Wiener algebras of operators, and applications to pseudodifferential operators

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Abstract

We introduce a Wiener algebra of operators on $L^2(\mathbb{R}^N)$ which contains, for example, all pseudodifferential operators in the Hörmander class $OPS^0_{0,0}$. A discretization based on the action of the discrete Heisenberg group associates to each operator in this algebra a band-dominated operator in a Wiener algebra of operators on $l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$. The (generalized) Fredholmness of these discretized operators can be expressed by the invertibility of their limit operators. This implies a criterion for the Fredholmness on $L^2(\mathbb{R}^N)$ of pseudodifferential operators in $OPS^0_{0,0}$ in terms of their limit operators. Applications to Schrödinger operators with continuous potential and other partial differential operators are given.

1 Introduction

In this paper, we consider pseudodifferential operators on $L^2(\mathbb{R}^N)$ with symbols in $S^0_{0,0}$. For $m \geq 0$, the Hörmander class $S^m_{0,0}$ consists of all functions $a \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$|a|_{r,t} := \sum_{|\alpha| \leq r, |\beta| \leq t} \sup_{(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N} |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \langle \xi \rangle^{-m} < \infty$$

for each choice of $r, t \in \mathbb{N}$. Here, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ is a multi-index, and we write ∂_x^α and ∂_ξ^α for the operator ∂^α, applied to the functions $x \mapsto a(x, \xi)$ and $\xi \mapsto a(x, \xi)$, respectively.

Let $a \in S^m_{0,0}$. The operator $Op(a)$ defined on the Schwartz space $S(\mathbb{R}^N)$ by

$$(Op(a)u)(x) := (2\pi)^{-N} \int_{\mathbb{R}^N} a(x, \xi) \hat{u}(\xi) e^{i\langle x, \xi \rangle} d\xi, \quad x \in \mathbb{R}^N \tag{1}$$

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is called the pseudodifferential operator with symbol a. The class of all pseudodifferential operators with symbol in $S^m_{0,0}$ is denoted by $OPS^m_{0,0}$.

The basic boundedness and compactness results for pseudodifferential operators are as follows.

Theorem 1.1 *Let $a \in S^0_{0,0}$.*
(a) The operator $Op(a)$ is bounded on $L^2(\mathbb{R}^N)$, and

$$\|Op(a)\|_{L^2} \leq C|a|_{2k_1,\, 2k_2} \quad \textit{whenever } 2k_1 > N \textit{ and } 2k_2 > N,$$

where C is a constant independent of a (but depending on k_1 and k_2).
(b) The operator $Op(a)$ is compact on $L^2(\mathbb{R}^N)$ if and only if

$$\lim_{(x,\xi)\to\infty} a(x,\, \xi) = 0.$$

Assertion (a) is known as the Calderon-Vaillancourt theorem. Its proof can be found in [16], for example. More comprehensive introductions into the world of pseudodifferential operators are [10, 12, 25, 26].

In this paper we are going to study the Fredholm properties of pseudodifferential operators in $OPS^0_{0,0}$. By definition, a linear bounded operator A on a Banach space X if Fredholm if both its kernel $\ker A$ and its cokernel $\operatorname{coker} A := X/(AX)$ have finite dimension. The standard approach to Fredholmness of pseudodifferential operators, which makes use of the composition formulas (see, for instance, [22, 25, 10, 16]), does not work for operators in $OPS^m_{0,0}$. So, new tools are needed, and we would like to convince the reader that the limit operators method is very promising among these tools.

Here is a short description of that method and of its results. We write each vector $\gamma \in \mathbb{Z}^{2N}$ as $(\gamma_1,\, \gamma_2) \in \mathbb{Z}^N \times \mathbb{Z}^N$ and set $U_\gamma := V_{\gamma_1} E_{\gamma_2} \in L^2(\mathbb{R}^N)$, where

$$(E_\alpha u)(x) := e^{i\langle\alpha,\, x\rangle} u(x) \quad \text{and} \quad (V_\beta u)(x) := u(x-\beta).$$

The operators U_γ are unitary. Note that these operators, together with the scalar unitary operators $e^{ir}I$ with r running through the integers, form a non-commutative group, the so-called *discrete Heisenberg group*. In particular,

$$U^*_\alpha = e^{i\langle\alpha_2,\, \alpha_1\rangle} U_{-\alpha}, \quad U_\alpha U_\beta = e^{i\langle\alpha_2,\, \beta_1\rangle} U_{\alpha+\beta}, \tag{2}$$

$$U^*_\alpha U_\beta = e^{i\langle\alpha_2,\, \alpha_1-\beta_1\rangle} U_{\beta-\alpha} = e^{i\langle\beta_2,\, \alpha_1-\beta_1\rangle} U^*_{\alpha-\beta} \tag{3}$$

where $\alpha := (\alpha_1,\, \alpha_2)$, $\beta := (\beta_1,\, \beta_2) \in \mathbb{Z}^N \times \mathbb{Z}^N$.

Further we denote the set of all sequences in $\mathbb{Z}^{2N}$ which tend to infinity by $\mathcal{H}$. In accordance with the notations from [18, 19], we call an operator $A_h \in L(L^2(\mathbb{R}^N))$ the *limit operator of $A \in L(L^2(\mathbb{R}^N))$ with respect to the sequence $h \in \mathcal{H}$* if

$$\text{s-lim}_{m\to\infty}(U^*_{h(m)} A U_{h(m)} = A_h \quad \text{and} \quad \text{s-lim}_{m\to\infty}(U^*_{h(m)} A^* U_{h(m)} = A^*_h. \tag{4}$$

The set $\sigma_{op}(A)$ of all limit operators of A will be called the *operator spectrum* of A. With these notions, we will prove the following.

Theorem 1.2 *A pseudodifferential operator A in $OPS^0_{0,0}$ is Fredholm if and only if each of its limit operators is invertible. In particular, the essential spectrum $\sigma_{ess}(A) := \sigma(A + K(L^2(\mathbb{R}^N)))$ of A is given by*

$$\sigma_{ess}(A) = \cup_{A_h \in \sigma_{op}(A)} \sigma(A_h)$$

where $\sigma(A_h)$ refers to the usual spectrum of the operator A_h.

In many important instances, the structure of the limit operators is much simpler than the structure of the operator itself, which allows one to obtain explicit and effective Fredholm conditions.

Our strategy to prove Theorem 1.2 is as follows. We introduce an algebra $\mathcal{W}(L^2(\mathbb{R}^N))$ of Wiener type, which consists of certain linear and bounded operators on $L^2(\mathbb{R}^N)$. This algebra contains $OPS^0_{0,0}$ as its subalgebra. Similar algebras of Wiener type were considered by Sjöstrand [23, 24] and Boulkhemair [4].

A suitable discretization associates to every operator in $\mathcal{W}(L^2(\mathbb{R}^N))$ a band-dominated operator acting on an appropriate $l^2(\mathbb{Z}^{2N})$-space. Moreover, these discretizations belong to an algebra $\mathcal{W}(l^2(\mathbb{Z}^{2N}))$ of Wiener type again, the elements of which are band-dominated operators on $l^2(\mathbb{Z}^{2N})$. Here we call an operator band-dominated if it is the norm limit of a sequence of band operators.

It turns out that an operator in $\mathcal{W}(L^2(\mathbb{R}^N))$ is Fredholm if and only if its discretization satisfies a generalized Fredholm condition called *$\mathcal{P}$-Fredholmness.* The $\mathcal{P}$-Fredholmness of band-dominated operators has been studied in [18, 19] by means of the limit operators method. Basically, the result is as follows: A band-dominated operator is $\mathcal{P}$-Fredholm if and only if each of its (appropriately defined) limit operators is invertible, *and if the norms of their inverses are uniformly bounded.*

In practice, it proves to be hard to verify the condition of uniform boundedness of the inverses of the limit operators. It is one of the main results of the present paper that this condition is redundant for band-dominated operators in the discrete Wiener algebra $\mathcal{W}(l^2(\mathbb{Z}^{2N}))$. That is, an operator in this algebra is $\mathcal{P}$-Fredholm if and only if each of its limit operators is invertible. Combining these devices, we obtain the Fredholm criterion for pseudodifferential operators stated in Theorem 1.2.

A similar strategy has been pursued for operators of convolution type on $L^p(\mathbb{R}^N)$ in [17]. The discretization used in [17] is based on the action of the commutative group $\mathbb{Z}^N$. It yields that the $\mathcal{P}$-Fredholmness of the discretized operator is equivalent to some kind of generalized Fredholmness of the operator itself. Thus, one needs a further property of the operator (for example, its local compactness) in order to guarantee that its generalized Fredholmness implies its common Fredholmness. In contrast to this situation, the discretization employed in this paper is much finer. It is based on the action of a discrete Heisenberg group, and it leads to a simultaneous discretization with respect to the variable in $L^2(\mathbb{R}^N)$ and to the co-variable in the Fourier image, which we call bi-discretization.

The paper is organized as follows. We start with the introduction of the discrete Wiener algebra $\mathcal{W}(l^2(\mathbb{Z}^{2N}))$ in Section 2. In particular, we will derive the announced criterion for operators in $\mathcal{W}(l^2(\mathbb{Z}^{2N}))$ to be $\mathcal{P}$-Fredholm. The bi-discretization is described in Section 3. It is applied to the study of the Fredholm properties of pseudodifferential operators in Section 4 (with the main result being Theorem 4.15), and several applications to more concrete classes of pseudodifferential operators are given in Section 5. Let us mention some of these classes explicitely. In Section 5.1, we consider operators in $OPS^0_{0,0}$ with slowly oscillating symbols. For operators in this class, all limit operators are either operators of multiplication by a bounded function, or operators of convolution. Thus, the invertibility of these operators can be effectively checked, and this yields an explicit description of the essential spectrum. The Fredholm theory of pseudodifferential operators in $OPS^m_{1,0}$ with symbols which are slowly oscillating with respect to the spatial variable x has been considered by Grushin [9].

