Projective completions of Jordan pairs
Part II. Manifold structures and symmetric spaces

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Abstract. We define *symmetric spaces* in arbitrary dimension and over arbitrary non-discrete topological fields $\mathbb{K}$, and we construct manifolds and symmetric spaces associated to topological *continuous quasi-inverse Jordan pairs* and *-triple systems*. This class of spaces, called *smooth generalized projective geometries*, generalizes the well-known (finite or infinite-dimensional) bounded symmetric domains as well as their "compact-like" duals. An interpretation of such geometries as models of Quantum Mechanics is proposed, and particular attention is paid to geometries that might be considered as "standard models" – they are associated to *associative continuous inverse algebras* and to *Jordan algebras of hermitian elements* in such an algebra.

Contents.

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Introduction

In finite dimensions, the theory of *Lie groups* is closely related to the theory of *symmetric spaces*. In infinite dimensions, the theory of Lie groups is by now developed in great generality, whereas for symmetric spaces there is not even a commonly accepted definition. Nevertheless, there is an interesting class of spaces, called *(infinite-dimensional) bounded symmetric domains*, for which one can develop a nice structure theory and which, without doubt, are honest symmetric spaces. Remarkably enough, the framework of their theory (developed by W. Kaup and H. Upmeier, cf. the monograph [Up85] and the literature given there) is not so much Lie but rather *Jordan* theoretic. Recently, also their "compact-like" dual symmetric spaces (the analog of the compact dual of a non-compact symmetric space in finite dimension) have attracted attention, the most important examples being infinite-dimensional Graßmannians of many kinds (cf. [PS86], [DNS89], [DNS90], [KA01], [MM01], [IM02]). These compact-like infinite-dimensional manifolds can be seen as a "projective completion" of the underlying Jordan triple system, in a similar way as an ordinary projective space $\mathbb{RP}^n$ can be seen as the projective completion of the affine space $\mathbb{R}^n$.

In the present work, which is the second part in a series of two papers started by [BN03], we will give a far-reaching generalization of the above mentioned theories. We will not only free the real theory from the Banach space set-up present in [Up85], but develop the theory in the context of any Hausdorff topological vector space as model space, over any non-discrete topological field. In fact, we even work over any topological ring having dense unit group. Compared with the approach from [Up85], our approach is more algebraic and less analytic, which makes it considerably simpler and more elementary. The algebraic results from Part I of this work ([BN03]) which we need are summarized in Chapter 4, and the basic notions of differential calculus and manifolds over general topological fields and rings from [BGN03] are recalled in Chapter 1. The reader who is only interested in the real or complex theory may everywhere replace $\mathbb{K}$ by $\mathbb{R}$ or $\mathbb{C}$, and he will see that all notions from calculus we use are the ones which he is used to.

We now give a more detailed description of the contents. In Chapter 2 the basic theory of symmetric spaces, in arbitrary dimension and over general base fields or rings (in which 2 is invertible), is developed. For several reasons, we believe that the correct starting point for the general theory is the approach to symmetric spaces by O. Loos ([Lo69]) – the main idea being to incorporate all symmetries σ_x with respects to points x in the symmetric space M into a smooth binary "multiplication map" $m : M \times M \to M$, $(x,y) \mapsto \sigma_x(y)$ which is non-associative, but has other nice algebraic properties. The analogy with the theory of Lie groups then becomes very close, and we get a good analog of the functor assigning to a Lie group its Lie algebra (Theorem 2.10). For further results on the differential geometry of symmetric spaces (including the *canonical connection* and its curvature) we refer to [Be03b]. One should *not* think of symmetric spaces as homogeneous spaces G/H – homogeneity is a rather special phenomenon, and the same holds for the existence of a locally diffeomorphic *exponential map* which cannot be guaranteed in general (see examples and discussion of exponential maps in Remarks 2.11, 3.5, 6.5).

In Chapter 3 we construct a class of symmetric spaces related to *continuous inverse Jordan algebras*; by definition, these are topological Jordan algebras over $\mathbb{K}$ having an open set of invertible elements and for which the *Jordan inverse map* is continuous. Once more, we closely follow the presentation from [Lo69] (cf. loc. cit. Section II.1.2.5); however, our general framework permits to treat completely new examples such as the *space of non-degenerate quadratic forms on* $\mathbb{K}^n$ which, for fields such as $\mathbb{K} = \mathbb{Q}$, is the prime example of a non-homogeneous symmetric space. For the case of Banach–Jordan algebras the symmetric space structure of the set of units has been studied by O. Loos in [Lo96].

Having recalled in Chapter 4 the algebraic construction and main properties of "generalized projective geometries" associated to 3*-graded Lie algebras* (which are the Lie theoretic counterpart of *Jordan pairs*), we are ready to state and to prove our first main result (Theorem 5.3): the generalized projective geometry is actually a smooth manifold (on which the so-called *projective group* acts by diffeomorphisms) if some natural conditions on the Jordan pair are fulfilled. Namely, the Jordan pair (V^+, V^-) shall be a topological Jordan pair over $\mathbb{K}$, the *set* $(V^+ \times V^-)^\times$ *of quasi-invertible pairs* shall be open in $V^+ \times V^-$, and the *Bergman-inverse mapping* $(V^+ \times V^-)^\times \times V^+ \times V^- \to V^+ \times V^-$ shall be continuous; then we say that (V^+, V^-) is a *continuous quasi-inverse Jordan pair* (Section 5.1). If this is the case, a "generalized quotient rule" (Section 1.7) permits to conclude that the quasi-inverse mapping actually is smooth (Proposition 5.2), which is a major step in the proof of Theorem 5.3. Our continuous quasi-inverse condition on the Jordan pair is not only sufficient, but also necessary for the associated generalized projective geometry to be a smooth manifold; thus Theorem 5.3 is the most general result that one might expect in this context. Of course, it contains the previously mentioned results in the Banach situation as special cases.

In Chapter 6, we return to symmetric spaces: a symmetric space structure on a generalized projective geometry (X^+, X^-) depends on an additional structure, namely on a fixed bijection

$X^+ \to X^-$ which is a *polarity* – in fact, this is familiar already from the classical projective spaces $X^+ = \mathbb{RP}^n$ or $X^+ = \mathbb{CP}^n$: they are turned into symmetric spaces only after the choice of a scalar product which distinguishes an identification of X^+ with the dual projective space X^- and thus determines isometry subgroups $\mathbb{P}\,\mathrm{O}_{n+1}$, resp. $\mathbb{P}\,\mathrm{U}_{n+1}$, of the projective group $\mathbb{P}\,\mathrm{GL}_{n+1}(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We prove that, under the general assumptions of Theorem 5.3, a *continuous* polarity $p : X^+ \to X^-$ is automatically smooth and gives rise to a symmetric space structure on the open set $M^{(p)}$ of non-isotropic points in X^+ (Theorem 6.2 (i)). We also calculate the associated Lie triple system (i.e., the curvature of the canonical connection; cf. [Be03b]): it is given by anti-symmetrising the corresponding Jordan triple product (Theorem 6.2 (ii)). This generalizes the *geometric Jordan-Lie functor* which has been defined in [Be00] for the finite-dimensional real case.

In Chapters 7, 8 and 9, we give applications and examples of the preceding results and explain some links with the (abundant) related work in mathematics and physics. On the one hand, Jordan algebras have been introduced by P. Jordan (cf. [JNW34]) in an attempt to lay algebraic foundations of quantum mechanics. On the other hand, research on the foundations of quantum mechanics lead by quite different arguments to the conclusion that "... quantum mechanical systems are those whose logics form some sort of projective geometries" ([Va85, p. 6]). In the hope to bring these two lines of thought together, the concept of "generalized projective geometry" has been introduced by the first named author in [Be02]. More recently, concepts of *delinearization of quantum mechanics* have been proposed in the context of (Banach) hermitian symmetric spaces, see [CGM03], where this program is motivated in the following way: "The true aim of the delinearization program is to free the mathematical foundations of quantum mechanics from any reference to linear structure and to linear operators. It appears very gratifying to be aware of how naturally geometric concepts describe the more relevant aspects of ordinary quantum mechanics, suggesting that the geometric approach could be very useful also in solving open problems in Quantum Theories." The close relation of the delinearization approach via hermitian symmetric spaces to Jordan theory has not been noticed in [CGM03] nor in the closely related paper [AS97]. In Chapter 9 we propose a "dictionary" between the language of generalized projective geometries (which is equivalent to the language of Jordan theory) and the language of quantum mechanics. We do not claim anything about the applicability of this dictionary to the "physical world"; all that we aim at is to propose a terminology that makes evident the structural analogy between quantum mechanics and the theory of generalized projective geometries.

Chapters 7 and 8 are devoted to what one might call "standard models of quantum mechanics" – these are the geometries corresponding to associative continuous inverse algebras, resp. to their Jordan sub-algebras of hermitian elements. These are (in general) infinite-dimensional geometries which, however, geometrically behave very much like a projective line (over a non-commutative base ring). A special feature of these geometries is that some of their associated symmetric spaces are "of group type", i.e. they are Lie groups, considered as symmetric spaces: all *orthogonal* and *unitary* groups associated to involutive continuous inverse algebras can be realized in this way.

In the final Chapter 10 we mention some further topics and open problems related to this work.

Notation. Throughout this paper, $\mathbb{K}$ denotes a commutative topological ring with unit 1 (i.e. $\mathbb{K}$ carries a topology such that the ring operations are continuous, the group $\mathbb{K}^\times$ of invertible elements is open and inversion $i: \mathbb{K}^\times \to \mathbb{K}$ is continuous) such that the group of units $\mathbb{K}^\times$ is dense in $\mathbb{K}$. We assume that 2 is invertible in $\mathbb{K}$. In particular, $\mathbb{K}$ may be any non-discrete topological field of characteristic different from 2 such as $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{C}_p$, ${}^*\mathbb{R}, \ldots$

If $\mathbb{K}$ is a topological ring, all $\mathbb{K}$-modules V are assumed to be topological modules, i.e. they carry a topology such that the structure maps $V \times V \to V$ and $\mathbb{K} \times V \to V$ are continuous. Moreover, we assume that all topological $\mathbb{K}$-modules are Hausdorff. The class of continuous mappings is denoted by C^0.

1. Calculus and manifolds

1.1. Differentiability in locally convex spaces. In order to motivate our general concept of differentiability, we recall the definition of differentiable mappings on locally convex spaces (cf. [Gl01a], [Ke74], [Ha82]): suppose E, F are real locally convex spaces (not necessarily complete), $U \subset E$ open and $f : U \to F$ continuous. Then f is called *of class* C^1 if, for all $x \in U$ and $h \in E$, the directional derivative

$$df(x;h) := \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

exists and $df : U \times E \to F$ is continuous. Inductively, one defines f to be of class C^{k+1} if df is of class C^k (cf. [Gl01a, Lemma 1.14] for this definition), and we denote by C^0 the class of continuous maps. For our purposes, the following equivalent characterization of the class C^1 will be useful:

Proposition 1.2. *The map $f : U \to F$ is of class C^1 if and only if there exists a map*

$$f^{[1]} : U \times E \times \mathbb{R} \supset U^{[1]} := \{(x,h,t) : \, x + th \in U\} \to F$$

of class C^0 such that for all $(x,h,t) \in U^{[1]}$,

$$f(x+th) - f(x) = t \cdot f^{[1]}(x,h,t).$$

Proof. Given $f^{[1]}$ as in the proposition, we get $df(x;h) = f^{[1]}(x,h,0)$, and df will be of class C^0 since so is $f^{[1]}$. Conversely, assume that f is C^1 and define $f^{[1]}$ by

$$f^{[1]}(x,h,t) := \begin{cases} \frac{f(x+th)-f(x)}{t}, & t \in \mathbb{R}^\times \\ df(x)h, & t = 0. \end{cases}$$

Then $f^{[1]}$ is of class C^0: this is seen by using, locally, the integral representation

$$f^{[1]}(x,h,t) = \int_0^1 df(x + sth)h \, ds$$

(Fundamental Theorem of Calculus, cf. [Gl01a, Th. 15]; note that no completeness assumption is necessary here: a priori, the integral from the right-hand side has to be taken in the completion of F, but as it actually equals $f^{[1]}(x,h,t)$, it belongs to F itself.) Now the continuity of $f^{[1]}$ follows by standard estimates (cf. [BGN03, Prop. 7.4] for the details). ■

1.3. General definition of the class C^1 over topological fields and rings. Now let $\mathbb{K}$ be a general topological ring having dense group of units $\mathbb{K}^\times$, let V, W be Hausdorff topological $\mathbb{K}$-modules and $U \subset V$ open. We say that a map $f : V \supset U \to W$ is $C^1(U,W)$ or just *of class* C^1 if there exists a C^0-map

$$f^{[1]} : U \times V \times \mathbb{K} \supset f^{[1]} := \{(x,v,t) | \, x \in U, x + tv \in U\} \to W,$$

such that

$$f(x+tv) - f(x) = t \cdot f^{[1]}(x,v,t)$$

whenever $(x,v,t) \in U^{[1]}$. The *differential of f at x* is defined by

$$df(x) : V \to W, \quad v \mapsto df(x)v := f^{[1]}(x,v,0).$$

By density of $\mathbb{K}^\times$ in $\mathbb{K}$, the map $f^{[1]}$ is uniquely determined by f and hence $df(x)$ is well-defined.

1.4. Definition of the classes C^k and C^∞. Let $f : V \supset U \to F$ be of class C^1. We say that f is $C^2(U,F)$ or *of class* C^2 if $f^{[1]}$ is C^1, in which case we define $f^{[2]} := (f^{[1]})^{[1]} : U^{[2]} \to F$, where $U^{[2]} := (U^{[1]})^{[1]}$. Inductively, we say that f is $C^{k+1}(U,F)$ or *of class* C^{k+1} if f is of class C^k and $f^{[k]} : U^{[k]} \to F$ is of class C^1, in which case we define $f^{[k+1]} := (f^{[k]})^{[1]} : U^{[k+1]} \to F$ with $U^{[k+1]} := (U^{[k]})^{[1]}$. The map f is called *smooth* or *of class* C^∞ if it is of class C^k for each $k \in \mathbb{N}_0$. – Note that $U^{[k+1]} = (U^{[1]})^{[k]}$ for each $k \in \mathbb{N}_0$, and that f is of class C^{k+1} if and only if f is of class C^1 and $f^{[1]}$ is of class C^k; in this case, $f^{[k+1]} = (f^{[1]})^{[k]}$.

1.5. Differentiation rules. We assume that $f : U \to W$ is of class C^k. Its differential is the C^0-map

$$df : U \times V \to W, \quad (x,v) \mapsto df(x)v = f^{[1]}(x,v,0);$$

the *directional derivative in direction* v is

$$\partial_v f : U \to W, \quad x \mapsto \partial_v f(x) := df(x)v.$$

We define also

$$Tf : U \times V \to W \times W, \quad (x,v) \mapsto (f(x), df(x)v).$$

Then the following holds (cf. [BGN03]):

(1) For all $x \in U$, $df(x) : V \to W$ is a $\mathbb{K}$-linear C^0-map.

(2) If f and g are composable and of class C^k, then $g \circ f$ is of class C^k, and $T(g \circ f) = Tg \circ Tf$.

(3) Multilinear maps of class C^0 are C^k and are differentiated as usual. In particular, if $f, g : U \to \mathbb{K}$ are C^1, then the product $f \cdot g$ is C^1, and $\partial_v(fg) = (\partial_v f)g + f\partial_v g$. Polynomial maps $\mathbb{K}^n \to \mathbb{K}^m$ are always C^∞ and are differentiated as usual.

(4) Inversion $i : \mathbb{K}^\times \to \mathbb{K}$ is C^∞, and $(di)(x)v = -x^{-2}v$. It follows that rational maps $\mathbb{K}^n \supset U \to \mathbb{K}^m$ are always C^∞ and are differentiated as usual.

(5) The cartesian product of two C^k-maps is C^k.

