Every smooth p-adic Lie group admits a compatible analytic structure

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Abstract. We show that every finite-dimensional p-adic Lie group of class C^k (where $k \in \mathbb{N} \cup \{\infty\}$) admits a C^k-compatible analytic Lie group structure. We also construct an exponential map for every $k+1$ times strictly differentiable (SC^{k+1}) ultrametric p-adic Banach-Lie group, which is an SC^1-diffeomorphism and admits Taylor expansions of all finite orders $\leq k$.

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Introduction

It is well-known that every finite-dimensional real Lie group of class C^k (where $1 \leq k \leq \infty$) admits a C^k-compatible analytic manifold structure making the group operations analytic ([3], [9], [17], [21], [22], [24]; also [16, §4.4]), and similar (slightly weaker) results are valid for real Banach-Lie groups [15]. It therefore suffices for all practical purposes to ignore the case of finite order differentiability and restrict attention to smooth real Lie groups, or to analytic real Lie groups whenever this is profitable.

In this paper, we prove analogous results for p-adic Lie groups:

Theorem A. *Let G be a finite-dimensional Lie group of class $C^k_{\mathbb{K}}$ (where $k \in \mathbb{N} \cup \{\infty\}$) over a valued field $\mathbb{K}$ which is a finite extension of $\mathbb{Q}_p$. Then there exists a $\mathbb{K}$-analytic manifold structure on G making it a $\mathbb{K}$-analytic Lie group, and which is $C^k_{\mathbb{K}}$-compatible with the given $C^k_{\mathbb{K}}$-manifold structure.*

Here, $C^k_{\mathbb{K}}$-maps are understood in the sense of [2], where a setting of differential calculus over arbitrary non-discrete topological fields $\mathbb{K}$ is described. We recall that mappings between open subsets of finite-dimensional $\mathbb{Q}_p$-vector spaces are $C^1_{\mathbb{Q}_p}$ if and only if they are strictly differentiable (where strict differentiability is a classical concept, [5]). Thus, taking $k = 1$, Theorem A subsumes as a special case that every strictly differentiable finite-dimensional p-adic Lie group admits a compatible analytic structure making it a p-adic Lie group in the usual sense, as considered in [7], [14], and [23]. We also obtain valuable structural information concerning ultrametric Banach-Lie groups (Proposition 2.1, Remark 2.2, Proposition 3.3, Corollary 4.3):

Theorem B. *Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathbb{K}$ be a complete ultrametric field. Assume that G is a Lie group of class $C^{k+1}_{\mathbb{K}}$ modelled on an ultrametric Banach space over $\mathbb{K}$, or that $\mathbb{K}$ is* *locally compact and G a finite-dimensional $C^k_{\mathbb{K}}$-Lie group over $\mathbb{K}$. Then G is complete, and there exists a diffeomorphism $\phi\colon U \to B_r(0)$ from an open subgroup $U \subseteq G$ onto an open ball $B_r(0) \subseteq L(G)$, with the following properties:*

(a) *The inverse images $U_s := \phi^{-1}(\overline{B}_s(0))$ of closed balls for $s \in \,]0, r[$ are open, normal subgroups of U. The family of open subgroups $(U_s)_{s<r}$ is a basis of identity neighbourhoods for G. If G is $C^2_{\mathbb{K}}$, then $(U_s)_{s<r}$ defines a filtration for U.*

(b) *If $\operatorname{char}(\mathbb{K}) = p > 0$, then $\mathbb{Z} \to U$, $n \mapsto x^n$ extends to a continuous homomorphism $\eta_x\colon \mathbb{Z}_p \to U$, for each $x \in U$.*

(c) *If $\operatorname{char}(\mathbb{K}) = 0$ and $\mathbb{K}$ is a valued extension field of $\mathbb{Q}_p$ for some p, then $\mathbb{Z} \to U$, $n \mapsto x^n$ extends to a continuous homomorphism $\eta_x : \mathbb{Z}_p \to U$ which actually is $C^1_{\mathbb{Q}_p}$, for each $x \in U$. The map $\log_G : U \to L(G)$, $\log_G(x) := \eta'_x(0)$ is an $SC^1_{\mathbb{K}}$-diffeomorphism[1] onto $B_r(0) \subseteq L(G)$, such that*

$$\exp_G := \log_G^{-1} : B_r(0) \to U$$

is an exponential map (see Definition 5.2) and an $SC^1_{\mathbb{K}}$-diffeomorphism. In the Banach case, let us assume now that G is not only $C^{k+1}_{\mathbb{K}}$ but $C^{k+2}_{\mathbb{K}}$ (or at least $k+1$ times strictly differentiable). Then each continuous homomorphism $\mathbb{Z}_p \to G$ is $C^k_{\mathbb{Q}_p}$, and we can achieve that, in local coordinates, $\exp_G$ admits Taylor expansions of all finite orders $\leq k$.

Classical construction of the analytic structure. The exponential map $\exp_G : L(G) \to G$ of a finite-dimensional smooth Lie group G over $\mathbb{R}$ can be obtained via $\exp_G(X) := \gamma_X(1)$, where γ_X is the uniquely determined integral curve starting in 1, to the left invariant vector field X_ℓ on G with $X_\ell(1) = X$. Here γ_X can also be described as the unique C^1-homomorphism $\mathbb{R} \to G$ with $\gamma'_X(0) = X$. Due to smooth dependence of solutions on parameters, $\exp_G$ is smooth and induces a local diffeomorphism at 0, giving rise to a chart for G ("coordinates of the first kind"). It turns out that the group operations are analytic with respect to this chart, because solutions to linear differential equations with analytic coefficients are analytic, and depend analytically on initial conditions and parameters. Using translations, the analytic structure can then be spread over all of G.

Difficulties in the ultrametric case. It is impossible to adapt the classical construction just described to the ultrametric case. For example, integral curves stop to be useful, since neither can we assume their existence (nor local uniqueness), nor do they need to define homomorphisms (nor local homomorphisms) if they exist. For instance, consider a smooth injective map $\gamma\colon \mathbb{Z}_p \to \mathbb{Q}_p$ such that $\gamma(0) = 0$ and $\gamma' \equiv 0$ [21, Exercise 29.G].[2] Any such γ provides an example of a smooth integral curve for the invariant vector field $X_\ell \equiv 0$ on

[1] A map between open subsets of normed spaces is called $SC^1_{\mathbb{K}}$ if it is strictly differentiable at each point. Every $C^2_{\mathbb{K}}$-map is $SC^1_{\mathbb{K}}$, and every $SC^1_{\mathbb{K}}$-map is $C^1_{\mathbb{K}}$ [11].

[2] Typical examples of such maps send a p-adic integer $\sum_{j=0}^{\infty} a_j p^j$ (where $a_j \in \{0, \ldots, p-1\}$) to $\sum_{j=0}^{\infty} a_j p^{k_j}$, where $0 < k_1 < k_2 < \cdots$ is an ascending sequence of integers going to ∞ sufficiently fast.

$(\mathbb{Q}_p, +)$, such that γ does not coincide with a homomorphism on any zero-neighbourhood in $\mathbb{Z}_p$, and fails to be analytic on any zero-neighbourhood. Furthermore, the germ of γ is not determined by the initial value problem, because also $\eta \equiv 0$ is a solution.[3]

Strategy of proof. In view of the difficulties just described, it is clear that a different strategy of proof is needed to construct an analytic structure on a finite-dimensional p-adic C^k-Lie group G, which strictly avoids any recourse to differential equations. Here, the crucial idea is to use Lazard's characterization of analytic p-adic Lie groups:

A topological group G can be given a (necessarily unique) finite-dimensional p-adic analytic Lie group structure if and only if G has an open, compact subgroup U such that:

L1 *U is a pro-p-group*;

L2 *U is finitely generated topologically, i.e., $U = \overline{\langle F \rangle}$ for a finite subset $F \subseteq U$; and*:

L3 *The set of p^2-th powers of elements of U contains the commutator subgroup of U, $[U, U] \subseteq \{x^{p^2} : x \in U\}$.*

(Cited from [23, p. 157]; cf. [14, A1, Thm. (1.9)]). Hence, to construct a compatible analytic structure on a finite-dimensional p-adic C^k-Lie group G, we need to master two tasks:

(a) Show that G satisfies conditions **L1–L3**.

(b) Show that Lazard's analytic Lie group structure is C^k-compatible with the given C^k-manifold structure.

Here, an open subgroup U of G satisfying **L1** and **L3** can be constructed as in the analytic case, based on Inverse Function Theorems for C^k-maps (see [11]) and Taylor expansions. In the Banach case, similar arguments also provide the filtration $(U_s)_{s<r}$ described in Theorem B (a), and show that $\{x^{p^n} : x \in U_s\} \subseteq U_{p^{-n}s}$ for each $n \in \mathbb{N}$ in the situation of Theorem B (b) and (c), entailing that the homomorphism $\mathbb{Z} \to U$, $z \mapsto x^z$ is continuous with respect to the p-adic topology and hence extends to a continuous homomorphism $\eta_x : \mathbb{Z}_p \to G$ (Section 2). We now assume that $\mathbb{K}$ is an extension field of $\mathbb{Q}_p$. To establish **L2**, the natural strategy is to try and introduce coordinates of the second kind on U. So far, we only know that *continuous* p-adic one-parameter groups exist. In a first step (Section 3), we show that the one-parameter groups η_x are $C^1_{\mathbb{Q}_p}$ in fact, which enables us to define $\log_G$ (as in Theorem B (c)), deduce its strict differentiability by iterating first order Taylor expansions of the p-th power map, and to define $\exp_G := \log_G^{-1}$. While it is not hard to see that every C^1-homomorphism between smooth Lie groups modelled on real locally convex spaces is automatically smooth [13], such a result is not available here.[4] It is therefore necessary to prove by hand that every one-parameter group $\mathbb{Z}_p \to G$ is $C^k_{\mathbb{Q}_p}$.

[3] Actually, the germs around 0 of smooth maps $\zeta : \mathbb{Z}_p \to \mathbb{Q}_p$ with $\zeta(0) = 0$ and $\zeta' \equiv 0$ form a $\mathbb{Q}_p$-vector space of dimension $2^{\aleph_0}$.

[4] In the p-adic case, one needs to control iterated difference quotient maps instead of mere higher differentials, making it impossible to adapt the proof.

This is the most difficult problem arising in our construction. We solve it in Section 4, where we express $\exp_G$ as a limit of iterates of the inverse of the p-th power map τ_p:

$$\exp_G(x) = \lim_{n\to\infty} \tau_p^{-n}(p^n x)$$

and deduce that (in local coordinates) $\exp_G$ admits Taylor expansions of all finite orders $\leq k$ from the fact that so does τ_p^{-1}. To this end, we substitute the k-th order Taylor expansions of each τ_p^{-1} in the n-th iterate $x \mapsto \tau_p^{-n}(p^n x)$ into each other, multiply out, and then prove by tracking each individual term of the resulting sum that the homogeneous terms of each given order converge, and that the polynomials resulting as the limits actually provide a Taylor expansion for $\exp_G$ with a remainder term vanishing of the required order. Now $\exp_G$ possessing a k-th order Taylor expansion, so do the one-parameter groups $\mathbb{Z}_p \to G$, $z \mapsto \exp_G(zx)$, whence they are $C^k_{\mathbb{Q}_p}$-maps by Schikhof's Converse to Taylor's Theorem for Curves ([19, Thm. 10.7], [20, Thm. 83.5]).

In the case of a finite-dimensional p-adic $C^k_{\mathbb{Q}_p}$-Lie group, it is now straightforward to verify condition **L2**, whence a p-adic analytic Lie group structure on G exists. It only remains to show its $C^k_{\mathbb{Q}_p}$-compatibility with the given structure. After adapting the Trotter product formula and other classical ideas from the analytic case to p-adic C^k-Lie groups, we prove compatibility by showing the existence of coordinates of the second kind which work simultaneously for both the analytic and the given C^k-Lie group structure (Section 5).

In an appendix, we prove various lemmas compiled in a preparatory Section 1.

Directions for further research. Beyond Theorem A, one would expect that every $C^{k+2}_{\mathbb{K}}$-Lie group G modelled on an ultrametric Banach space over a valued extension field $\mathbb{K}$ of $\mathbb{Q}_p$ admits a $C^k_{\mathbb{K}}$-compatible $\mathbb{K}$-analytic Lie group structure. Two steps are still missing to achieve this goal: 1. Show that $\exp_G$ is not only an $SC^1_{\mathbb{K}}$-diffeomorphism admitting finite order Taylor expansions, but actually a $C^k_{\mathbb{K}}$-diffeomorphism. 2. Show that $\exp_G$ induces an isomorphism of groups from some ball in $L(G)$, equipped with the Baker-Campbell-Hausdorff multiplication, onto a subgroup of G. Here, the quite complicated notion of higher order differentiability makes it technically difficult to perform Step 1. Possibly, Step 2 might be based on an adaptation of the construction of the analytic structure on uniformly powerful pro-p-groups via coordinates of the first kind, as described in [8, §9.4]. Recall that the finite extension fields of $\mathbb{Q}_p$ occurring in Theorem A are precisely the local fields[5] of characteristic 0 [25]. From this perspective, it is very natural to ask whether Theorem A remains valid for finite-dimensional Lie groups over local fields of positive characteristic. However, due to the lack of natural, preferred coordinate systems (and the impossibility of an analogue of Lazard's Theorem) in positive characteristic, it is not at all clear how one might approach the problem in this case.

[5] That is to say, the totally disconnected, locally compact, non-discrete topological fields.

1 Basic definitions and facts

In this section, we briefly recall some notations and basic facts from the papers [2] and [11], which are our basic references for differential calculus over non-discrete topological fields (see also [1] and [12]). Then, various (new) lemmas are formulated, most of which are based on Taylor expansions and generalize classical facts from multivariable calculus over $\mathbb{R}$. We recommend to take these lemmas on faith at this point; if desired, the proofs can be looked up later in Appendix A.

Definition 1.1 Let $\mathbb{K}$ be a non-discrete (Hausdorff) topological field, E and F be (Hausdorff) topological $\mathbb{K}$-vector spaces, and $f\colon U \to F$ be a map on an open subset of E. We call f C^0 if it is continuous; it is C^1 if it is C^0 and if there exists a (necessarily unique) continuous map $f^{[1]}\colon U^{[1]} \to F$ on $U^{[1]} := \{(x,y,t) \in U \times E \times \mathbb{K} \colon x + ty \in U\}$ such that $f^{[1]}(x,y,t) = t^{-1}(f(x+ty) - f(x))$ if $t \neq 0$. Inductively, f is called C^k for $k \in \mathbb{N}$ if f is C^1 and $f^{[1]}$ is C^{k-1}; we then let $f^{[k]} := (f^{[1]})^{[k-1]}$. The map f is C^∞ or smooth if it is C^k for all $k \in \mathbb{N}$. We also write $C^k_{\mathbb{K}}$ instead of C^k ($k \in \mathbb{N}_0 \cup \{\infty\}$) to emphasize the ground field.

If $E = \mathbb{K}$ here, we call $f\colon U \to F$ a *curve*. A curve $f\colon U \to F$ is C^1 if and only if there exists a continuous map $f^{<1>}\colon U \times U \to F$ such that $f^{<1>}(x,y) = (x-y)^{-1}(f(x) - f(y)) =: f^{>1<}(x,y)$ for all $x, y \in U$ such that $x \neq y$ [2, La. 6.1].

1.2 It can be shown that compositions of composable C^k-maps are C^k [2, Prop. 4.5]. Furthermore, for every C^k-map $f\colon E \supseteq U \to F$, the iterated directional derivatives

$$d^j f(x, y_1, \ldots, y_j) := (D_{y_1} \cdots D_{y_j} f)(x) \qquad \text{for } x \in U,\ y_1, \ldots, y_j \in E$$

exist for all $j \in \mathbb{N}$ such that $j \leq k$, and define continuous maps $d^j f : U \times E^j \to F$ such that $d^j f(x, \bullet) : E^j \to F$ is symmetric j-linear, for each $x \in U$. The associated homogeneous polynomials are denoted $\delta^j_x f : E \to F$, $\delta^j_x f(v) := d^j f(x, v, \ldots, v)$. We occasionally abbreviate $f'(x) := df(x, \bullet) \colon E \to F$. Every C^k-map $f : U \to F$ as before, where $k \in \mathbb{N}$, admits a k-th order Taylor expansion

$$f(x+ty) - f(x) = \sum_{j=1}^{k} t^j\, a_j(x,y) + t^k R_k(x,y,t) \qquad \text{for all } (x,y,t) \in U^{[1]},$$

for uniquely determined maps $a_j\colon U \times E \to F$ of class C^{k-j} which are F-valued forms of degree j in the second argument, and where $R_k\colon U^{[1]} \to F$ is a continuous map such that $R_k(x,y,0) = 0$ for all $(x,y) \in U \times E$ (see [2], Thm. 5.1 and Thm. 5.4, where R_k is denoted R_{k+1}). If f is C^ℓ actually with $\ell > k$, then R_k is $C^{\ell-k}$. Since $\delta^j_x f(y) = j!\, a_j(x,y)$, the Taylor expansion attains the familiar form when $\operatorname{char}(\mathbb{K}) = 0$.

1.3 If $\mathbb{K}$ is a valued field here (whose absolute value $|.| : \mathbb{K} \to [0, \infty[$ we shall always assume non-trivial), E a normed $\mathbb{K}$-vector space and F a polynormed $\mathbb{K}$-vector space (viz.

a topological $\mathbb{K}$-vector space F whose vector topology can be obtained from a family of continuous seminorms $\|.\|_\gamma$ on F), then it is also natural to consider

$$R\colon U \times U \to F, \qquad R(x,y) := R_k(x, y-x, 1)\,.$$

We call both R_k and R the k-th order Taylor remainder; no confusion is likely.

1.4 Let E and F be normed vector spaces over a valued field $\mathbb{K}$, and $f : U \to F$ be a map on an open subset of E. Given $x \in U$, f is called *strictly differentiable at x* if $\lim_{(y,z)\to(x,x)} \frac{\|f(z)-f(y)-f'(x).(z-y)\|}{\|z-y\|} = 0$ for some (necessarily unique) continuous linear map $f'(x)\colon E \to F$ (where $y \neq z$). The map f is *strictly differentiable* (or SC^1) if it is strictly differentiable at each $x \in U$. Then f is C^1, and $f'(x) = df(x, \bullet)$. Recursively, the map is called k times strictly differentiable (or SC^k) if it is SC^1 and $f^{[1]}$ is SC^{k-1}. The following can be shown (see [11]): If f is SC^k, then f is C^k. If f is C^{k+1}, then f is SC^k. If $\mathbb{K}$ is locally compact and E is finite-dimensional, then f is C^k if and only if f is SC^k.

1.5 If $(E, \|.\|)$ is a normed space over a valued field $(\mathbb{K}, |.|)$, $x \in E$ and $r > 0$, we write $B^E_r(x) := \{y \in E\colon \|y-x\| < r\}$ and $\overline{B}^E_r(x) := \{y \in E\colon \|y-x\| \leq r\}$, or simply $B_r(x)$ and $\overline{B}_r(x)$, when E is understood. If the absolute value $|.|$ on $\mathbb{K}$ satisfies the ultrametric inequality $|x+y| \leq \max\{|x|, |y|\}$, we shall call $(\mathbb{K}, |.|)$ an *ultrametric field.* In this case, if also $\|.\|$ is ultrametric, then each ball $B_r(0)$ or $\overline{B}_r(0)$ around 0 is an open and closed additive subgroup of E. Then $B_r(x) = x + B_r(0)$ is a coset and hence also open and closed, and $B_r(x) = B_r(y)$ for each $y \in B_r(x)$ (and likewise for $\overline{B}_r(x)$).

While in the case of a real normed space every non-zero vector v can be normalized, in general we cannot find $r \in \mathbb{K}$ such that $\|rv\| = 1$ when working over a valued field $(\mathbb{K}, |.|)$ whose value group $|\mathbb{K}^\times|$ is a proper subgroup of $]0, \infty[$. As a substitute for normalization, in many of the proofs we shall fix some element $a \in \mathbb{K}^\times$ with $|a| < 1$, and then consider $a^{-k}v$ where $k \in \mathbb{Z}$ is chosen such that $|a|^{k+1} \leq \|v\| < |a|^k$.

The following lemmas will be needed later (for mappings into Banach spaces).