In 5.2, we consider operators in $OPS^0_{0,0}$ the symbols of which are almost-periodic with respect to x. Here we use the limit operators method to get a simple proof of the following results: The class of these operators does not contain non-trivial compact operators, and an operator in this class is Fredholm if and only if it is invertible. For elliptic operators in this class, conditions for the invertibility are given in Shubin [20, 21], Fedosov and Shubin [8] and Coburn, Moyer and Singer [5]. These conditions are based upon the concept of the almost periodic index.

In 5.3, we will deal with operators with semi-almost periodic symbols, and in Sections 5.4 and 5.5 we consider operators of nonzero order. Finally, in 5.6, we are going to apply the results of Section 5.5 to describe the essential spectrum of some electromagnetic Schrödinger operators.

2 Operators in the discrete Wiener algebra

2.1 Band-dominated operators and their $\hat{\mathcal{P}}$-Fredholmness

We start this section with recalling the notions of rich band and band-dominated operators and the criterion for $\hat{\mathcal{P}}$-Fredholmness from [19]. The reader should take into account that we used the notion *invertibility at infinity* instead of $\hat{\mathcal{P}}$-*Fredholmness* in [19].

Given a Banach space X, a positive integer N and a real number $p \geq 1$, we let $l^p(\mathbb{Z}^N, X)$ stand for the Banach space of all sequences x on $\mathbb{Z}^N$ with values in X such that

$$\|f\|_p^p := \sum_{\alpha \in \mathbb{Z}^N} \|x_\alpha\|_X < \infty,$$

and we write $l^\infty(\mathbb{Z}^N, X)$ for the Banach space of all sequences $x : \mathbb{Z}^N \to X$ with

$$\|f\|_\infty := \sup_{\alpha \in \mathbb{Z}^N} \|x_\alpha\|_X < \infty.$$

Further, E^∞ stands for one of the Banach spaces $l^p(\mathbb{Z}^N, X)$ with $1 \le p \le \infty$, whereas E refers to one of the spaces $l^p(\mathbb{Z}^N, X)$ with $1 < p < \infty$.

Every function $a \in l^\infty(\mathbb{Z}^N, L(X))$ gives rise to a multiplication operator on E^∞ on defining

$$(ax)_\gamma := a_\gamma x_\gamma, \quad \gamma \in \mathbb{Z}^N.$$

We denote this operator by aI. Evidently, $aI \in L(E^\infty)$ and $\|aI\| = \|a\|_\infty$. A *band operator* on E^∞ is a finite sum of the form $\sum_\alpha a_\alpha \hat{V}_\alpha$ where $\alpha \in \mathbb{Z}^{2N}$, $a_\alpha \in l^\infty(\mathbb{Z}^N, L(X))$, and where $\hat{V}_\gamma$ is the shift operator

$$(\hat{V}_\gamma u)_\alpha := u_{\alpha-\gamma}, \quad \alpha \in \mathbb{Z}^N.$$

A *band-dominated operator* is the norm limit of a sequence of band operators. The band-dominated operators form a closed and symmetric subalgebra of $L(E^\infty)$ which we denote by $\mathcal{A}$.

Given $\gamma \in \mathbb{Z}^N$, let S_γ stand for the operator on E^∞ which sends a sequence f to the sequence g with $g_\gamma = f_\gamma$ and $g_\lambda = 0$ for $\lambda \neq \gamma$. For $n \ge 0$, define $\hat{P}_n$ as the sum $\sum_{|\gamma|_\infty \le n} S_\gamma$, and let $\hat{\mathcal{P}}$ stand for the family $(\hat{P}_n)_{n\ge 0}$. The operators $\hat{P}_n$ are projections which converge strongly to the identity operator if $p < \infty$.

Let $A \in L(E^\infty)$, and let $h : \mathbb{N} \to \mathbb{Z}^N$ be a sequence which tends to infinity. We say that the operator A_h is the *limit operator* of A with respect to the sequence h if

$$\lim_{n\to\infty} \|\hat{P}_k(\hat{V}_{-h(n)} A \hat{V}_{h(n)} - A_h)\| = \lim_{n\to\infty} \|(\hat{V}_{-h(n)} A \hat{V}_{h(n)} - A_h)\hat{P}_k\| = 0$$

for every $k \in \mathbb{N}$. Let further $\mathcal{H}$ denote the set of all sequences $h : \mathbb{Z}^N \to \mathbb{N}$ which tend to infinity, and let $\mathcal{A}^\$$ refer to the set of all operators $A \in \mathcal{A}$ enjoying the following property: Every sequence $h \in \mathcal{H}$ possesses a subsequence g for which the limit operator A_g exists. We refer to the operators in $\mathcal{A}^\$$ as *rich* band-dominated operators.

Further, we have to mention the notions of generalized compactness and generalized Fredholmness. We did not use these notions explicitely in [19], but a closer look will convince the reader that the definitions given in [19] are in full coincidence with these notions. An operator $K \in L(E^\infty)$ is called $\hat{\mathcal{P}}$*-compact* if

$$\|K\hat{P}_n - K\| \to 0 \quad \text{and} \quad \|\hat{P}_n K - K\| \to 0 \quad \text{as } n \to \infty.$$

By $K(E^\infty, \hat{\mathcal{P}})$ we denote the set of all $\hat{\mathcal{P}}$-compact operators on E^∞, and by $L(E^\infty, \hat{\mathcal{P}})$ the set of all operators $A \in L(E^\infty)$ for which both AK and KA are $\hat{\mathcal{P}}$-compact whenever K is $\hat{\mathcal{P}}$-compact. Then $L(E^\infty, \mathcal{P})$ is a closed subalgebra of $L(E^\infty)$ which contains $K(E^\infty, \hat{\mathcal{P}})$ as its closed ideal. Moreover, $K(E^\infty, \hat{\mathcal{P}})$ contains all compact operators if $1 < p < \infty$. An operator $A \in L(E^\infty, \hat{\mathcal{P}})$ is called *$\hat{\mathcal{P}}$-Fredholm* if it is invertible modulo operators in $K(E^\infty, \hat{\mathcal{P}})$. In case X has finite dimension, this is just the usual notion of a Fredholm operator. Now the main result of [19] can be stated as follows.

Theorem 2.1 *An operator $A \in \mathcal{A}^\$$ is $\hat{\mathcal{P}}$-Fredholm if and only if each of its limit operators is invertible and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_{op}(A)\} < \infty. \tag{5}$$

2.2 The Wiener algebra

The result of Theorem 2.1 takes a more satisfactory form for band-dominated operators which belong to the Wiener algebra, in which case the *uniform* boundedness of the inverses of the limit operators is not required.

Let $(a_\alpha)_{\alpha \in \mathbb{Z}^N}$ be a sequence of functions in $l^\infty(\mathbb{Z}^N, L(X))$ satisfying

$$\sum_{\alpha \in \mathbb{Z}^N} \|a_\alpha\|_\infty < \infty. \tag{6}$$

Then the series $\sum_{\alpha \in \mathbb{Z}^n} a_\alpha \hat{V}_\alpha$ converges in the norm of $L(E^\infty)$, and

$$\left\| \sum_{\alpha \in \mathbb{Z}^N} a_\alpha \hat{V}_\alpha \right\|_{L(E^\infty)} \leq \sum_{\alpha \in \mathbb{Z}^N} \|a_\alpha\|_\infty. \tag{7}$$

Let $\mathcal{W}$ stand for the set of all operators $A = \sum_{\alpha \in \mathbb{Z}^N} a_\alpha \hat{V}_\alpha$ with coefficient functions a_α satisfying (6). Provided with the usual operations and the norm

$$\|A\|_{\mathcal{W}} := \sum_{\alpha \in \mathbb{Z}^N} \|a_\alpha\|_\infty,$$

the set $\mathcal{W}$ becomes a Banach algebra, the so-called *Wiener algebra*. By (7), the Wiener algebra is continuously embedded into $L(E^\infty, \hat{\mathcal{P}})$ and, hence, into $\mathcal{A}$ for all choices of E^∞.

Later on, we will also have to deal with Wiener algebras of operators on $L^2(\mathbb{R}^N)$. In this setting, we will refer to the Wiener algebra $\mathcal{W}$ on the sequence spaces as the *discrete Wiener algebra*.

A basic basic property of the Wiener algebra is described in the following theorem the proof of which can be found in [13].

Theorem 2.2 *The Wiener algebra $\mathcal{W}$ is inverse closed in $L(E^\infty)$.*

This means that, if $A \in \mathcal{W}$ is invertible in $L(E^\infty)$, then $A^{-1} \in \mathcal{W}$.

Corollary 2.3 *Let $A \in \mathcal{W}$ be invertible on one of the spaces E^∞. Then A is invertible on all of these spaces, and the norms of the corresponding inverses are uniformly bounded.*

Indeed, if A is invertible on one of the spaces E^∞, then $A^{-1} \in \mathcal{W}$ by Theorem 2.2, and from $\|A^{-1}\|_{L(E^\infty)} \leq \|A^{-1}\|_{\mathcal{W}}$ we conclude that A^{-1} is the inverse for A on every of the spaces E^∞, and that the norm of A^{-1} in $L(E^\infty)$ is bounded by $\|A^{-1}\|_{\mathcal{W}}$. ■

2.3 Fredholmness of operators in the Wiener algebra

The intersection $\mathcal{W} \cap \mathcal{A}^\$$ is called the *rich Wiener algebra* and will be denoted by $\mathcal{W}^\$$. It is not hard to see and will used in the following proposition that the multiplication operators forming the diagonals of an operator A in the rich Wiener algebra are rich operators themselves.

Here is what can be said about limit operators of rich operators in the Wiener algebra.