(6) If $f : V_1 \times V_2 \supset U \to W$ is C^1, and for $(x_1,x_2) \in U$ we let

$$l_{x_1}(x_2) := r_{x_2}(x_1) := f(x_1,x_2),$$

then the *rule on partial derivatives* holds:

$$df(x_1,x_2)(v_1,v_2) = d(l_{x_1})(x_2)v_1 + d(r_{x_2})(x_1)v_2.$$

(7) ("Schwarz' Lemma") If f is of class C^2, then for all $x \in U$, $v, w \in V$,

$$\partial_v \partial_w f(x) = \partial_w \partial_v f(x).$$

Hence, if f is of class C^k and $x \in U$, then the map

$$d^k f(x) : V^k \to W, \quad (v_1, \ldots, v_k) \mapsto \partial_{v_1} \ldots \partial_{v_k} f(x)$$

is a symmetric multilinear C^0-map.

(8) There are several versions of Taylor's formula (see [BGN03]), but none of them will be used in this work.

1.6. Continuous inverse algebras. We will need various generalizations of the quotient rule (4). An associative $\mathbb{K}$-algebra A with unit $\mathbf{1}$ is called a *continuous inverse algebra* (c.i.a.) if the product $A \times A \to A$ is continuous, the unit group $A^\times$ is open in A and inversion $i : A^\times \to A$ is continuous. Writing

$$i(x+th) - i(x) = -x^{-1}(th)(x+th)^{-1} = t(-x^{-1}h(x+th)^{-1}),$$

we see that i actually is C^1 and $i^{[1]}(x,h,t) = -x^{-1}h(x+th)^{-1}$, whence $di(x)h = -x^{-1}hx^{-1}$. Iterating this argument, we see that i is C^∞.

1.7. The generalized quotient rule. For the second generalization of the quotient rule, assume $f : E \supset U \to \mathrm{End}(F)$ takes, on the open set $U \subset E$, values in the group $\mathrm{GL}(F)$ of (continuous) invertible linear self-maps of E. We do not want to fix a topology on $\mathrm{End}(F)$, and hence it makes no sense to assume f or the inversion map $j : \mathrm{GL}(F) \to \mathrm{GL}(F)$ to be continuous or differentiable. Instead, we assume that $\widetilde{f} : U \times F \to F$, $(x,v) \mapsto f(x)v$ is of class C^k and that

$$\widetilde{jf} : U \times F \to F, \quad (x,v) \mapsto f(x)^{-1}v$$

is of class C^0. We claim that then $\widetilde{jf}$ also is of class C^k. Indeed, for $k=1$ we have:

$$\begin{aligned}
&\widetilde{jf}((x,v)+s(h_1,h_2)) - \widetilde{jf}(x,v) \\
&= \widetilde{jf}((x,v)+s(h_1,h_2)) - \widetilde{jf}((x,v)+s(h_1,0)) + \widetilde{jf}((x,v)+s(h_1,0)) - \widetilde{jf}(x,v) \\
&= f(x+sh_1)^{-1}(v+sh_2) - f(x+sh_1)^{-1}v + f(x+sh_1)^{-1}v - f(x)^{-1}v \\
&= sf(x+sh_1)^{-1}h_2 + (f(x+sh_1)^{-1} - f(x)^{-1})v \\
&= sf(x+sh_1)^{-1}h_2 + f(x)^{-1}(f(x) - f(x+sh_1))f(x+sh_1)^{-1}v \\
&= sf(x+sh_1)^{-1}h_2 + f(x)^{-1}(\widetilde{f}(x, f(x+sh_1)^{-1}v) - \widetilde{f}(x+sh_1, f(x+sh_1)^{-1}v)) \\
&= sf(x+sh_1)^{-1}h_2 + sf(x)^{-1}(\widetilde{f})^{[1]}((x, f(x+sh_1)^{-1}v),(h_1,0),s)
\end{aligned}$$

which is the same as the product of s with

$$\begin{aligned}
(\widetilde{jf})^{[1]}((x,v),(h_1,h_2),s) &= f(x+sh_1)^{-1}h_2 + f(x)^{-1}(\widetilde{f})^{[1]}((x,f(x+sh_1)^{-1}v),(h_1,0),s) \\
&= \widetilde{jf}(x+sh_1,h_2) + \widetilde{jf}(x,(\widetilde{f})^{[1]}((x,\widetilde{jf}(x+sh_1,v)),(h_1,0),s)),
\end{aligned} \tag{1.1}$$

which, according to our assumptions, is a C^0-map. It follows that $\widetilde{jf}$ is C^1, and letting $s=0$, we get

$$d(\widetilde{jf})(x,v)(h_1,h_2) = f(x)^{-1}h_2 - f(x)^{-1}d\widetilde{f}(x,f(x)^{-1}v)(h_1,0).$$

Moreover, using Equation (1.1) together with the chain rule, we can iterate this argument, and it follows that $\widetilde{jf}$ is C^k if so is $\widetilde{f}$.

1.8. Manifolds. A *C^k-manifold with atlas (modeled on the topological $\mathbb{K}$-module E)* (where $k \in \mathbb{N}_0 \cup \{\infty\}$) is a topological space M together with an *E-atlas* $\mathcal{A} = \{(\varphi_i, U_i) \colon i \in I\}$. This means that U_i, $i \in I$, is a covering of M by open sets, and $\varphi_i \colon M \supset U_i \to \varphi_i(U_i) \subset E$ is a *chart*, i.e. a homeomorphism of the open set $U_i \subset M$ onto an open set $\varphi_i(U_i) \subset E$, and any two charts $(\varphi_i,U_i),(\varphi_j,U_j)$ are *C^k-compatible* in the sense that

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}|_{\varphi_j(U_i \cap U_j)} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

and its inverse φ_{ji} are of class C^k.

If the atlas $\mathcal{A}$ is *maximal* in the sense that it contains all compatible charts, then M is called a *C^k-manifold (modeled on E)*.

Smooth maps between manifolds (with or without atlas) are now defined as usual, and it is seen that C^k-manifolds (with or without atlas) form a category.

1.9. The tangent functor. Set-theoretically, M can be seen as the quotient of the following equivalence relation $S/\sim$, where

$$S := \{(i,x) | \, x \in \varphi_i(U_i)\} \subset I \times E,$$

and $(i,x) \sim (j,y)$ if $\varphi_i^{-1}(x) = \varphi_j^{-1}(y)$. We write $p = [i,x] \in M = S/\sim$. Then the tangent bundle is defined to be the quotient of the equivalence relation on the set

$$TS := S \times E \subset I \times E \times E$$

given by:

$$(i,x,v) \sim (j,y,w) \quad :\Longleftrightarrow \quad \varphi_j \circ \varphi_i^{-1}(x) = y, \quad d(\varphi_j \circ \varphi_i^{-1})(x)v = w.$$

All usual properties of the tangent bundle are now easily proved (cf. [BGN03]); in particular, there is a natural manifold structure (with atlas $T\mathcal{A}$) on TM such that the natural projection $\pi : TM \to M$ is smooth; the *tangent space* T_pM is defined to be the fiber $\pi^{-1}(p)$. If $f: M \to N$ is C^k, there is a well-defined *tangent map* $Tf : TM \to TN$, and we have the usual functorial properties (including compatibility with direct products: $T(M \times N) \cong TM \times TN$); thus T will be called the *tangent functor*.

1.10. The Lie bracket. Smooth sections of TM are called *vector fields*. There is a Lie bracket on the $\mathbb{K}$-module $\mathfrak{X}(M)$ of vector fields on M, given in a chart by

$$[X,Y](x) = dY(x)X(x) - dX(x)Y(x) \tag{1.2}$$

([BGN03, Th. 8.4]; note that the sign is a matter of convention). The Lie bracket is *natural* in the sense that, if (X,X') and (Y,Y') are φ-related under some smooth map φ, then so is $([X,Y],[X',Y'])$ ([BGN03, Lemma 8.5]). See [Be03b] for a conceptual definition of the Lie bracket and for a systematic exposition of differential geometry (especially, the theory of connections) in this framework.

2. Lie groups and symmetric spaces

2.1. Manifolds with multiplication. A *product* or *multiplication map* on a manifold M is a smooth binary map $m : M \times M \to M$, and *homomorphisms of manifolds with multiplication* are smooth maps that are compatible with the respective multiplication maps. *Left and right multiplication operators*, defined by $l_x(y) = m(x,y) = r_y(x)$, are partial maps of m and hence smooth self maps of M. Applying the tangent functor to this situation, we see that (TM, Tm) is again a manifold with multiplication, and tangent maps of homomorphisms are homomorphisms of the respective tangent spaces. The tangent map Tm is given by the formula

$$T_{(x,y)}m(\delta_x,\delta_y) = T_{(x,y)}m((\delta_x,0_y)+(0_x,\delta_y)) = T_x(r_y)\delta_x + T_y(l_x)\delta_y. \tag{2.1}$$

Formula (2.1) is nothing but the rule on partial derivatives (1.5.(6)) written in the language of manifolds. In particular, (2.1) shows that the canonical projection and the zero section,

$$\pi : TM \to M, \quad \delta_p \to p, \qquad z : M \to TM, \quad p \mapsto 0_p \tag{2.2}$$

are homomorphisms of manifolds with multiplication. We will always identify M with the subspace $z(M)$ of TM. Then (2.1) implies that the operator of left multiplication by $p = 0_p$ in TM is nothing but $T(l_p) : TM \to TM$, and similarly for right multiplications.

2.2. Lie groups. A *Lie group over* $\mathbb{K}$ is a smooth $\mathbb{K}$-manifold G carrying a group structure such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ are smooth. Homomorphisms of Lie groups are smooth group homomorphisms. Clearly, Lie groups and their homomorphisms form a category in which direct products exist.

Applying the tangent functor to the defining identities of the group structure (G,m,i,e), it is immediately seen that then $(TG, Tm, Ti, 0_{T_eG})$ is again a Lie group such that $\pi : TG \to G$ becomes a homomorphism of Lie groups and such that the zero section $z : G \to TG$ also is a homomorphism of Lie groups.

2.3. The Lie algebra of a Lie group. A vector field $X \in \mathfrak{X}(G)$ is called *left invariant* if, for all $g \in G$, $X \circ l_g = Tl_g \circ X$. In particular, $X(g) = X(l_g(e)) = T_e l_g X(e)$; thus X is uniquely determined by the value $X(e)$, and thus the map

$$\mathfrak{X}(G)^{l_G} \to T_eG, \quad X \mapsto X(e) \tag{2.3}$$

from the space of left invariant vector fields into T_eG is injective. It is also surjective: if $v \in T_eG$, then right multiplication with v in TG, $Tr_v : TG \to TG$ preserves fibers and hence defines a vector field

$$v^l : G \to TG, \quad g \mapsto T_g r_v(0_g) = Tm(g,v) = T_e l_g(v)$$

which is left invariant since right multiplications commute with left multiplications. Now, the space $\mathfrak{X}(G)^{l_G}$ is a Lie subalgebra of $\mathfrak{X}(M)$; this follows immediately from the naturality of the Lie bracket because X is left invariant if and only if the pair (X, X) is l_g-related for all $g \in G$. The space $\mathfrak{g} := T_eG$ with the Lie bracket defined by $[v,w] := [v^l, w^l]_e$ is called *the Lie algebra of* G.

Theorem 2.4.

(i) *The Lie bracket* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is* C^0.

(ii) *For every homomorphism* $f : G \to H$, *the tangent map* $\dot{f} := T_e f : \mathfrak{g} \to \mathfrak{h}$ *is a homomorphism of Lie algebras.*

Proof. (i) Pick a chart $\varphi : U \to V$ of G such that $\varphi(e) = 0$. Since $w^l(x) = Tm(x,w)$ depends smoothly on (x,w), it is represented in the chart by a smooth map (which again will be denoted by $w^l(x)$). But this implies that $[v^l, w^l](x) = d(w^l)(x)v^l(x) - d(v^l)(x)w^l(x)$ depends smoothly on v, w and x and hence $[v,w]$ depends smoothly on v, w.

(ii) First one has to check that the pair of vector fields $(v^l, (\dot{\varphi} v)^l)$ is f-related, and then the naturality of the Lie bracket implies that $\dot{f}[v,w] = [\dot{f}v, \dot{f}w]$. ■

The functor from Lie groups over $\mathbb{K}$ into C^0-Lie algebras over $\mathbb{K}$ will be called the *Lie functor over* $\mathbb{K}$.

2.5. Symmetric spaces. A *symmetric space over* $\mathbb{K}$ is a smooth manifold with a multiplication map $m : M \times M \to M$ such that, for all $x, y, z \in M$, writing also σ_x for the left multiplication l_x,

(M1) $m(x,x) = x$,

(M2) $m(x, m(x,y)) = y$, i.e. $\sigma_x^2 = \mathrm{id}_M$,

(M3) $m(x, m(y,z)) = m(m(x,y), m(x,z))$, i.e. $\sigma_x \in \mathrm{Aut}(M,m)$,

(M4) $T_x(\sigma_x) = -\mathrm{id}_{T_xM}$.

Homomorphisms of symmetric spaces are the corresponding homomorphisms of manifolds with multiplication. The left multiplication operator σ_x is, by (M1)–(M3), an automorphism of order two fixing x; it is called the *symmetry around* x. Since 2 is invertible in $\mathbb{K}$, Property (M4) says that, "infinitesimally", x is an isolated fixed point of the symmetry σ_x. If we have an implicit function theorem at our disposition, then this holds also locally (see [Ne02, Lemma 3.2] for the Banach case). In particular, in the finite-dimensional case over $\mathbb{K} = \mathbb{R}$, our definition is equivalent to the one by O. Loos in [Lo69].

Remark. It would be interesting to know whether there are real infinite-dimensional symmetric spaces for which x is not isolated in the set of fixed points of the symmetry σ_x. If there were a (infinite-dimensional real) Lie group G for which the unit element is not isolated in the space of elements of order 2, then we could take $M = G$ with $m(g,h) = gh^{-1}g$. ■

The group $G(M)$ generated by all products $\sigma_x\sigma_y$, $x, y \in M$, is a (normal) subgroup of $\mathrm{Aut}(M,m)$, called the *group of displacements.* A distinguished point $o \in M$ is called a *base point.* With respect to a base point, one defines the *quadratic representation*

$$Q : M \to G(M), \quad x \mapsto \sigma_x\sigma_o. \tag{2.4}$$

Proposition 2.6. *The tangent bundle* (TM, Tm) *of a symmetric space is again a symmetric space.*

Proof. We express the identities (M1)–(M3) by commutative diagrams to which we apply the tangent functor T. Since T commutes with direct products, we get the same diagrams and hence the laws (M1)–(M3) for Tm (cf. [Lo69, II.2] for the explicit form of the diagrams).

Next we prove (M4): first of all, note that the fibers of $\pi : TM \to M$ (i.e. the tangent spaces) are stable under Tm because π is a homomorphism. We claim that for $v,w \in T_pM$ the explicit formula $Tm(v,w) = 2v - w$ holds (i.e. the structure induced on tangent spaces is the canonical "flat" symmetric structure of an affine space). In fact, from (M3) for Tm we get $v = Tm(v,v) = T_p(\sigma_p)v + T_p(r_p)v = -v + T_p(r_p)v$, whence $T_p(r_p)v = 2v$ and

$$Tm(v,w) = T_p(\sigma_p)w + T_p(r_p)v = 2v - w.$$

Now fix $p \in M$ and $v \in T_pM$. We choose 0_p as base point in TM. Then $Q(v) = \sigma_v\sigma_{0_p}$ is, by (M3), an automorphism of (TM, Tm) such that $Q(v)0_p = \sigma_v(0_p) = 2v$. But

$$\frac{1}{2} : TM \to TM, \quad \delta_x \mapsto \frac{1}{2}\delta_x$$

also is an automorphism of (TM, Tm), as shows Formula (2.1). Therefore the automorphism group of TM acts transitively on fibers, and after conjugation of σ_v with $(\frac{1}{2}Q(v))^{-1}$ we may assume that $v = 0_p$. But in this case the proof of our claim is easy: we have $\sigma_{0_p} = T\sigma_p$, and since $T_p\sigma_p = -\operatorname{id}_{T_pM}$, the canonical identification $T_{0_p}(TM) \cong T_pM \oplus T_pM$ yields $T_{0_p}(\sigma_{0_p}) = (-\operatorname{id}_{T_pM}) \times (-\operatorname{id}_{T_pM}) = -\operatorname{id}_{T_{0_p}TM}$, whence (M4). ■

2.7. The algebra of derivations of M. A vector field $X : M \to TM$ on a symmetric space M is called a *derivation* if X is also a homomorphism of symmetric spaces. This can be rephrased by saying that $(X \times X, X)$ is m-related. The naturality of the Lie bracket therefore implies that the space $\mathfrak{g}$ of derivations is stable under the Lie bracket. It is also easily checked that it is a $\mathbb{K}$-submodule of $\mathfrak{X}(M)$, and hence $\mathfrak{g} \subset \mathfrak{X}(M)$ is a Lie-subalgebra.