Lemma 1.6 *Let E be a normed space over a valued field $\mathbb{K}$, F be a polynormed $\mathbb{K}$-vector space, and $f\colon U \to F$ be a $C^2_{\mathbb{K}}$-map on an open subset $U \subseteq E$. Then, for every $x_0 \in U$ and continuous seminorm $\|.\|_\gamma$ on F, there are $\delta > 0$ and $C > 0$ such that $B_{2\delta}(x_0) \subseteq U$ and*

$$\|f(x+y) - f(x) - f'(x).y\|_\gamma \leq C \cdot \|y\|^2 \quad \textit{for all } x \in B_\delta(x_0) \textit{ and } y \in B_\delta(0).$$

Lemma 1.7 *Let $(\mathbb{K}, |.|)$ be a valued field, E, F be normed spaces over $\mathbb{K}$, H be a polynormed $\mathbb{K}$-vector space, $U \subseteq E$ and $V \subseteq F$ open zero-neighbourhoods, $\|.\|_\gamma\colon H \to [0, \infty[$ a continuous seminorm, and $f\colon U \times V \to H$ be a mapping of class $C^2_{\mathbb{K}}$ such that, for certain continuous linear maps $\lambda\colon E \to H$ and $\mu\colon F \to H$, we have*

$$f(x,0) = f(0,0) + \lambda(x) \quad \textit{for all } x \in U \textit{ and} \tag{1}$$
$$f(0,y) = f(0,0) + \mu(y) \quad \textit{for all } y \in V. \tag{2}$$

Then there exists $\delta > 0$ and a constant $C > 0$ such that $B^E_\delta(0) \subseteq U$, $B^F_\delta(0) \subseteq V$ and

$$\|f(x,y) - f(0,0) - \lambda(x) - \mu(y)\|_\gamma \leq C\|x\|\,\|y\| \quad \textit{for all } x \in B^E_\delta(0) \textit{ and } y \in B^F_\delta(0).$$

Lemma 1.8 *Let E be a normed vector space over a valued field $\mathbb{K}$, F be a polynormed $\mathbb{K}$-vector space, $f: U \to F$ be a map on an open subset of E, and $k \in \mathbb{N}$. If f is of class $C^{k+1}_{\mathbb{K}}$ or if f is of class $C^k_{\mathbb{K}}$, $\mathbb{K}$ is complete and E finite-dimensional, then the map*

$$U \to \mathcal{L}^k(E,F)\,, \quad x \mapsto d^k f(x, \bullet)$$

is continuous.

Here, we equip the $\mathbb{K}$-vector space $\mathcal{L}^k(E,F)$ of continuous k-linear maps $\beta : E^k \to F$ with the vector topology determined by the family of seminorms $\|.\|_\gamma : \mathcal{L}^k(E,F) \to [0,\infty[$ defined via $\|\beta\|_\gamma := \inf\{C > 0 : (\forall x_1, \ldots, x_k \in E)\ \|\beta(x_1,\ldots,x_k)\|_\gamma \leq C\|x_1\|\cdots\|x_k\|\}$, where $\|.\|_\gamma : F \to [0,\infty[$ ranges through the continuous seminorms on F (the double use of the symbol $\|.\|_\gamma$ should not create confusion). We set $\mathcal{L}(E,F) := \mathcal{L}^1(E,F)$ and $\mathcal{L}(E) := \mathcal{L}(E,E)$.

Lemma 1.9 *Let E be a normed vector space over a valued field $\mathbb{K}$ of characteristic 0, F be a polynormed $\mathbb{K}$-vector space, $k \in \mathbb{N}$ and $f: U \to F$ be a map on an open subset of E such that f is $C^{k+1}_{\mathbb{K}}$, or such that f is of $C^k_{\mathbb{K}}$, $\mathbb{K}$ locally compact, and E finite-dimensional. Let $R: U \times U \to F$, $R(x,y) := f(y) - f(x) - \sum_{j=1}^k \frac{1}{j!}\,\delta^j_x f(y-x)$ be the k-th order Taylor remainder. Then, for any $z \in U$ and any continuous seminorm $\|.\|_\gamma$ on F, we have*

$$\lim_{(x,y)\to(z,z)} \frac{\|R(x,y)\|_\gamma}{\|x-y\|^k} = 0 \quad (\textit{where } x \neq y). \tag{3}$$

The following partial converse will be used essentially:

Lemma 1.10 *Let E be a normed vector space over an ultrametric field $\mathbb{K}$ of characteristic 0, F be an ultrametric Banach space over $\mathbb{K}$, $k \in \mathbb{N}$, and $f: U \to F$ be a function on an open subset $U \subseteq E$. We assume that f can be written in the form*

$$f(y) = f(x) + \sum_{j=1}^k a_j(x, y-x) + R(x,y) \quad \textit{for all } x, y \in U,$$

where $a_j: U \times E \to F$ is a continuous mapping for $j \in \{1,\ldots,k\}$ such that $a_j(x,\bullet): E \to F$ is a homogeneous polynomial of degree j for each $x \in U$, $R: U \times U \to F$ is a continuous mapping such that $R(z,z) = 0$, and (3) holds for each $z \in U$ and the norm $\|.\|_\gamma := \|.\|$ on F. Then $f \circ \eta$ is of class $C^k_{\mathbb{K}}$, for each $C^k_{\mathbb{K}}$-curve $\eta : \mathbb{K} \supseteq I \to U$. The maps a_j, $j = 1, \ldots, k$, are uniquely determined.

It is unknown whether any f as in Lemma 1.10 actually is $C^k_{\mathbb{K}}$, beyond the case $E = \mathbb{K}$.

Lemma 1.11 *Let E, F be normed vector spaces over an ultrametric field $\mathbb{K}$, with F ultrametric. Let $n \in \mathbb{N}$ and $\beta: E^n \to F$ be a continuous n-linear map. Then, for all $\varepsilon > 0$ and $u_i, v_i \in E$ such that $\|u_i\| \leq 1$, $\|v_i\| \leq 1$, and $\|u_i - v_i\| \leq \varepsilon$ for $i = 1, \ldots, n$, we have*

$$\|\beta(u_1,\ldots,u_n) - \beta(v_1,\ldots,v_n)\| \leq \|\beta\|\varepsilon\,.$$

Proof. $\|\beta(u_1,\ldots,u_n)-\beta(v_1,\ldots,v_n)\| = \|\sum_{k=1}^n \beta(v_1,\ldots,v_{k-1},u_k-v_k,u_{k+1},\ldots,u_n)\| \leq \max_k \|\beta(v_1,\ldots,v_{k-1},u_k-v_k,u_{k+1},\ldots,u_n)\| \leq \|\beta\|\varepsilon$. □

See [2] for the definition of C^k-manifolds over a topological field $\mathbb{K}$. Given $k \in \mathbb{N}\cup\{\infty\}$, a *Lie group of class* C^k modelled on a topological $\mathbb{K}$-vector space E is a group G, equipped with a C^k-manifold structure modelled on E with respect to which the group multiplication and inversion are C^k. We abbreviate $L(G) := T_1(G)$; if $k \geq 3$, then $L(G)$ has a topological Lie algebra structure with bracket $[X,Y] := [X_\ell, Y_\ell](1)$ obtained from the Lie bracket of the corresponding left invariant vector fields on G.

2 Filtrations and continuous one-parameter groups in ultrametric Banach-Lie groups

In this section, based on Taylor expansions and versions of the Inverse Function Theorem, we construct a basis for the filter of identity neighbourhoods in an ultrametric Banach-Lie group G, consisting of open subgroups of G which are diffeomorphic to balls in $L(G)$. In the process, we collect more and more information concerning the properties of these subgroups and the behaviour of various mappings on them. This enables us to prove the existence of a large supply of continuous p-adic one-parameter groups (which actually cover the above neighbourhoods). The later sections also hinge on the present investigations, which give us sufficient control over all relevant maps, their differentials and Taylor expansions to establish differentiability properties for one-parameter groups, the logarithm, and the exponential map.

In the following, $(\mathbb{K}, |.|)$ is a complete ultrametric field, with valuation ring $\mathbb{A} := \{z \in \mathbb{K}\colon |z| \leq 1\}$, the maximal ideal $\mathfrak{m} := \{z \in \mathbb{K}\colon |z| < 1\}$ of $\mathbb{A}$, and residue field $\mathbb{F} := \mathbb{A}/\mathfrak{m}$. We let $p := \mathrm{char}(\mathbb{F})$ be the characteristic of $\mathbb{F}$. If $\mathbb{K}$ is locally compact, we let $q := [\mathbb{A} : \mathfrak{m}]$ be the module of $\mathbb{K}$, and define $a := \max\{|z| : z \in \mathfrak{m}\}$; thus $|\mathbb{K}^\times| = a^{\mathbb{Z}}$. Starting with Proposition 2.1 (h), in the case where $\mathrm{char}(\mathbb{K}) = 0$, we shall assume for the rest of the proposition and all later results based on the proposition that the absolute value $|.|$ of $\mathbb{K}$ is non-trivial on $\mathbb{Q}$. Then $0 < |p| < 1$, and after replacing $|.|$ with a suitable power we may assume without loss of generality that $|p| = p^{-1}$, entailing that the closure of $\mathbb{Q}$ in $\mathbb{K}$ is $\mathbb{Q}_p$, equipped with its usual absolute value.

Proposition 2.1 *Let $k \in \mathbb{N}\cup\{\infty\}$, and E be an ultrametric Banach space over $\mathbb{K}$. If $\mathbb{K}$ is locally compact and E is finite-dimensional, define $k_+ := k$; otherwise, define $k_+ := k+1$. Let G be a Lie group of class $C^{k_+}_{\mathbb{K}}$ modelled on E, and $\|.\|$ be an ultrametric norm on $L(G) \cong E$ defining its topology. Then there exists a $C^{k_+}_{\mathbb{K}}$-diffeomorphism $\phi\colon U \to B_r(0)$ from an open subgroup U of G onto the open ball $B_r(0) \subseteq L(G)$ for some $r \in]0,1[$, such that $\phi(1) = 0$ and $T_1\phi = \mathrm{id}_{L(G)}$. Via*

$$\mu(x,y) := x * y := \phi(\phi^{-1}(x)\phi^{-1}(y)) \quad \textit{for} \;\; x,y \in B_r(0)$$

we obtain a group multiplication on $B_r(0)$ making ϕ an isomorphism of $C^{k_+}_{\mathbb{K}}$-Lie groups. Let $\alpha \in]0,1[$. Then, replacing $\|.\|$ with a positive multiple and shrinking r if necessary, it can be achieved that the following holds:

(a) *For each $x \in B_r(0)$ and $s \in]0,r]$, we have $x * B_s(0) = B_s(0) * x = x + B_s(0)$. In particular, the left uniform structure on $(B_r(0),*)$, its right uniform structure, and the uniform structure on $(B_r(0),+)$ all coincide, and $(B_r(0),*)$ is a complete topological group (whence so is G).*

(b) *$V_s := B_s(0)$ is an open, normal subgroup of $(B_r(0),*) =: V_r$, for each $s \in]0,r]$.*

(c) *For every $\varepsilon > 0$, there exists $\delta \in]0,r]$ such that $[V_s,V_s] \subseteq V_{\varepsilon s}$ for all $s \in]0,\delta]$. If $k_+ \geq 2$, then furthermore $[V_t,V_s] \subseteq V_{\alpha ts} \subseteq V_{ts}$, for all $s,t \in]0,r]$.*

(d) *The inversion map $\iota\colon V_r \to V_r$, $x \mapsto x^{-1}$ is a surjective isometry. For each $\varepsilon > 0$, there exists $\delta \in]0,r]$ such that $\|x^{-1}+x\| \leq \varepsilon\|x\|$ for all $x \in V_\delta$. If $k_+ \geq 2$, then furthermore $\|x^{-1}+x\| \leq \alpha\|x\|^2$ for all $x \in V_r$.*

(e) *Define $W_s := \overline{B}_s(0)$ for $s \in]0,r[$. Then analogues of* (b) *and* (c) *are valid for W_s and W_t in place of V_s and V_t (provided we require furthermore $s,t<r$ here).*

(f) *$V_s/V_{\alpha s}$ is abelian for each $s \in]0,r]$, and so is W_s/V_s for each $s \in]0,r[$.*

(g) *For all $\varepsilon > 0$, there exists $\delta \in]0,r]$ such that $\|xy-x-y\| \leq \varepsilon \max\{\|x\|,\|y\|\}$ for all $x,y \in B_\delta(0)$, writing $xy := x*y$ now. Hence $xy-x-y \in B_{\varepsilon s}(0)$, for all $s \in]0,\delta]$ and $x,y \in B_s(0)$. If $k_+ \geq 2$, we can achieve that $\|xy-x-y\| \leq \alpha\|x\|\,\|y\|$ for all $x,y \in V_r$, whence $\|xy-x-y\| < \alpha s^2$ and thus $xy \in x+y+B_{\alpha s^2}(0)$, for all $s \in]0,r]$ and $x,y \in V_s$.*

(h) *If $\mathbb{K}$ is locally compact, $L(G)$ is of finite, positive dimension d, and $\|.\|$ corresponds to the maximum norm on $\mathbb{K}^d$ with respect to a suitable choice of basis of $L(G)$, then we may achieve that $r = a^{j_0}$ for some $j_0 \in \mathbb{N}$ and*

$$V_{a^j}/V_{a^{j+1}} \cong (\mathbb{F}^d,+) \quad \text{for all } j \in \mathbb{N} \text{ such that } j \geq j_0.$$

In particular, $[V_{a^j} : V_{a^{j+1}}] = q^d$ for each $j \geq j_0$, and $U \cong V_r$ is a pro-p-group.

(i) *Let $\tau_p : V_r \to V_r$, $\tau_p(x) := x^p$ be the p-th power map. If $\operatorname{char}(\mathbb{K}) = p$ and $k_+ \geq 2$, we have $(V_s)^{\{p\}} := \tau_p(V_s) \subseteq V_{\alpha s^2}$ for all $s \in]0,r]$. If $\operatorname{char}(\mathbb{K}) = p$ and $k_+ = 1$, we have $(V_s)^{\{p\}} \subseteq V_{\alpha s}$ for each $s \in]0,r]$, and furthermore for each $\varepsilon > 0$ there exists $\delta \in]0,r]$ such that $(V_s)^{\{p\}} \subseteq V_{\varepsilon s}$ for all $s \in]0,\delta]$. If $\operatorname{char}(\mathbb{K}) = 0$, we have $(V_s)^{\{p\}} = V_{p^{-1}s} = pV_s$ for every $s \in]0,r]$, and $p^{-1}\tau_p\colon V_r \to V_r$ is a surjective isometry.*

(j) *$[V_s,V_s] \subseteq V_{p^{-2}s}$, for all $s \in]0,r]$.*

(k) *$\|x^n\| \leq \|x\|$, for every $x \in V_r$ and $n \in \mathbb{N}_0$.*

(l) *For all* $\varepsilon > 0$*, there exists* $\delta \leq r$ *such that* $\|x^n - nx\| \leq \varepsilon\|x\|$ *for all* $x \in V_\delta$ *and all* $n \in \mathbb{Z}$*. If* $k_+ \geq 2$*, then furthermore* $\|x^n - nx\| \leq \alpha\|x\|^2$*, for every* $x \in V_r$ *and* $n \in \mathbb{Z}$*.*

(m) *For each* $x \in V_r$*, the homomorphism* $\mathbb{Z} \to V_r$*,* $n \mapsto x^n$ *extends to a continuous homomorphism* $\eta_x\colon \mathbb{Z}_p \to V_r$*. Given* $x \in V_r$ *and* $z \in \mathbb{Z}_p$*, we abbreviate* $x^z := \eta_x(z)$*.*

Proof. (a) and (b): Let $\phi\colon P \to V \subseteq L(G)$ be a chart of G about 1, such that $\phi(1) = 0$ and $T_1\phi = \mathrm{id}_{L(G)}$. There is an open, symmetric identity neighbourhood $Q \subseteq P$ such that $QQ \subseteq P$. Then $W := \phi(Q) \subseteq V$ is an open zero-neighbourhood in $L(G)$, and

$$\mu\colon W \times W \to V, \quad \mu(x, y) := x * y := \phi(\phi^{-1}(x)\phi^{-1}(y))$$

and $\iota\colon W \to W$, $\iota(x) := \phi((\phi^{-1}(x))^{-1})$ are smooth maps such that $\mu(x, 0) = \mu(0, x) = x$, $d\mu((0, 0), (u, v)) = u + v$, and $d\iota(0, u) = -u$ for all $x \in W$, $u, v \in L(G)$. For each $x \in W$, the map $\mu_x := \mu(x, \bullet) : W \to V$ is a C^{k_+}-diffeomorphism onto its open image. Since $T_0(\mu_0) = \mathrm{id}_{L(G)}$, there is $r \in \,]0, 1[$ such that $B_r(0) \subseteq W$, $\mu_x(B_r(0)) = B_r(0)$ for each $x \in B_r(0)$, and

$$\mu_x(B_s(0)) = x + B_s(0) \qquad \text{for all } x \in B_r(0) \text{ and } s \in \,]0, r]; \tag{4}$$

see [11], Thm. 7.4 (a)′ and (b)′. Thus $x * B_s(0) = x + B_s(0)$. After shrinking r, also

$$B_s(0) * x = x + B_s(0) \quad \text{for all } x \in B_r(0) \text{ and } s \in \,]0, r], \tag{5}$$

by an analogous argument. Since $T_0(\iota) = -\mathrm{id}_{L(G)}$, shrinking r even more if necessary, we can achieve that furthermore ι is isometric and

$$\iota(B_s(0)) = B_s(0) \qquad \text{for all } s \in \,]0, r], \tag{6}$$

by [11], Prop. 7.1 (a)′ and (b)′. Because $x + B_s(0) = B_s(0)$ for $x \in B_s(0)$, we deduce from (4) and (6) that $B_s(0)$ is closed under the operations μ and ι (which correspond to multiplication and inversion in G), entailing that $\phi^{-1}(B_s(0))$ is a subgroup of G, whence $V_s := (B_s(0), \mu|_{B_s(0)\times B_s(0)}, \iota|_{B_s(0)})$ is a Lie group, for each $s \in \,]0, r]$. In particular, $U := \phi^{-1}(B_r(0))$ is a subgroup of G. Since $x * V_s = x + B_s(0) = V_s * x$ for all $x \in V_r$ by (4) and (5), the subgroup V_s of V_r is normal. The assertion concerning uniform structures is clear from (4) and (5). Now completeness transfers from the open (and hence closed) subgroup $(B_r(0), +)$ of $(L(G), +)$ to $(B_r(0), *)$, U, and G.

(c) Assume $k_+ \geq 2$ first. By Lemma 1.7, there exists a constant $C > 0$ and $\delta < r$ such that $\|xyx^{-1}y^{-1}\| \leq C\,\|x\|\,\|y\|$. For the norm $\|.\|' := \sigma\|.\|$, where $\sigma \geq \max\{1, C/\alpha\}$, the preceding inequality turns into $\|xyx^{-1}y^{-1}\|' \leq \frac{C}{\sigma}\|x\|'\|y\|' \leq \alpha\|x\|'\|y\|'$. Hence, after replacing $\|.\|$ with $\|.\|'$, the second assertion of (c) (and hence also the first) is satisfied.

Case $k_+ = 1$: Then $f\colon V_r \times V_r \to V_r$, $f(x, y) := xyx^{-1}y^{-1}$ is strictly differentiable, with $f(0, 0) = 0$ and $f'(0, 0) = 0$ (using the maximum norm on $L(G) \times L(G)$). Given $\varepsilon > 0$, we therefore find $\delta \in \,]0, r]$ such that $\|f(x, y)\| = \|f(x, y) - f(0, 0) - f'(0, 0).(x, y)\| \leq \varepsilon\|(x, y)\|$ for all $x, y \in V_\delta$. Then $f(V_s \times V_s) \subseteq V_{\varepsilon s}$ and thus $[V_s, V_s] \subseteq V_{\varepsilon s}$, for all $s \in \,]0, \delta]$.

(d) By the proof of (a) and (b), $\iota\colon V_r \to V_r$ is a surjective isometry. *Case* $k_+ \geq 2$: Since $\iota'(0) = -\mathrm{id}_{L(G)}$, we deduce from Lemma 1.6 that there are $\rho \in \,]0,r]$ and $C > 0$ such that $\|\iota(x) + x\| \leq C\|x\|^2$ for all $x \in V_\rho$. Replacing $\|.\|$ with a suitable multiple (as in the proof of (c)), we may assume that $C \leq \alpha$, whence the third (and hence also the second) assertion holds. *If* $k_+ = 1$, then ι is strictly differentiable with $\iota(0) = 0$ and $\iota'(0) = -\mathrm{id}_{L(G)}$, whence, given ε, there is $\delta \in \,]0,r]$ such that $\|\iota(x) + x\| \leq \varepsilon\|x\|$ for all $x \in V_\delta$.

(e) Using that $W_s = \bigcap_{b>s} V_b$ for each $s \in \,]0,r[$, (e) readily follows from (b) and (c).

(f) Let δ be as in (c), applied with $\varepsilon := \alpha$. Then $[V_s, V_s] \subseteq V_{\alpha s}$ for all $s \in \,]0,\delta]$, whence $V_s/V_{\alpha s}$ is abelian. Since $[W_s, W_s] \subseteq W_{\alpha s} \subseteq V_s$, also W_s/V_s is abelian. Now replace r with δ.

(g) Case $k_+ \geq 2$: Since $\mu(x,0) = x$ and $\mu(0,y) = y$ for all $x, y \in V_r$, applying Lemma 1.7 with $\lambda = \mu = \mathrm{id}_{L(G)}$ we find $\rho \in \,]0,r]$ and $C > 0$ such that $\|xy - x - y\| \leq C\|x\|\,\|y\|$ for all $x, y \in V_\rho$. After replacing $\|.\|$ with a suitable positive multiple, we may assume that $C \leq \alpha$ (cf. proof of (c)). Replacing now r with ρ, we have $\|xy - x - y\| \leq \alpha\|x\|\,\|y\|$ for all $x, y \in V_r$, whence the second assertion holds and hence also the first. If $k_+ = 1$, then μ is strictly differentiable with $\mu(0,0) = 0$ and $d\mu((0,0),(x,y)) = x + y$, from which the assertion readily follows.