Proposition 2.4 *Let $A \in \mathcal{W}^\$$ and let $h \subseteq \mathbb{Z}^N$ be a sequence tending to infinity. Then there is a subsequence g of h such that the limit operator A_g exists with respect to all spaces E^∞. This limit operator belongs to $\mathcal{W}$, and $\|A_g\|_{\mathcal{W}} \leq \|A\|_{\mathcal{W}}$.*

Proof. Let $A = \sum_{\alpha \in \mathbb{Z}^N} a_\alpha \hat{V}_\alpha$ with $\sum_{\alpha \in \mathbb{Z}^N} \|a_\alpha\| < \infty$. Since all diagonals a_α are rich multiplication operators, a Cantor diagonal argument yields the existence of a subsequence g of h such that the limit operators $(a_\alpha I)_g$ exist with respect to E^∞ for all α. These limit operators are again operators of multiplication by certain functions $a_{\alpha,g}$, and

$$\|a_{\alpha,g}\|_\infty = \|(a_\alpha I)_g\|_{L(E^\infty)} \leq \|a_\alpha\|_\infty,$$

which follows immediately from the definition of limit operators. Thus,

$$\sum_{\alpha \in \mathbb{Z}^N} \|a_{\alpha,g}\|_\infty < \infty,$$

and the operator $A_g := \sum_{\alpha \in \mathbb{Z}^N} a_{\alpha,g} \hat{V}_\alpha$ is correctly defined. This operator belongs to the Wiener algebra $\mathcal{W}$, and $\|A_g\|_{\mathcal{W}} \leq \|A\|_{\mathcal{W}}$. Now it is evident that A_g is indeed the limit operator of A with respect to the sequence g in each of the spaces E^∞. ■

The main result of this section is the following theorem which states that, for rich operators A in the Wiener algebra, the uniform boundedness condition from Theorem 2.1,

$$\sup\{\|(A_h)^{-1}\|, \, A_h \in \sigma(A)\} < \infty,$$

is automatically satisfied if all limit operators of A are invertible.

Theorem 2.5 *Let X be a reflexive Banach space. Then the following assertions are equivalent for every operator $A \in \mathcal{W}^\$$:*

(a) There is a space E such that A is $\hat{\mathcal{P}}$-Fredholm on E.

(b) There is a space E such that all limit operators of A are invertible on E.

(c) All limit operators of A are invertible on $l^\infty(\mathbb{Z}^N, X)$.

(d) All limit operators of A are invertible on $l^\infty(\mathbb{Z}^N, X)$, and the norms of their inverses are uniformly bounded.

(e) All limit operators of A are invertible on E^∞ for all spaces E^∞, and the norms of their inverses are uniformly bounded.

(f) The operator A is $\hat{\mathcal{P}}$-Fredholm on all spaces E.

Proof. $(a) \Rightarrow (b)$: This implication can be easily checked. See, for example, the simpler part of the proof of Theorem 2.16 in [19].
$(b) \Rightarrow (c)$: Let A_h be a limit operator of A with respect to the Banach space E. If A_h is invertible on E, then A_h^{-1} is in the Wiener algebra $\mathcal{W}$ by Proposition 2.4 and Theorem 2.2, and $A_h^{-1} \in L(l^\infty(\mathbb{Z}^N, X))$ by Corollary 2.3.
$(c) \Rightarrow (d)$: Let $\chi : \mathbb{R}^N \to [0, 1]$ be a continuous function which is identically 1 in a certain neighborhood of 0 and which vanishes outside the cube $[-1, 1]^N$. Further, given a positive integer k, define the function χ_k by $\chi_k(x) := \chi(x/k)$, and let T_k refer to the operator of multiplication by the restriction of the function χ_k onto $\mathbb{Z}^N$. We claim that there are constants $C > 0$ and $k \in \mathbb{N}$ such that

$$\|u\|_\infty \leq C\,(\|Au\|_\infty + \|T_k u\|_\infty) \quad \text{for all } u \in l^\infty(\mathbb{Z}^N, X). \tag{8}$$

The claim is evidently equivalent to the existence of constants C, k such that

$$1/C \leq \|Au\|_\infty + \|T_k u\|_\infty \quad \text{for all unit vectors } u \in l^\infty(\mathbb{Z}^N, X).$$

Assume, such constants do not exist. Then, for all $C > 0$ and $k \in \mathbb{N}$, there exists a vector $u_{k,C} \in l^\infty(\mathbb{Z}^N, X)$ with $\|u_{k,C}\|_\infty = 1$ such that

$$1/C > \|Au_{k,C}\|_\infty + \|T_k u_{k,C}\|_\infty$$

In particular, we can choose $C = k$, i.e. for each $k \in \mathbb{N}$, there is a $u_k \in l^\infty(\mathbb{Z}^N, X)$ with $\|u_k\|_\infty = 1$ such that

$$1/k > \|Au_k\|_\infty + \|T_k u_k\|_\infty. \tag{9}$$

From $\|u_k\|_\infty = 1$ and $\|T_k u_k\|_\infty < 1/k$ we conclude the existence of points $x_k \in \mathbb{Z}^N$ such that

$$\|u_k(x_k)\|_{L(X)} \geq 1/2 \quad \text{and} \quad |x_k| \to \infty.$$

Let h be the sequence $h(m) := x_m$. Since A is rich, there is a subsequence g of h for which the limit operator A_g exists. Let $v_m := \hat{V}_{-g(m)} u_{g(m)}$. Then, for arbitrary k, $m \in \mathbb{N}$,

$$\begin{aligned}
\|A_g T_k v_m\| &\leq \|(A_g - \hat{V}_{-g(m)} A \hat{V}_{g(m)}) T_k\| \, \|v_m\| + \|\hat{V} V_{-g(m)} A \hat{V}_{g(m)} T_k v_m\| \\
&\leq \|(A_g - \hat{V}_{-g(m)} A \hat{V}_{g(m)}) T_k\| \\
&\quad + \|(\hat{V}_{-g(m)} A \hat{V}_{g(m)} T_k - T_k \hat{V}_{-g(m)} A \hat{V}_{g(m)}) v_m\| \\
&\quad\quad + \|T_k \hat{V}_{-g(m)} A \hat{V}_{g(m)} v_m\| \\
&\leq \|(A_g - \hat{V}_{-g(m)} A \hat{V}_{g(m)}) T_k\| \\
&\quad + \|\hat{V}_{-g(m)} A \hat{V}_{g(m)} T_k - T_k \hat{V}_{-g(m)} A \hat{V}_{g(m)}\| + \|A u_{g(m)}\|.
\end{aligned} \tag{10}$$

Let $\varepsilon > 0$ be arbitrary. Then choose and fix k such that the second term on the right hand side of estimate (10) becomes less than ε for all m, which can be done due to Proposition 2.2 in [19]. Now choose $m > 1/\varepsilon$ so large that the first term in (10) also becomes less than ε. Since $\|A u_m\| < 1/m$ by (9), then the third term in (10) is less than ε, too. Thus,

$$\forall \, \varepsilon > 0 \, \exists \, k, \, m \in \mathbb{N} \; : \; \|A_g T_k v_m\|_\infty \leq 3 \, \varepsilon. \tag{11}$$

On the other hand, $\|v_m(0)\| = \|u_{g(m)}(g(m))\| \geq 1/2$, whence $\|T_k v_m\|_\infty \geq 1/2$. Thus, by (11), and since all limit operators of A are invertible by hypothesis,

$$1/2 \leq \|T_k v_m\|_\infty \leq \|A_g^{-1}\| \, \|A_g T_k v_m\|_\infty \leq 3 \, \varepsilon \, \|A_g^{-1}\|$$

whence

$$\|A_g^{-1}\| \geq 1/(6 \, \varepsilon) \quad \text{for all } \varepsilon > 0.$$

This is clearly impossible, and our claim (8) is proved. We will now employ (8) to prove the uniform boundedness of the inverses of the limit operators of A on $l^\infty(\mathbb{Z}^N, X)$.

From (8) we conclude that, for all $u \in l^\infty(\mathbb{Z}^N, X)$, $r \in \mathbb{N}$ and $l \in \mathbb{Z}^N$,

$$\|\hat{V}_l T_r u\|_\infty \leq C(\|A \hat{V}_l T_r u\|_\infty + \|T_k \hat{V}_l T_r u\|_\infty).$$

Let $h \in \mathcal{H}$ be a sequence for which the limit operator A_h exists. Since every $\hat{V}_l$ is an isometry, we get

$$\|T_r u\|_\infty \leq C(\|\hat{V}_{-h(m)} A \hat{V}_{h(m)} T_r u\|_\infty + \|\hat{V}_{-h(m)} T_k \hat{V}_{h(m)} T_r u\|_\infty). \tag{12}$$

Further, since $T_r u \in c_0(\mathbb{Z}^N, X)$ and $\hat{V}_{-h(m)} T_k \hat{V}_{h(m)} \to 0$ strongly on $c_0(\mathbb{Z}^N, X)$, we can pass to the limit as $m \to \infty$ in (12) to obtain

$$\|T_r u\|_\infty \leq C \|A_h T_r u\|_\infty \tag{13}$$

for all $u \in l^\infty(\mathbb{Z}^N, X)$ and $r \in \mathbb{N}$. For $r \to \infty$, the left hand side of (13) goes to $\|u\|_\infty$. For the right hand side, some more care is in order. Again from Proposition 2.2 in [19], we conclude that the right hand side of

$$| \, \|A_h T_r u\| - \|T_r A_h u\| \, | \leq \|A_h T_r - T_r A_h\| \, \|u\|$$

tends to zero as $r \to \infty$ (note that A_h is band dominated if A is so). Since $\|T_r A_h u\| \to \|A_h u\|$ as $r \to \infty$, this estimate implies that $\|A_h T_r u\| \to \|A_h u\|$ as $r \to \infty$. Thus, passage to the limit $r \to \infty$ in (13) gives

$$\|u\|_\infty \leq C\|A_h u\|_\infty \quad \text{for all } u \in l^\infty(\mathbb{Z}^N, X)$$

whence $\|A_h^{-1}\| \leq C$, i.e. the uniform boundedness of the inverses of the limit operators.