Let us fix a base point $o \in M$. The map $X \mapsto T\sigma_o \circ X \circ \sigma$ is a Lie algebra automorphism of $\mathfrak{X}(M)$ of order 2 which stabilizes $\mathfrak{g}$. We let

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-, \quad \mathfrak{g}^\pm = \{X \in \mathfrak{g} \mid T\sigma_o \circ X \circ \sigma_o = \pm X\}$$

be its associated eigenspace decomposition (recall that 2 is assumed to be invertible in $\mathbb{K}$). The space $\mathfrak{g}^+$ is a Lie subalgebra of $\mathfrak{X}(M)$, whereas $\mathfrak{g}^-$ is only closed under the triple bracket

$$(X,Y,Z) \mapsto [X,Y,Z] := [[X,Y],Z].$$

Proposition 2.8.

(i) *The space* $\mathfrak{g}^+(M)$ *is the kernel of the evaluation map* $\operatorname{ev}_o : \mathfrak{g} \to T_oM$, $X \mapsto X(o)$.

(ii) *Restriction of* ev_o *yields a bijection* $\mathfrak{g}^- \to T_oM$, $X \mapsto X(o)$.

Proof. (i) Assume $X \in \mathfrak{g}^+$. Then $T_o\sigma X(o) = X(\sigma_o(o)) = X(o)$ implies $-X(o) = X(o)$ and hence $X(o) = 0$. On the other hand, if $X(o) = 0$, then $X(\sigma_o(p)) = X(m(o,p)) = Tm(X(o), X(p)) = Tm(0_o, X(p)) = T\sigma_o X(p)$, whence $X \in \mathfrak{g}^+$.

(ii) By (i), $\mathfrak{g}^- \cap \ker(\operatorname{ev}_o) = \mathfrak{g}^- \cap \mathfrak{g}^+ = 0$, and hence $\operatorname{ev}_o : \mathfrak{g}^- \to T_oM$ is injective. It is also surjective: let $v \in T_oM$. Consider the map

$$\widetilde{v} = \frac{1}{2}Q(v) \circ z : M \to TM, \quad p \mapsto \frac{1}{2}Q(v)0_p = \frac{1}{2}Tm(v, Tm(0_o, 0_p)).$$

It is a composition of homomorphisms and hence is itself a homomorphism from M into TM. Moreover, as we have seen in the proof of Proposition 2.6, $\widetilde{v}(o) = v$. Thus we will be done if we can show that $\widetilde{v} \in \mathfrak{g}^-$. First of all, $\widetilde{v}$ is a vector field since $Q(v)\delta_p \in T_{m(o,m(o,p))}M = T_pM$ for all $p \in M$. Finally,

$$T\sigma_o \circ \widetilde{v} \circ \sigma_o = \frac{1}{2}T\sigma_o \circ Q(v) \circ z \circ \sigma_o = \frac{1}{2}Q(T\sigma_o v) \circ z = \frac{1}{2}Q(-v) \circ z = -\widetilde{v}.$$ ■

2.9. The Lie triple system of a symmetric space with base point. The space $\mathfrak{m} := T_oM$ with triple bracket given by

$$[u,v,w] := -R_o(u,v)w := [[\widetilde{u},\widetilde{v}],\widetilde{w}](o)$$

is called the *Lie triple system (Lts) associated to* (M,o). It satisfies the identities of an abstract Lie triple system over $\mathbb{K}$ (cf. [Lo69, p. 78/79]). The notation $R_o(u,v)w$ alludes to the fact that the triple Lie bracket indeed is the curvature tensor of a canonical connection on M (cf. [Lo69] for the finite-dimensional real case and [Be03b] for the general case). Since the base point o is arbitrary, we have indeed defined a tensor field R on M (in a chart it is easily seen that the dependence of R_o on o is smooth).

Theorem 2.10. *Let M be a symmetric space over $\mathbb{K}$ with base point o.*

(i) *The Lie triple bracket of the Lts $\mathfrak{m}$ associated to (M,o) is C^0.*

(ii) *If $\varphi : M \to M'$ is a homomorphism of symmetric spaces such that $\varphi(o) = o'$, then $\dot\varphi := T_o\varphi : \mathfrak{m} \to \mathfrak{m}'$ is an Lts homomorphism.*

Proof. One uses the same arguments as in the proof of Theorem 2.4. ∎

The functor from symmetric spaces with base point to C^0-Lie triple systems will be called the *Lie functor for symmetric spaces.* It contains the Lie functor for Lie groups in the following sense: if G is a Lie group, then $m(x,y) = xy^{-1}x$ defines on G the structure of a symmetric space (the condition (M4) here is equivalent to $Ti(e) = -\mathrm{id}_{T_eG}$ which is proved in the same way as usual), and as in [Lo69] it is seen that the Lts of G is given in terms of the Lie algebra of G by $\frac{1}{4}[[X,Y],Z]$.

2.11. On geodesics and exponential maps. If M is a finite-dimensional real or complex symmetric space and M_1 is a connected component of M, then the subgroup $G(M_1)$ of $G(M)$ generated by all products $\sigma_x\sigma_y$, $x,y \in M_1$, acts transitively on M_1. This follows from the existence of an *exponential map* in this case (cf. [Lo69]). In the general case, even for $\mathbb{K} = \mathbb{R}$, there is no exponential map, and the connected components need no longer be homogeneous. In the following, we give a brief account of the relevant definitions and explain the main arguments.

If M is a symmetric space over $\mathbb{K}$, we define a *geodesic* to be a non-constant homomorphism $\gamma : \mathbb{K} \to M$, where $\mathbb{K}$ carries the "canonical flat symmetric space structure" $m(v,w) = 2v - w$ which exists on any topological $\mathbb{K}$-module. We say that M is *geodesically connected* if any two points $p,q \in M$ can be joined by a broken geodesic, i.e. there exist points $p = p_0, \ldots, p_n = q$ such that p_i and p_{i+1} can be joined by a geodesic.

Proposition 2.12. *If M is geodesically connected, then the transvection group $G(M)$ acts transitively on M.*

Proof. We use the same arguments as in the real finite-dimensional case ([Lo69]): if $\gamma : \mathbb{K} \to M$ is a geodesic such that $\gamma(0) = p_i$ and $\gamma(1) = p_{i+1}$, we let $y := \gamma(\frac{1}{2})$ and $g := \sigma_y \circ \sigma_{p_i} \in G(M)$; then

$$g(p_i) = \sigma_y(p_i) = m(\gamma(\tfrac{1}{2}), \gamma(0)) = \gamma(m(\tfrac{1}{2}, 0)) = \gamma(1) = p_{i+1}.$$

Now the claim follows by a trivial induction on n. ∎

The crucial property used in the proof is that for two points, sufficiently close to each other, we can find a *midpoint.* The midpoint should be seen as a "square root" of one point with respect to the other; thus the lack of square roots in $\mathbb{K}$ is one obstruction for homogeneity of symmetric spaces, as is illustrated by the example of the projective space $\mathbb{QP}^n$ over $\mathbb{K} = \mathbb{Q}$. Note also that geodesic connectedness does not imply connectedness in the topological sense since already $\mathbb{K}$ may be totally disconnected as the example of the p-adic numbers $\mathbb{Q}_p$ shows.

We say that M *has an exponential map* if, for every $p \in M$ and $v \in T_pM$, there exists a unique geodesic $\varphi_v : \mathbb{K} \to M$ such that $\varphi_v(0) = p$ and $T_0\varphi_v(1) = v$ and such that the map

$$\mathrm{Exp} := \mathrm{Exp}_p : T_pM \to M, \quad v \mapsto \varphi_v(1)$$

is smooth. We say that M *is locally exponential* if M has an exponential map, and for all $p \in M$, Exp_p is a diffeomorphism of some neighborhood of 0 in T_pM onto some neighborhood of p in M. Then the set of all points that can be joined to a given point by a broken geodesic is open, and hence M is geodesically connected if M is topologically connected. It can be shown that, if $\mathbb{K} = \mathbb{R}$ and the model space of V is a Banach space, then M is locally exponential (one can use the same arguments as in [Lo69]) and hence $G(M)$ acts transitively on topological connected components. However, already for Fréchet symmetric spaces this is no longer true in general.

3. Symmetric spaces associated to continuous inverse Jordan algebras

3.1. Unit groups of continuous inverse algebras. It is clear that the unit group $A^\times$ of a continuous inverse algebra A (cf. Section 1.6) is a Lie group. The associated Lie algebra is A with the commutator bracket. We are going to explain a similar construction which arises when one tries to replace the commutator by the anti-commutator.

3.2. Continuous inverse Jordan algebras. A *Jordan algebra* is a commutative $\mathbb{K}$-algebra V such that the product $x \bullet y$ satisfies the identity $x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y)$. Our basic reference for Jordan algebras is [MC03]; see also [FK94]. We assume that V has a unit $\mathbf{1}$. Any associative algebra A with the anti-commutator $x \bullet y = \frac{xy+yx}{2}$ is a Jordan algebra; subalgebras of such Jordan algebras are called *special.* For x, y belonging to a Jordan algebra V one defines

$$L(x)y := x \bullet y, \quad Q(x) := 2L(x)^2 - L(x^2),$$

and

$$Q(x,y) := Q(x+y) - Q(x) - Q(y) = 2(L(x)L(y) + L(y)L(x) - L(xy)). \tag{3.1}$$

Then the *fundamental formula* holds:

$$Q(Q(x)y) = Q(x)Q(y)Q(x). \tag{3.2}$$

One defines the *Jordan inverse* j by

$$j : V^\times := \{x \in V \,|\, Q(x) \,\text{invertible}\,\} \to V, \quad x \mapsto j(x) := x^{-1} := Q(x)^{-1}x. \tag{3.3}$$

We say that V is a *continuous inverse Jordan algebra* (c.i.J.a.) if V is a topological Jordan algebra such that $V^\times$ is open in V and $j : V^\times \to V$ is C^0.

Proposition 3.3. *The Jordan inverse of a continuous inverse Jordan algebra is smooth, and its differential is given by*

$$dj(x)v = -Q(x)^{-1}v.$$

Proof. The fact that j is smooth follows from the generalized quotient rule (1.7) with $f : V \to \mathrm{End}(V)$, $x \mapsto Q(x)$ because the associated map $\widetilde{f} : (x,v) \mapsto Q(x)v$ is C^0 and polynomial, hence C^∞. However, in order to find the correct expression for the differential, we repeat the main steps of the calculation: for $(x,h,t) \in (V^\times)^{[1]}$,

$$\begin{aligned}
j(x+th) - j(x) &= Q(x+th)^{-1}(x+th) - Q(x)^{-1}x \\
&= tQ(x+th)^{-1}h + (Q(x+th)^{-1} - Q(x)^{-1})x \\
&= tQ(x+th)^{-1}h - Q(x)^{-1}(Q(x+th) - Q(x))Q(x+th)^{-1}x \\
&= tQ(x+th)^{-1}h - Q(x)^{-1}(Q(th) + Q(x,th))Q(x+th)^{-1}x \\
&= t\big(Q(x+th)^{-1}h - Q(x)^{-1}(tQ(h) + Q(x,h))Q(x+th)^{-1}x\big).
\end{aligned}$$

The expression following the scalar t is $j^{[1]}(x,h,t)$. Letting $t = 0$, we get

$$dj(x)h = Q(x)^{-1}h - Q(x)^{-1}Q(x,h)Q(x)^{-1}x.$$

Now,

$$Q(x,h)Q(x)^{-1}x = Q(x,h)x^{-1} = 2([L_x, L_{x^{-1}}] + L_{xx^{-1}})h = 2h$$

since L_x and $L_{x^{-1}} = Q(x)^{-1}L_x$ commute (cf. the "L-inverse formula" [MC03, III.6.1]) and $x^{-1} \bullet x = Q(x)^{-1}x^2 = Q(x)^{-1}Q(x)\mathbf{1} = \mathbf{1}$. It follows that $dj(x)h = -Q(x)^{-1}h$. ∎

Theorem 3.4. *If V is a continuous inverse Jordan algebra, then the set $M := V^\times$ of invertible elements of V is a symmetric space with product map*

$$m : M \times M \to M, \quad (x,y) \mapsto Q(x)y^{-1} = Q(x)Q(y)^{-1}y.$$

The quadratic map $Q : V \to \mathrm{End}(V)$ is a polynomial extension of the quadratic representation $Q : M \to G(M)$ associated to the symmetric space with base point $\mathbf{1}$. The Lie triple system on the tangent space $T_{\mathbf{1}}M \cong V$ at the base point $\mathbf{1} \in M$ is given by

$$-R(x,y)z = [[L(x), L(y)], L(z)]\mathbf{1} = [L(x), L(y)]z = x \bullet (y \bullet z) - y \bullet (x \bullet z).$$

Proof. (cf. [Lo96] for the Banach case) Using the fundamental formula (3.2), it is easily checked that $m(x,y)$ belongs to $V^\times$ if x, y belong to $V^\times$. Thus m is well-defined, and it is smooth since the Jordan inversion is smooth (Prop. 3.3).