(h) By (g), after shrinking r, we may assume that $\|xy - x - y\| < a \max\{\|x\|, \|y\|\}$ for all $x, y \in V_r$, and $r = a^{j_0}$ for some $j_0 \in \mathbb{N}$. The subgroup $V_{a^{j+1}}$ being normal in V_{a^j} by (b), for each $j \geq j_0$, we deduce that

$$xV_{a^{j+1}}\, yV_{a^{j+1}} = xyV_{a^{j+1}} = xy + V_{a^{j+1}} = x + y + B_{a^{j+1}}(0) \qquad \text{for all } x, y \in V_{a^j}$$

from (a) and the fact the $\|xy - x - y\| < a^{j+1}$ by the preceding. Hence the quotient $V_{a^j}/V_{a^{j+1}}$ of $(V_{a^j}, *)$ does not only coincide with the quotient $B_{a^j}(0)/B_{a^{j+1}}(0)$ of $(B_{a^j}(0), +)$ as a set (see (a)), but also as a group. Here $B_{a^j}(0)/B_{a^{j+1}}(0) \cong (\mathbb{A}/\mathfrak{m})^d \cong \mathbb{F}^d$. Since $(V_{p^{-j}})_{j \geq j_0}$ is a basis of identity neighbourhoods consisting of open normal subgroups of V_r such that $V_r/V_{p^{-j}}$ is a finite p-group, indeed V_r is a pro-p-group.

(i) If $\mathrm{char}(\mathbb{K}) = p$, we have $\tau_p'(0) = p\,\mathrm{id}_{L(G)} = 0$. Thus Lemma 1.6 provides $\delta < r$ and $C > 0$ such that $\|\tau_p(y)\| \leq C\|y\|^2$ for all $y \in B_\delta(0)$, when $k_+ \geq 2$. As in the proof of (c), we see that C can be chosen $\leq \alpha$ after replacing $\|.\|$ with a suitable positive multiple. Replacing r with δ, the assertion then holds. Since $\tau_p(0) = 0$ and $\tau_p'(0) = 0$, the assertion concerning the case $\mathrm{char}(\mathbb{K}) = p$ and $k_+ = 1$ readily follows from the strict differentiability of τ_p (possibly after shrinking r). Thus (i) holds in positive characteristic.

Now assume $\mathrm{char}(\mathbb{K}) = 0$. Since $(\tau_p)'(0) = p\,\mathrm{id}_{L(G)}$, there exists $\rho \in \,]0,r]$ such that $p^{-1}\tau_p\colon V_\rho \to V_\rho$ is a surjective isometry ([11], Prop. 7.1 (a)$'$ and (b)$'$). Now replace r by ρ.

(j) Replace r with δ from (c), applied with $\varepsilon := p^{-2}$.

(k) Let $x \in V_r$ and $n \in \mathbb{N}$. If $x \in V_s$ for some $s \in \,]0,r]$, then also $x^n \in V_s$, since V_s is a group. Hence $\|x^n\| = \inf\{s\colon x^n \in V_s\} \leq \inf\{s\colon x \in V_s\} = \|x\|$.

(l) Case $k_+ \geq 2$: It suffices to prove the second assertion. We assume $n \in \mathbb{N}_0$ first and proceed by induction. The case $n = 0$ is trivial. Now assume the assertion is correct for some $n \in \mathbb{N}_0$. Then, exploiting the ultrametric inequality, we obtain

$$\begin{aligned}\|x^{n+1} - (n+1)x\| &= \|x^n x - x^n - x + x^n - nx\| \\ &\leq \max\{\|x^n x - x^n - x\|, \|x^n - nx\|\} \leq \alpha\|x\|^2,\end{aligned}$$

using that $\|x^n x - x^n - x\| \leq \alpha \|x^n\| \, \|x\| \leq \alpha \|x\|^2$ by (g) and (k), and $\|x^n - nx\| \leq \alpha \|x\|^2$ by the induction hypothesis.

To complete the proof of the second assertion, observe that

$$\|x^{-n} + nx\| = \|(x^{-1})^n - nx^{-1} + nx^{-1} + nx\| \leq \max\{\alpha\|x^{-1}\|^2, |n| \cdot \|x^{-1} + x\|\} \leq \alpha\|x\|^2$$

for each $n \in \mathbb{N}$, using that $\|x^{-1}\| = \|x\|$ and $\|x^{-1} + x\| \leq \alpha\|x\|^2$ by (d), and $|n| \leq 1$ since $(\mathbb{K}, |.|)$ is an ultrametric field.

Case $k_+ = 1$: Given $\varepsilon > 0$, choose δ with the properties described in (d) and (g). Let $n \in \mathbb{N}$ first; the case $n = 1$ is trivial. For $x \in V_\delta$, we have $\|x^n - nx\| \leq \varepsilon\|x\|$ by induction and $\|x^n x - x^n - x\| \leq \varepsilon \max\{\|x^n\|, \|x\|\} = \varepsilon\|x\|$, by (g) and (k). Thus $\|x^{n+1} - (n+1)x\| = \|x^n x - x^n - x + x^n - nx\| \leq \max\{\|x^n x - x^n - x\|, \|x^n - nx\|\} \leq \varepsilon\|x\|$ indeed. Then also $\|x^{-n} + nx\| \leq \max\{\|(x^{-1})^n - nx^{-1}\|, \|nx^{-1} + nx\|\} \leq \max\{\varepsilon\|x^{-1}\|, \|x^{-1} + x\|\} \leq \varepsilon\|x\|$, by the choice of δ.

(m) Let $x \in V_r$. Given $\varepsilon \in]0, r]$, there exists $N \in \mathbb{N}$ such that $\alpha^N r < \varepsilon$ and $p^{-N} r < \varepsilon$. Then $x^n \in V_\varepsilon$ for each $n \in \mathbb{Z}$ such that $|n|_p \leq p^{-N}$. Indeed, any such n has the form $n = p^j m$, where $j \geq N$ and m is either 0 or coprime to p. Then $x^{p^j} = \tau_p^j(x) \in V_{\alpha^j r} \subseteq V_\varepsilon$ if $\mathrm{char}(\mathbb{K}) = p$, and $x^{p^j} = \tau_p^j(x) \in V_{p^{-j} r} \subseteq V_\varepsilon$ if $\mathrm{char}(\mathbb{K}) = 0$, by (i). Hence, V_ε being a group, we also have $x^n = (x^{p^j})^m \in V_\varepsilon$, as asserted. By the preceding, the homomorphism

$$\mathbb{Z} \to V_r, \quad n \mapsto x^n \tag{7}$$

is continuous at 0 with respect to the topology on $\mathbb{Z}$ induced by $\mathbb{Z}_p$ and therefore uniformly continuous, being a homomorphism. Now $(V_r, *)$ being complete by (a), we see that the homomorphism from (7) extends uniquely to a continuous homomorphism $\eta_x \colon \mathbb{Z}_p \to V_r$. □

Remark 2.2 In the preceding situation, define $w \colon V_r \to]0, \infty]$ via $w(0) := \infty$, $w(x) := \log_r(\|x\|)$ for $x \in V_r \setminus \{0\}$. If $k_+ \geq 2$, then w defines a filtration on $V_r \cong U$ (in the sense of [23], Part I, Chapter II, Defn. 2.1), by Proposition 2.1 (e). The subgroups of V_r associated to the filtration (see [23], Part I, Chapter II, §2) are $(V_r)_\lambda = W_{r^\lambda}$, for each $\lambda \in]0, \infty[$.

3 Differentiability of one-parameter groups

In this section, we show that the continuous one-parameter groups η_x constructed above (by p-adic interpolation of power maps) are actually of class $C^1_{\mathbb{Q}_p}$. This allows us to define a logarithm and an exponential map, which we then show to be strictly differentiable.

Lemma 3.1 *If $\phi \colon G \to H$ is a homomorphism between C^k-Lie groups over a non-discrete topological field and $\phi|_U$ is C^k on some open identity neighbourhood $U \subseteq G$, then ϕ is C^k.*

Proof. For each $x \in G$, we have $\phi|_{xU} = \lambda^H_{\phi(x)} \circ \phi|_U \circ \lambda^G_{x^{-1}}|^U_{xU}$ using the indicated left translations on G and H, which are C^k-maps. Hence $\phi|_{xU}$ is C^k. □

Lemma 3.2 *Let G be a Lie group of class $C^1_{\mathbb{K}}$ over a non-discrete topological field $\mathbb{K}$, and $\xi\colon P \to G$ a continuous homomorphism, defined on an open subgroup $P \subseteq \mathbb{K}$. Assume that*

$$(\phi \circ \xi)'(0) = \lim_{z \to 0} \frac{\phi(\xi(z))}{z}$$

exists for some chart $\phi\colon U \to U_1 \subseteq L(G)$ of G about 1 such that $\phi(1) = 0$. Then ξ is $C^1_{\mathbb{K}}$.

Proof. Let $W \subseteq U$ be a symmetric open identity neighbourhood such that $WW \subseteq U$, and $W_1 := \phi(W)$. Define $\mu\colon W_1 \times W_1 \to U_1$, $\mu(x,y) := x * y := \phi(\phi^{-1}(x)\phi^{-1}(y))$. Then $B := \xi^{-1}(W)$ is an open 0-neighbourhood in $\mathbb{K}$; we let $A \subseteq \mathbb{K}$ be an open, symmetric 0-neighbourhood such that $A + A \subseteq B$. Define $\zeta\colon B \to W_1$, $\zeta(z) := \phi(\xi(z))$. By hypothesis, $\zeta'(0) = \lim_{z\to 0} \frac{\zeta(z)}{z}$ exists, whence

$$\varepsilon\colon B \to L(G)\,, \quad \varepsilon(z) := \left\{ \begin{array}{ll} \frac{\zeta(z)}{z} & \text{if } z \neq 0 \\ \zeta'(0) & \text{if } z = 0 \end{array} \right.$$

is a continuous map. For any $x, y \in A$ such that $x \neq y$, we have

$$\begin{aligned} \zeta^{>1<}(x,y) &= \frac{1}{y-x}(\zeta(y) - \zeta(x)) = \frac{1}{y-x}(\zeta(x) * \zeta(y-x) - \zeta(x) * 0) \\ &= \mu^{[1]}\big((\zeta(x), 0), \big(0, (y-x)^{-1}\zeta(y-x)\big), y-x\big) \\ &= \mu^{[1]}((\zeta(x), 0), (0, \varepsilon(y-x)), y-x)\,, \end{aligned}$$

where the final expression also makes sense for $x = y$, and defines a continuous function $(\zeta|_A)^{<1>}\colon A \times A \to L(G)$. Thus $\zeta|_A$ is $C^1_{\mathbb{K}}$, whence so is $\xi|_A$ and hence ξ, by Lemma 3.1. □

Proposition 3.3 *If* $\operatorname{char}(\mathbb{K}) = 0$ *in the situation of Proposition* 2.1*, then, shrinking r if necessary, it can be achieved that, in addition to* (a)–(m)*, also the following holds:*

(n) *For all $\varepsilon > 0$, there exists $\delta \leq r$ such that $\|x^z - zx\| \leq \varepsilon\|x\|$, for all $x \in V_\delta$ and $z \in \mathbb{Z}_p$. If $k_+ \geq 2$, then furthermore $\|x^z - zx\| \leq \alpha\|x\|^2$, for all $x \in V_r$ and $z \in \mathbb{Z}_p$.*

(o) *For each $x \in V_r$, the homomorphism η_x is of class $C^1_{\mathbb{Q}_p}$, and $\eta_x'(0) \in V_r$.*

(p) *The mapping $V_r \to \mathcal{L}(L(G))$, $x \mapsto \tau_p'(x)$ is continuous, with $\tau_p'(0) = p\,\mathrm{id}_{L(G)}$ and $\|\tau_p'(x) - p\,\mathrm{id}_{L(G)}\| < p^{-1}$ for each $x \in V_r$, whence $p^{-1}\tau_p'(x)$ is a surjective isometry.*

(q) *The map $\log : V_r \to V_r$, $\log(x) := \eta_x'(0)$ is a surjective isometry and an $SC^1_{\mathbb{K}}$-diffeomorphism such that* $\log(0) = 0$ *and* $\log'(0) = \mathrm{id}_{L(G)}$*. We have* $\log(x^z) = z\log(x)$ *for all $x \in V_r$ and all $z \in \mathbb{Z}_p$.*

(r) *The map* $\exp := \log^{-1}\colon V_r \to V_r$ *is a surjective isometry and an $SC^1_{\mathbb{K}}$-diffeomorphism satisfying* $\exp(0) = 0$ *and* $\exp'(0) = \mathrm{id}_{L(G)}$*. For every $x \in V_r$, the map $\zeta : \mathbb{Z}_p \to V_r$, $z \mapsto \exp(zx)$ is a homomorphism of class $C^1_{\mathbb{Q}_p}$ such that $\zeta'(0) = x$, and it is uniquely determined by this property.*

Proof. (n) In view of the continuity of ξ_x and scalar multiplication $\mathbb{Z}_p \times L(G) \to L(G)$, assertion (n) readily follows from Proposition 2.1 (l).

(o) The limit $\lim_{z\to 0} z^{-1}x^z$ (with $0 \neq z \in \mathbb{Z}_p$) will exist uniformly in $x \in V_r$, provided the following limit exists uniformly in $x \in V_r$:

$$\lim_{n\to\infty} p^{-n}x^{p^n} . \tag{8}$$

To see this, let $\varepsilon > 0$. By (n), there exists $\delta \in \,]0, r]$ such that $\|x^z - zx\| \leq \varepsilon p^{-1}\|x\| \leq \varepsilon\|x\|$ for all $x \in V_\delta$ and $z \in \mathbb{Z}_p$. We may assume that $\delta = p^{-N}$ for some $N \in \mathbb{N}$. Let $x \in V_r$ and $0 \neq z \in \mathbb{Z}_p$ with $|z|_p \leq \delta$. Then $|z|_p = p^{-n}$ with $n \geq N$ and thus $z = p^n m$, where $m := p^{-n}z \in \mathbb{Z}_p$ has absolute value 1. Since $\|x^{p^n}\| = p^{-n}\|x\| < |z|_p r \leq \delta$, we have $\|(x^{p^n})^m - mx^{p^n}\| \leq \varepsilon\|x^{p^n}\| = \varepsilon p^{-n}\|x\|$. Noting that $x^z = (x^{p^n})^m$ (which is clear if $m \in \mathbb{Z}$ and follows for $m \in \mathbb{Z}_p$ by continuity), we see that $z^{-1}x^z = p^{-n}x^{p^n} + z^{-1}(x^z - mx^{p^n})$, where $\|z^{-1}(x^z - mx^{p^n})\| \leq |z|_p^{-1}\varepsilon p^{-n}\|x\| \leq \varepsilon$ and $n \geq N$. Thus indeed $z^{-1}x^z$ will converge uniformly if so does $p^{-n}x^{p^n}$. Let us show now that $p^{-n}x^{p^n}$ converges uniformly. Given ε, with N as before we have $\|x^{p^{n+1}} - px^{p^n}\| \leq \varepsilon p^{-1}\|x^{p^n}\|$ for all $n \geq N$ and thus

$$\|p^{-n-1}x^{p^{n+1}} - p^{-n}x^{p^n}\| \leq \varepsilon p^n\|x^{p^n}\| \leq \varepsilon \quad \text{for all } x \in V_r \text{ and all } n \geq N. \tag{9}$$

Writing $p^{-n_2}x^{p^{n_2}} - p^{-n_1}x^{p^{n_1}}$ as a telescopic sum, (9) and the ultrametric inequality yield

$$\left\|p^{-n_2}x^{p^{n_2}} - p^{-n_1}x^{p^{n_1}}\right\| \leq \max_{j=0,\ldots,n_2-n_1-1} \left\|p^{-n_1-j-1}x^{p^{n_1+j+1}} - p^{-n_1-j}x^{p^{n_1+j}}\right\| \leq \varepsilon$$

for all $x \in V_r$ and $n_1, n_2 \geq N$ (where $n_2 > n_1$, say), whence $p^{-n}x^{p^n}$ is uniformly Cauchy and hence converges uniformly in x.

By the preceding, the limit $\eta_x'(0) = \lim_{z\to 0} z^{-1}x^z = \lim_{n\to\infty} p^{-n}x^{p^n}$ exists for each $x \in V_r$. Since $\|p^{-n}x^{p^n}\| = \|x\|$ for all n, also $\|\eta_x'(0)\| \leq \|x\| < r$, whence $\eta_x'(0) \in V_r$.

(p) The map τ_p is strictly differentiable in both of the cases $k_+ = 1$ and $k_+ \geq 2$, whence $x \mapsto \tau_p'(x)$ is continuous by [11, La. 3.2]. Since $\tau_p'(0) = p\,\mathrm{id}_{L(G)}$, after shrinking r we therefore have $\|\tau_p'(x) - p\,\mathrm{id}_{L(G)}\| < p^{-1}$ for each $x \in V_r$ and thus $\|p^{-1}\tau_p'(x) - \mathrm{id}_{L(G)}\| < 1$, whence $p^{-1}\tau_p'(x)$ is a surjective isometry by [11], La. 7.2 and its proof.

(q) Let $x \in V_r$ be given; we want to show that log is strictly differentiable at x. To this end, assume $\varepsilon > 0$. The map

$$h\colon V_r \times V_r \to L(G)\,, \quad h(y,u) := \tau_p(y+u)$$

being $C^2_{\mathbb{K}}$ (if $k_+ \geq 2$), resp., $C^1_{\mathbb{K}}$ (if $k_+ = 1$), applying [11, La. 3.5] (resp., [11, La. 4.5]) to h around $(0,0)$, we find $\delta \in \,]0, r]$ such that

$$\|\tau_p(w) - \tau_p(v) - \tau_p'(y).(w-v)\| \leq \frac{\varepsilon}{p}\,\|w - v\| \tag{10}$$

for all $y, v, w \in V_\delta$. There exists $N \in \mathbb{N}$ such that $\tau_p^N(x) \in V_\delta$. Applying [11, La. 3.5] (resp., [11, La. 4.5]) to h around $(\tau_p^{N-1}(x), 0)$, we find $\delta_{N-1} \in \,]0, r]$ such that (10) holds for all

$y, v, w \in B_{\delta_{N-1}}(\tau_p^{N-1}(x))$. After shrinking δ_{N-1}, we can achieve that $\tau_p(B_{\delta_{N-1}}(\tau_p^{N-1}(x))) \subseteq V_\delta$ here. Proceeding in this way, we find $\delta_0, \delta_1, \ldots, \delta_{N-1}$ such that

$$\tau_p(B_{\delta_n}(\tau_p^n(x))) \subseteq B_{\delta_{n+1}}(\tau_p^{n+1}(x))$$

for all $n \in \{0, 1, \ldots, N-2\}$, and such that (10) holds for all $n \in \{0, 1, \ldots, N-1\}$ and all $y, v, w \in B_{\delta_n}(\tau_p^n(x))$.

Let $v, w \in B_{\delta_0}(x)$. Then $\tau_p^n(v) \in B_{\delta_n}(\tau_p^n(x))$ for all $n \in \{0, \ldots, N-1\}$ and thus furthermore $\tau_p^n(v) \in V_\delta$ for all $n \geq N$, and likewise for $\tau_p^n(x)$ and $\tau_p^n(w)$. By (10), we have

$$\tau_p(w) - \tau_p(v) - \tau_p'(x).(w-v) =: r_1, \qquad \text{where } \|r_1\| \leq \tfrac{\varepsilon}{p}\|w-v\|.$$

Thus $\|p^{-1}\tau_p(w) - p^{-1}\tau_p(v) - A_1.(w-v)\| \leq \varepsilon\, \|w-v\|$, where $A_1 := p^{-1}\tau_p'(x)$. More generally, $r_n := \tau_p^n(w) - \tau_p^n(v) - \tau_p'(\tau_p^{n-1}(x)).(\tau_p^{n-1}(w) - \tau_p^{n-1}(v))$ satisfies

$$\|r_n\| \leq \frac{\varepsilon}{p}\|\tau_p^{n-1}(w) - \tau_p^{n-1}(v)\| = \frac{\varepsilon}{p^n}\|w-v\| \tag{11}$$

for all $n \in \mathbb{N}$, by (10) and (i). Then

$$\|p^{-n}\tau_p^n(w) - p^{-n}\tau_p^n(v) - A_n.(w-v)\| \leq \varepsilon\, \|w-v\| \quad \text{for all } \ n \in \mathbb{N}, \tag{12}$$

where $A_n := p^{-n}\big(\tau_p'(\tau_p^{n-1}(x)) \circ \tau_p'(\tau_p^{n-2}(x)) \circ \cdots \circ \tau_p'(x)\big) \in \mathcal{L}(L(G))$. In fact,

$$\begin{aligned}
\tau_p^n(w) - \tau_p^n(v) &= \tau_p'(\tau_p^{n-1}(x)).(\tau_p^{n-1}(w) - \tau_p^{n-1}(v)) + r_n \\
&= \tau_p'(\tau_p^{n-1}(x)).\tau_p'(\tau_p^{n-2}(x)).(\tau_p^{n-2}(w) - \tau_p^{n-2}(v)) + \tau_p'(\tau_p^{n-1}(x)).r_{n-1} + r_n \\
&= p^n A_n(w-v) + \sum_{k=1}^{n-1}\big(\tau_p'(\tau_p^{n-1}(x)) \circ \cdots \circ \tau_p'(\tau_p^{n-k}(x))\big).r_{n-k} + r_n
\end{aligned}$$

shows that $p^{-n}\tau_p^n(w) - p^{-n}\tau_p^n(v) - A_n.(w-v)$ equals

$$\sum_{k=1}^{n-1} p^{-k}\big(\tau_p'(\tau_p^{n-1}(x)) \circ \cdots \circ \tau_p'(\tau_p^{n-k}(x))\big).p^{-(n-k)}r_{n-k} + p^{-n}r_n\,,$$

where each summand involving r_{n-k} has norm $\leq \varepsilon\|y-x\|$, because $\|r_{n-k}\| \leq \frac{\varepsilon}{p^{n-k}}\|y-x\|$ and the derivatives of τ_p have norm $\leq p^{-1}$, by (p). Since also $p^{-n}r_n$ has norm $\leq \varepsilon\|y-x\|$ by (11), the ultrametric inequality shows that (12) holds.

