D/N is a Lie group and a dense subgroup of C/N. A Lie group is complete, and thus D/N is closed in C/N, that is, $D/N = C/N$ for all $N \in \mathcal{N}(D)$. Therefore $D = C$ and thus $G = \widetilde{G}$. Hence G is complete and

$f: W \to V \times C$ is bijective. We can write $W = W_1 \times W_2$ such that $f = f_1 \times f_2$ where f_1 is an isomorphism from W_1 onto V and $f_2: W_2 \to C$ is a bijective morphism of topological abelian groups from a weakly complete vector group W_2 onto a compact group C. In particular, every point in C is on a one-parameter subgroup, that is $\exp: \mathfrak{L}(C) \to C$ is surjective.

But C, as a bijective image of a real vector space, is torsionfree and divisible. Thus $\widehat{C}$ is divisible and torsion free and so is a rational vector space, that is, a direct sum of copies of $\mathbb{Q}$, and therefore C is a power of copies of $\widehat{\mathbb{Q}}$. If $C = \widehat{\mathbb{Q}}^J$ for some set J, then $\mathfrak{L}(C)$ may be identified with $\mathfrak{L}(\widehat{\mathbb{Q}})^J$ as $\mathfrak{L}$ preserves limits, and $\exp_C$ may be identified with $(\exp_{\widehat{\mathbb{Q}}})^J$. But $\exp_{\widehat{\mathbb{Q}}}: \mathbb{R} \cong \mathfrak{L}(\widehat{\mathbb{Q}}) \to \widehat{\mathbb{Q}}$ is not surjective, because the nontrivial compact homomorphic images of $\mathbb{R}$ are isomorphic to $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ and $\mathbb{T} \not\cong \widehat{\mathbb{Q}}$ since $\widehat{\mathbb{T}} \cong \mathbb{Z} \not\cong \mathbb{Q} \cong \widehat{\widehat{\mathbb{Q}}}$. Thus $J = \emptyset$ and $C = \{0\}$. This means that $G = V$ and $f \cong W \to V$ is an isomorphism of topological groups. □

Lemma 3.7. *Assume that G is an abelian topological group with closed subgroups G_1 and H such that G_1 is either*
(a) *a weakly complete vector subgroup, or*
(b) *a compact subgroup.*
Assume further that $G_1 + H = G$, that $G_1 \cap H = \{0\}$, and that G/H is a proto-Lie group. Then $\mu: G_1 \times H \to G$, $\mu(v, h) = v + h$ is an isomorphism of topological groups.

Proof. Clearly μ is a bijective morphism of topological groups. We must show that its inverse is continuous. The morphism $\beta: G_1 \to G/H$, $\beta(v) = v + H$ is a continuous bijection. In Case (a), by Lemma 3.6 β is open. In Case (b) it is a homeomorphism since G_1 is compact. That is, $\beta^{-1}: G/G_0 \to G_0$ is continuous in both cases. Let $q: G \to G/G_0$ be the quotient map. Then $\alpha \stackrel{\text{def}}{=} \beta \circ q: G \to G_0$ is a morphism of topological groups, and $\mu^{-1}(g) = \big(\alpha(g), g - \alpha(g)\big)$ is likewise a morphism of topological groups. □

If G is a pro-Lie group and H a closed normal subgroup, then G/H is a proto-Lie group (see [8] Theorem 4.1, or [9], Proposition 6.1), but G/H may fail to be complete (see [10]).

Proposition 3.8. *Assume that G is an abelian proto-Lie group and that G_1 is a closed connected subgroup which is a finite dimensional Lie group. Then there is a closed subgroup H such that the morphism*

$$(v, h) \mapsto v + h : G_1 \times H \to G$$

is an isomorphism of topological groups.

Proof. We claim that it is no loss of generality to assume that G is a pro-Lie group. Indeed, let $\widetilde{G}$ be the completion of G. Since the subgroup G_1 is a finite dimensional Lie group, it is locally compact. Locally compact subgroups are complete. (see for instance [6], p. 777, Corollary A4.24.). Thus G_1 is also a closed subgroup of the pro-Lie group $\widetilde{G}$. If we can show that there is a closed subgroup $\widetilde{H}$ of $\widetilde{G}$ such that $(v,\widetilde{h}) \mapsto v+\widetilde{h} : G_1 \times \widetilde{H} \to \widetilde{G}$ is an isomorphism of topological groups, then setting $H \stackrel{\rm def}{=} G \cap \widetilde{H}$ we get a subgroup of G such that $(v,h) \mapsto g+h : G_1 \times H \to G$ is an isomorphism of topological groups. This proves our claim; from here on we shall assume that G is a pro-Lie group.

The subgroup G_1, being a finite dimensional Lie group, has no small subgroups. Hence there is a zero neighborhood U such that $\{0\}$ is the only subgroup contained in $G_1 \cap U$. Now let $N \in \mathcal{N}(G)$ be contained in U. Then $N \cap G_1 \subseteq U \cap G_1 = \{0\}$. Recall that G/N is a Lie group and $(G_1+N)/N$ is isomorphic to $G_1/(G_1 \cap N) \cong G_1$ by the Closed Subgroup Theorem for Projective Limits (see proof of Theorem 2.3(iii)). So $(G_1+N)/N$ is isomorphic to $\mathbb{R}^p \oplus \mathbb{T}^q$ for suitable integers and is a closed subgroup of G/N which is isomorphic to $\mathbb{R}^m \oplus \mathbb{T}^n \oplus D$ for integers m and n and a discrete subgroup D. It is therefore a direct summand algebraically and topologically, that is, there is a closed subgroup H of G containing N such that H/N is a complementary summand for $(G_1+N)/N$. Thus $G_1 + H = G$ and $G_1 \cap H \subseteq G_1 \cap N = \{0\}$. Then if $C \stackrel{\rm def}{=} \mathrm{comp}(G_1) = \{0\}$, it follows from Lemma 3.7(a), that $(v,h) \mapsto v+h : G_1 \times H \to G$ is an isomorphism of topological groups. Therefore, in the general case, $G/C \cong G_1/C \times (H+C)/C$ and thus there is a vector subgroup V of G such that $G \cong V \times (C+H)$; it remains to be observed that $C+H \cong C \times H$. But that is Lemma 3.7(b). □

Next we need a lemma on weakly complete vector spaces.