Property (M1) follows trivially from the definition of j, (M2) and (M3) follow by straightforward calculations from the fundamental formula (cf. also [Lo69, II.1.2.5]), and since $\sigma_{\mathbf{1}}(y) = y^{-1}$, we have $\sigma_x\sigma_{\mathbf{1}}(y) = Q(x)((y^{-1})^{-1}) = Q(x)y$, proving that the quadratic representation of M and the quadratic representation of the Jordan algebra coincide on $V^\times$. Next we prove (M4) (using Prop. 3.3):

$$T_x(\sigma_x) = T_x(\sigma_x\sigma_{\mathbf{1}}\sigma_{\mathbf{1}}) = T_x(Q(x) \circ j) = Q(x) \circ T_x j = -Q(x)Q(x)^{-1} = -\mathrm{id}.$$

In order to calculate the Lie triple system, we remark first that $TM = T(V^\times)$ is realized by the same construction as V, but with respect to the Jordan algebra $TV \cong V \times V$ with product being the tangent map of the Jordan product of V, i.e. $(x,x') \bullet (y,y') = (x \bullet y, x \bullet y' + x' \bullet y)$ – seen algebraically, this is the scalar extension of V by the ring of dual numbers over $\mathbb{K}$, $\mathbb{K}[\varepsilon] := \mathbb{K}[x]/(x^2) \cong \mathbb{K} \oplus \varepsilon\mathbb{K}$, $\varepsilon^2 = 0$. Taking the unit element $\mathbf{1}$ as base point, the tangent vector $v \in T_{\mathbf{1}}M$ corresponds to the element $\mathbf{1} + \varepsilon v \in TV$. Recall from the proof of Proposition 2.8 the vector field

$$\widetilde{v}(p) = \frac{1}{2}Tm(v, Tm(0_{\mathbf{1}}, 0_p)) = Tm(\tfrac{v}{2}, Tm(0_{\mathbf{1}}, 0_p)) = Q(\tfrac{v}{2})0_p.$$

With the preceding notation, $0_p = p + \varepsilon 0$, $v = \mathbf{1} + \varepsilon v$, and $\widetilde{v}$ is in the chart V described by

$$\widetilde{v}(p) = Q(\mathbf{1} + \varepsilon\tfrac{v}{2}).0_p = (2L(\mathbf{1} + \varepsilon\tfrac{v}{2})^2 - L((\mathbf{1} + \varepsilon\tfrac{v}{2})^2))0_p = (L(\mathbf{1}) + \varepsilon L(v))p = p + \varepsilon v \bullet p.$$

In other words, in the chart V, $\widetilde{v}$ is the *linear* vector field given by the operator $L(v) : V \to V$. But then Formula (1.2) shows that the commutator of two linear vector fields $L(x)$ and $L(y)$ is simply the (negative of) the usual bracket $[L(x), L(y)]$ of endomorphisms and hence the triple commutator is given by $[[L(x), L(y)], L(z)]$, proving that $[x,y,z] = [[L(x), L(y)], L(z)]\mathbf{1}$. From this the other formulas follow because $[L(x), L(y)]\mathbf{1} = x \bullet y - y \bullet x = 0$. ∎

One can prove a converse of Theorem 3.4: a symmetric space M which is open in a $\mathbb{K}$-module V and such that the quadratic map extends to a homogeneous quadratic polynomial, is essentially given by the preceding construction; in [Be00, Ch. II] (in the finite-dimensional real case) such spaces have been called *quadratic prehomogeneous symmetric spaces.*

3.5. Remark on the orbit structure for the action of $G(M)$. In general, $M = V^\times$ is far from being homogeneous under the action of the group $G(M)$: for instance, if $V = \mathrm{Sym}(n, \mathbb{K})$ is the Jordan algebra of symmetric $n \times n$-matrices over $\mathbb{K}$, then $V^\times$ is the space of non-degenerate quadratic forms on $\mathbb{K}^n$. The group $G(M)$ is contained in $\mathrm{GL}(V)$, acting in the usual way on the space of forms. It follows that the $G(M)$-orbits are included in congruence classes of forms, and hence the orbit structure is at least as complicated as the classification of (non-degenerate) quadratic forms over $\mathbb{K}$.

If V is a Banach Jordan algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then $V^\times$ is a Banach symmetric space, hence is locally exponential (Section 2.11). The exponential map at the base point $\mathbf{1}$ is given by the usual exponential series $e^v = \sum_{k=0}^{\infty} \frac{v^k}{k!}$ (where the power v^k is taken with respect to the Jordan product), and the topological connected components of $V^\times$ are homogeneous under the transvection group. It would be very interesting to understand the corresponding situation for p-adic Banach Jordan algebras (where the exponential series does not converge everywhere).

4. Geometries associated to Jordan pairs

In this chapter we review the algebraic theory from [BN03]; results quoted without further comment can be found there. Our basic reference for Jordan pairs is [Lo75].

4.1. Three-graded Lie algebras and Jordan pairs. A 3*-graded Lie algebra* (over $\mathbb{K}$) is a Lie algebra over $\mathbb{K}$ of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ such that $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$, i.e., $\mathfrak{g}_{\pm 1}$ are abelian subalgebras which are $\mathfrak{g}_0$-modules, in the following often denoted by $V^\pm$ or $\mathfrak{g}_\pm$, and $[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{g}_0$. Then the linear map $D : \mathfrak{g} \to \mathfrak{g}$ with $DX = iX$ $(X \in \mathfrak{g}_i)$ is a derivation, called the *grading element*, and if D is inner, $D = \mathrm{ad}(E)$, then the grading is called an *inner* 3*-grading*, and E is called an *Euler operator*. The pair (V^+, V^-) together with the trilinear maps

$$T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm, \quad (x,y,z) \mapsto -[[x,y],z] \tag{4.1}$$

is a *(linear) Jordan pair* over $\mathbb{K}$, i.e. it satisfies the identities, where we use the notation $T^\pm(X,Y)Z := T^\pm(X,Y,Z)$:

$$\begin{aligned} T^\pm(X,Y,Z) &= T^\pm(Z,Y,X), \\ T^\pm(X,Y)T^\pm(U,V,W) &= T^\pm(T^\pm(X,Y,U),V,W) - \\ &\qquad T^\pm(U,T^\mp(Y,X,V),W) + T^\pm(U,V,T^\pm(X,Y,W)). \end{aligned} \tag{4.2}$$

Conversely, every linear Jordan pair arises in this way.

4.2. The projective elementary group. Let $(\mathfrak{g}, D)$ be a 3-graded Lie algebra over $\mathbb{K}$. For $x \in \mathfrak{g}_{\pm 1}$, the operator $e^{\mathrm{ad}\, x} = \mathbf{1} + \mathrm{ad}\, x + \frac{1}{2}(\mathrm{ad}\, x)^2$ is a well defined automorphism of $\mathfrak{g}$. The group generated by these operators,

$$G := G(D) := \mathrm{PE}(\mathfrak{g}, D) := \langle e^{\mathrm{ad}\, x} \colon x \in \mathfrak{g}_{\pm 1} \rangle \subseteq \mathrm{Aut}(\mathfrak{g}), \tag{4.3}$$

is called the *projective elementary group of* $(\mathfrak{g}, D)$. With respect to the fixed 3-grading, automorphisms g of $\mathfrak{g}$ will often be written in "matrix form"

$$g = \begin{pmatrix} g_{11} & g_{10} & g_{1,-1} \\ g_{01} & g_{00} & g_{0,-1} \\ g_{-1,1} & g_{-1,0} & g_{-1,-1} \end{pmatrix}. \tag{4.4}$$

In particular, the generators of G are represented by the following matrices (where $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_{-1}$, $h \in H$):

$$e^{\mathrm{ad}\, x} = \begin{pmatrix} \mathbf{1} & \mathrm{ad}\, x & \frac{1}{2}\mathrm{ad}(x)^2 \\ 0 & \mathbf{1} & \mathrm{ad}\, x \\ 0 & 0 & \mathbf{1} \end{pmatrix}, \quad e^{\mathrm{ad}\, y} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \mathrm{ad}\, y & \mathbf{1} & 0 \\ \frac{1}{2}\mathrm{ad}(y)^2 & \mathrm{ad}\, y & \mathbf{1} \end{pmatrix} \quad h = \begin{pmatrix} h_{11} & & \\ & h_{00} & \\ & & h_{-1,-1} \end{pmatrix}. \tag{4.5}$$

The subgroups $U^\pm := U^\pm(D) := e^{\mathrm{ad}\, \mathfrak{g}_\pm}$ of G are abelian and generate G. We define the *automorphism group of* $(\mathfrak{g}, D)$ to be $\mathrm{Aut}(\mathfrak{g}, D) = \{g \in \mathrm{Aut}(\mathfrak{g}) : g \circ D = D \circ g\}$, and we further define subgroups $H := H(D)$ and $P^\pm := P^\pm(D)$ of G via

$$H := G(D) \cap \mathrm{Aut}(\mathfrak{g}, D) \quad \text{and} \quad P^\pm := HU^\pm = U^\pm H. \tag{4.6}$$

4.3. The projective completion. From now on we assume that the grading derivation D is inner, $D = \mathrm{ad}(E)$. We denote by

$$\mathcal{G} := \{\mathrm{ad}(F) : F \in \mathfrak{g}, \ \mathrm{ad}(F)^3 = \mathrm{ad}(F)\} \subset \mathrm{der}(\mathfrak{g}) \tag{4.7}$$

the space of all inner 3-gradings. By definition, the group H is the stabilizer of D in $G(D)$, and hence the homogeneous space $M := G(D)/H$ is just the orbit of D under the action of $G(D)$ on $\mathcal{G}$. One shows that $P^\pm$ is precisely the stabilizer group of the flag

$$\mathfrak{f}^\pm(\mathrm{ad}(E)): \quad 0 \subset \mathfrak{f}_1^\pm := \mathfrak{g}_{\pm 1} \subset \mathfrak{f}_0^\pm := \mathfrak{g}_{\pm 1} \oplus \mathfrak{g}_0 \subset \mathfrak{g}. \tag{4.8}$$

Flags of this type are called *inner 3-filtrations of* $\mathfrak{g}$, and the space of inner 3-filtrations is denoted by $\mathcal{F}$. The flags $o^\pm := \mathfrak{f}^\mp(\mathrm{ad}(E))$ are (for fixed E) called the *canonical base points in* $\mathcal{F}$, and we denote by

$$X^\pm := G(D).o^\pm \cong G(D)/P^\mp \subset \mathcal{F} \tag{4.9}$$

their $G(D)$-orbits. The maps

$$V^\pm \to X^\pm, \quad x \mapsto e^{\mathrm{ad}(x)}.o^\pm \tag{4.10}$$

are injective, called the *projective completion of* $V^\pm$. The reader may think of $X^\pm$ as a kind of "manifold" modeled on the $\mathbb{K}$-modules $V^\pm$: we will say that

$$\mathcal{A} := \{(g(V^+), g) : g \in G\}, \qquad \varphi_g : g(V^+) \to V^+, \quad g.x \mapsto x \tag{4.11}$$

is the *natural atlas of* X^+. The chart domains $g(V^+)$ carry a natural structure of an affine space over $\mathbb{K}$, depending only on the point $y := g.o^- \in X^-$. We then write $V_y := g(V^+)$ and denote for $x, z \in V_y$ by

$$\mu_r(x,y,z) := rz \tag{4.12}$$

the product rz in the $\mathbb{K}$-module V_y with zero vector x.

4.4. Transversality. The natural map from gradings to filtrations $\mathcal{G} \to \mathcal{F} \times \mathcal{F}$ and the corresponding map $M \to X^+ \times X^-$, $gH \mapsto (gP^-, gP^+)$ are injective. Two filtrations $(\mathfrak{f}, \mathfrak{e})$ are obtained from an inner grading $\mathrm{ad}(E)$ if and only if they are *transversal* or *complementary* in the sense that

$$\mathfrak{g} = \mathfrak{f}_1 \oplus \mathfrak{e}_0, \quad \mathfrak{g} = \mathfrak{e}_1 \oplus \mathfrak{f}_0$$

([BN03, Th. 3.6]); we write then $\mathfrak{e} \top \mathfrak{f}$.

4.5. Denominators and nominators. For $x \in V^+$ and $g \in \mathrm{Aut}(\mathfrak{g})$, we define

$$d_g(x) := (e^{-\mathrm{ad}(x)} g^{-1})_{11}, \quad c_g(x) := (g e^{\mathrm{ad}(x)})_{-1,-1}, \tag{4.13}$$

where the "matrix coefficients" h_{ij} are as in Equation (4.4). Then

$$d_g^+ := d_g : V^+ \to \mathrm{End}(V^+), \qquad c_g^+ := c_g : V^+ \to \mathrm{End}(V^-) \tag{4.14}$$

are quadratic polynomial maps, called the *denominator* and *co-denominator* of g (w.r.t. the fixed inner grading $\mathrm{ad}(E)$). In particular, if $g = e^{\mathrm{ad}(w)}$, $w \in V^-$, $x \in V^+$,

$$\begin{aligned} d_g(x) = B_+(x,w) &:= \mathrm{id}_{V^+} + \mathrm{ad}(x)\,\mathrm{ad}(w) + \frac{1}{4}\,\mathrm{ad}(x)^2\,\mathrm{ad}(w)^2 \in \mathrm{End}(V^+) \\ c_g(x) = B_-(w,x) &:= \mathrm{id}_{V^-} + \mathrm{ad}(w)\,\mathrm{ad}(x) + \frac{1}{4}\,\mathrm{ad}(w)^2\,\mathrm{ad}(x)^2 \in \mathrm{End}(V^-) \end{aligned} \tag{4.15}$$

are called the *Bergman operators*. For $x \in V^+$ and $g \in \mathrm{Aut}(\mathfrak{g})$, we define the *nominator of* g to be

$$n_g(x) := \mathrm{pr}_1(e^{-\mathrm{ad}(x)} g^{-1} E) = (e^{-\mathrm{ad}(x)} g^{-1})_{10}.E. \tag{4.16}$$

Then $n_g : V^+ \to V^+$ is a quadratic polynomial. In particular, for $g = e^{\mathrm{ad}(w)}$, $w \in V^-$,

$$n_g(x) = x - \frac{1}{2}\,\mathrm{ad}(w)^2 x = x - Q^+(x)w. \tag{4.17}$$

Theorem 4.6. *Let $g \in \mathrm{Aut}(\mathfrak{g})$ and $x \in V^+$. Then $g.x \in V^+$ if and only if $d_g(x)$ and $c_g(x)$ are invertible, and then the value $g.x \in V^+$ is given by*

$$g.x = d_g(x)^{-1} n_g(x). \qquad \blacksquare$$

In particular, for $g = e^{\mathrm{ad}(w)}$, $w \in V^-$, we get from Theorem 4.6

$$g(x) = B_+(x,w)^{-1}(x - Q^+(x)w). \tag{4.18}$$

In axiomatic Jordan theory, the last expression is denoted by x^w and is called the *quasi-inverse* (cf. [Lo75]). A pair $(x,y) \in V^+ \times V^-$ is called *quasi-invertible* if the Bergman operators $B_+(x,y)$ and $B_-(y,x)$ are invertible.

4.7. Jordan fractional quadratic maps. An $\mathrm{End}(V^+)$*-valued Jordan matrix coefficient (of type* $(1,1)$*, resp. of type* $(1,0)$*)* is a map of the type

$$q : V^\sigma \times V^\nu \to \mathrm{End}(V^+), \quad (x,y) \mapsto (e^{\mathrm{ad}(x)} g e^{\mathrm{ad}(y)} h)_{11}, \tag{4.19}$$

where $\sigma, \nu \in \{\pm\}$ and $g, h \in G$, resp.

$$p : V^\sigma \times V^\nu \to V^+, \quad (x,y) \mapsto (e^{\mathrm{ad}(x)} g e^{\mathrm{ad}(y)} h)_{10} E.$$

These maps are quadratic polynomials in x and in y, and nominators and denominators are partial maps of p and q by fixing one of the arguments to be zero. A *Jordan fractional quadratic map* is a map of the form

$$f : V^\sigma \times V^\nu \supset U \to V^+, \quad (x,y) \mapsto q(x,y)^{-1} p(x,y),$$

where q, p are Jordan matrix coefficients of type (1,1), resp. (1,0), and $U = \{(x,y) \in V^\sigma \times V^\nu : q(x,y) \in \mathrm{GL}(V^+)\}$.

Theorem 4.8. *The actions*

$$V^+ \times X^+ \to X^+ \quad \textit{and} \quad V^- \times X^+ \to X^+$$

are given, with respect to all charts from the atlas $\mathcal{A}$ (cf. Eqn. (2.6)*), by Jordan fractional quadratic maps. In other words, for all $g, h \in G$, the maps*

$$(v,y) \mapsto (h \circ \exp(v) \circ g).y, \quad (w,y) \mapsto (h \circ \exp(w) \circ g).y$$

are Jordan fractional quadratic, and the maps μ_r are, in all charts, given by Jordan fractional quadratic maps. $\blacksquare$

5. Smooth generalized projective geometries

5.1 Continuous quasi inverse Jordan pairs. Let (V^+, V^-) be a *topological Jordan pair* over the topological ring $\mathbb{K}$ (i.e. V^+, V^- are topological $\mathbb{K}$-modules such that the trilinear structure maps T^+, T^- are C^0). If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and the underlying locally convex spaces are Banach or Fréchet, then we speak of *Banach–*, resp., *Fréchet–Jordan pairs.* For topological Jordan pairs we introduce the following two conditions:

(C1) A topological Jordan pair is called a *continuous quasi-inverse Jordan pair* or a *(C1)-Jordan pair* if the set of quasi-invertible pairs,

$$(V^+ \times V^-)^\times = \{(x,y) \in V^+ \times V^- : B_+(x,y), B_-(y,x) \,\text{invertible}\,\},$$

is open in $V^+ \times V^-$, and the "Bergman inverse map"

$$(V^+ \times V^-)^\times \times V^+ \times V^- \to V^+ \times V^-, \quad (x,a,v,b) \mapsto (B_+(x,a)^{-1}v, B_-(a,x)^{-1}b)$$

is of class C^0.