$(d) \Rightarrow (e)$: The proof of this implication is based on the possibility to associate with every operator in the Wiener algebra a naturally defined adjoint operator. To make this point clear we will indicate the dependence of the Wiener algebra from the underlying Banach space X by writing $\mathcal{W}_X$ in place of $\mathcal{W}$. For $A = \sum_{\alpha \in \mathbb{Z}^N} a_\alpha V_\alpha \in \mathcal{W}_X$, we define its *Wiener adjoint* $A^\star$ as $\sum_{\alpha \in \mathbb{Z}^N} V_{-\alpha} a_\alpha^* I$, where $a_\alpha^*(x)$ is the usual Banach dual operator of $a_\alpha(x)$, acting on X^*. Clearly, we have $A^\star = \sum_{\alpha \in \mathbb{Z}^N} b_\alpha V_{-\alpha}$ where $b_\alpha(x) = a_\alpha^*(x + \alpha)$. This shows that $A^\star$ belongs to the Wiener algebra $\mathcal{W}_{X^*}$, and it is easy to check that the mapping $A \mapsto A^\star$ is an anti-linear isometry from $\mathcal{W}_X$ into $\mathcal{W}_{X^*}$ which satisfies $(AB)^\star = B^\star A^\star$ for all $A, B \in \mathcal{W}_X$. In particular, $I^\star = I$ and, if A is invertible in $\mathcal{W}_X$, then $A^\star$ is invertible in $\mathcal{W}_{X^*}$ and $(A^\star)^{-1} = (A^{-1})^\star$.

For the proof of the implication $(d) \Rightarrow (e)$, let now $A \in \mathcal{W}_X^\$$ be an operator with

$$C_\infty(A) := \sup\{\|A_h^{-1}\|_{L(l^\infty(\mathbb{Z}^N, X))} : A_h \in \sigma_{op}(A)\} < \infty. \tag{14}$$

The limit operators of $A^\star$ are just the Wiener adjoints of the limit operators of A. Thus, the invertibility of all limit operators of A implies the invertibility of all limit operators of $A^\star$. So we conclude from the already established implication $(c) \Rightarrow (d)$ that

$$C_\infty(A^\star) := \sup\{\|(A_h^\star)^{-1}\|_{L(l^\infty(\mathbb{Z}^N, X^*))} : A_h \in \sigma_{op}(A)\} < \infty.$$

Since the limit operators of $A^\star$ as well as their inverses belong to the Wiener algebra $\mathcal{W}_{X^*}$ (due to Proposition 2.4 and Theorem 2.2), the operators $A_h^\star$ also act as bounded and invertible operators on $c_0(\mathbb{Z}^N, X^*)$, and $\|(A_h^\star)^{-1}\|_{L(c_0(\mathbb{Z}^N, X^*))} \leq \|(A_h^\star)^{-1}\|_{L(l^\infty(\mathbb{Z}^N, X^*))}$. This shows that

$$C_0(A^\star) := \sup\{\|(A_h^\star)^{-1}\|_{L(c_0(\mathbb{Z}^N, X^*))} : A_h \in \sigma_{op}(A)\} < \infty. \tag{15}$$

The operator A, thought of as acting on $l^1(\mathbb{Z}^N, X)$, can be identified with the usual Banach dual operator of $A^\star \in L(c_0(\mathbb{Z}^N, X^*))$ (this is the place where we need the reflexivity of X). Hence,

$$C_1(A) := \sup\{\|A_h^{-1}\|_{L(l^1(\mathbb{Z}^N, X))} : A_h \in \sigma_{op}(A)\} = C_0(A^\star) < \infty.$$

Consequently, by the Riess-Thorin interpolation theorem (Theorem 1 and Remark 4 in Section 1.18.3 of [27]), we have for every $1 < p < \infty$ and $A_h \in \sigma(A)$,

$$\|A_h^{-1}\|^p_{L(l^p(\mathbb{Z}^N, X))} \leq \|A_h^{-1}\|^{p-1}_{L(l^\infty(\mathbb{Z}^N, X))} \|A_h^{-1}\|_{L(l^1(\mathbb{Z}^N, X))} \leq C_\infty(A)^{p-1} C_1(A),$$

which verifies the uniform boundedness of the norms of the inverses of the limit operators of A on all spaces $l^p(\mathbb{Z}^N, X)$ with $1 \le p \le \infty$. For $E^\infty = c_0(\mathbb{Z}^N, X)$, this result follows in the same way as we derived (15).

Finally, the implication $(e) \Rightarrow (f)$ is Theorem 2.1, and the implication $(f) \Rightarrow (a)$ is evident. ∎

Observe that the implication $(c) \Rightarrow (d)$ holds for arbitrary rich operators A and arbitrary (not necessarily reflexive) Banach spaces X.

Corollary 2.6 *Let X be a reflexive Banach space. Then the $\mathcal{P}$-essential spectrum of an operator $A \in \mathcal{W}^\$$ in the space E^∞ does not depend on E^∞, and*

$$\sigma_{\hat{\mathcal{P}}-ess}(A) = \cup \sigma_E(A_h)$$

where the union is taken over all limit operators A_h of A and where the $\hat{\mathcal{P}}$-essential spectrum $\sigma_{\hat{\mathcal{P}}-ess}(A)$ consists of all $\lambda \in \mathbb{C}$ for which the operator $A - \lambda I$ is not $\hat{\mathcal{P}}$-Fredholm.

If the space X is finite dimensional, then the $\mathcal{P}$-essential spectrum is the usual essential spectrum. The proof of the independence of the $\mathcal{P}$-essential spectrum of the underlying space follows from Theorem 2.2 and from the fact that limit operators of operators in the Wiener algebra belong to the Wiener algebra again.

3 Bi-discretization of operators on $L^2(\mathbb{R}^N)$

3.1 Bi-discretization

Let $f \in C_0^\infty(\mathbb{R}^N)$ be a non-negative function such $f(x) = f(-x)$ for all x, $f(x) = 1$ if $|x_i| \le 2/3$ for all $i = 1, \dots, N$ and that $f(x) = 0$ if $|x_i| \ge 3/4$ for at least one i. Define a non-negative function φ by

$$\varphi^2(x) := \frac{f(x)}{\sum_{\beta \in \mathbb{Z}^N} f(x - \beta)}, \quad x \in \mathbb{R}^N,$$

and set $\varphi_\alpha(x) := \varphi(x - \alpha)$ for $\alpha \in \mathbb{Z}^N$. The family (φ_α) forms a partition of unit on $\mathbb{R}^N$ in the sense that

$$\sum_{\alpha \in \mathbb{Z}^N} \varphi_\alpha^2(x) = 1 \quad \text{for all } x \in \mathbb{R}^N. \tag{16}$$

For $\gamma := (\alpha, \beta) \in \mathbb{Z}^N \times \mathbb{Z}^N$, we set $\phi_\gamma(x, \xi) := \varphi_\alpha(x)\varphi_\beta(\xi)$ and $\Phi_\gamma := Op(\phi_\gamma)$. These operators are compact by Theorem 1.1 (b), and (16) implies that

$$\begin{aligned} \sum_{\gamma \in \mathbb{Z}^{2N}} \Phi_\gamma^* \Phi_\gamma u &= \sum_{\beta \in \mathbb{Z}^N} Op(\varphi_\beta) \sum_{\alpha \in \mathbb{Z}^N} \varphi_\alpha^2 Op(\varphi_\beta) u \\ &= \sum_{\beta \in \mathbb{Z}^N} Op(\varphi_\beta)^2 u = F^{-1} \sum_{\beta \in \mathbb{Z}^N} \varphi_\beta^2 F u = u \end{aligned}$$

for all $u \in L^2(\mathbb{R}^N)$. Thus, the operator family $(\Phi_\gamma)_{\gamma \in \mathbb{Z}^{2N}}$ forms a partition of unit in the sense that

$$\sum_{\gamma \in \mathbb{Z}^{2N}} \Phi_\gamma^* \Phi_\gamma = I \tag{17}$$

where the series converges strongly on $L^2(\mathbb{R}^N)$. Analogously, one checks that $\sum_\gamma \Phi_\gamma \Phi_\gamma^* = I$. Moreover,

$$\|u\|_{L^2}^2 = \sum_{\gamma \in \mathbb{Z}^{2N}} \|\Phi_\gamma u\|_{L^2}^2 = \sum_{\gamma \in \mathbb{Z}^{2N}} \|\Phi_\gamma^* u\|_{L^2}^2 \tag{18}$$

for every $u \in L^2(\mathbb{R}^N)$ which follows easily from (17):

$$\|u\|_{L^2}^2 = \sum_{\gamma \in \mathbb{Z}^{2N}} \langle \Phi_\gamma^* \Phi_\gamma u,\, u\rangle = \sum_{\gamma \in \mathbb{Z}^{2N}} \langle \Phi_\gamma u,\, \Phi_\gamma u\rangle = \sum_{\gamma \in \mathbb{Z}^{2N}} \|\Phi_\gamma u\|_{L^2}^2.$$

One also easily checks that $\Phi_\gamma = U_\gamma \Phi_0 U_\gamma^*$ with the unitary operators U_γ introduced in the introduction.

We define the *bi-discretization* Gu of a function $u \in L^2(\mathbb{R}^N)$ by

$$(Gu)_\gamma := \Phi_0 U_\gamma^* u, \quad \gamma \in \mathbb{Z}^{2N},$$

i.e. we consider Gu as a vector-valued function on $\mathbb{Z}^{2N}$ with values in $L^2(\mathbb{R}^N)$. These functions form a Hilbert space $l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ with scalar product

$$\langle f,\, g\rangle := \sum_{\gamma \in \mathbb{Z}^{2N}} \langle f_\gamma,\, g_\gamma\rangle_{L^2(\mathbb{R}^N)}.$$

Proposition 3.1 *The operator $G : L^2(\mathbb{R}^N) \to l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ is an isometry. Its adjoint is given by*

$$G^* f = \sum_{\gamma \in \mathbb{Z}^{2N}} U_\gamma \Phi_0^* f_\gamma$$

where the series converges in $L^2(\mathbb{R}^N)$.