Lemma 3.9. *Let W be a weakly complete vector space and $\mathcal{F}$ a filterbasis of closed affine subspaces, that is, subsets of the form $g_j + V_j$ for a closed vector subspace V_j. Then $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. The set $\{V_j : j \in J\}$ is a filter basis. Indeed let $i,j \in J$, then there is a $k \in J$ such that $g_k + V_k \subseteq (g_i+V_i) \cap (g_j+V_j)$, since $\mathcal{F}$ is a filter basis. Therefore $g_i + V_i = g_k + V_i$ and $g_j + V_j = g_k + V_j$. Now $g_k + V_k \subseteq (g_k+V_i) \cap (g_k+V_j)$, and hence $V_k \subseteq V_i \cap V_j$. Let $V = \bigcap_{j \in J} V_j$. Then W/V is a weakly complete vector space and $\mathcal{F}/V = \{(g_j+V)+V_j/V : j \in J\}$ is a filter basis of closed affine subsets. It clearly suffices to show that $\mathcal{F}/V$ has a nonempty intersection. Thus we assume from here on that $V = \{0\}$, that is the filter basis $\mathcal{V} \stackrel{\rm def}{=} \{V_j : j \in J\}$ has the intersection $\{0\}$. But then $\lim \mathcal{V} = 0$ in W by [8], Lemma 6.69. This implies that $\mathcal{F}$ is a Cauchy filter: Let U be an identity neighborhood; then there is a $j \in J$ such that $V_j \subseteq U$. Then $(g_j+V_j) - (g_j+V_j) = V_j \subseteq U$. Since W is a complete topological vector space, every Cauchy filter basis converges. Let $g = \lim \mathcal{F}$. Since all $g_j + V_j$ are closed, we have $g \in g_j + V_j$ for all $j \in J$ and this completes the proof of the Lemma. □

In terms of a terminology that has been used for situations like this we can say that

weakly complete topological vector spaces are linearly compact.

(See for instance H. Leptin, Linear kompakte Moduln und Ringe I und II, Math. Z. **62** (1953), 79–90, respectively, Math. Z. **66** (1955), 241–267.)

Theorem 3.10. *Assume that G is a topological abelian group with a closed a weakly complete vector subgroup G_1. Then $G \cong G_1 \times G/G_1$.*

Proof. Let $\mathcal{S}$ be the set of all closed subgroups S of G satisfying the following conditions:
(i) $S \cap G_1$ is a vector group.
(ii) $(\forall g \in G)\, S \cap (g + G_1) \neq \emptyset$.

We claim that $(\mathcal{S}, \supseteq)$ is an inductive poset. For a proof of the claim let $\mathcal{T}$ be a chain in $\mathcal{S}$ and set $T \stackrel{\text{def}}{=} \bigcap \mathcal{T}$. We have to show that T satisfies (i) and (ii).

(i) We note $T \cap G_1 = \bigcap_{S\in\mathcal{T}} S \cap G_1$, and all $S \cap G_1$ are closed vector groups; hence their intersection is a closed vector group.

(ii) Let $g \in G$; we must show that $T \cap (g + G_1) \neq \emptyset$.

Now for each $S \in \mathcal{T}$ we find an $s_S \in S \cap (g+G_1)$ by (ii). Then $g+G_1 = s_G + G_1$. We claim that

$$(*) \qquad S \cap (g + G_1) = s_S + (S \cap G_1).$$

Indeed if $s \in S \cap (g+G_1)$, then $s \in g+G_1 = s_S + G_1$ and thus $s - s_S \in S \cap G_1$ and thus $s \in s_S + (S \cap G_1)$. Conversely, if $s \in S \cap G_1$, then $s_S + s \in S \cap (s_S + G_1) = S \cap (g + G_1)$.

By (i) we know that $S \cap G_1$ is a (closed) vector subgroup V_S of G_1. Thus from $(*)$ we obtain

$$(**) \qquad \emptyset \neq (S - g) \cap G_1 = s_S - g + V_S,$$

where $s_G - g \in G_1$. Now the family $\{s_S - g + V_S : S \in \mathcal{T}\}$ is a filter basis of closed affine subspaces of the vector group G_1. By Lemma 3.9, there is a

$$t \in \bigcap_{S\in\mathcal{T}} (s_S - g + V_S) \in G_1,$$

and thus $t + g \in \bigcap_{S\in\mathcal{T}} (s_S + V_S) = \bigcap_{S\in\mathcal{T}} S \cap (g + G_1) = T \cap (g + G_1)$.