(C2) We say that a topological Jordan pair (V^+, V^-) is a *(C2)-Jordan pair* or a *weak continuous quasi-inverse Jordan pair* if, for any $a \in V^-$, the set

$$U_a := \{x \in V^+ : B_+(x,a), B_-(a,x) \text{ invertible}\}$$

is open in V^+, and the "partial Bergman inverse map"

$$U_a \times V^+ \to V^+, \quad (x,v) \mapsto B_+(x,a)^{-1}v$$

is of class C^0, and if the dual condition, with V^+ and V^- interchanged, also holds.

It is clear that condition (C1) implies (C2). For instance, Banach–Jordan pairs are automatically (C1) since in this case the operators $B(x,a)$ belong to the Banach algebra $L(V)$ of continuous linear operators on V, and inversion in the Banach algebra $L(V)$ is smooth (Banach algebras are special cases of continuous inverse algebras, cf. 1.6).

Proposition 5.2. *In a* (C1)*-Jordan pair, the quasi-inversion map*

$$(V^+ \times V^-)^\times \to V^+ \times V^-, \quad (x,a) \mapsto (x^a, a^x) := (e^{\mathrm{ad}(a)}.x, e^{\mathrm{ad}(x)}.a) \tag{5.1}$$

is smooth, and in a (C2)*-Jordan pair, the partial maps*

$$U_a \to V^+, \quad x \mapsto e^{\mathrm{ad}(a)}.x, \qquad U_x \to V^-, \quad a \mapsto e^{\mathrm{ad}(x)}.a$$

are smooth.

Proof. Assume (V^+, V^-) satisfies Condition (C1). Following the notation from Section 1.8, we let

$$\begin{aligned}
f : V^+ \times V^- \to \mathrm{End}(V^+) \times \mathrm{End}(V^-), &\quad (x,a) \mapsto (B_+(x,a), B_-(a,x)),\\
\widetilde{f} : V^+ \times V^- \times V^+ \times V^- \to V^+ \times V^-, &\quad ((x,a),(x',a')) \mapsto f(x,a).(x',a'),\\
jf : (V^+ \times V^-)^\times \to \mathrm{GL}(V^+) \times \mathrm{GL}(V^-), &\quad (x,a) \mapsto (B_+(x,a)^{-1}, B_-(a,x)^{-1}),\\
\widetilde{jf} : (V^+ \times V^-)^\times \times V^+ \times V^- \to V^+ \times V^-, &\quad ((x,a),(x',a')) \mapsto (jf(x,a))(x',a')\\
&\qquad = (B_+(x,a)^{-1}x', B_-(a,x)^{-1}a').
\end{aligned}$$

Then $\widetilde{f}$ is a continuous polynomial, hence C^∞, and by (C1), $\widetilde{jf}$ is C^0. The generalized quotient rule (Section 1.8) implies then that $\widetilde{jf}$ is C^∞. We recall from Theorem 4.6 that for $x \in U_a$ we have

$$e^{\mathrm{ad}(a)}.x = B_+(x,a)^{-1}(x - Q^+(x)a) \in V^+.$$

We therefore see that the map

$$(x,a) \mapsto e^{\mathrm{ad}(a)}.x = B_+(x,a)^{-1}(x - Q^+(x)a) = \widetilde{jf}(x,a,x - Q^+(x)a, 0)$$

is C^∞, and that the quasi-inversion map is C^∞. The second claim is proved by similar arguments. ■

Theorem 5.3. (Manifold structure on $X^\pm$) *Let (V^+,V^-) be a topological* (C2)*-Jordan pair over the topological ring $\mathbb{K}$ and (X^+,X^-) its projective completion.*

(i) *There exist on $X^\pm$ structures of a smooth manifolds, modeled on the topological $\mathbb{K}$-modules V^+, resp., V^-, uniquely defined by the condition that the collection of charts $\mathcal{A}^\pm = (g(V^\pm), g \in G)$ defined in* Equation (4.11) *becomes an atlas of $X^\pm$.*

(ii) *The projective group G acts by diffeomorphisms of X^+ and of X^-. If $g \in G$ is such that $d_g(x)$ is invertible for some $x \in V^+$, then the set*

$$V_{(g)} := \{x \in V^+ : d_g(x) \in \mathrm{GL}(V^+), c_g(x) \in \mathrm{GL}(V^-)\} = \{x \in V^+ : g.x \in V^+\}$$

is open in V^+, and $g: V_{(g)} \to V, x \mapsto d_g(x)^{-1} n_g(x)$ is a smooth map whose differential at the point x is given by

$$dg(x) = d_g(x)^{-1}.$$

If, in addition, (V^+,V^-) satisfies (C1)*, then we have with respect to the manifold structure defined in Part* (i)*:*

(iii) *The actions $V^+ \times X^+ \to X^+$ and $V^- \times X^+ \to X^+$ are smooth.*

(iv) *The set $M \subset (X^+ \times X^-)$ of transversal pairs is open in $X^+ \times X^-$.*

(v) *For $r \in \mathbb{K}^\times$, the multiplication map*

$$\mu_r : (X^+ \times X^- \times X^+)^\top := \{(x,y,z) : (x,y),(z,x) \in M\} \to X^+$$

(cf. Equation (4.12)*) is defined on an open set and is smooth.*

Proof. We prove (i) for $X := X^+$. Uniqueness of the differentiable structure is clear since the sets $g(V^+)$, $g \in G$, cover X. In order to prove existence, we equip X with the final topology with respect to the maps (the finest topology for which all these maps are continuous)

$$\varphi_g : V^+ \to X, \quad v \mapsto g.v,$$

for $g \in G$, where φ_e is the inclusion $V^+ \subset X$. In other words, a subset $O \subseteq X$ is open if and only if all inverse images $\varphi_g^{-1}(O) = g^{-1}(O) \cap V^+$, $g \in G$, are open in V^+.

Step 1. G acts by homeomorphisms on X. This is immediate from the definition of the topology on X.

Step 2. Let us show that the induced topology on $V^+ \subset X$ is the original topology on V^+. Clearly, the intersection of an open set O of X with V^+ is open in V^+ because $O \cap V^+ = \mathrm{id}^{-1}(O) \cap V^+$. Conversely, assume that $U \subset V^+$ is open in V^+. We have to show that, for all $g \in G$, $g^{-1}(U) \cap V^+$ is open in V^+. If this set is empty, we are done; if not, pick $x \in g^{-1}(U) \cap V^+$. Then $g \circ e^{\mathrm{ad}(x)}.0 = g.x \in U$, and replacing g by $g \circ e^{\mathrm{ad}(x)}$ we may assume that $x = 0$. Now, every $g \in G$ such that $g.0 \in V^+$ admits a unique decomposition

$$g = e^{\mathrm{ad}(v)} h e^{\mathrm{ad}(w)}, \quad v \in V^+, h \in H, w \in V^-,$$

(cf. [BN03, Th. 1.12 (4)]). Hence

$$g^{-1}(U) \cap V^+ = \big(e^{-\mathrm{ad}(w)} h^{-1} e^{-\mathrm{ad}(v)}(U)\big) \cap V^+ = \big(e^{-\mathrm{ad}(w)} h^{-1}(U - v)\big) \cap V^+.$$

Now it suffices to show that $h^{-1}(U-v)$ is open in V^+ because $e^{\mathrm{ad}(w)}$, on its open domain of definition, is smooth, hence in particular continuous (Proposition 5.2). For this, we will use the following lemma:

Lemma 5.4. *Assume (V^+,V^-) is a topological* (C2)*-Jordan pair and let $B^+ \subset \mathrm{End}_{\mathbb{K}}(V^+)$ be the associative subalgebra generated by all Bergman operators $B^+(x,y)$, $x \in V^+, y \in V^-$. Then, for all $g \in G$ and for all $x \in V^+$, the denominator $d_g(x)$ belongs to B^+.*

Proof. We prove the lemma by induction on the "word length of g" which is, by definition, the smallest $k \in \mathbb{N}$ such that g has an expression of the form

$$g = e^{\mathrm{ad}(w_1)} e^{\mathrm{ad}(v_1)} \cdots e^{\mathrm{ad}(w_k)} e^{\mathrm{ad}(v_k)}, \quad v_i \in V^+, w_i \in V^-.$$

If $k = 1$, then, using the cocycle relation $d_{fh}(x) = d_h(x) d_f(h.x)$ which holds whenever $h.x \in V^+$ (cf. [BN03, Prop. 2.6]), we see that

$$d_g(x) = d_{e^{\mathrm{ad}(v_1)}}(x) d_{e^{\mathrm{ad}(w_1)}}(x + v_1) = B(x + v_1, w_1)$$

belongs to B^+ whenever $(x + v_1, w_1)$ is quasi-invertible. The set of such x is open in V^+ since our Jordan pair is (C2), and hence generates V^+ as a $\mathbb{K}$-module. Therefore the denominator $d_g : V^+ \to \mathrm{End}(V^+)$, being quadratic polynomial by 4.5, coincides with the quadratic polynomial $x \mapsto B(x + v_1, w_1)$, whence $d_g(x) \in B^+$ for all $x \in V^+$.

Now let $g \in G$ be arbitrary and assume that the claim holds for all elements of G of smaller word length than g. We write $g = \widetilde{g} \circ e^{\mathrm{ad}(w_k)} e^{\mathrm{ad}(v_k)}$ with $\widetilde{g}$ of word length smaller than the one of g. Then, again using the cocycle relations, we have

$$d_g(x) = d_{\widetilde{g} e^{\mathrm{ad}(w_k)}}(x + v_k) = B(x + v_k, w_k) \circ d_{\widetilde{g}}(e^{\mathrm{ad}(w_k)}(x + v_k))$$

whenever $(x + v_k, w_k)$ is quasi-invertible. By induction, the second factor $d_{\widetilde{g}}(e^{\mathrm{ad}(w_k)}(x + v_k))$ belongs to B^+ whenever $(x + v_k, w_k)$ is quasi-invertible. Hence $d_g(x)$ belongs to B^+ whenever $(x + v_k, w_k)$ is quasi-invertible. As above, note that the set of such x is open in V^+. Thus the denominator $d_g : V^+ \to \mathrm{End}(V^+)$ is a quadratic polynomial map taking, on a non-empty open set, values in the $\mathbb{K}$-module B^+; hence the whole image is in B^+, and the lemma is proved. ■

Note that the proof of the lemma immediately carries over to any Jordan pair such that each set U_a, $a \in V^-$, generates V^+ as a $\mathbb{K}$-module. However, for general Jordan pairs this property does not always hold – take e.g. the ring $\mathbb{K}[x]$, seen as a Jordan algebra over $\mathbb{K}$, where the unit group is far from generating $\mathbb{K}[x]$ as a $\mathbb{K}$-module.

Now, returning to the proof of the theorem, note that elements of B^+ are *continuous* linear operators on V^+ since so are all $B(x,y)$, $x \in V^+$, $y \in V^-$. Therefore, by Lemma 5.4, for all $h \in H$, $h_{11} = d_h(0)$ is continuous on V^+. But the action of h on V^+ is given by $h.x = h_{11}x$, and hence h acts continuously on V^+. This achieves the proof of Step 2. (Note that, in particular, we have shown that V^+ is open in X.)

Step 3. The transition functions are smooth. In fact, the transition functions are

$$\varphi_{bc} = c^{-1}b : V^+ \cap b^{-1}c(V^+) \to V^+ \cap c^{-1}b(V^+)$$

for $b, c \in G$. We have already seen that they are homeomorphisms. If the intersections are non-empty, we may as above decompose $g := c^{-1}b$ as a product $g = e^{\mathrm{ad}(v)} h e^{\mathrm{ad}(w)}$; the element $e^{\mathrm{ad}(v)}$ with $v \in V$ acts as a translation, hence smoothly, the element $e^{\mathrm{ad}(w)}$ with $w \in V^-$ acts smoothly according to Proposition 5.2, and the element $h \in H$ is a continuous linear map by Lemma 5.4 and hence also acts smoothly. Taken together, Step 2 and Step 3 show that X is a smooth manifold.

(ii) The proof of Step 3 above shows that elements $g \in G$ act smoothly on X. It only remains to show that the differential of g is related to the denominator via $dg(x) = d_g(x)^{-1}$. As above, we first reduce to the case $g \in P^-$ and $x = o$. Then we decompose $g = h e^{\mathrm{ad}(a)}$, $a \in V^-$, $h \in H$. By the chain rule and the cocycle rule for the denominators [BN03, Th. 2.10], it now suffices to prove the claim for h and $\exp(a)$ separately. Since h acts linearly on V^+ and $d_h(o) = h^{-1}$, we are done with the first case. As to $\exp(a)$, we have $d_{\exp(a)}(o) = B(o,a) = \mathrm{id}_{V^+}$. Hence we have to show that $d\exp(a)(o) = \mathrm{id}$. This follows from

$$e^{\mathrm{ad}(a)}.tv - e^{\mathrm{ad}(a)}.o = B(tv,a)^{-1}(tv - Q(tv)a) - 0 = t\,(B(tv,a)^{-1}(v - tQ(v)a)),$$

where the last term, divided by t, is a C^0-map of t and v taking value v for $t = 0$.

(iii) Recall that, according to [BN03, Th. 3.7], both actions are described in charts by Jordan fractional quadratic maps as defined in Section 4. Therefore it suffices to show that Jordan fractional quadratic maps are smooth: first of all, if the elements $g, h \in G$ appearing in the definition from 4.7 are trivial, then our claim amounts to saying that the quasi-inversion map is smooth, which is true in a (C1)-Jordan pair, according to Proposition 5.2. If g and h are not trivial, then they can be written as a composition of translations and quasi inverses which, according to step (i), act as diffeomorphisms. Hence all Jordan fractional quadratic maps are smooth.

(iv) $M \cap (V^+ \times V^-) = (V^+ \times V^-)^\times$ is open by Property (C1).

(v) The argument proving this claim is the same as for part (iii), using that also μ_r is given by Jordan fractional quadratic maps [BN03, Th. 4.3]. ∎

Theorem 5.5. *Assume (V^+, V^-) is a* (C2)*-Jordan pair. Then there are canonical G-equivariant bijections between the tangent bundle TX^+ of X^+ as a smooth manifold, the tangent bundle of X^+ as defined in* [BN03, Th. 2.1] *and the tangent bundle as defined in* [Be02] *via scalar extension by dual numbers.*

Proof. For all three models of the tangent bundle, the tangent space $T_o X^+$, as a $\mathbb{K}$-module, is isomorphic to V^+. Therefore in all models we get a homogeneous bundle of the kind $G \times_{P^-} V^+$, and we only have to show that the actions of the stabilizer group P^- on V^+ coincide in these three pictures. In the context of smooth manifolds, the group P^- acts on V^+ via the linear isotropy representation $\pi(p) = T_o(p)$. In the chart V^+, using Theorem 5.3(ii), we get $T_o p = dp(0) = d_p(0)^{-1} = p_{11}$. This is the representation of P^- used in the model for the tangent bundle in [BN03], and hence these two models coincide. Finally, for the model used in [Be02], as shown in [Be02, (7.3)], the action of U^- on V^+ is trivial, and the action of H commutes with $\varepsilon_{o,o'}$, so H acts on V^+ as group of automorphisms of the Jordan pair (V^+, V^-). This characterizes the representation of P^- used in the other two models, and hence all three models are isomorphic as G-bundles. ∎

6. Smooth polar geometries and associated symmetric spaces

6.1. Continuous inverse Jordan triple systems. Assume $(\mathfrak{g}, D)$ is a 3-graded Lie algebra with an involution θ (automorphism of order 2 reversing the grading). Then $V := \mathfrak{g}_1$ together with the trilinear map $T : V \times V \times V \to V$ defined by

$$T(x, y, z) := [[x, \theta(y)], z] \tag{6.1}$$

is a *Jordan triple system* (Jts) which, by definition, is a $\mathbb{K}$-module V with a trilinear map $T : V \times V \times V \to V$ satisfying the identities (4.2) with superscripts omitted. The map $V^+ \to V^-$ induced by θ is an *involution* of the "underlying Jordan pair" $(V^+, V^-) \cong (V, V)$, and in this way Jordan triple systems are equivalent to Jordan pairs with involution (cf. [Lo75]). (Note that T defines a Jts if and only if $-T$ defines a Jts; thus the sign in (6.1) is a matter of convention. Here we follow, as in [Be00], the convention that, in the real finite-dimensional case, *negative* triple systems shall correspond to *compact* symmetric spaces, see below.) A topological Jordan triple system is called (C1) or a *continuous quasi inverse Jts* if the underlying Jordan pair (V, V) is (C1) and the involution is continuous. (For Jordan triple systems, Condition (C2) is not very interesting.) Equivalently, (C1) means that the set $(V \times V)^\times$ is open in $V \times V$ and the Bergman inverse map $(V \times V)^\times \times V \to V$ is continuous.