We shall presently see that the sequence $(A_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{L}(L(G)), \|.\|)$. Let $A \in \mathcal{L}(L(G))$ denote the limit. Recall from (8) that $p^{-n}\tau_p^n(w) \to \log(w)$ and $p^{-n}\tau_p^n(v) \to \log(v)$ as $n \to \infty$. Letting pass $n \to \infty$ in (12), we find that

$$\|\log(w) - \log(v) - A.(w-v)\| \leq \varepsilon\, \|w-v\|\,.$$

As $v, w \in B_{\delta_0}(x)$ were arbitrary, this means that log is indeed strictly differentiable at x, with $\log'(x) = A$.

To see that the sequence $(A_n)_{n\in\mathbb{N}}$ converges, let $\varepsilon \in \,]0,1[$. Part (p) provides $\delta \in \,]0,r]$ such that $\|p^{-1}\tau_p'(y) - \mathrm{id}\| < \varepsilon$ for all $y \in V_\delta$. Let $N \in \mathbb{N}$ such that $p^{-N} \leq \delta$. Then $\|p^{-1}\tau_p'(\tau_p^n(x)) - \mathrm{id}\| < \varepsilon$ for each $n \geq N$, whence $p^{-1}\tau_p'(\tau_p^n(x))$ is contained in the ball $B_\varepsilon(\mathrm{id}) \subseteq \mathcal{L}(L(G))$, which is an open subgroup of $\mathrm{GL}(L(G))$. As a consequence, for any $n_1, n_2 \geq N$, with $n_2 > n_1$, we have $A_{n_2} \circ A_{n_1}^{-1} = p^{-1}\tau_p'(\tau_p^{n_2-1}(x)) \circ \cdots \circ p^{-1}\tau_p'(\tau_p^{n_1}(x)) \in B_\varepsilon(\mathrm{id})$. Thus $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach-Lie group $\mathrm{GL}(\mathcal{L}(G))$ (which is complete by (a)), and hence convergent.

If $x = 0$ here, then $\tau_p^n(x) = 0$ for each n and thus $A_n = \mathrm{id}$ for each n, entailing that $\log'(0) = \lim_{n\to\infty} A_n = \mathrm{id}$. Using [11], Prop. 7.1 (a)$'$, (b)$'$ and Thm. 7.3, we therefore find $\rho \in \,]0,r]$ such that $\log(V_\rho) = V_\rho$ and such that $\log\colon V_\rho \to V_\rho$ is a surjective isometry and an $SC^1_{\mathbb{K}}$-diffeomorphism. Now replace r with ρ.

Since $\eta_0 \equiv 0$, it is clear that $\log(0) = \eta_0'(0) = 0$. Finally, given $x \in V_r$, there exists $y \in V_r$ such that $\eta_y'(0) = \log(y) = x$. For $n, m \in \mathbb{Z}$, we have $(y^n)^m = y^{nm}$, whence $\eta_{y^n}(z) = \eta_y(nz)$ for all $z \in \mathbb{Z}_p$ by continuity of η_y and η_{y^n}, using that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Thus $\log(y^n) = \eta_{y^n}'(0) = n\eta_y'(0) = n\log(y)$ for all $n \in \mathbb{Z}$ and hence $\log(y^z) = z\log(y)$ for all $z \in \mathbb{Z}_p$, by continuity.

(r) It is immediate from (q) that exp is a surjective isometry, an $SC^1_{\mathbb{K}}$-diffeomorphism, $\exp(0) = 0$, and $\exp'(0) = \mathrm{id}$. Given $x \in V_r$ and $z \in \mathbb{Z}_p$, we have $zx = z\log(\exp(x)) = \log(\exp(x)^z)$ and thus $\exp(zx) = \exp(x)^z = \eta_{\exp x}(z) =: \zeta(z)$, which is a $C^1_{\mathbb{Q}_p}$-homomorphism $\mathbb{Z}_p \to V_r$ such that $\zeta'(0) = \log(\exp(x)) = x$. Suppose that also $\xi : \mathbb{Z}_p \to V_r$ is a $C^1_{\mathbb{Q}_p}$-homomorphism such that $\xi'(0) = x$. Set $y := \xi(1)$. Then $\xi(n) = y^n = \eta_y(n)$ for all $n \in \mathbb{Z}$ and thus $\xi = \eta_y$ by continuity, entailing that $x = \xi'(0) = \eta_y'(0) = \log(y)$. Hence $\exp(x) = y$ and $\zeta(z) = \exp(x)^z = y^z = \xi(z)$ for all $z \in \mathbb{Z}_p$, whence $\xi = \zeta$. □

4 Higher differentiability of one-parameter groups

In this section, we perform the most difficult step of our construction: We show that the exponential map $\exp\colon V_r \to V_r$ (from Proposition 3.3 (r)) admits Taylor expansions of all finite orders $\leq k$. As a consequence, every p-adic one-parameter subgroup of G will be C^k.

We start with a simple observation:

Lemma 4.1 *Let X, Y be metric spaces and $f_n : X \to Y$ homeomorphisms for $n \in \mathbb{N}$, which converge uniformly to a surjective isometry $f\colon X \to Y$. Then $f_n^{-1} \to f^{-1}$ uniformly.*

Proof. For each $y \in Y$, we have, using that f is an isometry:

$$d(f^{-1}(y), f_n^{-1}(y)) = d(y, f(f_n^{-1}(y))) = d(f_n(f_n^{-1}(y)), f(f_n^{-1}(y))) \leq \sup_{x\in Y} d(f_n(x), f(x))\,.$$

The assertion follows. □

Since $p^{-n}\tau_p^n(x) \to \log(x)$ uniformly on V_r by the proof of Proposition 3.3 (o), where $\log\colon V_r \to V_r$ is a surjective isometry, the preceding lemma shows that

$$\exp(x) = \lim_{n\to\infty} \tau_p^{-n}(p^n x) \quad \text{uniformly in } x \in V_r, \tag{13}$$

whence $\exp(x) = \lim_{n\to\infty} g^n(p^n x)$ with $g := (\tau_p)^{-1} : V_{p^{-1}r} \to V_r$, and $g^n := \overbrace{g \circ \cdots \circ g}^{n} :$ $V_{p^{-n}r} \to V_r$.

Having expressed exp in terms of a limit of iterates of a given map, the following proposition (our most difficult technical result) establishes Taylor expansions (and hence higher differentiability properties) for exp.

Proposition 4.2 *Let* $(\mathbb{K}, |.|)$ *be an ultrametric field extending* $(\mathbb{Q}_p, |.|_p)$*. Let* $(E, \|.\|)$ *be an ultrametric Banach space over* $\mathbb{K}$*,* $r > 0$*,* $k \in \mathbb{N}$*, and* $g\colon B_r(0) \to B_{pr}(0)$ *be a diffeomorphism of class* $C^{k+1}_{\mathbb{K}}$ *(resp., of class* $C^k_{\mathbb{K}}$ *if* $\mathbb{K}$ *is locally compact and* E *finite-dimensional), such that* $g(0) = 0$*,* $g'(0) = p^{-1}\mathrm{id}_E$*, the map* $B_r(0) \to B_r(0)$*,* $x \mapsto pg(x)$ *is a surjective isometry, and such that*

$$\|g'(x)\| = p \quad \text{for all } x \in B_r(0) \tag{14}$$

(where $B_r(0), B_{pr}(0) \subseteq E$*). Suppose that the surjective isometries*

$$B_r(0) \to B_r(0), \quad x \mapsto g^n(p^n x)$$

converge uniformly to a function $f\colon B_r(0) \to B_r(0)$*. Then* f *admits a* k*-th order expansion as described in Lemma* 1.10*, and thus* $f \circ \eta$ *is* $C^k_{\mathbb{K}}$ *for each* $C^k_{\mathbb{K}}$*-curve* $\eta\colon \mathbb{K} \supseteq W \to B_r(0)$*.*

Corollary 4.3 *If* G *is* $SC^{k_+}_{\mathbb{K}}$*, then* exp *from Proposition* 3.3 (r) *admits a* j*-th order expansion (over the ground field* $\mathbb{K}$*), for each integer* $j \le k$*, and* $\exp \circ \eta$ *is* $C^k_{\mathbb{K}}$ *for each* $C^k_{\mathbb{K}}$*-curve* $\eta\colon \mathbb{K} \supseteq W \to V_r$*. In particular,* $\eta_x\colon \mathbb{Z}_p \to V_r$*,* $z \mapsto x^z$ *and* $\mathbb{Z}_p \to V_r$*,* $z \mapsto \exp(zx)$ *are* $C^k_{\mathbb{Q}_p}$*, for each* $x \in V_r$*, and so is any continuous homomorphism* $\eta\colon \mathbb{Z}_p \to G$*.*

Proof. By the Inverse Function Theorem for SC^k-maps [11, Thm. 7.3], the map $g := (\tau_p)^{-1}\colon B_{p^{-1}r}(0) \to B_r(0)$ is $SC^{k_+}_{\mathbb{K}}$ and hence of class $C^{k_+}_{\mathbb{K}}$. The first and second assertion are therefore obvious from (13) and Proposition 4.2. For each $x \in V_r$, the map $\mathbb{A} \to V_r$, $z \mapsto \exp(zx)$ is $C^k_{\mathbb{K}}$ (and thus $C^k_{\mathbb{Q}_p}$), being a composition of exp and the $C^k_{\mathbb{K}}$-curve $z \mapsto zx$. As $\eta_x(z) = \exp(z\log(x))$ for $z \in \mathbb{Z}_p$, also η_x is $C^k_{\mathbb{Q}_p}$.

Finally, if $\eta : \mathbb{Z}_p \to G$ is a continuous homomorphism, then there exists $N \in \mathbb{N}$ such that $\eta(z) \in U$ for all $z \in \mathbb{Z}_p$ such that $|z|_p \le p^{-N}$. Define $x := \phi(\eta(p^N)) \in V_r$. Then $z \mapsto \phi(\eta(p^N z))$ and η_x are continuous homomorphisms $\mathbb{Z}_p \to V_r$ which agree at 1, thus on $\langle 1 \rangle = \mathbb{Z}$, and hence on all of $\mathbb{Z}_p$, by continuity. As a consequence, η is $C^k_{\mathbb{Q}_p}$ on $p^N\mathbb{Z}_p$ and hence $C^k_{\mathbb{Q}_p}$ on all of $\mathbb{Z}_p$, by Lemma 3.1. □

If $\mathbb{K}$ is locally compact and G finite-dimensional, then the $C^k_{\mathbb{K}}$-Lie group G is also $SC^k_{\mathbb{K}}$. Otherwise (when $k_+ = k+1$), we might assume that G is $C^{k+2}_{\mathbb{K}}$ to ensure that G is $SC^{k+1}_{\mathbb{K}}$.

Proof of Proposition 4.2. In view of the uniqueness assertion in Lemma 1.10, it suffices to show that every $x_0 \in B_r(0)$ has an open neighbourhood U_{x_0} on which f admits a k-th order expansion (by uniqueness, the individual a_j's on the sets $U_{x_0} \times E$ then combine to a well-defined map a_j on $B_r(0) \times E$).

Thus, fix $x_0 \in B_r(0)$ now. The proof proceeds in two stages. First, we show that certain limits exist, and define certain continuous maps b_j and a_j using these limits (Lemma 4.5 (a), Eqn. (26)). The maps a_j are the natural candidates for the coefficients of a k-th order expansion for f, and $b_j(x, \bullet)$ is the symmetric j-linear map corresponding to the homogeneous polynomial $a_j(x, \bullet)$. The second step, then, will be to show that these coefficients a_j can be used to expand f, with a remainder R vanishing of order k (Lemma 4.6). The decisive tool for the proof is a notational formalism (set up in the proof of Lemma 4.6) allowing us to track each individual term in the sum obtained by substituting n times the k-th order Taylor expansion of g into itself to express g^n (both the multilinear terms and each individual contribution to the remainder term).[6] We now begin the proof with a counterpart to the labelling of the contributions to the k-th order Taylor expansion of g^n just mentioned: We describe a corresponding notational scheme allowing us to label certain compositions of the differentials of g, which are then used to define the b_j's and a_j's.

Given $n \in \mathbb{N}$, consider an n-tuple $(s_1, \ldots, s_n)$, where, for each $\nu \in \{1, \ldots, n\}$,

$$s_\nu \colon \{1, \ldots, m_\nu\} \to \{1, 2, \ldots, k\}$$

is a mapping on $\{1, \ldots, m_\nu\}$ for some $m_\nu \in \mathbb{N}$, such that

$$m_1 = 1 \qquad \text{and} \qquad m_\nu = \sum_{i=1}^{m_{\nu-1}} s_{\nu-1}(i) \quad \text{for } \nu \in \{2, \ldots, n\}.$$

Abbreviate $m_{n+1} := \sum_{i=1}^{m_n} s_n(i)$. Given $x \in B_r(0)$, we recursively define continuous $m_{\nu+1}$-linear maps

$$g^x_{(s_1,\ldots,s_\nu)} \colon E^{m_{\nu+1}} \to E$$

for $\nu = 1, \ldots, n$ using higher differentials of g via

$$\begin{aligned} g^x_{(s_1)} &:= \frac{1}{s_1(1)!} d^{s_1(1)} g(f(px), \bullet)\,; \\ g^x_{(s_1,\ldots,s_\nu)} &:= g^x_{(s_1,\ldots,s_{\nu-1})} \circ \prod_{i=1}^{m_\nu} \frac{1}{s_\nu(i)!} d^{s_\nu(i)} g(f(p^\nu x), \bullet)\,. \end{aligned}$$

For example, if $s_1(1) := 2$, $s_2(1) := 1$, and $s_2(2) := 3$, then $m_2 = 2$, $m_3 = 4$,

$$\begin{aligned} g^x_{(s_1)}(u_1, u_2) &= \frac{1}{2} d^2 g\big(f(px), u_1, u_2\big), \qquad \text{and} \\ g^x_{(s_1,s_2)}(u_1, u_2, u_3, u_4) &= \frac{1}{2} d^2 g\Big(f(px),\, dg(f(p^2x), u_1),\, \frac{1}{3!} d^3 g(f(p^2x), u_2, u_3, u_4)\Big) \end{aligned}$$

for all $x \in B_r(0)$ and $u_1, u_2, u_3, u_4 \in E$.

[6] Of course, based on the symmetry of higher differentials, many of these terms coincide and could be combined in one term, but this does not seem useful in the present context and would only make the presentation longer and more complicated.

We let S_n be the set of all $s = (s_1, \ldots, s_n)$ as before (with variable $m_1, \ldots, m_{n+1}$ depending on s), and $S_{n,j} := \{(s_1, \ldots, s_n) \in S_n \colon \sum_{i=1}^{m_n} s_n(i) = j\}$, for each $j \in \{1, \ldots, k\}$. For $n \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$, we define

$$h_{j,n} \colon B_r(0) \times E^j \to E, \quad h_{j,n}(x, u_1, \ldots, u_j) := \sum_{s \in S_{n,j}} g_s^x(p^n u_1, \ldots, p^n u_j)\,.$$

Then $h_{j,n}$ is continuous, and $h_{j,n}(x, \bullet) \colon E^j \to E$ is j-linear.

The mappings $B_r(0) \to \mathcal{L}^j(E, E)$, $x \mapsto d^j g(x, \bullet)$ being continuous for each $j \in \{1, \ldots, k\}$ (Lemma 1.8), we find $\rho_0 \in \,]0, r]$ such that $C_j := \sup\{\|(j!)^{-1} d^j g(x, \bullet)\| \colon x \in B_{\rho_0}(0)\} < \infty$ for each $j \in \{1, \ldots, k\}$. There is $N_0 \in \mathbb{N}$ such that $p^{-N_0} r \le \rho_0$ and thus $f(p^n x_0) \in B_{\rho_0}(0)$ for all $n > N_0$. For each $n \in \{1, \ldots, N_0\}$, we find $\rho_n \in \,]0, r]$ such that $C_{n,j} := \sup\{\|(j!)^{-1} d^j g(x, \bullet)\| \colon x \in B_{\rho_n}(f(p^n x_0))\} < \infty$ for each $j \in \{1, \ldots, k\}$. There is $\rho \in \,]0, r]$ such that $f(p^n x) \in B_{\rho_n}(f(p^n x_0))$ for all $x \in B_\rho(x_0)$ and $n \in \{1, \ldots, N_0\}$. Choosing $\rho \le \rho_0$, we can achieve that furthermore $f(p^n x) \in B_{\rho_0}(0)$ for all $n > N_0$ and all $x \in B_\rho(x_0)$, using that f is isometric and $B_{\rho_0}(0) = B_{\rho_0}(f(p^n x_0))$.

After replacing the given norm $\|.\|$ with a suitable positive multiple (and adapting r, ρ and the ρ_n's accordingly), we may assume that $C_j \le 1$ and $C_{n,j} \le 1$, for all $j \ge 2$ and $n \in \{1, \ldots, N_0\}$. Then

$$\|(j!)^{-1} d^j g(f(p^n x), \bullet)\| \le 1, \quad \text{for all } j \in \{2, \ldots, k\},\, n \in \mathbb{N} \text{ and } x \in B_\rho(x_0). \tag{15}$$

It is important to estimate the norms of the multilinear maps g_s^x. Given $s \in S_{n,j}$, where $j \ge 2$, there exists a largest integer $\ell_s \in \{1, \ldots, n\}$ such that $s_{\ell_s}(i) > 1$ for some $i \in \{1, \ldots, m_{\ell_s}\}$ (with notation as above). We now show by induction on $n \in \mathbb{N}$:

Lemma 4.4 $\|g_s^x(p^n u_1, \ldots, p^n u_j)\| \le p^{-\ell_s} \|u_1\| \cdots \|u_j\|$ *holds, for all* $j \in \{2, \ldots, k\}$, $x \in B_\rho(x_0)$, $u_1, \ldots, u_j \in E$, *and* $s \in S_{n,j}$.

Proof. If $n = 1$, then $s_1(1) = j$, $\ell_s = 1$. Using (15), we get

$$\|g_s^x(pu_1, \ldots, pu_j)\| = \|(j!)^{-1} d^j g(f(px), pu_1, \ldots, pu_j)\| \le p^{-j} \|u_1\| \cdots \|u_j\| \le p^{-\ell_s} \|u_1\| \cdots \|u_j\|.$$

Now let $n \ge 2$ and suppose the lemma has been proven up to $n-1$. We have

$$g_s^x(p^n u_1, \ldots, p^n u_j) = g^x_{(s_1, \ldots, s_{\ell_s})}(p^{\ell_s} v_1, \ldots, p^{\ell_s} v_j)$$

with $v_i := pg'(f(p^{\ell_s + 1})) \cdots pg'(f(p^n x)).u_i$ of norm $\|v_i\| \le \|u_i\|$, for $i = 1, \ldots, j$.

Special case: If $\ell_s = 1$ or $\ell_s \ge 2$ and $(s_1, \ldots, s_{\ell_s - 1}) \in S_{\ell_s - 1, 1}$, then indeed

$$\begin{aligned}\|g_s^x(p^n u_1, \ldots, p^n u_j)\| &= \|g'(f(px)) \cdots g'(f(p^{\ell_s - 1} x))(j!)^{-1} d^j g(f(p^{\ell_s} x), p^{\ell_s} v_1, \ldots, p^{\ell_s} v_j)\| \\ &\le p^{\ell_s - 1} p^{-j\ell_s} \|v_1\| \cdots \|v_j\| \le p^{-(j-1)\ell_s} \|u_1\| \cdots \|u_j\| \le p^{-\ell_s} \|u_1\| \cdots \|u_j\|.\end{aligned}$$

If we are *not* in the special situation just described, then $\ell_s \geq 2$ and $t := (s_1, \ldots, s_{\ell_s - 1}) \in S_{\ell_s - 1, j'}$ with $j' := m_{\ell_s} \in \{2, \ldots, j-1\}$. Hence

$$\begin{aligned}
\|g_s^x(p^n u_1, \ldots, p^n u_j)\| &\leq \|g_t^x\| \cdot \left(\prod_{i=1}^{j'} \|(s_{\ell_s}(i)!)^{-1} d^{s_{\ell_s}(i)} g(f(p^{\ell_s} x), \bullet)\| \right) \|p^{\ell_s} v_1\| \ldots \|p^{\ell_s} v_j\| \\
&\leq \|g_t^x\| \|p^{j'} p^{-j\ell_s} \|u_1\| \cdots \|u_j\| \leq p^{-\ell_t} p^{j'(\ell_s - 1)} p^{j'} p^{-j\ell_s} \|u_1\| \cdots \|u_j\| \\
&\leq p^{-\ell_t - (j-j')\ell_s} \|u_1\| \cdots \|u_j\| \leq p^{-\ell_s} \|u_1\| \cdots \|u_j\| .
\end{aligned}$$

Here, passing to the second line we used (14) and (15). For the next inequality, we used that $\|g_t^x\| \leq p^{-\ell_t} p^{j'(\ell_s - 1)}$, by the induction hypothesis. The lemma is established. □

From Lemma 4.4 (case $j \geq 2$) and (14) (used if $j = 1$), we deduce:

$$\|g_s^x(p^n \bullet, \ldots, p^n \bullet)\| \leq 1, \quad \text{for all } x \in B_\rho(x_0),\ n \in \mathbb{N},\ j \in \{1, \ldots, k\}, \text{ and } s \in S_{n,j}. \tag{16}$$

Hence also

$$\|h_{j,n}(x, \bullet)\| \leq 1 \quad \text{for all } x \in B_\rho(x_0),\ j \in \{1, \ldots, k\} \text{ and } n \in \mathbb{N}, \tag{17}$$

in view of the ultrametric inequality.