This completes the proof that $(\mathcal{S}, \supseteq)$ is inductive. Using Zorn's Lemma, let H be a minimal member of $\mathcal{S}$. We claim that $H \cap G_1 = \{0\}$. Suppose that the claim were false. Then $H_1 \stackrel{\text{def}}{=} H \cap G_1$ is a nonzero weakly complete vector group. Let N be a vector subgroup of H_1, such that H_1/N is a finite dimensional vector group (for instance, one isomorphic to $\mathbb{R}$). Now G/N as a quotient of a pro-Lie group, is a proto-Lie group by [8], Theorem 4.1, or by [9], Proposition 6.1. Then by Proposition 3.8, there is a closed subgroup S of H containing N such that $(H_1/N) + (S/N) = H/N$, and $(H_1/N) \cap (S/N) = \{N\}$. Thus $H_1 \cap S = N$ is a vector subgroup and $G_1 + S = G_1 + H = G$ and so the subgroup S of H/N satisfies (i) and (ii) above. The minimality of H then entails $S = H$ and thus $N = H_1 \cap S = H_1$ and that is a contradiction to the choice of N. This proves our claim that there is a closed subgroup H of G such that $G = G_1 + H$ and

$G_1 \cap H = \{0\}$. Thus the function

$$(v, h) \mapsto v + h : G_1 \times H \to G$$

is an isomorphism of topological groups by Lemma 3.7. □

Another way of expressing the preceding theorem in a category theoretical fashion is this:

Weakly complete vector groups are injectives in the category of abelian pro-Lie groups.

Recall from the Vector Group Splitting Lemma 3.3 and Definition 3.4 that every abelian pro-Lie group has a vector group complement. A topological abelian group in which the filter basis of open subgroups converges to the identity is called *protodiscrete.*

Theorem 3.11. (Vector Group Splitting Theorem for Abelian Pro-Lie Groups) *Let G be an abelian pro-Lie group and V a vector group complement. Then there is a closed subgroup H such that*

(i) $(v, h) \mapsto v + h : V \times H \to G$ *is an isomorphism of topological groups,*

(ii) H_0 *is compact and equals* $\operatorname{comp} G_0$ *and* $\operatorname{comp}(H) = \operatorname{comp}(G)$*; in particular,* $\operatorname{comp}(G) \subseteq H$.

(iii) $H/H_0 \cong G/G_0$*, and this group is prodiscrete.*

(iv) $G/\operatorname{comp}(G) \cong V \times S$ *for some protodiscrete abelian group without nontrivial compact subgroups.*

(v) G *has a characteristic closed subgroup* $G_1 = G_0 \operatorname{comp}(G)$ *which is isomorphic to* $V \times \operatorname{comp}(H)$ *such that* G/G_1 *is prodiscrete without nontrivial compact subgroups.*

(vi) *The exponential function* $\exp_G$ *of* $G = V \oplus H$ *decomposes as*

$$\exp_G = \exp_V \oplus \exp_H \quad \textit{where} \quad \exp_V \colon \mathfrak{L}(V) \to V \quad \textit{is an isomorphism}$$

of weakly complete vector groups and $\exp_H = \exp_{\operatorname{comp}(G_0)} \colon \mathfrak{L}\big(\operatorname{comp}(G_0)\big) \to \operatorname{comp}(G_0)$ *is the exponential function of the unique largest compact connected subgroup; here* $\mathfrak{L}\big(\operatorname{comp}(G_0)\big) = \operatorname{comp}(\mathfrak{L})(G)$ *is the set of relatively compact one-parameter groups of* G.

(vii) *The arc component* G_a *of* G *is* $V \oplus H_a = V \oplus \operatorname{comp}(G_0)_a = \operatorname{im} \mathfrak{L}(G)$. *Moreover, if* $\mathfrak{h}$ *is a closed vector subspace of* $\mathfrak{L}(G)$ *such that* $\exp \mathfrak{h} = G_a$*, then* $\mathfrak{h} = \mathfrak{L}(G)$.

Proof. (i) By Theorem 3.10, H exists such that (i) is satisfied.

(ii) Let us write $G = V \times H$. Then $G_0 = V \times H_0$. Since $V \times \{0\}$ is a vector group complement, $G = (V \times \{0\}) \oplus \operatorname{comp}(G_0)$ algebraically and topologically. Then the projection of G_0 onto H_0 along V maps the compact subgroup $\operatorname{comp}(G_0)$ onto H_0. Thus H_0 is compact. So $\{0\} \times H_0 \subseteq \operatorname{comp}(G_0)$, and since $G_0 = V \times H_0$, the factor group $\operatorname{comp}(G_0)/(\{0\} \times H_0)$ is isomorphic to a subgroup of V. Since V as a vector group has no nontrivial compact subgroup, $\operatorname{comp}(G) = \{0\} \times H_0$ follows.

If $G = V \times H$ then $\operatorname{comp}(G) = \operatorname{comp}(V) \times \operatorname{comp}(H) = \{0\} \times \operatorname{comp}(H)$.

(iii) Retaining the convention $G = V \times H$ after (i), we have $G_0 = V \times H_0$. Then

$$G/G_0 = \frac{V \times H}{V \times H_0} \cong \big((V \times H)/(V \times \{0\})\big)/\big((V \times H_0)/(V \times \{0\})\big) \cong H/H_0.$$

By the Closed Subgroup Theorem for pro-Lie groups **CL**, H is a pro-Lie group. Since H_0 is compact, H/H_0 is complete by [8], Theorem 4.28(iii), or by [16], p. 206, Theorem 11.18 and p. 242, Lemma 13.13. Thus by Lemma [8] 3.31, or by [9], 4.6, H/H_0 is a prodiscrete group.

(iv) Again we write $G = V \times H$ and have $\operatorname{comp}(G) = \{0\} \times \operatorname{comp}(H)$. Thus $G/\operatorname{comp}(G) = \frac{V \times H}{\{0\} \times \operatorname{comp}(H)} \cong V \times H/\operatorname{comp}(H)$. By Theorem 2.3(iv), $H/\operatorname{comp}(H)$ is compactfree.