6.2. Polarities. Every involution θ of a given inner 3-graded Lie algebra $(\mathfrak{g}, D)$ induces a bijection

$$p : X^+ \to X^-, \quad gP^- \to \theta(g)P^+.$$

We say that p is a *polarity* because it is an anti-automorphism of the generalized projective geometry (X^+, X^-) (in the sense of [Be02, Ch. 3]) and the corresponding *space of non-isotropic* *points*

$$M^{(p)} = \{x \in X^+ : (x, p(x)) \in M\} \tag{6.2}$$

contains the base point o^+ and hence is non-empty. The *multiplication map*

$$m : M^{(p)} \times M^{(p)} \to M^{(p)}, \quad (x, y) \mapsto \mu_{-1}(x, p(x), y) \tag{6.3}$$

is well-defined and satisfies the algebraic identities (M1)–(M3) of a symmetric space (cf. [Be02, 4.1], [BN03, 4.2]). Note that, if we identify $X := X^+$ with X^- via the polarity p, then by definition of $M^{(p)}$,

$$M^{(p)} \to (X \times X)^\top \cap \operatorname{diag}(X \times X), \quad x \mapsto (x, x) \tag{6.4}$$

is a bijection, and hence in the chart $V = V^+ \subset X^+ = X$,

$$M^{(p)} \cap V = \{x \in V : B(x, x) \,\text{invertible}\,\}. \tag{6.5}$$

This set is open in V if (V, T) is (C1).

Theorem 6.3. *Assume that (V, T) is a* (C1)*-Jordan triple system.*

(i) *The associated set $M^{(p)}$ of non-isotropic points is an open submanifold of X containing the base point o, and together with the multiplication map defined by* Equation (6.3) *it is a symmetric space. Moreover, for all $x \in M^{(p)}$, x is an isolated fixed point of the symmetry $\sigma_x = m(x, \cdot)$.*

(ii) *The Lie triple system associated to $(M^{(p)}, o)$ is the vector space $V = V^+$ together with the bracket given by*

$$[X, Y, Z] = T(X, Y, Z) - T(Y, X, Z).$$

Proof. (i) According to Theorem 5.3 (iv), M is open in $X^+ \times X^-$. Since p is C^0, $M^{(p)} = \{x \in X^+ : (x, p(x)) \in M\}$ is open in X^+.

As mentioned above, the identities (M1), (M2), (M3) hold already in the purely algebraic context of any generalized projective geometry (topological or not) with polarity. Let us prove (M4): the involution σ_x is given by the element $(-1)_{x,p(x)}$ of the group G and hence acts as a diffeomorphism. W.l.o.g. we may assume that $x = o$; then in the chart V this diffeomorphism is given by $-\operatorname{id}_V$, and hence (M4) holds. Moreover, 0 is the only fixed point of $\sigma_o = -\operatorname{id}_V$ in the open neighborhood $M \cap V$ of o in M.

It only remains to show that μ is smooth. This follows from the fact that $\mu(x, y) = \mu_{-1}(x, x, y)$ (when identifying X^+ with X^-), and μ_{-1} is smooth by Theorem 5.3 (v).

(ii) Theorem 5.5 allows us to use the realization of $TX^\pm$ from [BN03, 2.4]; in particular, we see that in the chart $V = V^+$, vector fields $Y \in \mathfrak{g}$ are realized by quadratic polynomial maps $\widetilde{Y}^+ : V^+ \to V^+$. We identify $v \in \mathfrak{g}_1$ with the constant vector field on V^+ taking value v. Then $\theta(v)$ is a homogeneous quadratic vector field on V^+, and hence $\widetilde{v} = v + \theta(v)$ is the unique vector field in $\mathfrak{g}^\theta$ anti-fixed by $(-\operatorname{id})_*$ such that $\widetilde{v}(o) = v$ (here $o = o^+$). Hence the Lie triple product is given by

$$\begin{aligned}[u, v, w] &= [[\widetilde{u}, \widetilde{v}], \widetilde{w}]_o = [[u + \theta(u), v + \theta(v)], w + \theta(w)]_o \\ &= [[u, \theta(v)], w]_o + [[\theta(u), v], w]_o = T(u, v, w) - T(v, u, w).\end{aligned}$$

■

6.4. Remark on the orbit structure of $M^{(p)}$. The space $M = (X^+ \times X^-)^\top$ is always a homogeneous symmetric space $M \cong G/H$ with G and H as in 4.2, but $M^{(p)}$, which can be seen as the intersection of M with the diagonal in $X \times X$ (cf. Equation (6.4)), is in general not homogeneous under its transvection group. A typical example for this situation is given by the projective spaces over $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{Q}_p$: here $G/H \cong \mathrm{GL}_{n+1}(\mathbb{K}) / \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_1(\mathbb{K})$, but $\mathbb{KP}^n$ is not homogeneous under $\mathrm{O}(n+1, \mathbb{K})$.

6.5. Remark on the exponential mapping. Assume V is a Banach Jts over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then V is (C1), and the symmetric space $M^{(p)}$ is a Banach symmetric space and hence is locally exponential with $\mathrm{Exp} = \mathrm{Exp}_o$. The explicit formula for Exp is obtained as in [Be00, Ch. X.4]: for all $x, y \in V$, the series

$$\cosh(x)y := \sum_{k=0}^{\infty} \frac{Q(x)^k}{(2k)!} y, \qquad \sinh(x) := \sum_{k=0}^{\infty} \frac{Q(x)^k x}{(2k+1)!}$$

converge absolutely and define analytic mappings $\cosh : V \to \mathrm{End}(V)$, $\sinh : V \to V$. The domain $D := \cosh^{-1}(\mathrm{GL}(V))$ is open in V and non-empty since $\cosh(0) = \mathrm{id}_V$. Then, for $x \in V$, the exponential image $\exp(x)$ belongs to $M \cap V$ if and only if $x \in D$, and we have

$$\exp(x) = \tanh(x) := \cosh(x)^{-1} \sinh(x)$$

(cf. [Be00, Th. X.4.1]; the proof carries over to the Banach case without any changes). As for the case of prehomogeneous symmetric spaces (Section 3.5), it would be very interesting to have analogous results in the p-adic Banach case (where the series cosh and sinh do no longer converge everywhere).

6.6. Remark on classification. It goes without saying that a classification of continuous quasi inverse Jordan pairs or -triple systems is out of reach. In the finite-dimensional complex or real case, *simple* objects can be classified (work of O. Loos, E. Neher and others; cf. [Be00, Ch. IV and XII] for precise references). On finds that in fact essentially *all classical and about half of the exceptional real and complex simple symmetric spaces are obtained in the form* $M^{(p)}$; this list is far too long to be given here (see [Be00, Ch. XII]). For other base fields, so far very little is known. In infinite dimensions over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, various classifications of certain simple objects are known (cf. [MC03], [Up85], [Ka83] (simple JH^*-triples) , [dlH72] (irreducible Riemannian symmetric spaces)). – In the following two chapters we will specialize our theory to two important types of Jordan algebras, namely to *associative algebras* and to *Jordan algebras of hermitian elements.*

7. The projective line over an associative algebra

7.1. Associative algebras as Jordan pairs. In this chapter, A is an associative algebra with unit $\mathbf{1}$ over a commutative ring $\mathbb{K}$ having $\frac{1}{2} \in \mathbb{K}$. Then A is a Jordan algebra with Jordan product $a \bullet b = \frac{ab+ba}{2}$ and a Jordan triple system with triple product $T(x,y,z) = xyz + zyx$. It follows that the Bergman operator is given by

$$B(x,y)z = (\mathbf{1} - xy)z(\mathbf{1} - yx) = l(\mathbf{1} - xy)r(\mathbf{1} - yx)z \tag{7.1}$$

where $l(a)$ and $r(a)$ are left-, resp. right multiplication by a in A. Thus (x,y) is quasi-invertible if and only if $\mathbf{1} - xy$ and $\mathbf{1} - yx$ are invertible, and then $B(x,y)^{-1}z = (\mathbf{1} - xy)^{-1} z (\mathbf{1} - yx)^{-1}$. If $\mathbb{K}$ is a topological ring and A is a continuous inverse algebra (c.i.a.), then the set of quasi-invertible pairs is open in $A \times A$, and the Bergman-inverse map is continuous. Therefore A is then a (C1)-Jordan triple system and hence (A, A) is a (C1)-Jordan pair.

7.2. The three-graded picture. The $\mathbb{K}$-Lie algebra $\mathfrak{g} := \mathfrak{gl}_2(A)$ of 2×2-matrices with coefficients in A has a natural 3-grading

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

which is given by the Euler operator

$$E := \tfrac{1}{2} I_{1,1} := \tfrac{1}{2} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{7.2}$$

This 3-grading has a natural involution given by

$$\theta(X) = FXF, \qquad F := \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{7.3}$$

From the commutator relation

$$[[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}], \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & xyz + zyx \\ 0 & 0 \end{pmatrix}$$

it follows that the Jordan triple system associated to these data is A with $T(x,y,z) = xyz + zyx$. Next we are going to describe another model of the geometry associated to this Jordan triple system.

7.3. The projective line. If A is an associative $\mathbb{K}$-algebra, we consider $W := A \times A$ as a right A-module; elements of W are written as column vectors. The *projective line over* A is, by definition, the space

$$\mathbb{P} := A\mathbb{P} := \{E \subset A \times A \mid E \cong A, \exists F \cong A : W = E \oplus F\}$$

of A-submodules E that are isomorphic to A and admit a complement which is also isomorphic to A (cf. [BN03, Section 8.7], [BlHa01] or [H95]). Elements of $\mathbb{P}$ can be written in the form

$$E = \Big[\begin{pmatrix} x \\ y \end{pmatrix} \Big] := \Big\{ \begin{pmatrix} xa \\ ya \end{pmatrix} \mid a \in A \Big\}$$

where $\binom{x}{y}$ is a base vector of E over A. For $(E,F) \in \mathbb{P} \times \mathbb{P}$ we write $E \top F$ if $W = E \oplus F$, and we let $(\mathbb{P} \times \mathbb{P})^\top = \{(E,F) \in \mathbb{P} \times \mathbb{P} \mid E \top F\}$. Then the map

$$\mathcal{P} := \{p \in \mathrm{End}_A(W) \mid p^2 = p, \mathrm{im}(p) \cong A, \ker(p) \cong A\} \to (\mathbb{P} \times \mathbb{P})^\top, \quad p \mapsto (\ker(p), \mathrm{im}(p))$$

is a bijection. As "canonical" base point in $(\mathbb{P} \times \mathbb{P})^\top$ we choose $(o^+, o^-) = (A \times 0, 0 \times A) = ([\binom{\mathbf{1}}{0}], [\binom{0}{\mathbf{1}}])$ which corresponds to the projection $p = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}$. The group $\mathrm{GL}_2(A)$ acts transitively on the projective line $\mathbb{P}$ and on the set $(\mathbb{P} \times \mathbb{P})^\top$. Another base point is given by $[\binom{\mathbf{1}}{\mathbf{1}}], [\binom{\mathbf{1}}{-\mathbf{1}}]$. The matrix transforming the canonical base point into the new one is the *Cayley transform*

$$C = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}. \tag{7.4}$$

7.4. Affine charts of the projective line. Every pair $(E,F) \in (\mathbb{P} \times \mathbb{P})^\top$ defines a linearization of $\mathbb{P}$: the set $F^\top$ of elements that are transversal to F is an affine space over $\mathbb{K}$ (not over A in general), and taking E as origin, $F^\top$ is turned into a $\mathbb{K}$-module. This module is (non-canonically) isomorphic to A. For the canonical base point (o^+, o^-) we fix such an imbedding of A into $\mathbb{P}$:

$$\Gamma : A \to \mathbb{P}, \quad z \mapsto \Gamma_z := \Big[\begin{pmatrix} z \\ \mathbf{1} \end{pmatrix} \Big].$$

Note that Γ_z is the graph of the left translation $l_z : A \to A$, $a \mapsto za$. In this picture, the action of $\mathrm{GL}_2(A)$ is described by usual fractional linear transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_z = \Gamma_{(az+b)(cz+d)^{-1}} \tag{7.5}$$

if $cz + d$ is invertible. In particular, the matrix F from Equation (7.3) represents inversion in A, and $I_{1,1}$ (Equation (7.2)) represents multiplication by the scalar -1. The imbedding $\Gamma : A \to \mathbb{P}$ does not only depend on the base point (o^+, o^-) but also on the fixed normalization of its representatives; however, the *sets* $\Gamma(A)$ and $\Gamma(A^\times)$ depend only on (o^+,o^-). For $\Gamma(A^\times)$ a more intrinsic description is given by

$$\Gamma(A^\times) = \Gamma(A) \cap \{E \in \mathbb{P} \mid E \top I_{1,1}(E)\}, \tag{7.6}$$

and the projective transformation induced by $I_{1,1}$ indeed depends only on (o^+,o^-) (in fact, we have seen above that $I_{1,1}$ is induced by multiplication by the scalar -1 in the $\mathbb{K}$-module defined by the pair (o^+,o^-) and hence its effect on $\mathbb{P}$ depends only on (o^+,o^-)). Moreover, the symmetric space structure on $\Gamma(A^\times)$ also depends only on the pair (o^+,o^-), whereas the group structure cannot be defined in terms of (o^+,o^-) alone.

7.5. Imbedding of the projective line into the three-graded model. For every projection $p: W \to W$, $\mathrm{ad}(p): \mathfrak{g} \to \mathfrak{g}$ is an inner 3-grading, and for every $E = \mathrm{im}(p) \in \mathbb{P}$, we get the corresponding flag $(\mathfrak{f}_1 \subset \mathfrak{f}_0 \subset \mathfrak{g}) \in \mathcal{F}$ which only depends on E. This defines a commutative diagram of maps

$$\begin{array}{ccc} \mathcal{P} \cong (\mathbb{P}\times\mathbb{P})^\top & \subset & \mathbb{P}\times\mathbb{P} \\ \downarrow & & \downarrow \\ \mathcal{G} \cong (\mathcal{F}\times\mathcal{F})^\top & \subset & \mathcal{F}\times\mathcal{F} \end{array} \tag{7.7}$$

which are all $\mathrm{GL}_2(A)$-equivariant, and the vertical arrows are injective ([BN03, Theorem 8.4]). In particular, the natural map $\mathbb{P} \to \mathcal{F}$ is an injection, and it is a bijection when restricted to the "(geometric) connected components of the base point" which are the orbits of the respective base points under the elementary projective group $G = \mathbb{P}E_2(A)$, where

$$\begin{aligned} E_2(A) &= \langle P^+, P^- \rangle \quad \subset \mathrm{GL}_2(A), \\ P^+ &= \Big\{ \begin{pmatrix} \mathbf{1} & x \\ 0 & \mathbf{1} \end{pmatrix} \mid x \in A \Big\}, \quad P^- = \Big\{ \begin{pmatrix} \mathbf{1} & 0 \\ y & \mathbf{1} \end{pmatrix} \mid y \in A \Big\}. \end{aligned} \tag{7.8}$$

Note that the matrix

$$J := \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -\mathbf{1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix} \tag{7.9}$$

belongs to $E_2(A)$ and satisfies $J.o^+ = o^-$. It follows that in both models we have $X^+ = X^-$ as sets. Moreover, since all base points in $(\mathbb{P}\times\mathbb{P})^\top$ are conjugate under $\mathrm{GL}_2(A)$, the same results hold also for all other geometric connected components of $\mathbb{P}$.