Proof. The isometry of G follows from (18) since

$$\|Gu\|_{l^2}^2 = \sum_{\gamma \in \mathbb{Z}^{2N}} \|\Phi_0 U_\gamma^* u\|_{L^2}^2 = \sum_{\gamma \in \mathbb{Z}^{2N}} \|U_\gamma \Phi_0 U_\gamma^* u\|_{L^2}^2 = \sum_{\gamma \in \mathbb{Z}^{2N}} \|\Phi_\gamma u\|_{L^2}^2 = \|u\|_{L^2}^2.$$

Further, one has

$$\begin{aligned} \langle Gu,\, f\rangle_{l^2} &= \sum_{\gamma \in \mathbb{Z}^{2N}} \langle (Gu)_\gamma,\, f_\gamma\rangle_{L^2} = \sum_{\gamma \in \mathbb{Z}^{2N}} \langle \Phi_0 U_\gamma^* u,\, f_\gamma\rangle_{L^2} \\ &= \sum_{\gamma \in \mathbb{Z}^{2N}} \langle u,\, U_\gamma \Phi_0^* f\rangle_{L^2} = \langle u,\, G^* f\rangle_{L^2} \end{aligned}$$

for every $u \in L^2(\mathbb{R}^N)$ and $f \in l^2(\mathbb{Z}^{2N},\, L^2(\mathbb{R}^N))$. ■

Thus, $G^*G = I$, and the operator $Q := GG^*$ is an orthogonal projection on $l^2(\mathbb{Z}^{2N},\, L^2(\mathbb{R}^N))$. We denote its range by $\operatorname{Im} Q$. Then

$$G : L^2(\mathbb{R}^N) \to \operatorname{Im} Q$$

is a unitary operator, and every operator $A \in L(L^2(\mathbb{R}^N))$ is unitarily equivalent to the operator

$$A_G := GAG^*|_{\operatorname{Im} Q}.$$

We extend A_G to an operator $\Gamma(A)$ acting on all of $l^2(\mathbb{Z}^{2N},\, L^2(\mathbb{R}^N))$ by setting

$$\Gamma(A) := A_G Q + I - Q = GAG^* + I - Q.$$

Clearly,

$$G^*\Gamma(A)G = G^*(GAG^* + I - GG^*)G = A.$$

3.2 Bi-discretization and Fredholmness

We will now examine the relation between the Fredholmness of an operator on $L^2(\mathbb{R}^N)$ and the $\hat{\mathcal{P}}$-Fredholmness of its discretization.

Proposition 3.2 *(a) The operators $\hat{P}_n Q$ and $Q\hat{P}_n$ are compact for every $k \in \mathbb{N}$.*
(b) The projection Q belongs to $L(l^2, \hat{\mathcal{P}})$.
(c) For every $A \in L(L^2(\mathbb{R}^N))$, the operator $\Gamma(A)$ belongs to $L(l^2, \hat{\mathcal{P}})$.
(d) Let $K \in L(l^2(\mathbb{Z}^{2N},\, L^2(\mathbb{R}^N)))$ be a $\hat{\mathcal{P}}$-compact operator of the form $K = QKQ$. Then G^*KG is compact.
(e) The operator $A \in L(L^2(\mathbb{R}^N))$ is invertible (Fredholm) if and only if the operator $\Gamma(A) \in L(l^2(\mathbb{Z}^{2N},\, L^2(\mathbb{R}^N)))$ is invertible ($\hat{\mathcal{P}}$-Fredholm).

Proof. (a) It is sufficient to verify the compactness of all operators $S_\gamma Q$ and QS_γ. A straightforward calculation yields

$$S_\gamma Q = \sum_{\beta \in \mathbb{Z}^{2N}} T_\gamma \Phi_0 U_\gamma^* U_\beta \Phi_0^* R_\beta \tag{19}$$

where we wrote

$$R_\beta : \operatorname{Im} S_\beta \to L^2(\mathbb{R}^N), \quad (\ldots,\, 0,\, f_\beta,\, 0,\, \ldots) \mapsto f_\beta$$

and

$$T_\gamma : L^2(\mathbb{R}^N) \to \operatorname{Im} S_\gamma, \quad f_\gamma \mapsto (\ldots,\, 0,\, f_\gamma,\, 0,\, \ldots),$$

for a moment. Since, with certain constants $c_{\gamma\beta}$,

$$\Phi_0 U_\gamma^* U_\beta \Phi_0^* = c_{\gamma\beta} \Phi_0 U_{\gamma-\beta}^* \Phi_0^* = c_{\gamma\beta} U_{\gamma-\beta}^* \Phi_{\gamma-\beta} \Phi_0^* = 0$$

if β is sufficiently large, the sum (19) has only a finite number of non-vanishing items. Each of these items is compact because Φ_0 is compact. Thus, $S_\gamma Q$ and $QS_\gamma = (S_\gamma Q)^*$ are compact.

(b) It is easy to check that the operator Q belongs to $L(l^2, \hat{\mathcal{P}})$ if and only if, for every $k \in \mathbb{N}$,

$$\|\hat{P}_k Q(I - \hat{P}_n)\| \to 0 \quad \text{and} \quad \|(I - \hat{P}_n)Q\hat{P}_k\| \to 0$$

as $n \to \infty$. These conditions follow immediately from the compactness of $\hat{P}_k Q$ and $Q\hat{P}_k$ and from the *-strong convergence of the $\hat{P}_n$ to the identity.

(c) As in the previous step, we have to show that, for every $k \in \mathbb{N}$,

$$\|\hat{P}_k \Gamma(A)(I - \hat{P}_n)\| \to 0 \quad \text{and} \quad \|(I - \hat{P}_n)\Gamma(A)\hat{P}_k\| \to 0$$

as $n \to \infty$. Let us check the first condition. We have

$$\hat{P}_k \Gamma(A)(I - \hat{P}_n) = \hat{P}_k QGAG^*(I - \hat{P}_n) + \hat{P}_k(I - \hat{P}_n) - \hat{P}_k Q(I - \hat{P}_n).$$

The first and the third term in this sum tend to zero in the norm since $\hat{P}_k Q$ is compact and since the $I - \hat{P}_n$ converge strongly to 0. The second term is zero whenever $n > k$.

(d) If K is $\hat{\mathcal{P}}$-compact, then $\|K(I - \hat{P}_n)\| \to 0$. Consequently,

$$\|G^*K(I - \hat{P}_n)G\| = \|G^*KGG^*(I - \hat{P}_n)G\| = \|G^*KG(I - G^*\hat{P}_n G)\| \to 0.$$

Since

$$G^*\hat{P}_n G = \sum_{\alpha \in [-n,n]^{2N} \cap \mathbb{Z}^{2N}} \Phi_\alpha^* \Phi_\alpha$$

and $\Phi_\alpha^* \Phi_\alpha$ is compact, the operator G^*KG is the norm limit of compact operators and, hence, compact.

(e) Since A and A_G are unitarily equivalent, the operator A is invertible (Fredholm) if and only if A_G is invertible (Fredholm). We claim that the latter happens if and only if the operator $\Gamma(A)$ is invertible ($\hat{P}$-Fredholm).

Let A_G be invertible on $\operatorname{Im} Q$, and let B be its inverse. Then, clearly, $QBQ + I - Q$ is the inverse of $\Gamma(A)$. Conversely, if C is the inverse of $\Gamma(A)$, then QCQ is the inverse of A_G, since $\Gamma(A)Q = Q\Gamma(A)Q = Q\Gamma(A)$.

Let now A_G be Fredholm, and let B be a regularizer of A_G, i.e. the operators $A_G B - I$ and $BA_G - I$ are compact. Then the operators

$$\begin{aligned}&\Gamma(A)(QBQ + I - Q) - I \\ &\quad = (QA_G Q + I - Q)(QBQ + I - Q) - I = QA_G BQ - Q = Q(A_G B - I)Q\end{aligned}$$

and $(QBQ + I - Q)\Gamma(A) - I$ are compact and, hence, also $\hat{\mathcal{P}}$-compact, whence the $\hat{\mathcal{P}}$-Fredholmness of $\Gamma(A)$. Let, conversely, $\Gamma(A)$ be a $\hat{\mathcal{P}}$-Fredholm operator.

Thus, there are an operator $B \in L(l^2, \hat{\mathcal{P}})$ and $\hat{\mathcal{P}}$-compact operators K, L such that

$$\Gamma(A)B = I + K \quad \text{and} \quad B\Gamma(A) = I + L.$$

We multiply both equalities from both sides by Q. Since $\Gamma(A)$ commutes with Q, we get

$$Q\Gamma(A)QBQ = Q + K' \quad \text{and} \quad QBQ\Gamma(A)Q = Q + L' \tag{20}$$

with $\hat{\mathcal{P}}$-compact operators K' and L' satisfying

$$K' = QK'Q \quad \text{and} \quad L' = QL'Q.$$

Multiplying (20) by G^* from the left band by G from the right hand side we find

$$AG^*BG = I + G^*K'G \quad \text{and} \quad G^*BGA = I + G^*L'G.$$

The operators $G^*K'G$ and $G^*L'G$ are compact by assertion (d). ■

3.3 Bi-discretization and limit operators

Our next goal is to relate the limit operators of operators A on $L^2(\mathbb{R}^N)$ with the limit operators of its discretization $\Gamma(A)$ on $L(l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)))$. The latter ones are defined as in Section 2.1 (with p, N and X replaced by 2, $2n$ and $L^2(\mathbb{R}^N)$). Given $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^{2N} = \mathbb{Z}^N \times \mathbb{Z}^N$, we define a unitary operator $\hat{T}_\gamma$ on $l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ by $(\hat{T}_\gamma u)_\alpha := e^{i\langle\gamma_2, \alpha_1\rangle} u_\alpha$.

Lemma 3.3 *Let $\gamma \in \mathbb{Z}^{2N}$. Then*

$$\hat{V}_{-\gamma} G = \hat{T}_\gamma G U_\gamma^* \quad \textit{and} \quad G^* \hat{V}_\gamma = U_\gamma G^* \hat{T}_\gamma^*$$

on $L^2(\mathbb{R}^N)$ and on $l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$, respectively.

Proof. Let $f \in L^2(\mathbb{R}^N)$ and $\alpha \in \mathbb{Z}^N \times \mathbb{Z}^N$. Then

$$\begin{aligned}(\hat{V}_{-\gamma} G U_\gamma f)_\alpha &= (GU_\gamma f)_{\alpha+\gamma} = \Phi_0 U^*_{\alpha+\gamma} U_\gamma f \\ &= e^{i\langle\gamma_2, \alpha_1\rangle} \Phi_0 U_\gamma f = e^{i\langle\gamma_2, \alpha_1\rangle} (Gf)_\alpha \\ &= (\hat{T}_\gamma G f)_\alpha\end{aligned}$$

where we used (3). Hence, $\hat{V}_{-\gamma} G U_\gamma = \hat{T}_\gamma G$ on $L^2(\mathbb{R}^N)$, which implies the assertions. ■

Lemma 3.4 *Every sequence $h \in \mathcal{H}$ possesses a subsequence g such that the functions*

$$f_m : \mathbb{Z}^N \to \mathbb{T}, \quad \alpha \mapsto e^{i\langle g(m),\, \alpha\rangle} \tag{21}$$

converge uniformly on $\mathbb{Z}^N$ as $m \to \infty$.