By (iii) above, H/H_0 is prodiscrete. As a quotient of the prodiscrete group H/H_0, the quotient $H/\operatorname{comp}(H) \cong (H/H_0)/(\operatorname{comp}(H)/H_0)$ is a protodiscrete group by [8], Proposition 3.30(b), or by [9], 6.1.

(v) $\operatorname{comp}(G)$ is a closed characteristic subgroup and by (iv) the factor group $G/\operatorname{comp}(G)$ decomposes into a direct product $V \times S$ in which $V \times \{0\}$ is the connected component and thus is characteristic. The kernel G_1 of the composition of the quotient morphism $G \to G/\operatorname{comp}(G)$ and the projection $G/\operatorname{comp}(G) \to S$ is a closed characteristic subgroup equal to $G_0 \operatorname{comp}(G)$ and $G/G_1 \cong S$. Applying (i) to G_1 we get $G_1 \cong V \times \operatorname{comp}(G)$.

(vi) follows immediately from (i) and Definition 5.4(i) in view of the fact that for any topological vector space V the exponential function $\exp_V\colon \mathfrak{L}(V) \to V$, $\exp_G X = X(1)$, is an isomorphism of topological vector spaces as all one parameter subgroups are of the form $X = r \mapsto r{\cdot}v$ for a unique vector $v = v_X$. See also Lemma 5.15.

(vii) Since $G = V \oplus H$ is a direct product decomposition we have $G_a = V_a \oplus H_a$. But V, as the additive topological group of a topological vector space is arcwise connected, and $H_a = (H_0)_a = \operatorname{comp}(G_0)_a$. By [6], p. 389, Theorem 8.30(ii), we have $\operatorname{comp}(G_0)_a = \mathfrak{L}\big(\operatorname{comp}(G_0)\big) = \mathfrak{L}(H)$. Thus from (vi) we get $G_a = \exp_G \mathfrak{L}(G)$.

Now let $\mathfrak{h}$ be a closed subalgebra of $\mathfrak{L}(G)$. If $\mathfrak{h} \neq \mathfrak{L}(G)$, then $\exp_G \mathfrak{h} \neq E(G) = G_a$ by Corollary 4.21(i). □

For *locally compact* abelian groups 3.11(i) yields a core result of their structure theory; it is presented practically in every source book on locally compact abelian groups (see for instance [6], p. 348, Theorem 7.57). The present proof is new even for locally compact abelian groups.

The examples in 1.1(iv),(v), (vi), and (vii) illustrate certain limitations of this main result. The examples in 1.1(iv) and (vii) show how prodiscrete elementwise compact groups may look; neither has a compact open subgroup and therefore both fail to be locally compact. The examples in (iv) are torsion free and divisible, then examples in (vii) have a dense proper torsion subgroup and are divisible. The examples in 1.1(v) and (vi) are compactfree; those in (v) are divisible, those in (vi) have no nondegenerate divisible subgroups. Thus unlike in the locally compact case, we cannot expect that inside the factor H, the subgroup $\mathrm{comp}(H)$ is open nor that the factor group G/G_0 has an open compact subgroup.

It is easy to mix these examples. There are compact abelian groups in which the component does not split (see [6], p. 373, Example 8.11).

Theorem 3.11 completely elucidates the structure of the identity component G_0, it largely clarifies the structure of G_1 (although $\mathrm{comp}(G)$ is best understood in the locally compact case), and it reduces the more subtle problems on G to the compactfree prodiscrete factor group G/G_1. One should recall the Example in Proposition 1.2 which typically might occur as a prodiscrete factor group.

Corollary 3.12. *Let G be an abelian pro-Lie group and V a vector group complement. Then the following statements are equivalent:*

(i) *G/G_0 is locally compact.*

(ii) *G/V is locally compact.*

(iii) $\mathrm{comp}(G/V)$ *is locally compact and open in G/V.*

(iv) *There is a locally compact subgroup H of G containing* $\mathrm{comp}(G)$ *as an open subgroup such that such that $(v,h)\mapsto v+h : V\times H\to G$ is an isomorphism of topological groups.*

Proof. (i)$\Longleftrightarrow$(ii): We have $G/G_0 \cong (G/V)/(G_0/V)$ and G_0/V is compact by Lemma 3.3. The quotient of a locally compact group is locally compact, and the extension of a locally compact group by a locally compact group is locally compact.

(iv)$\Longrightarrow$(iii): Since $\mathrm{comp}(G)=\mathrm{comp}(H)$ this is clear.

(iii)$\Longrightarrow$(ii): Trivial.

(ii)$\Longrightarrow$(iv): The locally compact abelian group G/V has a compact identity component $(G/V)_0=\mathrm{comp}(G/V)_0$ and thus has a compact open subgroup C. Let K be the full inverse image of C in G. Then K is an almost connected open subgroup of G to which the Vector Group Splitting Theorem applies. Thus K is the direct product of V and the unique maximal compact subgroup $\mathrm{comp}(K)$ of K. Then $\big(G/\mathrm{comp}(K)\big)_0 = K/\mathrm{comp}(K)\cong V$, this is an open divisible subgroup of $G/\mathrm{comp}(K)$. But then $G/\mathrm{comp}(K)$ is the direct sum of $\big(K/\mathrm{comp}(K)\big)_0$ and a discrete group $H/\mathrm{comp}(K)$ with a closed subgroup H of G containing $\mathrm{comp}(K)$ as an open subgroup. In particular, $\mathrm{comp}(K)\subseteq\mathrm{comp}(H)$ and $\mathrm{comp}(H)$ is open in H. From

$$\begin{aligned}\big(G/\mathrm{comp}(K)\big) &\cong \big(K/\mathrm{comp}(K)\big)\times\big(H/\mathrm{comp}(K)\big) \text{ and}\\ K &\cong V\times\mathrm{comp}(K) \text{ we derive}\\ G &\cong V\times H.\end{aligned}$$