7.6. Manifold structures. Now assume that $\mathbb{K}$ is a topological ring and A is a c.i.a. over $\mathbb{K}$. As we have seen in Section 7.1, A is then a (C1)-Jordan triple system, and hence the projective completion $X^+ \cong G/P^-$ of A carries a natural manifold structure satisfying all properties from Theorem 5.3. Using the imbedding from Section 7.5, by transport of structure, the component $G.o^- \subset \mathbb{P}$ can be equipped with the same structure, and since $\mathbb{P}$ is a disjoint union of geometric connected components which are conjugate under $\mathrm{GL}_2(A)$, we get a natural manifold structure on all of $\mathbb{P}$. This manifold structure agrees with the one that is obtained by taking $\Gamma(A) \subset \mathbb{P}$ as "base chart" and then constructing directly, via the action of $\mathrm{GL}_2(A)$, an atlas on $\mathbb{P}$ in the same way as we did for X^+ in Chapter 5. This is an immediate consequence of the $\mathrm{GL}_2(A)$-equivariance of the diagram (7.7).

7.7. Symmetric space structures. Associated to the given base point $(o^+,o^-) \in \mathbb{P}\times\mathbb{P}$, there are three natural involutions of G, given by conjugation with the matrices $I_{1,1}, F, J$, respectively. The first two are related to each other via the Cayley transform C and give rise to the symmeric space $A^\times \cong \Gamma(A^\times) \subset \mathbb{P}$. The third one gives rise the "c-dual symmetric space of $A^\times$" which is isomorphic to $A[i]^\times / A^\times$, where $A[i] := A \otimes_{\mathbb{K}} (\mathbb{K}[x]/(x^2+1))$ is the "complexification" of A (cf. [Be00]).

8. The hermitian projective line

8.1. The space of hermitian elements. Assume A is as in Section 7.1 and $* : A \to A$ is an involution ($\mathbb{K}$-linear antiautomorphism of order 2). We define the spaces of *hermitian*, resp. *anti-hermitian elements* by

$$\mathrm{Herm}(A,*) := \{a \in A \,|\, a^* = a\}, \qquad \mathrm{Aherm}(A,*) := \{a \in A \,|\, a^* = -a\}.$$

Then $\mathrm{Herm}(A,*)$ is a Jordan-subalgebra of A, and $\mathrm{Aherm}(A,*)$ is a Jordan-sub triple system of A. Recall that 2 is assumed to be invertible in $\mathbb{K}$, so $A = \mathrm{Herm}(A) \oplus \mathrm{Aherm}(A)$. (If $\mathbb{K} = \mathbb{R}$ and A is an algebra over $\mathbb{C}$ such that $*$ is $\mathbb{C}$-anti-linear, then $i\,\mathrm{Aherm}(A,*) = \mathrm{Herm}(A,*)$; more generally, this holds whenever there is an element $j \in Z(A)$ such that $j^2 = \pm\mathbf{1}$ and $j^* = -j$.) We are going to describe Linear Algebra models for the geometries associated to the Jordan pairs $(\mathrm{Herm}(A,*), \mathrm{Herm}(A,*))$ and $(\mathrm{Aherm}(A,*), \mathrm{Aherm}(A,*))$. They will be closely related to the $*$-*unitary group*

$$\mathrm{U}(A,*) := \{a \in A^\times \,|\, a^{-1} = a^*\}.$$

8.2. The $*$-symplectic and the $*$-pseudo unitary group. If $*$ is an involution of A, then by a direct calculation one checks that the $\mathbb{K}$-linear map

$$\Phi_1 : M_2(A) \to M_2(A), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix} \tag{8.1}$$

is an involutive anti-automorphism of the associative algebra $M_2(A)$. If A is commutative and $* = \mathrm{id}$, then $\Phi_1(A)$ is the matrix $\widetilde{A}$ adjoint to A via the relation $A\widetilde{A} = \widetilde{A}A = \det(A)\mathbf{1}$, and then the map Φ_1 appears also as "symplectic involution" in the context of the Cayley–Dickson process, cf. [MC03, II.2.9]. We can define three other involutions of $M_2(A)$ by

$$\Phi_2(X) := I_{1,1}\Phi_1(X)I_{1,1}, \qquad \Phi_3(X) := F\Phi_1(X)F, \qquad \Phi_4(X) := J\Phi_1(X)J^{-1}. \tag{8.2}$$

With $X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the explicit formulas are:

$$\Phi_2(X) = \begin{pmatrix} d^* & b^* \\ c^* & a^* \end{pmatrix}, \quad \Phi_3(X) = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}, \quad \Phi_4(X) = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}. \tag{8.3}$$

If $\Phi = \Phi_j$, $j = 1,2,3,4$, is any of these involutions, we obtain an involutive automorphism of $\mathrm{GL}_2(A)$ by

$$\widetilde{\Phi}_j : \mathrm{GL}_2(A) \to \mathrm{GL}_2(A), \quad g \mapsto \Phi_j(g)^{-1}$$

and an involutive Lie algebra automorphism

$$\dot{\Phi}_j : \mathfrak{gl}_2(A) \to \mathfrak{gl}_2(A), \quad X \mapsto -\Phi_j(X).$$

We define the $*$-*symplectic* and the $*$-*pseudo unitary group* via

$$\mathrm{Sp}(A,*) := \mathrm{U}(A \times A, \Phi_1) = \{g \in \mathrm{GL}_2(A) : \Phi_1(g) = g^{-1}\},$$
$$\mathrm{U}(A,A,*) := \mathrm{U}(A \times A, \Phi_2) = \{g \in \mathrm{GL}_2(A) : \Phi_2(g) = g^{-1}\},$$

and the corresponding Lie algebras

$$\mathfrak{sp}(A,*) := \{X \in \mathfrak{gl}_2(A) : \Phi_1(X) = -X\},$$
$$\mathfrak{u}(A,A,*) := \{X \in \mathfrak{gl}_2(A) : \Phi_2(X) = -X\}.$$

Since $-\Phi_j(I_{1,1}) = I_{1,1}$ for $j = 1, 2$, these two Lie algebras are stable under the grading derivation $\mathrm{ad}(I_{1,1})$ of $\mathfrak{gl}_2(A)$ and hence are themselves 3-graded Lie algebras which, moreover, are stable under conjugation by the matrix F. It follows that the Jordan triple system corresponding to these involutive 3-graded Lie algebras is given by restricting the one from $\mathfrak{gl}_2(A)$ to $-\Phi_j$-invariants. Now,

$$-\Phi_1 \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^* \\ 0 & 0 \end{pmatrix}, \qquad -\Phi_2 \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x^* \\ 0 & 0 \end{pmatrix},$$

and hence the Jts associated to $\mathfrak{sp}(A,*)$ is $\mathrm{Herm}(A,*)$ and the Jts associated to $\mathfrak{u}(A,A,*)$ is $\mathrm{Aherm}(A,*)$.

8.3. The (anti-) hermitian projective line. Next we are going to describe the geometries associated to $\mathrm{Herm}(A,*)$ and to $\mathrm{Aherm}(A,*)$. We have to extend the involutions $*$ and $-*$ of A to globally defined maps $\mathbb{P} \to \mathbb{P}$. The idea is simply to send an element $\mathrm{im}(p) \in \mathbb{P}$, where $p \in \mathcal{P}$, to the element $\ker(\Phi_j(p))$, $j = 1, 2$. This is well-defined:

Lemma 8.4. *Let V be a right A-module and $\mathcal{R}$ be the set of all complemented right A-submodules of V and assume $\varphi : \mathrm{End}_A(V) \to \mathrm{End}_A(V)$ is a $\mathbb{K}$-linear anti-automorphism. Then the map*

$$\widetilde{\varphi} : \mathcal{R} \to \mathcal{R}, \quad \mathrm{im}(p) \mapsto \ker(\varphi(p))$$

(where p is a projection onto $\mathrm{im}(p)$) is a well-defined bijection satisfying

$$\widetilde{\varphi}(g.E) = \varphi(g)^{-1}.\widetilde{\varphi}(E), \quad g \in \mathrm{GL}_A(V) = \mathrm{End}_A(V)^\times.$$

Moreover, if $V = A \times A$ and $\varphi = \Phi_j$, $j = 1, 2, 3$, then $\mathbb{P}$ is stable under $\widetilde{\varphi}$.

Proof. First of all, if $p^2 = p$, then also $(\varphi(p))^2 = \varphi(p)$, hence $\varphi(p)$ is a projection. If p and q are projections such that $\mathrm{im}(p) = \mathrm{im}(q)$, then there exists $g \in \mathrm{GL}_A(V)$ such that $q = p \circ g$. Hence $\ker(\varphi(q)) = \ker(\varphi(g) \circ \varphi(p)) = \ker(\varphi(p))$ since $\varphi(g)$ is bijective. Thus $\widetilde{\varphi}$ is well-defined. Clearly, $\widetilde{\varphi}$ is bijective with inverse $\widetilde{\varphi^{-1}}$.

The transformation property under g follows from

$$\varphi(g)^{-1}.\ker(\varphi(p)) = \ker(\varphi(g)^{-1}\varphi(p)\varphi(g)) = \ker(\varphi(gpg^{-1})) = \widetilde{\varphi}(\mathrm{im}(gpg^{-1})) = \widetilde{\varphi}(g.\mathrm{im}(p)).$$

Now let $\varphi = \Phi_j$, $j = 1, 2$, and $\mathrm{im}(p) \in \mathbb{P}$. Then there exists $g \in \mathrm{GL}_2(A)$ with $gpg^{-1} = \left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, whence $\varphi(p) = \varphi(g)\varphi\left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{smallmatrix}\right)\varphi(g)^{-1} = \varphi(g)\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)\varphi(g)^{-1}$, which has kernel $\varphi(g)(A \times 0) \cong A$. For $j = 3$, it suffices to note that the matrices F and $I_{1,1}$ are conjugate to each other (cf. Section 8.5), and hence also $\widetilde{\Phi}_2$ and $\widetilde{\Phi}_3$ are conjugate to each other. ■

The Lemma shows that $\widetilde{\Phi}_j$ for $j = 1, 2$ is induced by the automorphism $\widetilde{\Phi}_j : E_2(A) \to E_2(A)$ (which is well-defined since, by (8.1), the unipotent groups $P^\pm$ defined in (7.8) are stable under $\widetilde{\Phi}_j$, $j = 1, 2$), i.e. $\widetilde{\Phi}_j$ is given by

$$\widetilde{\Phi}_j : \mathbb{P} \to \mathbb{P}, \quad g.o^+ \to \widetilde{\Phi}_j(g).o^+. \tag{8.4}$$

We say that an element $E \in \mathbb{P}$ is

- *hermitian* if $\widetilde{\Phi}_1(E) = E$,
- *anti-hermitian* if $\widetilde{\Phi}_2(E) = E$,
- *unitary* if $\widetilde{\Phi}_3(E) = E$.

Assume $E = \Gamma_z = [\left(\begin{smallmatrix} z \\ \mathbf{1} \end{smallmatrix}\right)] = \mathrm{im}(p)$ with $p = \left(\begin{smallmatrix} 0 & z \\ 0 & \mathbf{1} \end{smallmatrix}\right)$. Then $\Phi_1(p) = \left(\begin{smallmatrix} \mathbf{1} & -z^* \\ 0 & 0 \end{smallmatrix}\right)$ has kernel $[\left(\begin{smallmatrix} z^* \\ \mathbf{1} \end{smallmatrix}\right)]$. Thus the restriction of $\widetilde{\Phi}_1$ to $A = \Gamma(A)$ is the involution $*$, and E is hermitian if and only if z is hermitian. Similarly, we see that E is anti-hermitian if and only if z is anti-hermitian. Finally, E is unitary if and only if $[\left(\begin{smallmatrix} z \\ \mathbf{1} \end{smallmatrix}\right)] = [\left(\begin{smallmatrix} \mathbf{1} \\ z^* \end{smallmatrix}\right)]$. First of all, this implies that z must be invertible in

A, and then the condition $[\binom{z}{\mathbf{1}}] = [\binom{(z^*)^{-1}}{\mathbf{1}}]$ is equivalent to $z^{-1} = z^*$, i.e. to the unitarity of z.

The sets $\mathbb{P}_h$, resp. $\mathbb{P}_{ah}$ of hermitian, resp., (anti-)hermitian elements in $\mathbb{P}$ is called the *(anti-)hermitian projective line*; the set $\mathbb{P}_u$ of unitary elements is called the *unitary projective line.* The projective completion of $\mathrm{Herm}(A,*)$, resp. of $\mathrm{Aherm}(A,*)$ are the imbeddings

$$\Gamma : \mathrm{Herm}(A,*) \to \mathbb{P}_h, \qquad \Gamma : \mathrm{Aherm}(A,*) \to \mathbb{P}_{ah}.$$

This geometric picture can be imbedded into the three-graded picture simply by restricting the imbedding (7.7) to $\widetilde{\Phi}_j$-invariants.

8.5. The Cayley transform. The matrices F and $I_{1,1}$ are conjugate in $\mathrm{GL}_2(A)$ via C: $F = C^{-1} I_{1,1} C$. It follows that $C^{-1}(\mathbb{P}_{ah}) = \mathbb{P}_u$, i.e. the anti-hermitian and the unitary projective line are isomorphic. In particular, the unitary group $U(A,*)$ is injected into $\mathbb{P}_{ah}$ via

$$U(A,*) \to \mathbb{P}_{ah}, \quad z \mapsto C(\Gamma_z)$$

If $z - e$ is invertible, then the last term equals $\Gamma_{(z+e)(z-e)^{-1}}$ and it belongs to $\Gamma_{\mathrm{Aherm}(A)}$.

8.6. Manifold structures and symmetric spaces. If A is a c.i.a. over a topological ring $\mathbb{K}$ and $*$ is continuous, then $\mathrm{Herm}(A,*)$ and $\mathrm{Aherm}(A,*)$ are (C1)-Jordan triple systems. The corresponding manifold structure on the geometric models is again simply obtained by seeing everything as submanifolds fixed under $\widetilde{\Phi}_j$ in the models corresponding to A. The natural polarities given by the matrix $I_{1,1}$, resp. by F, define symmetric spaces: as explained in the preceding section, the unitary group arises as the space of non-isotropic points in the anti-hermitian projective line $\mathbb{P}_{ah}$; in particular, $\mathrm{U}(A,*)$ is a symmetric space. Moreover, with respect to the underlying manifold structure, also the group multiplication in $\mathrm{U}(A,*)$ is smooth (the calculation is exactly the same as the one for the orthogonal group $\mathrm{O}_n(\mathbb{R})$ in the Cayley chart), and hence $\mathrm{U}(A,*)$ is a Lie group. The natural symmetric space realized in the hermitian projective line is the space of invertible elements in the Jordan algebra $\mathrm{Herm}(A,*)$ (already encountered in Chapter 3), resp. its c-dual symmetric space. Since the set-up is almost the same as the one in [Be96] (where the special case $A = \mathrm{End}(V)$, $* =$ adjoint, was considered; cf. Example 8.7), we can refer to [Be96] and to [Be00] for further details of the calculations.