Proof. Set $r_{-1} := h$, and let $\gamma : \mathbb{N} \to \mathbb{Z}^N$ be an enumeration of $\mathbb{Z}^N$. By the compactness of the unit circle $\mathbb{T}$, there is a subsequence r_0 of r_{-1} such that

$$e^{i\langle r_0(m),\, \gamma_0\rangle} \to f(\gamma_0) \in \mathbb{T} \quad \text{as } m \to \infty$$

and

$$|e^{i\langle r_0(m),\, \gamma_0\rangle} - f(\gamma_0)| < 2 \quad \text{for all } m \in \mathbb{Z}^N.$$

We proceed in this way and get, for every positive integer n, a subsequence r_n of r_{n-1} such that

$$e^{i\langle r_n(m),\, \gamma_n\rangle} \to f(\gamma_n) \in \mathbb{T} \quad \text{as } m \to \infty$$

and

$$|e^{i\langle r_n(m),\, \gamma_n\rangle} - f(\gamma_n)| < 2^{-n} \quad \text{for all } m \in \mathbb{Z}^N.$$

Set $g(n) := r_n(n)$. Since g is (with exception of a finite number of entries) a subsequence of each sequence r_n, we have $g \in \mathcal{H}$,

$$e^{i\langle g(m),\, \gamma_n\rangle} \to f(\gamma_n) \quad \text{as } m \to \infty$$

and

$$|e^{i\langle g(m),\, \gamma_n\rangle} - f(\gamma_n)| < 2^{-n} \quad \text{for all } m \in \mathbb{Z}^N \text{ and } n \in \mathbb{N}.$$

We claim that the functions f_m converge uniformly to the function $f : \mathbb{Z}^N \to \mathbb{T}$ defined in this way. Given $\varepsilon > 0$, choose $K \in \mathbb{N}$ such that $2^{-K} < \varepsilon$, and then choose $M \in \mathbb{N}$ such that

$$|e^{i\langle g(m),\, \gamma_n\rangle} - f(\gamma_n)| < \varepsilon \quad \text{for all } m \geq M \text{ and } n \leq K.$$

Then $|e^{i\langle g(m),\, \alpha\rangle} - f(\alpha)| < \varepsilon$ for all $m \geq M$ and $\alpha \in \mathbb{Z}^N$. ∎

Proposition 3.5 *Let $A \in L(L^2(\mathbb{R}^N))$ be such that the limit operator A_h with respect to the sequence $h \in \mathcal{H}$ exists. Then there is a subsequence g of h such that the limit operator $\Gamma(A)_g$ of $\Gamma(A)$ exists and that the operators $\Gamma(A)_g$ and $\Gamma(A_h)$ are unitarily equivalent.*

Proof. Let $h \in \mathcal{H}$ be a sequence such that the limit operator A_h exists. By the preceding lemma, there is a subsequence g of h such that the functions (21) converge uniformly on $\mathbb{Z}^{2N}$ to a certain function $f_g : \mathbb{Z}^{2N} \to \mathbb{T}$. Let the operator $T_g : l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)) \to l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ be defined by $(T_g u)_\alpha := f_g(\alpha_1) u_\alpha$. Since all values of f_g are unimodular, the operator T_g is unitary. Moreover, from the uniform convergence of the functions (21) to f_g we conclude that

$$\|\hat{T}_{g(m)} - T_g\| = \sup_{\alpha \in \mathbb{Z}^{2N}} |e^{i\langle g(m),\, \alpha\rangle} - f_g(\alpha)| \to 0 \quad \text{as } m \to \infty.$$

Now we have, by Lemma 3.3,

$$\hat{V}_{-g(m)} G A G^* \hat{V}_{g(m)} = \hat{T}_{g(m)} G U^*_{g(m)} A U_{g(m)} G^* \hat{T}_{g(m)},$$

and the right hand side of this equality converges *-strongly to $T_g G A_h G^* T_g^*$. Hence, the limit operator $(GAG^*)_g$ exists, and

$$(GAG^*)_g = T_g G A_h G^* T_g^*. \tag{22}$$

Choosing $A = I$, we see in particular that every sequence h which tends to infinity possesses a subsequence g such that the limit operator Q_g of $Q = GG^*$ exists and that this limit operator is equal to $T_g Q T_g^*$. Of course, one can choose the same subsequence g as in (22). Consequently, the limit operator of $\Gamma(A) = GAG^* + I - Q$ with respect to g also exists, and

$$\begin{aligned} \Gamma(A)_g &= (GAG^*)_g + (I-Q)_g \\ &= T_g G A_h G^* T_g^* + T_g (I-Q) T_g^* \;=\; T_g \Gamma(A_h) T_g^*. \end{aligned} \tag{23}$$

This proves the assertion. ■

4 Fredholmness of pseudodifferential operators

We are now going to single out a class of operators on $L^2(\mathbb{R}^N)$ which become band-dominated operators in the rich Wiener algebra after bi-discretization. This will enable us to derive Fredholm criteria for these operators. Particular examples of operators which belong to this class are provided by the pseudodifferential operators with symbol in $S^0_{0,0}$.

4.1 A Wiener algebra on $L^2(\mathbb{R}^N)$

We define a Wiener algebra of operators on $L^2(\mathbb{R}^N)$ by imposing conditions on the decay of the norms $\|\Phi_\alpha A \Phi^*_{\alpha-\gamma}\|$.

Definition 4.1 *Let A be a linear (at this moment not necessarily bounded) operator on $L^2(\mathbb{R}^N)$. We say that A belongs to $\mathcal{W}(L^2(\mathbb{R}^N))$ if*

$$\|A\|_{\mathcal{W}(L^2(\mathbb{R}^N))} := \sum_{\gamma \in \mathbb{Z}^{2N}} \sup_{\alpha \in \mathbb{Z}^{2N}} \|\Phi_\alpha A \Phi^*_{\alpha-\gamma}\|_{L(L^2(\mathbb{R}^N))} < \infty.$$

The class $\mathcal{W}(L^2(\mathbb{R}^N))$ contains sufficiently many interesting operators. Actually we will see that all pseudodifferential operators with symbol in $S^0_{0,0}$ belong to $\mathcal{W}(L^2(\mathbb{R}^N))$. To check this, we need some auxiliary results. The following proposition is proved in [16], Proposition 5.5.2.

Proposition 4.2 *Let $A = Op(a) \in OPS^0_{0,0}$, and let (φ_α) be a partition of unit satisfying* (16)*. Then, for all $\alpha, \beta \in \mathbb{Z}^N$ and $k_1, k_2 > N/2$,*

$$\|\varphi_\alpha A \varphi_\beta I\|_{L(L^2(\mathbb{R}^N))} \le C \langle \beta - \alpha \rangle^{-2k_1} |a|_{2k_1, 2k_2} \tag{24}$$

with a constant $C > 0$ independent of α, β and a (but depending on k_1 and k_2).

Proposition 4.3 *Let $A = Op(a) \in OPS^0_{0,0}$ and $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^N \times \mathbb{Z}^N$. Then*

$$\|\Phi_\alpha A \Phi^*_\beta\|_{L(L^2(\mathbb{R}^N))} \leq C|a|_{2k+2m,\,2k+2m} \langle \alpha_1 - \beta_1 \rangle^{-k} \langle \alpha_2 - \beta_2 \rangle^{-k}$$

whenever $2k > N$, $m \in \mathbb{N}$ is large enough, and with a constant $C > 0$ independent of a and of α and β (but depending on k and m).

Proof. Applying Proposition 4.2 to the operator $B := Op(\varphi_{\alpha_2}) A Op(\varphi_{\beta_2})$, we get

$$\begin{aligned} \|\Phi_\alpha A \Phi^*_\beta\|_{L(L^2(\mathbb{R}^N))} &= \|\varphi_{\alpha_1} Op(\varphi_{\alpha_2}) A Op(\varphi_{\beta_2}) \varphi_{\beta_1} I\|_{L(L^2(\mathbb{R}^N))} \\ &\leq C \langle \alpha_1 - \beta_1 \rangle^{-2k} |\mathrm{sym}_B|_{2k,\,2k} \end{aligned}$$

for all $2k > N$. By Theorem 4.2.1 from [16], $|\mathrm{sym}_B|_{2k,\,2k} \leq C|a|_{2k+2m,\,2k+2m}$ whenever $2m > N$. Thus,

$$\|\Phi_\alpha A \Phi^*_\beta\|_{L(L^2(\mathbb{R}^N))} \leq C \langle \alpha_1 - \beta_1 \rangle^{-2k} |a|_{2k+2m,\,2k+2m}. \tag{25}$$

Similarly, writing $FOp(a)F^{-1}$ (with F denoting the Fourier transform on $L^2(\mathbb{R}^N)$) as the pseudodifferential operator with double symbol $\tilde{a}(x, y, \xi) := a(-\xi, y)$, and estimating the right hand side of the estimate

$$\begin{aligned} \|\Phi_\alpha A \Phi^*_\beta\|_{L(L^2(\mathbb{R}^N))} &= \|F \Phi_\alpha A \Phi^*_\beta F^{-1}\|_{L(L^2(\mathbb{R}^N))} \\ &= \|Op(\varphi_{\alpha_1}) \varphi_{\alpha_2} F A F^{-1} \varphi_{\beta_2} Op(\varphi_{\alpha_2})\|_{L(L^2(\mathbb{R}^N))} \\ &\leq \|\varphi_{\alpha_2} F A F^{-1} \varphi_{\beta_2} I\|_{L(L^2(\mathbb{R}^N))} \end{aligned}$$

by using Theorem 4.3.2 from [16] and the Calderon-Vaillancourt theorem, we obtain

$$\|\Phi_\alpha A \Phi^*_\beta\|_{L(L^2(\mathbb{R}^N))} \leq C \langle \alpha_2 - \beta_2 \rangle^{-2k} |a|_{2k+2m,\,2k+2m}. \tag{26}$$

for every $2k > N$ and for every m which is sufficiently large (recall that φ is an even function by hypothesis). Multiplying (25) by (26) and taking square roots, we get the assertion. ■

Corollary 4.4 $OPS^0_{0,0} \subseteq \mathcal{W}(L^2(\mathbb{R}^N))$.