As we have $\mathrm{comp}(G)=\mathrm{comp}(H)$, the implication (ii)$\Longrightarrow$(iv) is proved. □

The structure theory results we discussed permit us to derive results on the duality of abelian pro-Lie groups. For any topological abelian group G we let $\widehat{G}=\mathrm{Hom}(G,\mathbb{T})$ denote its dual with the compact open topology. (See e.g. [6], Chapter 7.) There is a natural morphism of abelian groups $\eta_G\colon G\to\widehat{\widehat{G}}$ given by $\eta_G(g)(\chi) = \chi(g)$ which may or may not be continuous; information regarding this issue is to be found for instance in [6], pp. 298ff., notably in Theorem 7.7 on p. 300. We shall call a topological abelian group *reflexive* if $\eta_G: G \to \widehat{\widehat{G}}$ is an isomorphism of topological groups; G is also said *to have duality* (see [6], p. 305). In [1] Banaszczyk exhibited an example of a prodiscrete abelian group G which is not reflexive even though η_G is bijective. Let $q: \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the quotient morphism. By [6], p. 297, Proposition 7.5(iii), for every topological vector space E the morphism $\mathrm{Hom}(E, q): E' \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R}) \to \mathrm{Hom}(E, T) = \widehat{E}$, where $\mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ denotes the vector space of all continuous linear functionals endowed with the compact open topology, is an isomorphism of topological groups. If E is a topological vector space with its finite locally convex topology, then the compact open topology on $E' \cong \widehat{E}$ is the weak $*$-topology and $\widehat{E}$ is a weakly complete vector group: see [6], p.324, Lemma 7.28. If V is a weakly complete vector group, then the compact open topology on $V' \cong \widehat{V}$ agrees with the finest locally convex vector space topology on the vector space V': See [6], pp. 325–327, Theorem 7.30(ii). Both E and V are reflexive: See [6], pp. 325, 326, Theorem 7.30(i, ii).

Let A and B be topological abelian groups. Then $\widehat{A \times B}$ is naturally isomorphic to $\widehat{A} \times \widehat{B}$, and if A and B are reflexive, then $A \times B$ is reflexive. (Cf. [6], p. 306, Proposition 7.10.)

Corollary 3.13. *Assume that G is an abelian pro-Lie group. Let V be a vector group complement and H a closed subgroup according to* Theorem 3.11 *such that $(v, h) \mapsto v + h : V \times H \to G$ is an isomorphism of topological groups. Then $\widehat{G} \cong \widehat{V} \times \widehat{H}$, where $\widehat{V}$ is a real vector space with its finest locally convex topology, and G has duality iff $G/V \cong H$ has duality. This is the case if G/V is locally compact* (Corollary 3.12).
In particular, a connected *abelian pro-Lie group G has duality, and its character group is a direct product of a real vector group with its finest locally convex topology and a torsion free discrete group.*

Proof. In view of the preceding reminders and the duality theory of compact connected abelian groups (see [6], Chapter 8, notably p. 369, Corollary 8.5) we conclude the corollary immediately from the main Theorem 3.11. ◻

Accordingly, the connected abelian pro-Lie groups are exactly the character groups of direct sums of real vector groups and torsion free abelian groups, where we endow the vector group component with its finest locally convex vector space topology and the torsion free component with its discrete topology.

Corollary 3.13 reduces the issue of reflexivity of abelian pro-Lie groups to groups whose connected component is compact. Much remains to be done in this regard.

4. Special topological-algebraic properties

The Vector Group Splitting Theorem tells us that each abelian pro-Lie group G is built up in a lucid fashion from a weakly complete vector group and a more special abelian pro-Lie group H which in turn is an extension of the characteristic closed subgroup $\mathrm{comp}(G) = \mathrm{comp}(H)$ by a prodiscrete compactfree factor group $H/\mathrm{comp}(H) \cong G/\big(G_0\,\mathrm{comp}(G)\big)$. The Examples 1.1(iv) and (vii) (and the groups easily manufactured from these by passing to products, subgroups and quotients indicate that we are not to expect very explicit information on $\mathrm{comp}(G)$ without further hypotheses, and a similar statement holds for pro-discrete compactfree groups (see 1.1(v), (vi)).

A topological space is called a *Polish space* if it is completely metrizable and second countable. Recall that it is said to be *σ-compact*, if it is a countable union of compact subspaces. It is said to be *separable* if it has a dense countable subset.

Countable products of Polish spaces are Polish. For instance, any product $\prod_{n\in\mathbb{N}} L_n$ of a countable sequence of second countable Lie groups is a Polish pro-Lie group; this applies in particular to $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{Z}^{\mathbb{N}}$.

A mixture of topological and algebraic properties of topological groups is exemplified by the concepts introduced in 2.4, to which we return presently.

Remark 4.1. (i) Every almost connected locally compact group is compactly generated.

(ii) Every compactly generated topological group is σ-compact.

(iii) A topological group whose underlying space is a Baire space and which is σ-compact is a locally compact topological group.

(iv) A σ-compact Polish group is locally compact.

(v) A compactly generated Baire group is locally compact.

Proof. (i) Let K be a compact neighborhood of the identity. Then $\langle K\rangle$ is an open subgroup which has finite index in G. Let F be any finite set which meets each coset modulo $\langle K\rangle$. Then $K \cup F$ is a compact generating set of G.