Lemma 4.5 *For each $j \in \{1, \ldots, k\}$, we have:*

(a) *For every $x \in B_\rho(x_0)$ and $u_1, \ldots, u_j \in E$, the limit*

$$b_j(x, u_1, \ldots, u_j) := \lim_{n \to \infty} h_{j,n}(x, u_1, \ldots, u_j)$$

exists in E.

(b) *For each $x \in B_\rho(x_0)$, the map $b_j(x, \bullet)\colon E^j \to E$ is j-linear, of norm $\|b_j(x, \bullet)\| \leq 1$.*

(c) *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\|b_j(x, \bullet) - h_{j,n}(x, \bullet)\| \leq \varepsilon \qquad \text{for all } x \in B_\rho(x_0) \text{ and } n \geq N.$$

(d) *The map $b_j\colon B_\rho(x_0) \times E^j \to E$ is continuous.*

Proof. We observe first that (d) holds if so does (c). Indeed, given $R > 0$ and $\varepsilon' > 0$, apply (c) with $\varepsilon := \varepsilon'/R^j$. Then, for every $x \in B_\rho(x_0)$, $u_1, \ldots, u_j \in B_R(0)$ and $n \geq N$, we have $\|b_j(x, u_1, \ldots, u_j) - h_{j,n}(x, u_1, \ldots, u_j)\| \leq \varepsilon R^j = \varepsilon'$, showing that $h_{j,n} \to b_j$ uniformly on $B_\rho(x_0) \times B_R(0)^j$. Hence b_j is continuous on $B_\rho(x_0) \times B_R(0)^j$ for each $R > 0$ and hence continuous on all of $B_\rho(x_0) \times E^j$. It remains to prove (a)–(c).

The case $j = 1$. For each $n \in \mathbb{N}$, the set $S_{n,1}$ has one element s only, and we have

$$h_{1,n}(x, \bullet) = g_s^x(p^n \bullet) = pg'(f(px)) \circ \cdots \circ pg'(f(p^n x)) \quad \text{for each } x \in B_\rho(x_0). \tag{18}$$

Given $\varepsilon > 0$, say $\varepsilon \in \,]0,1[$ without loss of generality, there exists $\sigma \in \,]0,r]$ such that

$$\|pg'(y) - \mathrm{id}_E\| < \varepsilon \quad \text{for all } y \in B_\sigma(0) \subseteq E. \tag{19}$$

Choose $N \in \mathbb{N}$ such that $p^{-N}r \leq \sigma$; then $f(p^n x) \in B_\sigma(0)$ for all $x \in B_r(0)$ and all $n \geq N$. For all $x \in B_\rho(x_0)$ and $n_0, n_1 \geq N$, where $n_1 \geq n_0$ without loss of generality, we have

$$h_{1,n_0}(x,\bullet)^{-1} \circ h_{1,n_1}(x,\bullet) = pg'(f(p^{n_0+1}x)) \circ \cdots \circ pg'(f(p^{n_1}x)) = \mathrm{id}_E + A$$

for some $A \in \mathcal{L}(E)$ with $\|A\| < \varepsilon$, using that $B_\varepsilon^{\mathcal{L}(E)}(\mathrm{id}_E)$ is a subgroup of $\mathcal{L}(E)^\times$. Hence

$$\begin{aligned} \|h_{1,n_1}(x,\bullet) - h_{1,n_0}(x,\bullet)\| &= \|h_{1,n_0}(x,\bullet) \circ A\| \leq \|h_{1,n_0}(x,\bullet)\| \cdot \|A\| \\ &\leq \|A\| < \varepsilon \quad \text{for all } x \in B_\rho(x_0) \text{ and } n_0, n_1 \geq N, \end{aligned} \tag{20}$$

using (17). Thus $(h_{1,n}(x,\bullet))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(E)$ for $x \in B_\rho(x_0)$, and hence converges to some $b_1(x,\bullet) \in \mathcal{L}(E)$. Since $\|h_{1,n}(x,\bullet)\| \leq 1$ by (17), we also have $\|b_1(x,\bullet)\| \leq 1$, whence (a) and (b) hold for $j = 1$. Taking $n := n_0$ and letting $n_1 \to \infty$ in (20), we see that $\|b_1(x,\bullet) - h_{1,n}(x,\bullet)\| \leq \varepsilon$ for all $x \in B_\rho(x_0)$ and $n \geq N$, whence (c) holds.

Now let $j \geq 2$. If we can show that, for every $\varepsilon' > 0$, there is $N \in \mathbb{N}$ such that

$$\|h_{j,n_1}(x,\bullet) - h_{j,n_0}(x,\bullet)\| \leq \varepsilon' \quad \text{for all } x \in B_\rho(x_0) \text{ and } n_0, n_1 \geq N, \tag{21}$$

then (a), (b) and (c) follow exactly as in the case $j = 1$. To establish (21), and hence to complete the proof of the lemma, it suffices to find $N \in \mathbb{N}_0$ such that

$$\left\{ \begin{array}{l} h_{j,n_1}(x,u_1,\ldots,u_j) - h_{j,n_0}(x,u_1,\ldots,u_j) \in \overline{B}_\varepsilon(0) \\ \text{for all } x \in B_\rho(x_0),\ n_0, n_1 \geq N, \text{ and } u_1,\ldots,u_j \in E \text{ of norm} \leq 1, \end{array} \right. \tag{22}$$

where $\varepsilon := p^{-j}\varepsilon'$ (cf. [18, p. 59]). To this end, assuming w.l.o.g. $\varepsilon < 1$, we choose $\sigma > 0$ and $N \in \mathbb{N}$ as in the proof of the case $j = 1$, with $p^{-N} < \varepsilon$ now. To see that (22) holds with this N, let $x \in B_\rho(x_0)$, and $u_1, \ldots, u_j \in E$ be vectors of norm ≤ 1. For $n \geq N$ and $s \in S_{n,j}$ with $\ell_s > N$, we have

$$\|g_s^x(p^n u_1, \ldots, p^n u_j)\| \leq p^{-N}\|u_1\| \cdots \|u_j\| \leq \varepsilon\,,$$

by Lemma 4.4. Hence, using that $\overline{B}_\varepsilon(0)$ is an additive subgroup of E, we obtain

$$h_{j,n}(x,u_1,\ldots,u_n) \in \sum_{s\in S_{n,j,N}} g_s^x(p^n u_1,\ldots,p^n u_j) \;+ \overline{B}_\varepsilon(0)\,, \tag{23}$$

where $S_{n,j,N} := \{s \in S_{n,j} \colon \ell_s \leq N\}$. Abbreviating

$$S_{\ell,j}^* := \{s \in S_{\ell,j} \colon \ell_s = \ell\} \quad \text{and} \quad A_{\ell,n} := pg'(f(p^{\ell+1}x)) \circ \cdots \circ pg'(f(p^n x)) \in \mathcal{L}(E)$$

for $\ell \in \{1, \ldots, N\}$ (where $A_{\ell,n} = \mathrm{id}_E$ if $\ell = n$), the above sum can be re-written as

$$\sum_{s\in S_{n,j,N}} g_s^x(p^n u_1,\ldots,p^n u_j) = \sum_{\ell=1}^{N} \sum_{s\in S_{\ell,j}^*} g_s^x(p^\ell A_{\ell,n}u_1,\ldots,p^\ell A_{\ell,n}u_j)\,. \tag{24}$$

If $n_1 \geq n_0 \geq N$, $\ell \in \{1, \ldots, N\}$, and $s \in S^*_{\ell,j}$, then $B := pg'(f(p^{n_0+1}x)) \circ \cdots \circ pg'(f(p^{n_1}x)) \in B^{\mathcal{L}(E)}_\varepsilon(\mathrm{id}_E)$ (cf. proof of (20)) and thus

$$\begin{aligned} & g^x_s(p^\ell A_{\ell,n_1}u_1, \ldots p^\ell A_{\ell,n_1}u_j) - g^x_s(p^\ell A_{\ell,n_0}u_1, \ldots p^\ell A_{\ell,n_0}u_j) \\ &= g^x_s(p^\ell A_{\ell,n_0}Bu_1, \ldots p^\ell A_{\ell,n_0}Bu_j) - g^x_s(p^\ell A_{\ell,n_0}u_1, \ldots p^\ell A_{\ell,n_0}u_j) \in \overline{B}_\varepsilon(0) \end{aligned} \tag{25}$$

by Lemma 1.11, in view of (16), $\|A_{\ell,n_0}\| \leq 1$, $\|B\| \leq 1$, and $\|B - \mathrm{id}_E\| < \varepsilon$. Combining (23), (24) and (25), we see that

$$\begin{aligned} & h_{j,n_1}(x, u_1, \ldots, u_j) - h_{j,n_0}(x, u_1, \ldots, u_j) \\ & \in \sum_{\ell=1}^N \sum_{s \in S^*_{\ell,j}} \big(g^x_s(p^\ell A_{\ell,n_1}u_1, \ldots p^\ell A_{\ell,n_1}u_j) - g^x_s(p^\ell A_{\ell,n_0}u_1, \ldots p^\ell A_{\ell,n_0}u_j)\big) + \overline{B}_\varepsilon(0) = \overline{B}_\varepsilon(0)\,, \end{aligned}$$

whence (22) holds. This completes the proof. □

For each $j \in \{1, \ldots, k\}$, we define $a_j \colon B_\rho(x_0) \times E \to E$ via

$$a_j(x, u) := b_j(x, u, \ldots, u) \qquad \text{for } x \in B_\rho(x_0),\ u \in E. \tag{26}$$

Then a_j is continuous, and $a_j(x, \bullet)$ is a homogeneous polynomial of degree j, for each $x \in B_\rho(x_0)$. To complete the proof of Proposition 4.2, we show:

Lemma 4.6 $R \colon B_\rho(x_0) \times B_\rho(x_0) \to E$, $R(x, y) := f(y) - f(x) - \sum_{j=1}^k a_j(x, y - x)$ *is a continuous mapping such that*

$$f(y) = f(x) + \sum_{j=1}^k a_j(x, y - x) + R(x, y) \quad \textit{for all } x, y \in B_\rho(x_0),$$

$R(x, x) = 0$ *for all* $x \in B_\rho(x_0)$, *and*

$$\lim_{(x,y)\to(z_0,z_0)} \frac{\|R(x, y)\|}{\|x - y\|^k} = 0 \qquad \textit{for all } z_0 \in B_\rho(x_0) \ (\textit{where } x \neq y). \tag{27}$$

Proof. First, we set up a notational formalism enabling us to explicitly label each term in the sum obtained by expanding every factor g of g^n into its k-th order Taylor expansion:

$$g^n(p^n y) - g^n(p^n x) = \sum_{s \in S^0_n} H^{x,y}_{n,s}(p^n(y - x), \ldots, p^n(y - x)) \quad \text{for all } x, y \in B_r(0). \tag{28}$$

Let $P \colon B_r(0) \times B_r(0) \to E$, $P(x, y) := g(y) - g(x) - \sum_{j=1}^k \frac{1}{j!} d^j g(x, y - x, \ldots, y - x)$ be the remainder term of the k-th order Taylor expansion of g. To enable a unified notation for differentials and remainder terms, for $x, y \in B_r(0)$ and $j \in \{0, \ldots, k\}$ we define

$$c^{x,y}_j := \begin{cases} \frac{1}{j!} d^j g(x, \bullet) \in \mathcal{L}^j(E, E) & \text{if } j \in \{1, \ldots, k\}; \\ P(x, y) \in E & \text{if } j = 0. \end{cases}$$

In the following, expressions like $c_0^{x,y}(u_i,\ldots,u_j)$ with $i>j$ have to be read as $c_0^{x,y}$; they denote elements of E (not functions). Given $n\in\mathbb{N}$, we let S_n^0 be the set of all $(s_1,\ldots,s_n)$ where $s_\nu\colon\{1,\ldots,m_\nu\}\to\{0,1,\ldots,k\}$ for $\nu=1,\ldots,n$ for certain $m_\nu\in\mathbb{N}_0$ such that $m_1=1$ and $\sum_{i=1}^{m_\nu}s_\nu(i)=m_{\nu+1}$ for $\nu=1,\ldots,n-1$; set $m_{n+1}:=\sum_{i=1}^{m_n}s_n(i)$. Given $x,y\in B_r(0)$, $n\in\mathbb{N}$, and $s=(s_1,\ldots,s_\nu)\in S_\nu^0$, where $\nu\in\mathbb{N}$ with $\nu\leq n$, we define

$$H_{n,s}^{x,y}(u_1,\ldots,u_{m_{\nu+1}}):=c_{s_1(1)}^{g^{n-1}(p^nx),g^{n-1}(p^ny)}(u_1,\ldots,u_{m_{\nu+1}})\quad\text{for } u_1,\ldots,u_{m_{\nu+1}}\in E$$

if $\nu=1$, and recursively for $\nu\geq 2$

$$\begin{aligned}&H_{n,s}^{x,y}(u_1,\ldots,u_{m_{\nu+1}}) :=\\ &H_{n,(s_1,\ldots,s_{\nu-1})}^{x,y}\big(c_{s_\nu(1)}^{g^{n-\nu}(p^nx),g^{n-\nu}(p^ny)}(u_1,\ldots,u_{s_\nu(1)}),\ c_{s_\nu(2)}^{g^{n-\nu}(p^nx),g^{n-\nu}(p^ny)}(u_{s_\nu(1)+1},\ldots,u_{s_\nu(1)+s_\nu(2)}),\\ &\qquad\ldots,c_{s_\nu(m_\nu)}^{g^{n-\nu}(p^nx),g^{n-\nu}(p^ny)}(u_{m_{\nu+1}-s_\nu(m_\nu)+1},\ldots,u_{m_{\nu+1}})\big)\,.\end{aligned}$$

For example, if $n=2$, $m_1=1$, $m_2=3$ and $s:=(s_1,s_2)$ with $s_1(1):=3$, $s_2(1):=1$, $s_2(2)=0$ and $s_2(3):=2$, then $m_3=3$ and

$$H_{n,s}^{x,y}(u_1,u_2,u_3)=\frac{1}{3!}\,d^3g\Big(g(p^2x),\,dg(p^2x,u_1),\,P(p^2x,p^2y),\,\frac{1}{2}d^2g(p^2x,u_2,u_3)\Big)\,.$$

Using the notational formalism, when $k=n=2$ we calculate

$$\begin{aligned}&g^2(p^2y)-g^2(p^2x)=g(g(p^2y))-g(g(p^2x))\\ &=\ g'(g(p^2x)).(g(p^2y)-g(p^2x))+\frac{1}{2}d^2g(g(p^2x),g(p^2y)-g(p^2x),g(p^2y)-g(p^2x))\\ &\qquad+P(g(p^2x),g(p^2y))\\ &=\ \sum_{s\in S_2^0}H_{2,s}^{x,y}(p^2(y-x),\ldots,p^2(y-x))\,;\end{aligned}$$

here, we used the Taylor expansion of g around $g(p^2x)$ to pass to the second line, and then re-wrote $g(p^2y)-g(p^2x)$ using the Taylor expansion of g around p^2x, to pass to the third (the reader might write down all 13 summands to check this). Likewise, expanding each of the n factors of g^n in turn, a moment's reflection shows that (28) holds, for all $n\in\mathbb{N}$.

We now fix $\varepsilon\in\,]0,1[$ and $z_0\in B_\rho(x_0)$ for the rest of the proof. Keeping the notation set up before Lemma 4.4, we find $\delta_0\in\,]0,\rho_0]$ such that $\|P(x,y)\|\leq\varepsilon\|x-y\|^k$ for all $x,y\in B_{\delta_0}(0)$. There is $N_1\geq N_0$ such that $p^{-N_1}r\leq\delta_0$. For each $n\in\{1,\ldots,N_1\}$, we find $\delta_n\in\,]0,r]$ such that $\|P(x,y)\|\leq\varepsilon\|y-x\|^k$ for all $x,y\in B_{\delta_n}(f(p^nz_0))$. If $n\leq N_0$, we assume that $\delta_n\leq\rho_n$ here. Next, we find $\delta\in\,]0,\min\{\rho,\varepsilon\}[$ (whence $\delta<1$ in particular) such that $f(p^nx)\in B_{\delta_n}(f(p^nz_0))$ for each $n\in\{1,\ldots,N_1\}$ and each $x\in B_\delta(z_0)\subseteq B_\rho(x_0)$. Note that $f(p^nx)\in B_{\delta_0}(0)$ for each $n\geq N_1$ and each $x\in B_r(0)$. Our goal is to show that

$$\|R(x,y)\|\leq\varepsilon\,\|y-x\|^k\quad\text{for all } x,y\in B_\delta(z_0),\tag{29}$$

thus establishing (27). Since $g^n(p^n x) \to f(x)$ uniformly in x by hypothesis, there exists $N_2 \geq N_1$ such that

$$\|g^{n-\nu}(p^n x) - f(p^\nu x)\| \leq \delta_\nu \quad \text{for all } x \in B_r(0),\ \nu \in \{1, \ldots, N_1\}, \text{ and } n \geq N_2\,, \tag{30}$$

whence $g^{n-\nu}(p^n x) \in B_{\delta_\nu}(f(p^\nu z_0))$ if $x \in B_\delta(z_0)$. On the other hand, for every $n \geq N_2$ and $\nu \in \mathbb{N}$ such that $N_1 < \nu \leq n$, we have $\|g^{n-\nu}(p^n x)\| = p^{-\nu}\|x\| < p^{-\nu} r < p^{-N_1} r \leq \delta_0$. As a consequence, we have

$$\|P(g^{n-\nu}(p^n x), g^{n-\nu}(p^n y))\| \leq \varepsilon \|g^{n-\nu}(p^n x) - g^{n-\nu}(p^n y)\|^k \leq \varepsilon p^{-\nu k} \|y - x\|^k \tag{31}$$

for all $x, y \in B_\delta(z_0)$, $n \geq N_2$ and $\nu \in \{1, \ldots, n\}$. Let $x \in B_\delta(z_0)$, $n \geq N_2$, and $\nu \in \{1, \ldots, n\}$. If $\nu \leq N_0$, then $g^{n-\nu}(p^n x) \in B_{\delta_\nu}(f(p^\nu z_0)) \subseteq B_{\rho_\nu}(f(p^\nu z_0)) \subseteq B_{\rho_\nu}(f(p^\nu x_0))$; if $\nu > N_0$, then $\|g^{n-\nu}(p^n x)\| = p^{-\nu}\|x\| < p^{-N_0} r \leq \rho_0$ and thus $g^{n-\nu}(p^n x) \in B_{\rho_0}(0)$. Hence

$$\|(j!)^{-1} d^j g(g^{n-\nu}(p^n x), \bullet)\| \leq 1 \quad \text{for all } x \in B_\delta(z_0),\ 2 \leq j \leq k,\ n \geq N_2, \text{ and } 1 \leq \nu \leq n. \tag{32}$$

To establish (29), we prove estimates on the norms $\|H^{x,y}_{n,s}\|$, which will enable us to get rid of all summands involving remainder terms, or which are multilinear of order exceeding k:

Claim 1. *For any $n \geq N_2$, $\nu \in \{1, \ldots, n\}$, $x, y \in B_\delta(z_0)$, and $s \in S^0_\nu$, we have*

$$\|H^{x,y}_{n,s}\| \leq p^{\nu\, m_{\nu+1}}\,. \tag{33}$$

If $s \in S^0_\nu \setminus S_\nu$ here, then furthermore

$$\|H^{x,y}_{n,s}\| \leq \varepsilon \|y - x\|^k p^{\nu\, m_{\nu+1}}\,. \tag{34}$$

Claim 2. *For any $n \geq N_2$, $x, y \in B_\delta(z_0)$ and $s \in S^0_n$ with $s \notin S_n$ or $m_{n+1} > k$, we have*

$$\|H^{x,y}_{n,s}(p^n(x - y), \ldots, p^n(y - x))\| \leq \varepsilon \|y - x\|^k\,. \tag{35}$$

Once these claims are proved, for $n \geq N_2$ and $x, y \in B_\delta(z_0)$ we shall simply have

$$\begin{aligned} g^n(p^n y) - g^n(p^n x) &= \sum_{s \in S^0_n} H^{x,y}_{n,s}(p^n(y - x), \ldots, p^n(y - x)) \\ &\in \sum_{j=1}^{k} \widetilde{h}_{j,n}(x, y - x, \ldots, y - x) + \overline{B}_{\varepsilon\|y-x\|^k}(0) \end{aligned} \tag{36}$$

with $\widetilde{g}^x_s := H^{x,x}_{n,s}$ for $s \in S_n$ and $\widetilde{h}_{j,n}(x, u_1, \ldots, u_j) := \sum_{s \in S_{n,j}} \widetilde{g}^x_s(p^n u_1, \ldots, p^n u_j)$.