In general, there are many other polarities which are not isomorphic to the natural ones, and hence there are other symmetric spaces that can be realized inside $\mathbb{P}_h$ or $\mathbb{P}_{ah}$. In [Be00, XI.5] they have been called *conformally equivalent*, and for the classical series in finite dimension over $\mathbb{K} = \mathbb{R}$ a classification has been given. Roughly, one considers the set of all $\alpha \in \mathrm{Aut}(\mathfrak{g})$ such that $F^* \circ \alpha$ is a grading-reversing involution (where F^* is conjugation by F); it is called the *structure variety of* $\mathrm{Herm}(A,*)$*, resp. of* $\mathrm{Aherm}(A,*)$ (cf. [Be00, Section IV.2]). It contains, for instance, all "modifications" or "isotopes" given by

$$\alpha = \begin{pmatrix} 0 & H \\ H^{-1} & 0 \end{pmatrix}, \tag{8.5}$$

where H is an invertible element in $\mathrm{Herm}(A,*)$. Then one has to classify G-orbits in the structure variety. In finite dimension over the reals, topological connected components of the structure variety are homogeneous under G, and thus the task is relatively easy. In infinite dimension, or over other base fields or -rings, it seems possible that continuous families of non-isomorphic modifications may exist. This is an interesting topic for future research.

8.7. Example: algebras of endomorphisms. Let V be a $\mathbb{K}$-module equipped with a bilinear symmetric or skew-symmetric form $b : V \times V \to \mathbb{K}$ which is non-degenerate in the sense that the map $\sigma: V \to V^* := \mathrm{Hom}(V,\mathbb{K}), v \mapsto b(v,\cdot)$ is bijective. Let $A = \mathrm{End}(V)$ and define for $X \in \mathrm{End}(V)$ the adjoint $X^* \in \mathrm{End}(V)$ by $X^*.v := \sigma^{-1}(\sigma(v) \circ X)$. Of course, in a topological context one has to add further assumptions in order to ensure that A is a c.i.a. and that the adjoint map is continuous; e.g., one may assume that $\mathbb{K}$ is a topological field and V finite-dimensional over $\mathbb{K}$, or that we are in a Hilbert-space setting. Then $\Phi_1(X)$ is the adjoint of $X \in \mathrm{End}(V \oplus V)$ w.r.t. the bilinear form on $V \oplus V$ given by

$$\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}. \tag{8.6}$$

In particular, if b is a scalar product over $\mathbb{K} = \mathbb{R}$, then $\mathfrak{sp}(A,*)$ really is the symplectic Lie algebra $\mathfrak{sp}(V \times V, \mathbb{R})$. This is essentially the context considered in [Be96] (see also [Be00, Ch. VIII.4]). As is seen by elementary Linear Algebra (cf. loc. cit.), $\widetilde{\Phi}_1 : \mathbb{P} \to \mathbb{P}$ is then the "orthocomplement map" with respect to (8.6) (where $\mathbb{P}$ is the Grassmannian of subspaces of type V in $V \oplus V$ having complement of type V), and hence the hermitian projective line corresponds to the "Lagrangian variety with respect to the symplectic form", and the anti-hermitian projective line corresponds to "Lagrangians with respect to the quadratic neutral form" into which the orthogonal group $\mathrm{O}(V, b)$ can be imbedded.

9. A quantum mechanical interpretation

As explained in the introduction, there is a strong structural analogy between the mathematics considered in this work and the axiomatics of quantum mechanics. In the following, we give some examples for this structural analogy by proposing a "dictionary" between the language of generalized projective geometries and the language of quantum mechanics. This dictionary is by no means complete – we do not attack topics such as spectral theory of our observables or the use of unbounded operators. However, it seems that the theory of Jordan pairs and -triple systems is rich and flexible enough to incorporate such aspects; we intend to investigate these questions in future work. Our references for classical, linear Quantum Theory are [Th81] and [Va85]. According to [Th81, p. 33], the "Basic Assumption of Quantum Theory" is formulated as follows: "The observables and states of a system are described by hermitian elements a of a C^*-algebra A and by states on A." Let us see what this assumption implies if one tries to interprete it on the level of the projective completion of the algebra of hermitian elements. Consequently, we will start with the observables and not with the states.

9.1. Observables. The *space of observables* is the space X^+ of a generalized projective geometry (X^+, X^-). The space X^- may be called the "space of non-observables" or the "space of observers". As *standard model* we may take the hermitian projective line $X^+ = \mathbb{P}_h$ over an (infinite-dimensional) associative involutive c.i.a $(A,*)$. In this case, X^+ and X^- are canonically isomorphic (the isomorphism is a *canonical null-system* in the sense of [Be03a]). For a general approach, it seems not necessary to assume that $\mathbb{K} = \mathbb{C}$.

9.2. States and pure states. A *state* is an *intrinsic subspace of* X^+, i.e. a subset $Y \subset X^+$ which appears *linearly* (i.e. as an affine subspace) with respect to *any* affinization $y \in X^-$. Such subspaces correspond to *inner ideals of* V^+ in Jordan theory (cf. [Be02, 2.7.(4)], [BL04]). A *pure state* is an *intrinsic line*, i.e. a proper intrinsic subspace which is minimal for inclusion. The *superposition* of two pure states is the intrinsic subspace generated by the two lines. Under some additional assumptions, pure states correspond to *division idempotents* of the Jordan pair, and spaces of certain states form again a generalized projective geometry (cf. [Ka01] for results that point into this direction). Pure states correspond to *rank-one elements* (cf. [Lo94] for the notion of "rank"), and they are closely related to *chains* in the sense of Chain Geometry (cf. [H95]).

9.3. The Hamilton operator. A *Hamilton operator* is a polarity $p : X^+ \to X^-$ (cf. Section 6.1). A Hamilton operator is called *free* if the polarity p is an *inner polarity* in the sense of [Be03a]. In the standard model, there exists a free Hamilton operator p_0, given by the matrix F (called the "natural polarity" in Section 8.6). Then a general Hamilton operator can be seen as a deformation or *modification* of the free one as explained in Section 8.6; in particular, via Equation (8.5) every invertible hermitian element H leads to new Hamilton operator that needs not be conjugate to p_0. Note that the canonical identification $X^+ = X^-$ $(= \mathbb{P}_h)$ in the standard model is *not* a Hamilton operator because it is a null-system.

9.4. The time dependent Schrödinger equation. The *time dependent Schrödinger equation* is a dynamical differential equation canonically associated to the Hamilton operator p. Of course, here one thinks first of the *geodesic differential equation* in the symmetric space $M := M^{(p)} \subset X^+$ associated to the Hamilton operator p. (Every symmetric space carries a canonical torsionfree connection ∇ (cf. [Be03b]), and a geodesic is simply a smooth map $\alpha : \mathbb{K} \supset I \to M$ which is compatible with connections. In a chart, the geodesic equation is as usual $\alpha''(t) = C_{\alpha(t)}(\alpha'(t), \alpha'(t))$ where C is the Christoffel tensor of ∇ in the chart.) However, as pointed out in [AS97], the Schrödinger evolution should rather be seen as a Hamiltonian flow and not as a solution of a second order differential equation. But it is possible to reconcile these two aspects inside the category of generalized projective geometries because the tangent geometry (TX^+, TX^-, Tp) is again of the same type, and here the geodesic flow of $M^{(p)}$ appears as flow of a vector field, namely of the *spray associated to the canonical connection of* $M^{(p)}$ (cf. [Be03b]).

9.5. The time independent Schrödinger equation. An *eigenstate* of the Hamilton operator p is an intrinsic line which at the same time is a geodesic on $M^{(p)}$. They correspond to division tripotents of the Jordan triple system associated to p. A *complete system of eigenstates* corresponds to a *frame* of the Jordan triple system. The *time independent Schrödinger equation* consists in decomposing a given tripotent with respect to a frame.

9.6. Quantization. Note that some models of special and general relativity such as Minkowski space and the de Sitter- and anti-de Sitter model (and more general *causal symmetric spaces*) can be realized via generalized projective geometries ([Be96], [Be00]). It would be tempting to interprete a quantization of such spaces as a sort of representation of these finite-dimensional geometries in an infinite-dimensional geometry.

10. Prospects

10.1. Generalizations. The differential calculus developed in [BGN03] works in more general contexts, called "C^0-concepts", than the one of topological rings and modules. For instance, we may consider the class of rational mappings defined on Zariski-open sets in finite-dimensional vector spaces over an arbitrary infinite field $\mathbb{K}$ and define the class C^1 as in Section 1.3, where now C^0 means "rational". Essentially all results of the present work carry over to this more general framework (details are left to the reader). In particular, all finite-dimensional Jordan algebras, -triple systems and -pairs over arbitrary infinite fields are automatically "continuos (quasi-) inverse" since the formulas for (Bergman-) inversion clearly are rational. Thus, in finite dimensions over infinite fields, the projective completion is always a "smooth rational manifold" in the sense of [BGN03], and our construction yields "smooth rational symmetric spaces". All notions of differential geometry from [Be03b] continue to make sense in this setting.

10.2. Lie group actions. In the context of Theorem 5.3, one would like the projective group G to be a Lie group acting smoothly on the projective completion $X^{\pm}$. However, in general it seems impossible to define a Lie group structure on G because G is defined by generators, and it is very hard to find a good atlas for the subgroup H. In the real or complex Banach set-up, this problem can be avoided by taking instead of G and H the "much bigger" groups $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{g}, D)$ which are Banach Lie groups, and then realizing X^+ as a quotient manifold under the action of $\mathrm{Aut}(\mathfrak{g})_0$. This is the strategy used in [Up85]; it needs a fair amount of non-trivial functional analysis and does not carry over to more general situations.

Nevertheless, the problem remains wether in our general set-up it is possible to find some extension of G to a Lie group $\widetilde{G}$ acting smoothly on $X^{\pm}$. For instance, in the case of the standard models (Sections 7 and 8) this is the case: in case of the projective line we may take $\widetilde{G} = \mathrm{GL}_2(A)$ which is indeed a Lie group (if A is a c.i.a., then the algebra $M_n(A)$ of $n \times n$-matrices with entries in A is a c.i.a (cf. [Bos90], [Gl02]), and hence $\mathrm{GL}_n(A)$ is a Lie group), and in case of the (anti-) hermitian projective line we may take $\widetilde{G} = \mathrm{Sp}(A,*)$, resp. $\widetilde{G} = \mathrm{U}(A,A,*)$ which are unitary groups associated to an involutive c.i.a. and hence, as we have seen in Section 8.6, are Lie groups. We intend to investigate the problem of Lie group extensions of general projective groups in future work.

References

[AS97] Ashtekar, A. and T.A. Schilling, *Geometrical Formulation of Quantum Mechanics*, preprint 1997, arXive: gr-qc/9706069.

[Be96] Bertram, W., *On some causal and conformal groups*, J. Lie Theory **6** (1996), 215 – 247.

[Be00] Bertram, W., "The Geometry of Jordan and Lie Structures," Lecture Notes in Math. **1754**, Springer-Verlag, Berlin, 2000.

[Be02] —, *Generalized projective geometries: general theory and equivalence with Jordan structures*, Advances in Geometry **2** (2002), 329–369.

[Be03a] —, *The geometry of null-systems, Jordan algebras and von Staudt's theorem*, Ann. Inst. Fourier **53** (1) (2003), 193 – 225.

[Be03b] —, *Differential geometry over general base fields and -rings. Part I: First and scond order theory*, preprint, Nancy 2003.

[BGN03] Bertram, W., Glöckner, H. and K.-H. Neeb, *Differentiable calculus, manifolds and Lie groups over arbitrary infinite fields*, to appear in Expos. Math., arXive: math.GM/0303300.

[BL04] Bertram, W. and H. Löwe, *Incidence geometries associated to Jordan pairs*, in preparation.

[BN03] Bertram, W., and K.-H. Neeb, *Projective completions of Jordan pairs. Part I. Geometries associated to 3-graded Lie algebras*, to appear in: J. of Algebra, arXive: math.RA/0306272.

[BlHa01] Blunck, A. and H. Havlicek, *The connected components of the projective line over a ring*, Advances in Geometry **1** (2001), 107–117.

[Bo90] Bost, J.-B., *Principe d'Oka, K-theorie et systèmes dynamiques non-commutatifs*, Invent. Math. **101** (1990), 261–333.

[CGM03] Cirelli, R., Gatti M. and A. Manià, *The pure state space of quantum mechanics as hermitian symmetric space*, J. of Geometry and Physics **45** (2003), p. 267 – 284.

[DNS89] Dorfmeiser, J., E. Neher, and J. Szmigielski, *Automorphisms of Banach manifolds associated with the KP-equation*, Quart. J. Math. Oxford (2) **40** (1989), 161–195.

[DNS90] —, *Banach manifolds and their automorphisms associated with groups of type C_∞ and D_∞*, in "Lie algebras and related topics," Proc. Res. Conf., Madison/WI (USA) 1988, Contemp. Math. **110** (1990), 43–65.

[FK94] Faraut, J. and A. Koranyi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford 1994.

[Gl01a] Glöckner, H., *infinite-dimensional Lie groups without completeness restrictions*, in: Strasburger, A. et al. (Eds.) "Geometry and analysis on Lie groups," Banach Center Publications, Vol. **55**, Warsawa 2002; 53–59.

[Gl02] —, *Algebras whose groups of units are Lie groups*, Studia Math. **153:2** (2002), 147–177.

[dlH72] de la Harpe, P., "Classical Banach Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space," Lecture Notes in Math. **285**, Springer-Verlag, Berlin, 1972.

[Ha82] Hamilton, R., *The inverse function theorem of Nash and Moser*, Bull of the Amer. Math. Soc. **7** (1982), 65–222.

[H95] Herzer, A., *Chain geometries.* In: F. Buekenhout (editor), “Handbook of Incidence Geometry,” Elsevier 1995.

[IM02] Isidro, J. M. and M. Mackey, *The manifold of finite rank projections in the algebra $\mathcal{L}(H)$ of bounded linear operators*, preprint, Santiago 2002.

[JNW34] Jordan, P., von Neumann, J. and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35** (1934), 29 – 64.

[Ka83] Kaup, W., *Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension I, II*, Math. Annalen **257**(1981), 463–486; **262**(1983), 57–75.

[Ka01] —, *On Grassmannians associated with JB^*-triples*, Math. Z. **236** (2001), 567–584.

[Ke74] Keller, H. H., “Differential Calculus in Locally Convex Spaces,” Lecture Notes in Math., Springer-Verlag, 1974.

[Lo69] Loos, O., “Symmetric Spaces I,” Benjamin, New York, 1969.

[Lo75] —, “Jordan Pairs,” Springer LNM 460, Berlin, 1975.

[Lo94] —, *Decomposition of projective spaces defined by unit-regular Jordan pairs*, Comm. Alg. **22** (10) (1994), 3925–3964.

[Lo96] —, *On the set of invertible elements in Banach–Jordan algebras*, Results in Math. **29** (1996), 111-114.

[MM01] Mackey, M., and P. Mellon, *Compact-like manifolds associated to JB^*-triples*, manuscripta math. **106** (2001), 203–212.

[MC03] McCrimmon, K., “A Taste of Jordan Algebras,” Springer-Verlag, New York 2003.

[Ne02] Neeb, K.-H., *A Cartan-Hadamard theorem for Banach-Finsler Manifolds*, Geom. Dedicata **95** (2002), 115–156.

[PS86] Pressley, A., and G. Segal, “Loop Groups,” Oxford University Press, Oxford, 1986.

[Th81] Thirring, W., “A course in Mathematical Physics. Vol. 3: Quantum Mechanics of Atoms and Molecules,” Springer-Verlag, New York 1981.

[Up85] Upmeier, H., “Symmetric Banach Manifolds and Jordan C^*-algebras,” North Holland Mathematics Studies, 1985.

[Va85] Varadarajan, V.S., “Geometry of Quantum Theory,” Springer-Verlag, New York 1985.

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