(ii) If K is a compact generating set of G, then $C \stackrel{\mathrm{def}}{=} KK^{-1}$ is a compact generating set satisfying $C^{-1} = C$; then $G = \langle C\rangle = \bigcup_{n=1}^{\infty} C^n$.

(iii) A Baire space cannot be the union of a countable set of nowhere dense closed subsets. A topological group containing a compact set with nonempty interior is locally compact.

(iv) By the Baire Category Theorem (see [2], Chapter 9, §5, n° 3, Théorème 1), every Polish space is a Baire space.

(v) is clear from the preceding. □

The following remarks are straightforward from the definitions, from Proposition 2.6, and Theorem 3.11.

Remark 4.2. Let $G \cong V \times H$ be an abelian pro-Lie group for a vector group complement V, and let H be as in Theorem 3.11. Then following statements are equivalent:
(i) G is Polish iff both V and H are Polish.
(ii) G is σ-compact iff V, $\mathrm{comp}(G)$ and $H/\mathrm{comp}(H)$ are σ-compact.
(iii) G is compactly generated iff V and $H/\mathrm{comp}(H)$ are compactly generated and $\mathrm{comp}(G)$ is compact.
(iv) G is separable iff V and H are separable. □

These simple remarks lend some urgency to a more detailed understanding of the situation of weakly complete vector spaces; we shall turn to this topic in the next section.

Remark 4.3. For a discrete abelian group, the following statements are equivalent:
(i) G is finitely generated free.
(ii) G is isomorphic to a closed additive subgroup of $\mathbb{R}^n$ for some natural number n.
(iii) G is isomorphic to a closed additive subgroup of $\mathbb{R}^J$ for some set J.
(iv) G is isomorphic to a closed additive subgroup of a weakly complete vector space.

Proof. For the equivalence of (i) and (ii) see for instance [6], p. 625 Theorem A1.12(i). Trivially (ii) $\Longrightarrow$(iii) $\Longrightarrow$(iv).

Assume (iv), that is, that G is a closed discrete subgroup of a weakly complete vector group W. Since G is discrete, there is an identity neighborhood U_1 of W such that $W \cap U_1 = \{0\}$. Let U be an open identity neighborhood of W such that $U+U+U+U \subseteq U_1$. Since $\lim \mathcal{N}(W) = 0$ there is a $V \in \mathcal{N}(W)$ such that $V \subseteq U$ and thus $U + V \subseteq U + U$. By replacing U by $U + V$ where necessary we assume that $U + V = U$ and $U + U \subseteq U_1$. If $u \in (G - U) \cap U$ then $u = g - u'$ for some $0 \neq g \in G$ and $u' \in U$; thus $g = u + u' \in G \cap U + U \subseteq G \cap U_1 = \{0\}$. Thus V is the complement of $(G \setminus \{0\}) - U$ in $G + V$. Thus $G \cong (G + V)/V$ is a discrete hence closed subgroup of the finite dimensional vector space W/V. □

This result provides an alternative proof of the fact that the prodiscrete free abelian group F of infinite rank in 4.2 cannot be discrete.

The weak topology of a locally convex vector space is that which is induced by the weak-* topology induced by its injection into its double dual. By the duality of vector spaces and weakly complete vector spaces, the weak topology and the weak-* topology agree. In [14] the second author showed that a locally convex topological vector space is complete in its weak topology if an only if every discrete subgroup is finitely generated.

Proposition 4.4. *Let G be an abelian compactfree pro-Lie group.*

(i) *If $\mathcal{N}(G)$ has basis $\mathcal{M}$ of subgroups such that G/M is compactly generated, then there are sets I, J, and K such that G is isomorphic to a closed subgroup of a product $\mathbb{R}^I \times \mathbb{Z}^J$ and hence also of a weakly complete vector space $\mathbb{R}^K$.*

(ii) *If G is compactly generated, then* (i) *applies.*

(iii) *The group G is compactly generated and Polish iff it is isomorphic to $\mathbb{R}^m \times \mathbb{Z}^n$ iff it is locally compact.*

Proof. (i) We assume that for $N \in \mathcal{M}$ we have $G/N = V_N \oplus F_N \oplus \operatorname{tor} G/N$ were V_N is a finite dimensional vector group, F_N is finitely generated free and $\operatorname{tor} G/N$ is the finite torsion group of G/N. It follows that G may be identified with a closed subgroup of $P = \prod_{N\in\mathcal{M}} G/N = V \times F \times C$ where $V \cong \prod_{N\in\mathcal{M}} V_N$ is a weakly complete vector group, $F \cong \prod_{N\in\mathcal{M}} F_N$ and $C \cong \prod_{N\in\mathcal{M}} \operatorname{tor} G/N$. Then $\operatorname{comp}(P) = \{0\} \times \{0\} \times C$ and $G \cap \operatorname{comp}(P) = \{0\}$ since G is compactfree. The projection $P \to V \times F$ is a proper, hence closed morphism, with kernel $\operatorname{comp}(P)$, mapping G onto a closed subgroup of F which is isomorphic to $G/\big(G \cap \operatorname{comp}(P)\big) \cong G$. Since V is a product of copies of $\mathbb{R}$ and F is a product of copies of Z, assertion (i) is proved.

(ii) If G is compactly generated $N \in \mathcal{N}(G)$ then N is a closed subgroup such that G/N is compactly generated, hence is of the form specified in the proof of (i).