Proof of Claim 1. Fix $n \geq N_2$; the proof is by induction on $\nu \in \{1, \ldots, n\}$. Assume $\nu = 1$ first; thus $s = (s_1)$. If $s_1(1) = 0$, then $\|H^{x,y}_{n,s}\| = \|P(g^{n-1}(p^n x), g^{n-1}(p^n y))\| \leq \varepsilon p^{-k}\|y - x\|^k \leq \varepsilon\|y - x\|^k$ by (31), whence (34) holds and also (33), as $\varepsilon\|y - x\|^k \leq 1 = p^{m_2}$, because $m_2 = 0$. If $j := s_1(1) \geq 1$, then $\|H^{x,y}_{n,s}\| = \|(j!)^{-1} d^j g(g^{n-1}(p^n x), \bullet)\| \leq p \leq p^{1m_2}$ by (14) and (32), whence (33) holds (we need not check (34), because $s \in S_1$).

Induction step. Let $\nu \in \{2, \ldots, n\}$, and suppose the assertion is correct for ν replaced with $\nu - 1$. Given $s \in S^0_\nu$, abbreviate $t := (s_1, \ldots, s_{\nu-1})$. There are four cases:

Case 1: $m_\nu = 0$. Then also $m_{\nu+1} = 0$, and $H^{x,y}_{n,s} = H^{x,y}_{n,t}$ with $t \in S^0_{\nu-1} \setminus S_{\nu-1}$ and thus $\|H^{x,y}_{n,s}\| = \|H^{x,y}_{n,t}\| \leq \varepsilon \|y - x\|^k p^0$ by induction, whence (34) and (33) hold.

Case 2: $m_\nu > 0$, $s \in S^0_\nu \setminus S_\nu$, *and* $t \in S_{\nu-1}$. Then $s_\nu^{-1}(\{0\})$ is a non-empty set; let $j \in \{1, \ldots, m_\nu\}$ be its number of elements. As $H^{x,y}_{n,t}$ is continuous m_ν-linear of norm $\|H^{x,y}_{n,t}\| \leq p^{(\nu-1)m_\nu}$ by induction and $H^{x,y}_{n,s}$ is obtained by inserting j times $P(g^{n-\nu}(p^n x), g^{n-\nu}(p^n y))$ and $(m_\nu - j)$ times multilinear maps of norms $\leq p$ into $H^{x,y}_{n,t}$ (see (14), (32)), we get

$$\begin{aligned} \|H^{x,y}_{n,s}\| &\leq \|H^{x,y}_{n,t}\| \cdot p^{m_\nu - j} \cdot \|P(g^{n-\nu}(p^n x), g^{n-\nu}(p^n y))\|^j \leq p^{(\nu-1)m_\nu} p^{m_\nu - j} \varepsilon^j p^{-\nu k j} \|y - x\|^{kj} \\ &\leq p^{\nu(m_\nu - j)} \varepsilon \|y - x\|^k \leq p^{\nu\, m_{\nu+1}} \varepsilon \|y - x\|^k , \end{aligned}$$

using that $\|P(g^{n-\nu}(p^n x) g^{n-\nu}(p^n y))\| \leq \varepsilon p^{-\nu k} \|y - x\|^k$ by (31), $\varepsilon \leq 1$, $\|y - x\| \leq 1$, and $m_{\nu+1} = \sum_{i=1}^{m_\nu} s_\nu(i) = \sum_{i \in s_\nu^{-1}(\mathbb{N})} s_\nu(i) = (m_\nu - j) + \sum_{i \in s_\nu^{-1}(\mathbb{N})} (s_\nu(i) - 1) \geq m_\nu - j$. Thus (34) and (33) hold.

Case 3: $m_\nu > 0$ *and* $t \in S^0_{\nu-1} \setminus S_{\nu-1}$. Then, by induction, $H^{x,y}_{n,t}$ is continuous m_ν-linear, of norm $\leq \varepsilon \|y - x\|^k p^{(\nu-1)m_\nu}$. Let $j \in \{0, \ldots, m_\nu\}$ be the number of zeros of s_ν. Since $H^{x,y}_{n,s}$ is obtained from $H^{x,y}_{n,t}$ by inserting $(m_\nu - j)$ times multilinear maps of norms $\leq p$ and j times the element $P(g^{n-\nu}(p^n x), g^{n-\nu}(p^n y))$ of norm $\leq p^{-\nu}$ (cf. (31)), we obtain $\|H^{x,y}_{n,s}\| \leq \|H^{x,y}_{n,t}\| \cdot p^{m_\nu - j} p^{-\nu j} \leq \varepsilon \|y - x\|^k p^{(\nu-1)m_\nu} p^{m_\nu - j} p^{-\nu j} \leq \varepsilon \|y - x\|^k p^{\nu(m_\nu - j)} \leq \varepsilon \|y - x\|^k p^{\nu\, m_{\nu+1}}$.

Case 4: $s \in S_\nu$. Then $t \in S_{\nu-1}$ and $1 \leq m_\nu \leq m_{\nu+1}$. By induction, $H^{x,y}_{n,t}$ is a continuous m_ν-linear mapping of norm $\|H^{x,y}_{n,t}\| \leq p^{(\nu-1)m_\nu}$. Therefore $\|H^{x,y}_{n,s}\| \leq \|H^{x,y}_{n,t}\| \cdot \prod_{i=1}^{m_\nu} \|(s_\nu(i)!)^{-1} d^{s_\nu(i)} g(g^{n-\nu}(p^n x), \bullet)\| \leq p^{(\nu-1)m_\nu} p^{m_\nu} = p^{\nu\, m_\nu} \leq p^{\nu\, m_{\nu+1}}$. This completes the proof of Claim 1. □

Proof of Claim 2. Let $n \geq N_2$, $x, y \in B_\delta(z_0)$ and $s \in S^0_n$. If $s \in S^0_n \setminus S_n$, then indeed $\|H^{x,y}_{n,s}(p^n(x-y), \ldots, p^n(x-y))\| \leq \varepsilon \|y-x\|^k p^{n m_{n+1}} \|p^n(y-x)\|^{m_{n+1}} = \varepsilon \|y-x\|^k \|y-x\|^{m_{n+1}} \leq \varepsilon \|y - x\|^k$, using (34). If $s \in S_n$ and $m_{n+1} > k$, then $\|H^{x,y}_{n,s}(p^n(y - x), \ldots, p^n(y - x))\| \leq \varepsilon p^{n m_{n+1}} \|p^n(y - x)\|^{m_{n+1}} = \varepsilon \|y - x\|^{m_{n+1} - k} \|y - x\|^k \leq \varepsilon \|y - x\|^k$, using (33). □

To prove (29), we now fix $x, y \in B_\delta(z_0)$ for the rest of the proof, assuming without loss of generality that $x \neq y$ (the omitted case being trivial). Thus $\overline{\varepsilon} := \varepsilon \, \|y - x\|^k > 0$. There exists $N_3 \geq N_2$ such that $\|f(x) - g^n(p^n x)\| \leq \overline{\varepsilon}$, $\|f(y) - g^n(p^n y)\| \leq \overline{\varepsilon}$, and

$$\|b_j(x, y - x, \ldots, y - x) - h_{j,n}(x, y - x, \ldots, y - x)\| \leq \overline{\varepsilon} \qquad \text{for } j \in \{1, \ldots, k\}, \tag{37}$$

for all $n \geq N_3$. By the preceding and (36), we have

$$f(y) - f(x) \in g^n(p^n y) - g^n(p^n x) + \overline{B}_{\overline{\varepsilon}}(0) = \sum_{j=1}^{k} \widetilde{h}_{j,n}(x, y - x, \ldots, y - x) + \overline{B}_{\overline{\varepsilon}}(0)$$

for all $n \geq N_3$ and thus

$$R(x,y) \in \sum_{j=1}^{k} \big(\widetilde{h}_{j,n}(x, y-x, \ldots, y-x) - h_{j,n}(x, y-x, \ldots, y-x)\big) + \overline{B}_{\overline{\varepsilon}}(0)\,,$$

by (37). Hence (29) will hold if we can show that

$$\|h_{j,n}(x, y-x, \ldots, y-x) - \widetilde{h}_{j,n}(x, y-x, \ldots, y-x)\| \leq \overline{\varepsilon} \tag{38}$$

for all $j \in \{1, \ldots, k\}$ and all sufficiently large n.

Assume $j = 1$ first. We recall that $h_{1,n}(x, \bullet) = pg'(f(px)) \circ \cdots \circ pg'(f(p^n x))$ and $\widetilde{h}_{1,n}(x, \bullet) = pg'(g^{n-1}(p^n x)) \circ \cdots \circ pg'(g^0(p^n x))$. There is $\sigma \in \,]0, r]$ such that $\|pg'(z) - \mathrm{id}_E\| < \overline{\varepsilon}$ for all $z \in B_\sigma(0)$, and $N_4 \geq N_3$ such that $p^{-N_4} r < \sigma$. Then $\|pg'(f(p^n x)) - \mathrm{id}_E\| < \overline{\varepsilon}$ and $\|pg'(g^{n-\nu}(p^n x)) - \mathrm{id}_E\| < \overline{\varepsilon}$ for all $n > N_4$ and $\nu \in \{N_4, \ldots, n\}$. Recalling (17), we deduce for each $n > N_4$ that

$$\begin{aligned} h_{1,n}(x, y-x) &= h_{1,n-1}(x,\, pg'(f(p^n x)).(y-x)) \\ &= h_{1,n-1}(x, y-x) + h_{1,n-1}(x,\, (pg'(f(p^n x)) - \mathrm{id}_E).(y-x))) \\ &\in h_{1,n-1}(x, y-x) + \overline{B}_{\overline{\varepsilon}}(0)\,. \end{aligned}$$

Repeating this argument, we arrive at $h_{1,n}(x, y-x) \in h_{1,N_4}(x, y-x) + \overline{B}_{\overline{\varepsilon}}(0)$ for all $n > N_4$. Similarly, we see that $\widetilde{h}_{1,n}(x, y-x) \in pg'(g^{n-1}(p^n x)) \circ \cdots \circ pg'(g^{n-N_4}(p^n x)).(y-x) + \overline{B}_{\overline{\varepsilon}}(0)$, for all $n > N_4$. Hence

$$h_{1,n}(x, x-y) - \widetilde{h}_{1,n}(x, y-x) \in J_n.(y-x) + \overline{B}_{\overline{\varepsilon}}(0) \quad \text{for all } n > N_4, \text{ where}$$

$J_n := pg'(f(px)) \circ \cdots \circ pg'(f(p^{N_4} x)) - pg'(g^{n-1}(p^n x)) \circ \cdots \circ pg'(g^{n-N_4}(p^n x))$. Because $g^{n-\nu}(p^n x) \to f(p^\nu x)$ as $n \to \infty$, because g' is continuous and also the composition map $\mathcal{L}(E) \times \mathcal{L}(E) \to \mathcal{L}(E)$ is continuous, we see that $J_n \to 0$ as $n \to \infty$, whence there exists $N \geq N_4$ such that $\|J_n\| \leq \overline{\varepsilon}$ for all $n \geq N$. Thus (38) holds for $j = 1$ and all $n \geq N$.

Now assume that $j \in \{2, \ldots, k\}$. Repeating the proof of (23) with $\overline{\varepsilon}$ instead of ε, we see that $h_{j,n}(x, u_1, \ldots, u_j) \in \sum_{s \in S_{n,j,N}} g_s^x(p^n u_1, \ldots, p^n u_j) + \overline{B}_{\overline{\varepsilon}}(0)$ for all $j \in \{2, \ldots, k\}$, $n \geq N$, and $u_1, \ldots, u_j \in E$ of norm ≤ 1 (in view of our choice of σ and N_4). In view of (32), copying the proof of Lemma 4.4 we see that $\|\widetilde{g}_s^x(p^n \bullet, \ldots, p^n \bullet)\| \leq p^{-\ell_s}$ for all $n \geq N_2$, $j \in \{2, \ldots, k\}$ and $s \in S_{n,j}$, entailing that

$$\widetilde{h}_{j,n}(x, u_1, \ldots, u_j) \in \sum_{s \in S_{n,j,N}} \widetilde{g}_s^x(p^n u_1, \ldots, p^n u_j) + \overline{B}_{\overline{\varepsilon}}(0)$$

for all $j \in \{2, \ldots, k\}$, $n \geq N$, and $u_1, \ldots, u_j \in E$ of norm ≤ 1. Setting $\widetilde{A}_{\ell,n} := pg'(g^{n-\ell-1}(p^n x)) \circ \cdots \circ pg'(p^n x)$ for $\ell \in \{1, \ldots, N\}$, we can write $\widetilde{g}_s^x(p^n u_1, \ldots, p^n u_j) = \widetilde{g}_{n,t}^x(p^{\ell_s} \widetilde{A}_{\ell_s,n} u_1, \ldots, p^{\ell_s} \widetilde{A}_{\ell_s,n} u_j)$, where $t := (s_1, \ldots, s_{\ell_s})$ and $\widetilde{g}_{n,t}^x := H_{n,t}^{x,x}$. As a consequence, $h_{j,n}(x, u_1, \ldots, u_j) - \widetilde{h}_{j,n}(x, u_1, \ldots, u_j)$ is contained in

$$\sum_{\ell=1}^{N} \sum_{s \in S_{\ell,j}^*} \Big(g_s^x(p^\ell A_{\ell,n} u_1, \ldots, p^\ell A_{\ell,n} u_j) - \widetilde{g}_{n,s}^x(p^\ell \widetilde{A}_{\ell,n} u_1, \ldots, p^\ell \widetilde{A}_{\ell,n} u_j) \Big) + \overline{B}_{\overline{\varepsilon}}(0)\,,$$

with $A_{\ell,n}$ as in the proof of Lemma 4.5. Henceforth, we fix $u_i := y - x$ for $i = 1, \ldots, k$. To establish (38), we now only need to show that there exists $N_5 \geq N$ such that

$$g_s^x(p^\ell A_{\ell,n}u_1, \ldots, p^\ell A_{\ell,n}u_j) - \widetilde{g}_{n,s}^x(p^\ell \widetilde{A}_{\ell,n}u_1, \ldots, p^\ell \widetilde{A}_{\ell,n}u_j) \in \overline{B}_{\overline{\varepsilon}}(0) \tag{39}$$

for all $j \in \{2, \ldots, k\}$, $\ell \in \{1, \ldots, N\}$, $s \in S_{\ell,j}^*$, and $n \geq N_5$. Recall from the case $j = 1$ that $\|pg'(z) - \mathrm{id}_E\| < \overline{\varepsilon}$ for all $z \in E$ such that $|z| \leq p^{-N}r$ (because $N \geq N_4$). Also recall that $\|\widetilde{g}_{n,s}^x(p^\ell \bullet, \ldots, p^\ell \bullet)\| \leq 1$ and $\|pg'(z)\| \leq 1$ for all z. Replacing each of $pg'(g^0(p^n x)), \ldots, pg'(g^{n-N-1}(p^n x))$ with id_E in turn, we therefore obtain with Lemma 1.11 (applied $n - N$ times) that

$$\widetilde{g}_{n,s}^x(p^\ell \widetilde{A}_{\ell,n}u_1, \ldots, p^\ell \widetilde{A}_{\ell,n}u_j) \in \widetilde{g}_{n,s}^x(p^\ell B_{\ell,n}u_1, \ldots, p^\ell B_{\ell,n}u_j) + \overline{B}_{\overline{\varepsilon}}(0)\,, \tag{40}$$

where $B_{\ell,n} := pg'(g^{n-\ell-1}(p^n x)) \circ \cdots \circ pg'(g^{n-N}(p^n x)) \in \mathcal{L}(E)$. By the same argument,

$$g_s^x(p^\ell A_{\ell,n}u_1, \ldots, p^\ell A_{\ell,n}u_j) \in g_s^x(p^\ell A_{\ell,N}u_1, \ldots, A_{\ell,N}u_j) + \overline{B}_{\overline{\varepsilon}}(0)$$

for all j, ℓ, s, and n as before. Since $g^{n-\nu}(p^n x) \to f(p^\nu x)$ as $n \to \infty$ for all $\nu \in \mathbb{N}$, we see that $B_{\ell,n} \to A_{\ell,N}$ as $n \to \infty$ and $g_{n,s}^x \to g_s^x$, for all ℓ and s as before. We therefore find $N_5 \geq N$ such that, for all $n \geq N_5$, we have

$$\|g_s^x - \widetilde{g}_{n,s}^x\| \leq \overline{\varepsilon} p^{-j\ell} \tag{41}$$

and $\|B_{\ell,n} - A_{\ell,N}\| \leq \overline{\varepsilon}$ for all $j \in \{2, \ldots, k\}$, $\ell \in \{1, \ldots, N\}$, and $s \in S_{\ell,j}$. Then

$$g_s^x(p^\ell A_{\ell,N}u_1, \ldots, p^\ell A_{\ell,N}u_j) - \widetilde{g}_{n,s}^x(p^\ell B_{\ell,n}u_1, \ldots, p^\ell B_{\ell,n}u_j) \in \overline{B}_{\overline{\varepsilon}}(0)$$

for all $n \geq N_5$, by (41) and repeated application of Lemma 1.11. Combining this with (40), we see that (39) holds and hence also (38), (29), and (27). This completes the proof of Lemma 4.6 and thus also the proof of Proposition 4.2. □

5 Existence and compatibility of an analytic structure

In this section, we complete the programme sketched in the Introduction. We prove that every finite-dimensional p-adic C^k-Lie group satisfies the hypotheses of Lazard's Theorem, and show that Lazard's analytic structure is C^k-compatible with the given C^k-manifold structure. In a final step, we then pass from p-adic Lie groups to the case of Lie groups over finite extension fields of $\mathbb{Q}_p$.

Proposition 5.1 *If $\mathbb{K} = \mathbb{Q}_p$ in the situation of Proposition* 3.3*, G is finite-dimensional, and the given norm on $L(G)$ is a maximum norm with respect to some basis $e_1, \ldots, e_d$ of $L(G)$, then we can achieve that, in addition to* (a)–(r)*, also the following holds:*

(s) *For any* $s = p^{-j} \leq r$, *the map*

$$\psi\colon (\mathbb{Z}_p)^d \to W_{p^{-1}s} = V_s, \quad \psi(z_1, \dots, z_d) := \exp(z_1 p^{j+1} e_1) * \cdots * \exp(z_d p^{j+1} e_d)$$

is a $C^k_{\mathbb{Q}_p}$*-diffeomorphism.*

(t) *For any* $s \in \,]0, r]$, *the open subgroup* $\phi^{-1}(V_s) \cong V_s$ *of* G *satisfies Lazard's conditions* **L1–L3**, *whence* G *can be given a finite-dimensional* p*-adic analytic manifold structure compatible with the given topology and making it a* p*-adic analytic Lie group* $\widetilde{G}$.

(u) *The* p*-adic analytic manifold structure on* $\widetilde{G}$ *is* $C^k_{\mathbb{Q}_p}$*-compatible with the* $C^k_{\mathbb{Q}_p}$*-manifold structure on* G, *i.e.,* $\mathrm{id}\colon \widetilde{G} \to G$ *is a* $C^k_{\mathbb{Q}_p}$*-diffeomorphism.*

Proof. (s) By Proposition 2.1 (h), we have $r = p^{-j_0}$ for some $j_0 \in \mathbb{N}$. The map $f\colon (\mathbb{Z}_p)^d \to W_{p^{-1}r} = V_r$, $f(z_1, \dots, z_d) := \exp(z_1 p^{j_0+1} e_1) \cdots \exp(z_d p^{j_0+1} e_d)$ is $C^k_{\mathbb{Q}_p}$ by Corollary 4.3. Since $f'(0)(z_1, \dots, z_d) = p^{j_0+1}(z_1 e_1 + \cdots + z_d e_d)$, the linear map $p^{-j_0-1} f'(0)$ is a bijective isometry. Using the Inverse Function Theorem ([11] Prop. 7.14 and Thm. 7.3), we find $i \in \mathbb{N}_0$ such that f induces a $C^k_{\mathbb{Q}_p}$-diffeomorphism from $p^i \mathbb{Z}_p^d = B^{\mathbb{Z}_p^d}_{p^{-i+1}}(0)$ onto an open subset of $L(G)$, and such that $p^{-j_0-1} f$ is an isometry from the latter set onto $B^{L(G)}_{p^{-i+1}}(0)$. After replacing r with $p^{-i} r$, the assertion then holds.