Indeed, for $A \in OPS^0_{0,0}$, and with $\gamma := (\gamma_1, \gamma_2)$ and $\alpha := (\alpha_1, \alpha_2)$, the preceding proposition implies

$$\sum_{\gamma \in \mathbb{Z}^{2N}} \sup_{\alpha \in \mathbb{Z}^{2N}} \|\Phi_\alpha A \Phi^*_{\alpha-\gamma}\|_{L(L^2(\mathbb{R}^N))} \leq C|a|_{2k+2m,\,2k+2m} \sum_{\gamma \in \mathbb{Z}^{2N}} \langle \gamma_1 \rangle^{-k} \langle \gamma_2 \rangle^{-k},$$

which is finite if k is chosen large enough. ■

Here are some basic properties of $\mathcal{W}(L^2(\mathbb{R}^N))$.

Proposition 4.5 *(a)* $\mathcal{W}(L^2(\mathbb{R}^N)) \subset L(L^2(\mathbb{R}^N))$, *and*

$$\|A\|_{L(L^2(\mathbb{R}^N))} \le \|A\|_{\mathcal{W}(L^2(\mathbb{R}^N))} \quad \text{for all } A \in \mathcal{W}(L^2(\mathbb{R}^N)).$$

(b) When provided with the norm $A \mapsto \|A\|_{\mathcal{W}(L^2(\mathbb{R}^N))}$ *and with the involution* $A \mapsto A^*$ *(= the Hilbert space adjoint of* A*), the set* $\mathcal{W}(L^2(\mathbb{R}^N))$ *becomes a unital involutive Banach algebra.*

Proof. (a) The boundedness of $A \in \mathcal{W}(L^2(\mathbb{R}^N))$ as well as the norm estimate can be obtained as follows, where we employ (17) and (18) several times:

$$\begin{aligned}
\|Au\|^2 &= \sum_{\gamma\in\mathbb{Z}^{2N}} \|\Phi_\gamma A u\|^2 \\
&= \sum_{\gamma\in\mathbb{Z}^{2N}} \left\| \Phi_\gamma A \sum_{\delta\in\mathbb{Z}^{2N}} \Phi_\delta^* \Phi_\delta u \right\|^2 \\
&\le \sum_{\gamma\in\mathbb{Z}^{2N}} \left(\sum_{\alpha\in\mathbb{Z}^{2N}} \|\Phi_\gamma A \Phi_{\gamma-\alpha}^*\| \, \|\Phi_{\gamma-\alpha} u\| \right)^2 \\
&\le \sum_{\gamma\in\mathbb{Z}^{2N}} \left(\sum_{\alpha\in\mathbb{Z}^{2N}} k_A(\alpha) \|\Phi_{\gamma-\alpha} u\| \right)^2 \\
&\le \sum_{\gamma\in\mathbb{Z}^{2N}} \left(\sum_{\alpha\in\mathbb{Z}^{2N}} k_A(\gamma-\alpha) \|\Phi_{\alpha} u\| \right)^2
\end{aligned}$$

with $k_A(\alpha) := \sup_{\gamma\in\mathbb{Z}^N} \|\Phi_\gamma A \Phi_{\gamma-\alpha}^*\|$. Since k_A is in $l^1(\mathbb{Z}^N)$,

$$\|Au\|^2 \le \left(\sum_{\gamma\in\mathbb{Z}^{2N}} k_A(\gamma) \right)^2 \sum_{\alpha\in\mathbb{Z}^{2N}} \|\Phi_\alpha u\|^2 = \|A\|^2_{\mathcal{W}(L^2(\mathbb{R}^N))} \|u\|^2,$$

whence assertion (a).
(b) Let A, $B \in \mathcal{W}(L^2(\mathbb{R}^N))$. Then, clearly,

$$\|\alpha A\|_{\mathcal{W}(L^2(\mathbb{R}^N))} = |\alpha| \, \|A\|_{\mathcal{W}(L^2(\mathbb{R}^N))}$$

and

$$\|A+B\|_{\mathcal{W}(L^2(\mathbb{R}^N))} \le \|A\|_{\mathcal{W}(L^2(\mathbb{R}^N))} + \|B\|_{\mathcal{W}(L^2(\mathbb{R}^N))}.$$

For the product AB, one finds

$$\|AB\|_{\mathcal{W}(L^2(\mathbb{R}^N))} = \sum_{\gamma\in\mathbb{Z}^{2N}} \sup_{\alpha\in\mathbb{Z}^{2N}} \|\Phi_\alpha A B \Phi_{\alpha-\gamma}^*\|$$

$$
\begin{aligned}
&= \sum_{\gamma\in\mathbb{Z}^{2N}} \sup_{\alpha\in\mathbb{Z}^{2N}} \left\| \sum_{\theta\in\mathbb{Z}^{2N}} \Phi_\alpha A \Phi^*_{\alpha-\theta} \Phi_{\alpha-\theta} B \Phi^*_{\alpha-\gamma} \right\| \\
&\leq \sum_{\gamma\in\mathbb{Z}^{2N}} \sum_{\theta\in\mathbb{Z}^{2N}} k_A(\theta) k_B(\gamma-\theta) \\
&\leq \|A\|_{\mathcal{W}(L^2(\mathbb{R}^N))} \|B\|_{\mathcal{W}(L^2(\mathbb{R}^N))}.
\end{aligned}
$$

Further, since $\|\Phi_\gamma A \Phi^*_\delta\| = \|\Phi_\gamma A^* \Phi^*_\delta\|$, the operators A and A^* belong to the Wiener algebra $\mathcal{W}(L^2(\mathbb{R}^N))$ only simultaneously, and one has

$$
\|A\|_{\mathcal{W}(L^2(\mathbb{R}^N))} = \|A^*\|_{\mathcal{W}(L^2(\mathbb{R}^N))}.
$$

That the identity operator belongs to $\mathcal{W}(L^2(\mathbb{R}^N))$ follows from Corollary 4.4. Finally, if (A_n) is a Cauchy sequence in $\mathcal{W}(L^2(\mathbb{R}^N))$ then, by part (a), it is also a Cauchy sequence in $L(L^2(\mathbb{R}^N))$, hence convergent. Let $A \in L(L^2(\mathbb{R}^N))$ denote the limit of this sequence. Given $\varepsilon > 0$, choose M such that $\|A_n - A_m\|_{\mathcal{W}(L^2(\mathbb{R}^N))} < \varepsilon$ for all $m, n \geq M$. Letting m go to infinity in this inequality, we get the convergence of the A_m to A with respect to the norm in the Wiener algebra. ■

Next we consider bi-discretizations of operators in the Wiener algebra. For notational convenience, we denote the discrete Wiener algebra $\mathcal{W}$ on $l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ introduced in Section 2.2 by $\mathcal{W}(l^2(\mathbb{Z}^{2N}))$ in what follows.

Proposition 4.6 (a) *Let $A \in \mathcal{W}(L^2(\mathbb{R}^N))$. Then the operators GAG^* and $\Gamma(A)$ belong to the Wiener algebra $\mathcal{W}(l^2(\mathbb{Z}^{2N}))$.*
(b) *Let $B \in \mathcal{W}(l^2(\mathbb{Z}^{2N}))$. Then the operator G^*BG belongs to the Wiener algebra $\mathcal{W}(L^2(\mathbb{R}^N))$.*

Proof. (a) Let $u \in l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N))$ and $\alpha \in \mathbb{Z}^{2N}$. Then

$$
\begin{aligned}
(GAG^*u)_\alpha &= (GA \sum_{\gamma\in\mathbb{Z}^{2N}} U_\gamma \Phi_0^* u_\gamma)_\alpha = \Phi_0 U_\alpha^* A \sum_{\gamma\in\mathbb{Z}^{2N}} U_\gamma \Phi_0^* u_\gamma \\
&= \sum_{\gamma\in\mathbb{Z}^{2N}} \Phi_0 U_\alpha^* A U_{\alpha-\gamma} \Phi_0^* u_{\alpha-\gamma} = \sum_{\gamma\in\mathbb{Z}^{2N}} \Phi_0 U_\alpha^* A U_{\alpha-\gamma} \Phi_0^* (\hat{V}_\gamma u)_\alpha,
\end{aligned}
$$

which shows that $GAG^* \in \mathcal{W}(l^2(\mathbb{Z}^{2N}))$. When applied to the operator $A = I$ (which is in $\mathcal{W}(L^2(\mathbb{R}^N))$ by Proposition 4.5), this inclusion implies in particular that $Q = GG^* \in \mathcal{W}(l^2(\mathbb{Z}^{2N}))$. Clearly, the discrete Wiener algebra $\mathcal{W}(l^2(\mathbb{Z}^{2N}))$ also contains the identity operator, whence the first assertion.
(b) Let $B \in \mathcal{W}(l^2(\mathbb{Z}^{2N}))$ be given by

$$
B = \sum_{\beta\in\mathbb{Z}^{2N}} b_\beta \hat{V}_\beta \quad \text{with} \quad \|B\|_{\mathcal{W}(l^2(\mathbb{Z}^{2N}))} = \sum_{\beta\in\mathbb{Z}^{2N}} \|b_\beta\| < \infty
$$

with multiplication operators b_β. Further, let $\alpha,\, \gamma \in \mathbb{Z}^{2N}$ and $u \in L^2(\mathbb{R}^N)$. Then