(iii) If G is Polish and compactly generated, then it is locally compact by 4.1(i),(iv) and thus, being compactfree, is isomorphic to $\mathbb{R}^m \times \mathbb{Z}^n$. □

Corollary 4.5. *Any compactly generated, compactfree prodiscrete group is isomorphic to a closed subgroup of a group $\mathbb{Z}^J$. If it is not of finite rank, then is not isomorphic to a subgroup of $\mathbb{Z}^{\mathbb{N}}$.* □

This situation is illustrated by the Example in Proposition 1.2.

In the proof of Proposition 2.6 it was only needed that the Lie group quotients of the abelian Lie group G in question were compactly generated. This together with Proposition 4.4 yields at once:

Corollary 4.6. *If G is an abelian pro-Lie group whose Lie group quotients are compactly generated, then* $\operatorname{comp}(G)$ *is compact and* $G/\operatorname{comp}(G)$ *is embeddable in a weakly complete vector group. This applies, in particular, to all compactly generated abelian pro-Lie groups.* □

5. Weakly complete vector spaces

We begin with an observation showing that the idea of *compactly topologically generated* pro-Lie groups may not be very restrictive.

Remark 5.1. A weakly complete vector group is compactly topologically generated. A group of the form $\mathbb{Z}^J$ for any set J is compactly topologically generated.

Proof. For the purposes of the proof we may and will assume that $W = \mathbb{R}^J$ for some set J. For any subset I of J we identify $\mathbb{R}^I$ naturally with a subgroup of $\mathbb{R}^J$. The dual $E \stackrel{\rm def}{=} \widehat{W}$ may and will be identified with $\mathbb{R}^{(J)}$, the set of all $f\colon J \to \mathbb{R}$ with finite support, in such a fashion that $f \in E$ and $g \in W$ gives us $\langle f, g\rangle = \sum_{j\in J} f(j)g(j)$.

Let $K = \{\delta_j \in \mathbb{R}^J : j \in J\} \cup \{0\}$. Let V be a cofinite dimensional vector subspace of W. Then $V^\perp$ is a finite dimensional vector subspaces of the dual $E \stackrel{\rm def}{=} \widehat{W}$. Let $\mathrm{Fin}(J)$ denote the set of finite subsets of J. Since $E = \bigcup_{I\in\mathrm{Fin}(J)} \mathbb{R}^{(I)}$ and since $V^\perp$ is finite dimensional, there is an $I \in \mathrm{Fin}(J)$ such that $V^\perp \subseteq \mathbb{R}^{(I)}$ and thus $V \subseteq \left(\mathbb{R}^{(I)}\right)^\perp = R^{J\setminus I}$. Hence $K \setminus V = \{\delta_i : i \in I\}$ is finite. Therefore K is compact. On the other hand, $W = \overline{\mathbb{R}^{(J)}} = \overline{\langle [0,1]\cdot K\rangle}$ and $[0,1]\cdot K$ is a compact subset of $\mathbb{R}^{(J)}$. Hence W is compactly topologically generated.

Since $\delta_j \in \mathbb{Z}^J \subseteq \mathbb{R}^J$, the assertion on $\mathbb{Z}^J$ follows analogously, as $\mathbb{Z}^J = \overline{\mathbb{Z}^{(J)}} = \overline{\langle K\rangle}$. □

Lemma 5.2. *For a weakly complete topological vector space W, the following statements are equivalent:*
(A) *W is σ-compact.*
(B) *W is locally compact.*
(C) *W is finite dimensional.*
(D) *W is compactly generated.*

Proof. The equivalence of (B) and (C) is common knowledge. Locally compact connected groups are compactly generated by 4.1(i) and so (B) implies (D); and (D) implies (A) by 4.1(ii).

In order to prove that (A) implies (B), let W be a weakly complete σ-compact vector space. Its dual is a vector space E and W is finite dimensional iff E is finite dimensional. Suppose that E is infinite dimensional. Selecting from a basis an infinite countable subset we get a subspace F with a countable basis. Then $W/F^\perp$ is isomorphic to the dual of $F \cong \mathbb{R}^{(\mathbb{N})}$ and therefore $W/F^\perp$ is a homomorphic image of W which is isomorphic to $\mathbb{R}^{\mathbb{N}}$ and therefore is a Polish topological vector space. Since it is also σ-compact as a homomorphic image of a σ-compact group, it is locally compact by 5.2. But then it is finite dimensional, a contradiction. □

Let W be a weakly complete vector space. The dual E of W is a real vector space; let J be a basis of E. Every linear functional of E is given by a function $J \to \mathbb{R}$ and thus, by the Duality Theorem [8] 6.7, $W \cong \mathbb{R}^J$. The cardinal $\mathrm{card}\,J$ is called the *topological dimension* of W. (See [7].)

Lemma 5.3. *For a weakly complete topological vector space W, the following statements are equivalent:*
(i) $W \cong \mathbb{R}^J$ *with* $\mathrm{card}\,J \le \aleph_0$.
(ii) *W is locally compact or is isomorphic to $\mathbb{R}^{\mathbb{N}}$.*
(iii) *W is finite dimensional or is isomorphic to $\mathbb{R}^{\mathbb{N}}$.*

(iv) *W is second countable.*
(v) *W is first countable.*
(vi) *W is Polish.*

Proof. By the remarks preceding the lemma, for each cardinal $\aleph$, there is, up to isomorphism of topological vector spaces and of topological groups one and only one weakly complete vector space, namely, $\mathbb{R}^\aleph$. Conditions (i), (ii), (iii) are ostensibly all equivalent to saying that $\aleph$ is countable. The weight $w(W)$, that is the smallest cardinal representing the cardinality of a basis for the topology of $W \cong \mathbb{R}^\aleph$ is $\aleph_0$ if $\aleph$ is countable, and is $\aleph$ if $\aleph$ is infinite (see e.g. [6], pp. 763, 764, Exercise EA4.3), so (iv) is likewise equivalent to (ii), and implies (v). If the weakly complete vector space W is first countable, then the filter basis $\mathcal{I}(W)$ of cofinite dimensional closed vector subspaces has a countable basis, and thus $W \cong \lim_{V\in\mathcal{I}(W)} W/V$ is a closed vector subspace of $\prod_{V\in\mathcal{I}(W)} W/V \cong \mathbb{R}^\mathbb{N}$ and thus (vi) implies (iv). If (iv) is satisfied then the complete topological vector space W is metrizable (see e.g. [6], p. 772, Theorem A4.16) and thus (vi) follows; trivially (vi) implies (iv). □