(t) Let $s \in \,]0, r]$; since $V_s = V_{p^{-j}}$ with j chosen such that $p^{-j-1} < s \leq p^{-j}$, we may assume without loss of generality that $s = p^{-j}$ for some $j \in \mathbb{N}$. By Proposition 2.1 (h)–(j), V_s is a pro-p-group such that $V^{\{p^2\}} = V_{p^{-2}s} \supseteq [V_s, V_s]$. With notation as in (s), we have $V_s = \psi(\mathbb{Z}_p^d) = \overline{\langle x_1 \rangle} \cdots \overline{\langle x_d \rangle}$, where $x_\nu := \exp(p^{j+1} e_\nu)$ for $\nu = 1, \dots, d$. Hence $V_s = \overline{\langle x_1, \dots, x_d \rangle}$ is finitely generated topologically. Thus conditions **L1–L3** (formulated in the Introduction) are satisfied by V_s. Then also $U = \psi^{-1}(V_r)$ satisfies these conditions, and thus G can be made a p-adic analytic Lie group $\widetilde{G}$. The proof of (u) requires further preparation; we give it later in this section. □

Definition 5.2 Let G be a C^1-Lie group modelled on a topological $\mathbb{Q}_p$-vector space. A map $\psi\colon \Omega \to G$ on a balanced 0-neighbourhood $\Omega \subseteq L(G)$ (viz. $\mathbb{Z}_p \Omega = \Omega$) is an *exponential map* if $\psi(\Omega)$ is an identity neighbourhood, ψ is continuous at 0, and $\zeta_x\colon \mathbb{Z}_p \to G$, $\zeta_x(z) := \psi(zx)$ is a homomorphism of class $C^1_{\mathbb{Q}_p}$ with $\zeta_x'(0) = x$, for each $x \in \Omega$.

Remark 5.3 (a) Note that, if $\psi\colon \Omega \to G$ is an exponential map and $x \in \Omega$ with $\psi(x) = 1$, then also $\psi(nx) = 1$ for each $n \in \mathbb{Z}$ and hence $\psi(zx) = 1$ for each $z \in \mathbb{Z}_p$, by continuity, entailing that $x = \frac{d}{dt}\big|_{t=0} \psi(tx) = 0$. Hence ψ is injective and can therefore be considered as a bijection from Ω onto the identity neighbourhood $\psi(\Omega) \subseteq G$.

(b) If $\psi_i\colon \Omega_i \to G$ are exponential maps for $i \in \{1, 2\}$, then $W := \psi_1(\Omega_1) \cap \psi_2(\Omega_2)$ is an identity neighbourhood in G. Given $w \in W$, there exist elements $x_i \in \Omega_i$ such that $\psi_i(x_i) = w$. Then $\psi_1(nx_1) = \psi_1(x_1)^n = \psi_2(x_2)^n = \psi_2(nx_2)$ for each $n \in \mathbb{Z}$ and thus $\psi_1(tx_1) = \psi_2(tx_2)$ for all $t \in \mathbb{Z}_p$, by continuity, entailing that $x_1 = \frac{d}{dt}\big|_{t=0} \psi_1(tx_1) = \frac{d}{dt}\big|_{t=0} \psi_2(tx_2) = x_2$.

Thus $\Omega := \psi_1^{-1}(W) = \psi_2^{-1}(W)$, and $\psi_1|_\Omega = \psi_2|_\Omega$. Since ψ_1 is continuous, Ω is a 0-neighbourhood. Given $x \in \Omega$, we have $\psi(x_1) = \psi(x_2)$ and thus $\psi_1(zx) = \psi_2(zx)$ for each $z \in \mathbb{Z}_p$, entailing that $\mathbb{Z}_p x \subseteq \Omega$. Thus Ω is balanced.

(c) If G is a p-adic C^k-Lie group admitting an exponential map $\exp_G\colon \Omega \to G$, we let $\Gamma(G)$ be the set of germs $[\xi]$ at 0 of continuous homomorphisms $\xi : W \to G$ defined on some open subgroup $W \subseteq \mathbb{Q}_p$. Then

$$\theta_G\colon L(G) \to \Gamma(G), \quad \theta_G(x) = [t \mapsto \exp_G(tx)] \tag{42}$$

is a bijection [Since $x = \frac{d}{dt}\big|_{t=0} \exp_G(tx)$ for $x \in L(G)$, the map θ_G is injective. To see that θ_G is surjective, let $[\xi] \in \Gamma(G)$, where $\xi : W \to G$. Since ξ is continuous, we find $n \in \mathbb{N}$ such that $p^n \in W$ and $\xi(p^n) \in \exp_G(\Omega)$. Hence $\xi(p^n) = \exp_G(x)$ for some $x \in \Omega$, whence $\xi(mp^n) = \exp_G(mx)$ for all $m \in \mathbb{Z}$ and actually for all $m \in \mathbb{Z}_p$, by continuity. Thus $\xi(z) = \exp_G(zp^{-n}x)$ for all z in the 0-neighbourhood $p^n\mathbb{Z}_p$ in $\mathbb{Q}_p$, whence $[\xi] = \theta_G(p^{-n}x)$.] We give $\Gamma(G)$ the p-adic topological vector space structure making θ_G an isomorphism of topological vector spaces. Apparently, the map θ_G only depends on the germ of $\exp_G$ at 0 and therefore is independent of the choice of exponential map, by (b).

If G is an ultrametric p-adic Banach-Lie group of class $C^{k_+}_{\mathbb{Q}_p}$ and $\phi\colon U \to V_r \subseteq L(G)$ and $\exp\colon V_r \to V_r$ are as in Proposition 3.3, then clearly $\exp_G\colon V_r \to G$, $x \mapsto \phi^{-1}(\exp(x))$ is an exponential map for G. We define $\log_G := (\exp_G|^U)^{-1} = \log \circ \phi^{-1}\colon U \to V_r \subseteq L(G)$. Given $x \in U^{\{p^n\}} = \phi^{-1}(V_{p^{-n}r})$ (see Proposition 2.1 (i)), we let $x^{p^{-n}} := \phi^{-1}(\tau_p^{-n}(\phi(x)))$; this is the unique element in U with p^n-th power x. Using a version of Trotter's Product Formula, we now show that the topological vector space $\Gamma(G)$ is determined by the topological group underlying G.

Lemma 5.4 *In the preceding situation, we have:*

(a) $x + y = \lim_{n\to\infty} \log_G\big((\exp_G(p^n x)\exp_G(p^n y))^{p^{-n}}\big)$ *for all* $x, y \in V_r$.

(b) *If* $[\xi_1]$, $[\xi_2] \in \Gamma(G)$ *and* $[\xi_1] + [\xi_2] = [\xi_3]$, *there is* $\delta > 0$ *such that* $\xi_1(t)$, $\xi_2(t)$ *and* $\xi_3(t)$ *are defined for all* $t \in \mathbb{Z}_p$ *with* $|t|_p \leq \delta$, *each of them is an element of* U, *and* $\xi_3(t) = \lim_{n\to\infty}(\xi_1(p^n t)\xi_2(p^n t))^{p^{-n}}$.

(c) *If* $\widetilde{G}$ *is a p-adic ultrametric Banach-Lie group of class* $C^{k_+}_{\mathbb{Q}_p}$ *such that* $\widetilde{G}$ *and* G *have the same underlying topological group, then* $\Gamma(G) = \Gamma(\widetilde{G})$ *as a set and as a topological vector space over* $\mathbb{Q}_p$. *Furthermore, the map* $G \to \widetilde{G}$, $x \mapsto x$ *is an isomorphism of* $SC^1_{\mathbb{Q}_p}$*-Lie groups.*

Proof. (a) Since $G \cong V_r$, we may assume that $G = V_r$. Note that, for $v \in V_{p^{-n}r}$, we have $\exp(p^n(\log(v^{p^{-n}}))) = (\exp(\log(v^{p^{-n}})))^{p^n} = (v^{p^{-n}})^{p^n} = v = \exp(\log(v))$, whence $p^n \log(v^{p^{-n}}) = \log(v)$ and thus $\log(v^{p^{-n}}) = p^{-n}\log(v)$. Therefore

$$\log((\exp(p^n x)\exp(p^n y))^{p^{-n}}) = p^{-n}\log(\exp(p^n x)\exp(p^n y)) \quad \text{for all } x, y \in V_r,\ n \in \mathbb{N}. \tag{43}$$

Now $f\colon \mathbb{Z}_p \to L(G)$, $f(z) := \log(\exp(zx)\exp(zy))$ being C^1 and thus strictly differentiable at 0, with $f'(0) = x+y$, given $\varepsilon > 0$ we find $\delta > 0$ such that $\|f(z) - z(x+y)\| \leq \varepsilon |z|_p$ for all $z \in \mathbb{Z}_p$ such that $|z|_p \leq \delta$. Using (43), we obtain $\|\log((\exp(p^n x)\exp(p^n y))^{p^{-n}}) - (x-y)\| = p^n \|f(p^n) - p^n(x+y)\| \leq \varepsilon$ for all $n \in \mathbb{N}$ such that $p^{-n} \leq \delta$. Hence (a) holds.

(b) Let $v_i := \theta_G^{-1}([\xi_i])$ for $i \in \{1,2,3\}$. Then $v_3 = v_1 + v_2$, the map θ_G being linear. There is $\delta > 0$ such that $\delta\|v_i\| < r$ for each $i \in \{1,2,3\}$, $\xi_i(t)$ is defined for all $t \in \mathbb{Z}_p$ with $|t|_p \leq \delta$, and $\xi_i(t) = \exp_G(tv_i)$. Using Part (a) and the continuity of $\exp_G$, for any t as before we obtain $\xi_3(t) = \exp_G(tv_3) = \exp_G(tv_1 + tv_2) = \lim_{n\to\infty}(\exp(p^n tv_1)\exp(p^n tv_2))^{p^{-n}} = \lim_{n\to\infty}(\xi_1(p^n t)\xi_2(p^n t))^{p^{-n}}$, as asserted.

(c) Because the definition of the set $\Gamma(G)$ only involves the topological group structure of G, we have $\Gamma(G) = \Gamma(\widetilde{G})$ as a set. Given $z \in \mathbb{Q}_p$ and $[\xi] \in \Gamma(G)$, we have $z[\xi] = [t \mapsto \xi(zt)]$ both in $\Gamma(G)$ and $\Gamma(\widetilde{G})$. Thus the scalar multiplication maps of the two $\mathbb{Q}_p$-vector spaces coincide. We let $\widetilde{U} = \widetilde{\phi}^{-1}(\widetilde{V}_{\widetilde{r}}) \subseteq \widetilde{G}$ be an open subgroup of $\widetilde{G}$ playing a role analogous to that of $U \subseteq G$; after shrinking $\widetilde{r}$, we may assume without loss of generality that $\widetilde{U} \subseteq U$. Then $\widetilde{U}^{\{p^n\}}$ is an open subgroup of G for each $n \in \mathbb{N}$, and each $x \in \widetilde{U}^{\{p^n\}}$ has a unique p^n-th root y in $\widetilde{U}$, and also a unique p^n-th root $x^{p^{-n}}$ in U; by uniqueness of $x^{p^{-n}}$ in U, we have $x^{p^{-n}} = y \in \widetilde{U}$. Now assume that $[\xi_1], [\xi_2] \in \Gamma(G)$ are given, with $[\xi_1] + [\xi_2] = [\xi_3]$ in $\Gamma(G)$, $[\xi_1] + [\xi_2] = [\xi_4]$ in $\Gamma(\widetilde{G})$. Part (b) allows us to calculate the respective sum both in $\Gamma(G)$ and $\Gamma(\widetilde{G})$: there is $\delta > 0$ such that $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ and $\xi_4(t)$ are defined for all $t \in \mathbb{Q}_p$ with $|t|_p \leq \delta$, all of them are elements of $\widetilde{U}$ (hence of U), and

$$\xi_3(t) = \lim_{n\to\infty}(\xi_1(p^n t)\xi_2(p^n t))^{p^{-n}} = \xi_4(t)\,,$$

using that, as just explained, the p^n-th roots in U and $\widetilde{U}$ occurring here coincide. By the preceding, the germs at 0 of ξ_3 and ξ_4 coincide. Thus $[\xi_3] = [\xi_4]$, whence the sum $[\xi_1] + [\xi_2]$ is the same in $\Gamma(G)$ and $\Gamma(\widetilde{G})$. Hence $\Gamma(G)$ and $\Gamma(\widetilde{G})$ coincide as $\mathbb{Q}_p$-vector spaces.

To see that $\Gamma(G)$ and $\Gamma(\widetilde{G})$ coincide as topological vector spaces, consider the open 0-neighbourhoods $\Omega := \theta_G(V_r) \subseteq \Gamma(G)$ and $\widetilde{\Omega} := \theta_{\widetilde{G}}(\widetilde{V}_{\widetilde{r}}) \subseteq \Gamma(\widetilde{G})$. The maps $\mathrm{Log}_G := \theta_G|_{V_r}^{\Omega} \circ \log_G\colon U \to \Omega$ and $\mathrm{Log}_{\widetilde{G}} := \theta_{\widetilde{G}}|_{\widetilde{V}_{\widetilde{r}}}^{\widetilde{\Omega}} \circ \log_{\widetilde{G}}\colon \widetilde{U} \to \widetilde{\Omega}$ are SC^1-diffeomorphisms. For each $x \in U \cap \widetilde{U}$, there is a unique continuous homomorphism $\gamma_x\colon \mathbb{Z}_p \to G$ such that $\gamma_x(1) = x$ (namely, $\gamma_x = \phi^{-1} \circ \eta_{\phi(x)}$). Since $\exp_G(n\log_G(x)) = \exp_G(\log_G(x))^n = x^n$ for each $n \in \mathbb{Z}$, we deduce that $\gamma_x(z) = \exp_G(z\log_G(x))$ for all $z \in \mathbb{Z}_p$, and likewise $\gamma_x(z) = \exp_{\widetilde{G}}(z\log_{\widetilde{G}}(x))$. As a consequence,

$$\mathrm{Log}_G(x) = \theta_G(\log_G(x)) = [t \mapsto \exp_G(t\log_G(x))] = [\gamma_x] = \theta_{\widetilde{G}}(\log_{\widetilde{G}}(x)) = \mathrm{Log}_{\widetilde{G}}(x)\,,$$

entailing that $x \mapsto \mathrm{Log}_{\widetilde{G}}(\mathrm{Log}_G^{-1}(x)) = x$ is an SC^1-diffeomorphism (and hence a homeomorphism) from the open 0-neighbourhood $Q := \mathrm{Log}_G(U \cap \widetilde{U})$ in $\Gamma(G)$ onto the open identity neighbourhood $\mathrm{Log}_{\widetilde{G}}(U \cap \widetilde{U})$ in $\Gamma(\widetilde{G})$. As a consequence, $\Gamma(G) = \Gamma(\widetilde{G})$ as a topological $\mathbb{Q}_p$-vector space. Since $\mathrm{Log}_G|_{U\cap\widetilde{U}}^{Q} = \mathrm{Log}_{\widetilde{G}}|_{U\cap\widetilde{U}}^{Q}$ is an SC^1-diffeomorphism both on $U \cap \widetilde{U}$, considered as an open subset of G, and as an open subset of $\widetilde{G}$, we readily deduce that both the homomorphism $\mathrm{id}\colon G \to \widetilde{G}$ and its inverse $\mathrm{id}\colon \widetilde{G} \to G$ are of class SC^1. $\square$

Proof of Proposition 5.1, completed. (u) By Proposition 5.1 (t), there is a p-adic analytic manifold structure on G, compatible with the given topology, which makes G a finite-dimensional, p-adic analytic Lie group $\widetilde{G}$. By Lemma 5.4 (c), the analytic Lie group structure on $\widetilde{G}$ is C^1-compatible with the given C^k-manifold structure on G. By Part (s) of Proposition 5.1, the map

$$\psi\colon (\mathbb{Z}_p)^d \to V_r, \quad \psi(z_1, \ldots, z_d) := \zeta_1(z_1)\zeta_2(z_2)\cdots\zeta_d(z_d)$$

is a C^k-diffeomorphism, where $\zeta_\nu\colon \mathbb{Z}_p \to G$, $\zeta_\nu := \exp_G(zp^{j_0+1}e_\nu)$ for $\nu \in \{1, \ldots, d\}$. Each ζ_ν is, in particular, a continuous homomorphism and hence analytic as a map into $\widetilde{G}$ by Cartan's Theorem ([23], Part II, Chapter V, §9, Thm. 2). Hence ψ is analytic as a map into $\widetilde{G}$. Now, ψ being a C^1-diffeomorphism onto U considered as an open subset of G, the map ψ also is a C^1-diffeomorphism onto U considered as an open subset of $\widetilde{G}$, the two manifold structures being C^1-compatible. Being a C^1-diffeomorphism and analytic, $\psi\colon (\mathbb{Z}_p)^d \to U \subseteq \widetilde{G}$ is an analytic diffeomorphism (and hence a C^k-diffeomorphism), as a consequence of the Inverse Function Theorem for analytic maps [23, p. 73]. Thus both G and $\widetilde{G}$ induce the same C^k-manifold structure on the open identity neighbourhood U, whence the homomorphisms $\mathrm{id}\colon G \to \widetilde{G}$ and $\mathrm{id}\colon \widetilde{G} \to G$ are C^k, being C^k on U (Lemma 3.1). $\square$

We now prove our main result, Theorem A (from the Introduction):

Proof of Theorem A. Being a $C^k_{\mathbb{K}}$-Lie group, G can also be considered as a $C^k_{\mathbb{Q}_p}$-Lie group. Thus Proposition 5.1 provides a finite-dimensional p-adic analytic manifold structure on G making it a p-adic analytic Lie group $\widetilde{G}$, which is $C^k_{\mathbb{Q}_p}$-compatible with the given $C^k_{\mathbb{Q}_p}$-manifold structure on G. Then $L(\widetilde{G}) = T_1\widetilde{G}$ can be identified with $L(G)$, considered as $\mathbb{Q}_p$-vector space, in a natural way. Given $x \in G$, consider the inner automorphism $I_x : G \to G$, $I_x(y) := xyx^{-1}$ of the $C^k_{\mathbb{K}}$-Lie group G and the corresponding $\mathbb{K}$-linear tangent map $\mathrm{Ad}_x := L(I_x) := T_1(I_x) : L(G) \to L(G)$. Obviously the same mapping Ad_x is obtained when considering I_x as an automorphism of $\widetilde{G}$, and so the given $\mathbb{K}$-vector space structure on $L(G)$ is compatible with the adjoint action of $\widetilde{G}$. Furthermore, the Lie bracket on $L(G)$ as the Lie algebra of $\widetilde{G}$ is $\mathbb{K}$-bilinear with respect to the given $\mathbb{K}$-vector space structure on $L(G)$. To see this, note that the image of the $\mathbb{Q}_p$-analytic homomorphism $h : \widetilde{G} \to \mathrm{GL}_{\mathbb{Q}_p}(L(G))$, $h(x) := \mathrm{Ad}_x$ is contained in the closed subgroup $\mathrm{GL}_{\mathbb{K}}(L(G))$, whence the image of $\mathrm{ad} := L(h) : L(\widetilde{G}) \to \mathrm{gl}_{\mathbb{Q}_p}(L(G))$ is contained in the corresponding Lie subalgebra $\mathrm{gl}_{\mathbb{K}}(L(G))$ of $\mathbb{K}$-linear endomorphisms. Thus $\mathrm{ad}(x).y = [x, y]$ is $\mathbb{K}$-linear in y for each $x \in L(G)$, and hence so it is in x, by antisymmetry of the Lie bracket. Applying [7], Chapter III, §4.2, Cor. 2 to Thm. 2, we now obtain a unique $\mathbb{K}$-analytic manifold structure on G making it a $\mathbb{K}$-analytic Lie group $\widehat{G}$ with $\mathbb{K}$-Lie algebra $L(G)$, and $\mathbb{Q}_p$-analytically compatible with the p-adic analytic structure on $\widetilde{G}$. We let $\exp_{\widehat{G}}\colon \Omega \to \widehat{G}$ be a $\mathbb{K}$-analytic exponential map for the $\mathbb{K}$-analytic Lie group $\widehat{G}$, in the sense of [7], Chapter III, §4.3, Definition 1, defined on an open $\mathbb{A}$-submodule Ω of $L(G)$ (where, as before, $\mathbb{A} = \{z \in \mathbb{K} \colon |z| \leq 1\}$). By *loc.cit.*, Thm. 4 (i), there exists $\varepsilon > 0$ such that $\exp_{\widehat{G}}((t+s)x) = \exp_{\widehat{G}}(tx)\exp_{\widehat{G}}(sx)$ for all $x \in \Omega$ and all $s, t \in \mathbb{K}$ such that $|s|, |t| \leq \varepsilon$. After replacing Ω by $p^n\Omega$ with n sufficiently large, we may assume that $\varepsilon = 1$ here, entailing that $\exp_{\widehat{G}}(zx) = (\exp_{\widehat{G}}(x))^z$ for all $z \in \mathbb{Z}$ and hence also for all $z \in \mathbb{Z}_p$. Since $\exp'_{\widehat{G}}(0) = \mathrm{id}_{L(G)}$, after shrinking Ω we may assume that $\exp_{\widehat{G}}$ is a $\mathbb{K}$-analytic diffeomorphism onto an open subset of $\widehat{G}$. Note that, by the last and penultimate property, $\exp_{\widehat{G}}$ also is an exponential map in the sense of Definition 5.2 for the $C^1_{\mathbb{Q}_p}$-Lie group underlying $\widehat{G}$, and hence for G. On the other hand, Proposition 3.3 (r) provides an exponential map $\exp_G := \phi^{-1} \circ \exp \colon V_r \to U$ for G, such that

$$\zeta_x \colon \mathbb{A} \to G, \quad z \mapsto \exp_G(zx)$$

is of class $C^k_{\mathbb{K}}$, for each $x \in V_r$ (Corollary 4.3). Since both $\exp_{\widehat{G}}$ and $\exp_G$ are exponential maps for G, considered as an $C^1_{\mathbb{Q}_p}$-Lie group, we deduce from Remark 5.3 (b) that $\exp_G$ and $\exp_{\widehat{G}}$ coincide on $Q := B^{L(G)}_s(0) \subseteq \Omega$ for some $s \in \,]0, r]$. As a consequence, for every $x \in Q$ the map $\zeta_x = \exp_G(\bullet x) = \exp_{\widehat{G}}(\bullet x)$ is $C^k_{\mathbb{K}}$ both as a map into G, and as a map into $\widehat{G}$. We pick a basis $e_1, \ldots, e_d \in Q$ of the $\mathbb{K}$-vector space $L(G)$. Using the Inverse Function Theorem for $C^k_{\mathbb{K}}$-maps and the Inverse Function Theorem for $\mathbb{K}$-analytic maps, we find $n \in \mathbb{N}_0$ such that

$$(p^{-n}\mathbb{A})^d \to G, \quad (z_1, \ldots, z_d) \mapsto \zeta_{e_1}(z_1) \cdots \zeta_{e_d}(z_d)$$

is both a $C^k_{\mathbb{K}}$-diffeomorphism onto an open subset of G, and a $\mathbb{K}$-analytic diffeomorphism onto the corresponding subset of $\widehat{G}$. Hence $\mathrm{id} : G \to \widehat{G}$ is $C^k_{\mathbb{K}}$ on some open identity neighbourhood and thus $C^k_{\mathbb{K}}$, and likewise for $\mathrm{id} \colon \widehat{G} \to G$. □