Lemma 5.4. *For a weakly complete topological vector space W, the following statements are equivalent:*
(a) *W is separable.*
(b) *W contains a dense vector subspace of countable linear dimensions over $\mathbb{R}$.*
(c) *W is isomorphic as a topological vector space to $\mathbb{R}^J$ with* $\operatorname{card} J \le 2^{\aleph_0}$.
They are implied by the equivalent statements of Lemma 5.3.

Proof. A second countable space is always separable: It suffices to pick a point in every set of a countable basis for the topology: this yields a countable dense set. What remains therefore is to see the equivalence of (a), (b), and (c). We may safely assume that W is infinite dimensional, since the finite dimensional case is clear.

(a)$\Longrightarrow$(b): Let C be a countable dense subset of $\mathbb{R}^J$. Then the real linear span of C is dense vector subspace of $\mathbb{R}^J$ whose linear dimension is countable.

(b)$\Longrightarrow$(c): Assume that $\iota: \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}^J$ is a linear map between vector spaces such that and $\overline{\operatorname{im}(\iota)} = \mathbb{R}^J$. We give $\mathbb{R}^{(\mathbb{N})}$ the finest locally convex topology. The vector space dual of $\mathbb{R}^{(\mathbb{N})}$ may be identified with $\mathbb{R}^\mathbb{N}$, and that of $\mathbb{R}^J$ with $\mathbb{R}^{(J)}$. The morphism ι is both an epic (and a monic) in the category of (Hausdorff) topological vector spaces. Its adjoint morphism $\iota': \mathbb{R}^{(J)} \to \mathbb{R}^\mathbb{N}$ is a monic (and epic) and is therefore an injection (with dense image). Thus $\operatorname{card}(J) \le \dim_\mathbb{R} \mathbb{R}^\mathbb{N} = 2^{\aleph_0}$.

(c)$\Longrightarrow$(a): Let $W = \mathbb{R}^J$ with $\operatorname{card}(J) = 2^{\aleph_0}$. We shall show that W is separable; since $\mathbb{R}^I$ with $\operatorname{card}(I) \le \operatorname{card}(J)$ is a homomorphic image of $\mathbb{R}^J$, this will imply the implication. The topological vector space dual of $\mathbb{R}^J$ may be identified with $\mathbb{R}^{(J)}$ and then there is a linear bijection $\beta: \mathbb{R}^{(J)} \to \mathbb{R}^\mathbb{N}$. If we give $\mathbb{R}^{(J)}$ the finest locally convex topology and $\mathbb{R}^\mathbb{N}$ the product topology, then β is an epic (and a monic) in the category of topological vector spaces and thus its adjoint $\beta': \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}^J$ has a dense image (and is injective). Even in the finest locally convex topology, $\mathbb{Q}^{(\mathbb{N})}$ is dense in $\mathbb{R}^{(\mathbb{N})}$, and $\mathbb{Q}^{(\mathbb{N})}$ is countable. Hence $\mathbb{R}^J$ is separable as asserted. □

For the following theorem we recall the definition of the characteristic closed subgroup $G_1 = G_0 \operatorname{comp}(G) \cong V \times \operatorname{comp}(G)$ of an abelian pro-Lie group in Theorem 3.11(v) for a vector group complement V (see Definition 3.4(ii)).

Theorem 5.5. (The Compact Generation Theorem for Abelian Pro-Lie Groups) (i) *For a compactly generated abelian pro-Lie group G the characteristic closed subgroup* $\operatorname{comp}(G)$ *is compact and the characteristic closed subgroup G_1 is locally compact.*

(ii) *In particular, every vector group complement V is isomorphic to a euclidean group $\mathbb{R}^m$.*

(iii) *The factor group G/G_1 is a compactly generated prodiscrete group without compact subgroups. If G/G_1 is Polish, then G is locally compact and*

$$G \cong \mathbb{R}^m \times \operatorname{comp}(G) \times \mathbb{Z}^n.$$

Proof. By Theorem 3.11, $G \cong V \times H$ such that H_0 is compact. The factors V and H are compactly generated as homomorphic images of G. By Lemma 5.2, $V \cong \mathbb{R}^n$ for some nonnegative integer n. By Proposition 2.6, $\operatorname{comp}(H)$ is compact and by 3.11, $\operatorname{comp}(G) = \operatorname{comp}(H)$ and $H_0 \subseteq \operatorname{comp}(H)$. Thus $G_1 \cong V \times \operatorname{comp}(G)$ is locally compact. Also, $H/\operatorname{comp}(H)$ is totally disconnected, and by Theorem 4.28(iii), or by [16], p. 206, Theorem 11.18 and p. 242, Lemma 13.13, this quotient is a pro-Lie group and hence is pro-discrete by [8], Proposition 4.23 or by [9], Proposition 4.5. If the factor group G/G_1 is Polish, then it is finitely generated free by 4.4(iii), and the remainder follows. □

It is not known whether an abelian prodiscrete compactfree group is finitely generated free.

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