A Proofs of the lemmas from Section 1

Proof of Lemma 1.6. The map f being C^2, we have a second order Taylor expansion

$$f(x + ty) - f(x) - tdf(x, y) = t^2 a_2(x, y) + t^2 R_2(x, y, t)$$

for $(x, y, t) \in U^{[1]}$, with remainder $R_2 \colon U^{[1]} \to F$ (see [2], Thm. 5.1 and Prop. 5.3). Let $\|.\|_\gamma$ be a continuous seminorm on F. Since $R_2(x_0, 0, 0) = 0$ and $a_2(x_0, 0) = 0$, there exists $\rho \in \,]0, 1]$ such that $B_{2\rho}(x_0) \subseteq U$,

$$\|R_2(x, y, t)\|_\gamma \leq 1 \qquad \text{for all } x \in B_\rho(x_0),\ y \in B_\rho(0), \text{ and } |t| < \rho,$$

and $\|a_2(x, y)\|_\gamma \leq 1$ for all $x \in B_\rho(x_0)$ and $y \in B_\rho(0)$. Pick $a \in \mathbb{K}^\times$ such that $|a| < 1$; define $\delta := \rho^2|a| < \rho$ and $C := 2/(\rho|a|)^2$. Let $x \in B_\delta(x_0)$ and $y \in B_\delta(0)$. If $y = 0$, then $\|f(x + y) - f(x) - df(x, y)\|_\gamma \leq C\|y\|^2$ trivially. If $y \neq 0$, there exists $k \in \mathbb{Z}$ such that $|a|^{k+1} \leq \rho^{-1}\|y\| < |a|^k$. Then $\|a^{-k}y\| < \rho$ and $|a^k| \leq |a|^{-1}\rho^{-1}\|y\| < \rho$, and thus

$$f(x + y) - f(x) = f(x + a^k a^{-k}y) - f(x) = df(x, y) + a^{2k}a_2(x, a^{-k}y) + a^{2k}R_2(x, a^{-k}y, a^k)$$

where $\|a^{2k}a_2(x,a^{-k}y)+a^{2k}R_2(x,a^{-k}y,a^k)\|_\gamma \leq |a|^{2k}(\|a_2(x,a^{-k}y)\|_\gamma+\|R_2(x,a^{-k}y,a^k)\|_\gamma) \leq 2|a|^{2k} \leq 2|a|^{-2}\rho^{-2}\|y\|^2 = C\|y\|^2$. □

Proof of Lemma 1.7. We may assume that U and V are balanced. Then, for every $x \in U$, $y \in V$ and $s,t \in \mathbb{K}$ such that $|s|,|t| \leq 1$, the first order Taylor expansion of f about $(tx,0)$ shows that

$$f(tx,sy) = \underbrace{f(tx,0)}_{=f(0,0)+\lambda(tx)} + s\,df((tx,0),(0,y)) + sR_1((tx,0),(0,y),s)\,, \tag{44}$$

where $df\colon U\times V\times E\times F \to H$ and the remainder $R_1\colon (U\times V)^{[1]} \to H$ are of class $C^1_\mathbb{K}$ (see [2]). Using the first order Taylor expansions of df and R_1 about $((0,0),(0,y))$ and $((0,0),(0,y),s)$, with remainders P_1 and Q_1, respectively, we deduce from (44) that

$$\begin{aligned} f(tx,sy) &= f(0,0)+\lambda(tx)+s\underbrace{df((0,0),(0,y))}_{=\mu(y)} + st\underbrace{d(df)((0,0,0,y),(x,0,0,0))}_{=d^2f((0,0),(0,y),(x,0))} \\ &\quad + st\,P_1((0,0,0,y),(x,0,0,0),t) + s\underbrace{R_1((0,0),(0,y),s)}_{=0} \\ &\quad + st\,dR_1((0,0,0,y,s),(x,0,0,0,0)) + st\,Q_1((0,0,0,y,s),(x,0,0,0,0),t) \\ &= f(0,0)+\lambda(tx)+\mu(sy)+\beta(tx,sy)+st\,g(x,y,s,t)\,, \end{aligned} \tag{45}$$

where inessential brackets were suppressed in the notation, the 6th term vanishes due to (2), the map $\beta\colon E\times F\to H$, $\beta(u,v) := d^2f((0,0),(0,v),(u,0))$ is continuous bilinear, and where the 5th, 7th and 8th terms are combined in an apparent way in the form $st\,g(x,y,s,t)$, where $g: U\times V\times \overline{B}^\mathbb{K}_1(0)\times\overline{B}^\mathbb{K}_1(0) \to H$ is continuous, and $g(0,0,0,0)=0$. We find $\sigma \in]0,1]$ such that $B^E_\sigma(0)\subseteq U$, $B^F_\sigma(0)\subseteq V$, and $\|g(x,y,s,t)\|\leq 1$ for all $x\in B^E_\sigma(0)$, $y \in B^F_\sigma(0)$, and $s,t\in\mathbb{K}$ such that $|s|,|t|\leq\sigma$. Let $a\in\mathbb{K}^\times$ be an element such that $|a|<1$. Assume that $x\in E$ and $y\in F$ such that $\|x\|,\|y\| < \sigma^2|a| =: \delta$. If $x=0$ or $y=0$, then $\|f(x,y)-f(0,0)-\lambda(x)-\mu(y)\|_\gamma = 0$ by (1) and (2), respectively. Otherwise, we find uniquely determined numbers $k,\ell\in\mathbb{Z}$ such that $|a|^{k+1}\leq\frac{\|x\|}{\sigma}<|a|^k$ and $|a|^{\ell+1}\leq\frac{\|y\|}{\sigma}<|a|^\ell$. Then $|a|^k\leq\frac{\|x\|}{|a|\sigma}<\sigma$ and $\|a^{-k}x\|<\sigma$; similarly, $|a|^\ell<\sigma$ and $\|a^{-k}y\|<\sigma$. Choosing $t:=a^k$ and $s:=a^\ell$ in (45), we obtain

$$\begin{aligned} \|f(x,y)-f(0,0)-\lambda(x)-\mu(y)\|_\gamma &= \|f(a^k(a^{-k}x),a^\ell(a^{-\ell}y))-f(0,0)-\lambda(x)-\mu(y)\|_\gamma \\ &\leq \|\beta(x,y)\|_\gamma + |a|^k|a|^\ell\|g(a^{-k}x,a^{-\ell}y,a^\ell,a^k)\|_\gamma \\ &\leq \|\beta\|_\gamma\|x\|\,\|y\| + \frac{\|x\|}{|a|\sigma}\,\frac{\|y\|}{|a|\sigma} \leq C\,\|x\|\,\|y\|, \end{aligned}$$

with $C:=\|\beta\|_\gamma + (|a|\sigma)^{-2}$. □

The proof of Lemma 1.8 will be based on the following observation:

Lemma A.1 *Let X and E be normed spaces over a valued field $\mathbb{K}$, F be a polynormed $\mathbb{K}$-vector space, $U\subseteq X$ an open subset, $n\in\mathbb{N}$, and $f: U\times E^n\to F$ be a $C^1_\mathbb{K}$-map such* *that $f(x,\bullet)\colon E^n \to F$ is n-linear, for each $x \in U$. Then the map $\phi\colon U \to \mathcal{L}^n(E,F)$, $\phi(x) := f(x,\bullet)$ is continuous.*

Proof. Using the first order Taylor expansion of f, we can write

$$f(x+tz,y)-f(x,y) = td_1f(x,y,z)+tR(x,z,t,y) \quad \text{for all } (x,z,t)\in U^{[1]} \text{ and } y \in E^n,$$

where $d_1f(x,y,z) := df((x,y),\,(z,0))$ is linear in z and $R\colon U^{[1]}\times E^n \to F$ is a continuous map such that $R(x,z,t,y)=0$ whenever $t=0$. Since $f(x,\bullet)$ is n-linear, apparently so is $d_1f(x,\bullet,z)$, and then also $R(x,z,t,\bullet)$ (this is clear for $t\neq 0$, and follows for $t=0$ by continuity). Thus $d_1f(x,\bullet)$ is a continuous $(n{+}1)$-linear map. Now assume that $x\in U$, $\varepsilon>0$, and that $\|.\|_\gamma$ is a continuous seminorm on F. Then there is $\delta\in\,]0,1]$ such that $B^X_\delta(x)\subseteq U$ and $\|R(x,z,t,y_1,\ldots,y_n)\|_\gamma\leq 1$ for all $z\in B^X_\delta(0)$, $|t|\leq\delta$, and $y_1,\ldots,y_n\in B^E_\delta(0)$. Pick $a\in\mathbb{K}^\times$ such that $|a|<1$. Let $\rho:=\min\{\varepsilon(1+2\|d_1f(x,\bullet)\|_\gamma)^{-1}, \frac{\varepsilon}{2}(|a|\delta)^{n+1}, |a|\delta^2\}$. Let $0\neq z\in B^X_\rho(0)$ and $y_1,\ldots,y_n\in E$. If some $y_j=0$, then $(\phi(x+z)-\phi(x))(y_1,\ldots,y_n)=0$. Otherwise, we find $k_0,k_1,\ldots,k_n\in\mathbb{Z}$ such that $|a|^{k_0+1}\leq\|z\|\delta^{-1}<|a|^{k_0}$ and $|a|^{k_j+1}\leq\|y_j\|\delta^{-1}<|a|^{k_j}$ for $j=1,\ldots,n$. Thus $\|a^{-k_0}z\|<\delta$ and $\|a^{-k_j}y_j\|<\delta$ and hence, writing $y:=(y_1,\ldots,y_n)$,

$$\begin{aligned} f(x+z,y)-f(x,y) &= f(x+a^{k_0}a^{-k_0}z,y)-f(x,y) = d_1f(x,y,z)+a^{k_0}R(x,a^{-k_0}z,a^{k_0},y)\\ &= d_1f(x,y,z)+a^{k_0+k_1+\cdots+k_n}R(x,a^{-k_0}z,a^{k_0},a^{-k_1}y_1,\ldots,a^{-k_n}y_n), \end{aligned} \tag{46}$$

where $\|d_1f(x,y,z)\|_\gamma\leq\|d_1f(x,\bullet)\|_\gamma\cdot\|z\|\cdot\|y_1\|\cdots\|y_n\|\leq\frac{\varepsilon}{2}\|y_1\|\cdots\|y_n\|$. Since

$$|a^{k_0+k_1+\cdots+k_n}|\leq|a|^{-(n+1)}\delta^{-(n+1)}\|z\|\cdot\|y_1\|\cdots\|y_n\|\leq\frac{\varepsilon}{2}\|y_1\|\cdots\|y_n\|$$

and $\|R(x,a^{-k_0}z,a^{k_0},a^{-k_1}y_1,\ldots,a^{-k_n}y_n)\|_\gamma\leq 1$, also the final term in (46) has norm $\leq\frac{\varepsilon}{2}\|y_1\|\cdots\|y_n\|$. Thus $\|(\phi(y{+}z){-}\phi(x))(y_1,\ldots,y_n)\|_\gamma\leq\varepsilon\|y_1\|\cdots\|y_n\|$ for all $y_1,\ldots,y_n\in E$ and hence $\|\phi(x+z)-\phi(x)\|_\gamma\leq\varepsilon$, for all $z\in B^X_\rho(0)$. Thus ϕ is continuous. □

Proof of Lemma 1.8. If f is of class C^{k+1}, then $f^{[k]}$ is of class C^1 and hence so is $d^kf\colon U\times E^k\to F$, being a partial map of $f^{[k]}$. Since $d^kf(x,\bullet)$ is k-linear for each $x\in U$, Lemma A.1 shows that $U\to\mathcal{L}^k(E,F)$, $x\mapsto d^kf(x,\bullet)$ is continuous.

If f is merely C^k but $\mathbb{K}$ a complete valued field and E finite-dimensional, then we pick a basis $e_1,\ldots,e_n$ of E and recall that $\mathbb{K}^n\to E$, $(t_1,\ldots,t_n)\mapsto\sum_{j=1}^n t_je_j$ is an isomorphism of topological vector spaces ([6], Chapter I, §2, No. 3, Thm. 3). The continuity of ϕ now follows from the continuity of the maps $U\to F$, $x\mapsto d^kf(x,e_{i_1},\ldots,e_{i_k})$ and the (easily verified) fact that $\mathcal{L}^k(E,F)\to F^{\{1,\ldots,n\}^k}$, $\beta\mapsto(\beta(e_{i_1},\ldots,e_{i_k}))^n_{i_1,\ldots,i_k=1}$ is an isomorphism of topological vector spaces. □

Proof of Lemma 1.9. Let $z\in U$, $\|.\|_\gamma$ be a continuous seminorm on F, and $\varepsilon\in\,]0,1]$.

If f is of class C^{k+1}, then an apparent adaptation of the proof of Lemma 1.6 based on the

$(k+1)$-th order Taylor expansion of f gives $\delta \in \,]0,1]$ and $C > 0$ such that $B_{2\delta}(z) \subseteq U$ and

$$\left\| f(x+y) - f(x) - \sum_{j=1}^{k} \frac{1}{j!} d^j f(x, y, \ldots, y) \right\|_\gamma \leq C\|y\|^{k+1} \quad \text{for all } x \in B_\delta(z) \text{ and } y \in B_\delta(0).$$

Set $\sigma := \frac{\delta\varepsilon}{1+C} < \delta$. Then $\|R(x,y)\|_\gamma \leq C\|y-x\|^{k+1} \leq \varepsilon\|y-x\|^k$ for all $x, y \in B_\sigma(z)$. We deduce that (3) holds.

If f is C^k, $\mathbb{K}$ is locally compact and E finite-dimensional, let $R_k\colon U^{[1]} \to F$ be the remainder of the k-th order Taylor expansion. Choose $\rho > 0$ such that $\overline{B}_{2\rho}(z) \subseteq U$. Pick $a \in \mathbb{K}^\times$ such that $|a| < 1$. Since $R_k(x, y, 0) = 0$ for all $x \in U$ and $y \in E$, using the compactness of $\overline{B}_\rho(z) \times \overline{B}_1(0)$ we find $\delta \in \,]0, \rho]$ such that $\|R_k(x,y,t)\|_\gamma \leq \varepsilon|a|^k$ for all $(x,y,t) \in \overline{B}_\rho(z) \times \overline{B}_1(0) \times B_\delta(0) \subseteq U^{[1]}$. Given $x, y \in B_{|a|\delta/2}(z)$, with $x \neq y$ to avoid trivialities, there exists $\ell \in \mathbb{Z}$ such that $|a|^{\ell+1} \leq \|y-x\| < |a|^\ell$. Since $R(x,y) = R(x, x+(y-x)) = R(x, x + a^\ell a^{-\ell}(y-x)) = a^{k\ell} R_k(x, a^{-\ell}(y-x), a^\ell)$, where $|a^\ell| \leq |a|^{-1}\|y-x\| < \delta$ and $\|a^{-\ell}(y-x)\| < 1$, we deduce that $\|R(x,y)\|_\gamma = |a|^{k\ell}\|R_k(x, a^{-\ell}(y-x), a^\ell)\|_\gamma \leq |a|^{-k}\|y-x\|^k\varepsilon|a|^k = \varepsilon\|y-x\|^k$. Hence (3) holds. □

Proof of Lemma 1.10. For each $j = 1, \ldots, k$, and $x \in U$, let $b_j(x, \bullet)\colon E^j \to F$ be the symmetric j-linear map associated with $a_j(x, \bullet)$; since $b_j(x, \bullet)$ can be obtained from $a_j(x, \bullet)$ by polarization (cf. [4]), it is easy to see that $b_j\colon U \times E^j \to F$ is continuous. If $\eta\colon I \to U$ is a C^k-curve, defined on an open subset $I \subseteq \mathbb{K}$, we have the Taylor expansion

$$\eta(s) - \eta(t) = \sum_{i=1}^{k} c_i(t)(s-t)^j + r(t,s)(s-t)^k\,, \tag{47}$$

where $c_i := \frac{1}{i!}\eta^{(i)}$ is continuous, and $r\colon I \times I \to E$ is a continuous map vanishing on the diagonal. Substituting (47) into $f(y) - f(x) = \sum_{j=1}^{k} b_j(x, y-x, y-x, \ldots, y-x) + R(x,y)$, we find that

$$f(\eta(s)) - f(\eta(t)) = \sum_{\ell=1}^{k} g_\ell(t)(s-t)^\ell + (s-t)^k\rho(t,s)\,, \tag{48}$$

where
$$g_\ell(t) = \sum_{j=1}^{k} \sum_{\substack{i_1, \ldots, i_j \in \{1, \ldots, k\} \\ \text{s.t. } i_1 + \cdots + i_j = \ell}} b_j(\eta(t), c_{i_1}(t), \ldots, c_{i_j}(t))$$

and ρ is a sum of terms of the following form: Firstly, we have summands of the form

$$h(t,s) = (s-t)^{i_1 + \cdots + i_j - k} b_j(\eta(t), c_{i_1}(t), \ldots, c_{i_j}(t))\,,$$

where $j \in \{1, \ldots, k\}$ and $i_1, \ldots, i_j \in \{1, \ldots, k\}$ such that $i_1 + \cdots + i_j > k$; any such summand is a continuous map $I \times I \to F$, and vanishes on the diagonal. Second, we have summands obtained by substituting one or several remainder terms $(s-t)^j r(t,s)$ into $b_j(x, \bullet)$; using that $b_j(x, \bullet)$ is symmetric, such summands can be written in the form

$$h(t,s) = b_j(\eta(t), r(t,s), z_1(t,s), \ldots, z_{j-1}(t,s))\,,$$

where each $z_i(t,s)$ is either $(s-t)^k r(t,s)$, or $(s-t)^\ell c_\ell(t)$ for some ℓ. Again, any such $h\colon I\times I\to F$ is continuous, and vanishes on the diagonal. Finally, ρ involves a summand $P\colon I\times I\to F$ defined via

$$P(t,s):=\begin{cases}(s-t)^{-k}R(\eta(t),\eta(s)) & \text{if } s\neq t;\\ 0 & \text{if } s=t.\end{cases}$$

Then, by definition, P vanishes on the diagonal. To see that P is continuous, we only need to show that $P(s,t)\to P(s_0,s_0)=0$ whenever $(s,t)\to(s_0,s_0)$, with $s\neq t$ (cf. [10], Exerc. 3.2.B). But this is the case: If $s\neq t$ are such that $\eta(t)=\eta(s)$, then $P(t,s)=0$. If $\eta(t)\neq\eta(s)$ on the other hand, then

$$\|P(s,t)\|=\frac{\|R(\eta(t),\eta(s)\|}{\|\eta(t)-\eta(s)\|^k}\cdot\left\|\frac{\eta(t)-\eta(s)}{t-s}\right\|^k,$$

where the first term tends to 0 since R is a k-th order remainder, and where the second term can be written as $\|\eta^{<1>}(t,s)\|^k$ (with notation as in Definition 1.1 above), where $\eta^{<1>}: I\times I\to F$ is continuous. Hence also the product tends to 0. Being a sum of continuous maps vanishing on the diagonal, also $\rho\colon I\times I\to F$ is continuous and vanishes on the diagonal. Thus (48) shows that the curve $f\circ\eta$ admits a Taylor expansion as described in [20], Thm. 83.5 (resp., Prop. 27.2 (γ), resp., Prop. 28.4 if $k\leq 2$), whence $f\circ\eta$ is of class C^k by the cited theorem (resp., proposition).[7]

Choosing η as the curve $\eta(t):=x+ty$ for given elements $x\in U$ and $y\in E$, we obtain $f(x+ty)-f(x)=f(\eta(t))-f(\eta(0))=\sum_{j=1}^k t^j a_j(x,y)+t^k\rho(0,t)$. Since $\rho(0,\bullet)$ is continuous and $\rho(0,0)=0$, the coefficients $a_j(x,y)$ in the preceding formula are uniquely determined by the function $t\mapsto f(x+ty)-f(x)$, and hence by f (see [2, La. 5.2]). □

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