$$\begin{aligned}
\Phi_\alpha G^* B G \Phi^*_{\alpha-\gamma} u &= \sum_{\delta \in \mathbb{Z}^{2N}} \Phi_\alpha U_\delta \Phi_0^* (BG\Phi^*_{\alpha-\gamma} u)_\delta \\
&= \sum_{\delta \in \mathbb{Z}^{2N}} \Phi_\alpha U_\delta \Phi_0^* \sum_{\beta \in \mathbb{Z}^{2N}} b_\beta(\delta)(G\Phi^*_{\alpha-\gamma} u)_{\delta-\beta} \\
&= \sum_{\delta \in \mathbb{Z}^{2N}} \Phi_\alpha U_\delta \Phi_0^* \sum_{\beta \in \mathbb{Z}^{2N}} b_\beta(\delta) \Phi_0 U^*_{\delta-\beta} \Phi^*_{alpha-\gamma} u \\
&= \sum_{\delta \in \mathbb{Z}^{2N}} \Phi_\alpha \Phi_\delta^* \sum_{\beta \in \mathbb{Z}^{2N}} U_\delta b_\beta(\delta) U^*_{\delta-\beta} \Phi_{\delta-\beta} \Phi^*_{\alpha-\gamma} u
\end{aligned}$$

whence

$$\begin{aligned}
\|\Phi_\alpha G^* B G \Phi^*_{\alpha-\gamma}\| &\le \sum_{\delta \in \mathbb{Z}^{2N}} \|\Phi_\alpha \Phi_\delta^*\| \sum_{\beta \in \mathbb{Z}^{2N}} \|b_\beta\| \, \|\Phi_{\delta-\beta} \Phi^*_{\alpha-\gamma}\| \\
&= \sum_{\beta \in \mathbb{Z}^{2N}} \|b_\beta\| \sum_{\delta \in \mathbb{Z}^{2N}} \|\Phi_\alpha \Phi_\delta^*\| \, \|\Phi_{\delta-\beta} \Phi^*_{\alpha-\gamma}\|.
\end{aligned}$$

We write all indices as $\alpha = (\alpha_1,\, \alpha_2) \in \mathbb{Z}^N \times \mathbb{Z}^N$ and use Proposition 4.3 to get

$$\begin{aligned}
&\sum_{\delta \in \mathbb{Z}^{2N}} \|\Phi_\alpha \Phi_\delta^*\| \, \|\Phi_{\delta-\beta} \Phi^*_{\alpha-\gamma}\| \\
&\quad \le C \sum_{\delta \in \mathbb{Z}^{2N}} \langle \alpha_1 - \delta_1 \rangle^{-k} \langle \alpha_2 - \delta_2 \rangle^{-k} \langle \gamma_1 + \delta_1 - \alpha_1 - \beta_1 \rangle^{-k} \langle \gamma_2 + \delta_2 - \alpha_2 - \beta_2 \rangle^{-k} \\
&\quad = C \sum_{\delta_1 \in \mathbb{Z}^N} \langle \alpha_1 - \delta_1 \rangle^{-k} \langle \gamma_1 + \delta_1 - \alpha_1 - \beta_1 \rangle^{-k} \times \\
&\qquad\qquad \times \sum_{\delta_2 \in \mathbb{Z}^N} \langle \alpha_2 - \delta_2 \rangle^{-k} \langle \gamma_2 + \delta_2 - \alpha_2 - \beta_2 \rangle^{-k}.
\end{aligned}$$

If k is large enough, then the sequence $x \mapsto \langle x \rangle^{-k}$ belongs to $l^1(\mathbb{Z}^N)$. Since $l^1(\mathbb{Z}^N)$ is closed under convolution, there is a sequence $f \in l^1(\mathbb{Z}^N)$ such that

$$\|\Phi_\alpha G^* B G \Phi^*_{\alpha-\gamma}\| \le C \sum_{\beta \in \mathbb{Z}^{2N}} \|b_\beta\| f(\gamma_1 - \beta_1)\, f(\gamma_2 - \beta_2).$$

The sequence $g : (x_1,\, x_2) \mapsto f(x_1) f(x_2)$ belongs to $l^1(\mathbb{Z}^{2N})$. Hence, by the convolution theorem,

$$\|\Phi_\alpha G^* B G \Phi^*_{\alpha-\gamma}\| \le C \sum_{\beta \in \mathbb{Z}^{2N}} \|b_\beta\| \, g(\gamma - \beta) = h(\gamma)$$

with a certain function $h \in l^1(\mathbb{Z}^{2N})$ independent of α and γ. This estimate implies the assertion (b). ∎

Proposition 4.7 *The algebra $\mathcal{W}(L^2(\mathbb{R}^N))$ is inverse closed in $L(L^2(\mathbb{R}^N))$, i.e. if $A \in \mathcal{W}(L^2(\mathbb{R}^N))$ is invertible in $L(L^2(\mathbb{R}^N))$, then $A^{-1} \in \mathcal{W}(L^2(\mathbb{R}^N))$.*

Proof. Let $A \in \mathcal{W}(L^2(\mathbb{R}^N))$ be invertible on $L^2(\mathbb{R}^N)$. Then $\Gamma(A)$ belongs to $\mathcal{W}(l^2(\mathbb{Z}^{2N}))$ by Proposition 4.6 (a), and it is invertible in $L(l^2(\mathbb{Z}^{2N}, L^2(\mathbb{R}^N)))$ by Proposition 3.2. The well known inverse closedness of the discrete Wiener algebra ([13]) implies that $\Gamma(A)^{-1} \in \mathcal{W}(l^2(\mathbb{Z}^{2N}))$. Since

$$G^*\Gamma(A)^{-1}GA = G^*\Gamma(A)^{-1}GAG^*G = G^*\Gamma(A)^{-1}\Gamma(A)QG = I,$$

one has $G^*\Gamma(A)^{-1}G = A^{-1} \in \mathcal{W}(L^2(\mathbb{R}^N))$ by Proposition 4.6 (b). ■

4.2 Fredholmness of operators in $\mathcal{W}(L^2(\mathbb{R}^N))$

Operators on $L^2(\mathbb{R}^N)$ which possess a rich operator spectrum are defined in complete analogy to the discrete setting. More precisely: We let $\mathcal{W}^\$(L^2(\mathbb{R}^N))$ stand for the set of all operators A in the Wiener algebra $\mathcal{W}(L^2(\mathbb{R}^N))$ with the following property: every sequence $h \in \mathcal{H}$ possesses a subsequence g such that the limit operator A_g with respect to this sequence exists. It can be easily checked that $\mathcal{W}^\$(L^2(\mathbb{R}^N))$ is a closed and unital subalgebra of $\mathcal{W}(L^2(\mathbb{R}^N))$.

Proposition 4.8 *Let $A \in \mathcal{W}^\$(L^2(\mathbb{R}^N))$. Then GAG^* and $\Gamma(A)$ belong to the algebra $\mathcal{W}^\$(l^2(\mathbb{Z}^{2N}))$, and*

$$\begin{aligned}
\sigma_{op}(GAG^*) &= \{T_gGA_hG^*T_g^* : A_h \in \sigma_{op}(A)\}, \\
\sigma_{op}(\Gamma(A)) &= \{T_g\Gamma(A_h)T_g^* : A_h \in \sigma_{op}(A)\}.
\end{aligned}$$

Proof. Let $k \in \mathcal{H}$. Since A has a rich operator spectrum, there is a subsequence h of k such that A_h exists. By the Proposition 3.5, there is a subsequence g of h such that the limit operators $(GAG^*)_g$ and $\Gamma(A)_g$ exist. Hence, GAG^* and $\Gamma(A)$ are rich, too. The description of the corresponding operator spectra follows immediately from (22) and (23). ■

Theorem 4.9 *Let $A \in \mathcal{W}^\$(L^2(\mathbb{R}^N))$. Then A is a Fredholm operator if and only if all limit operators of A are invertible, and the essential spectrum of A is the union of all spectra of its limit operators.*

Proof. It is easy to see that, if A is a Fredholm operator, then all limit operators of A are invertible. Let, conversely, all limit operators of A be invertible. Then, by Propositions 4.8 and 3.2 (e), all limit operators of $\Gamma(A)$ are invertible. Consequently, $\Gamma(A)$ is a $\mathcal{P}$-Fredholm operator by Theorem 2.5. By Proposition 3.2 (e) again, A is a Fredholm operator. ■

Let $\mathcal{A}^\$(L^2(\mathbb{R}^N))$ denote the closure in $L(L^2(\mathbb{R}^N))$ of the rich Wiener algebra $\mathcal{W}^\$(L^2(\mathbb{R}^N))$. Further we agree upon calling a family of operators *uniformly invertible* if each member of the family is invertible and if the norms of their inverses are uniformly bounded.

Theorem 4.10 *An operator $A \in \mathcal{A}^\$(L^2(\mathbb{R}^N))$ is Fredholm on $L^2(\mathbb{R}^N)$ if and only if all limit operators of A are uniformly invertible on $L^2(\mathbb{R}^N)$.*

Proof. Let (A_n) be a sequence of operators in $\mathcal{W}^\$(L^2(\mathbb{R}^N))$ which converges to A in the norm. By $\mathcal{B}$ we denote the smallest C^*-subalgebra of $L(L^2(\mathbb{R}^N))$ which contains all operators A_n and the ideal $K(L^2(\mathbb{R}^N))$ of the compact operators, and we write $\mathcal{H}_\mathcal{B}$ for the set of all sequences h in $\mathcal{H}$ such that the limit operator B_h exists for every operator $B \in \mathcal{B}$. Then the mappings

$$W_h : \mathcal{A}/K(L^2(\mathbb{R}^N)) \to L(L^2(\mathbb{R}^N)), \quad A + K(L^2(\mathbb{R}^N)) \mapsto A_h$$

are correctly defined C^*-algebra homomorphisms for $h \in \mathcal{H}_\mathcal{B}$. Employing a Cantor diagonal argument is is also not hard to verify that

$$\sigma_{op}(B) = \{W_h(B) : h \in \mathcal{H}_\mathcal{B}\} \quad \text{for every } B \in \mathcal{B}.$$

Let now the limit operators of A be uniformly invertible. Then, by Neumann series, all limit operators of all operators A_n are uniformly invertible if only n is large enough. By Theorem 4.9, this implies that all operators A_n with n large enough are Fredholm or, equivalently, their cosets modulo the compact operators are invertible. Moreover, these cosets are even uniformly invertible which follows easily from the second assertion of 4.9 (or, likewise, from the symbol calculus developed in [18]). Since the cosets of A_n converge to the coset of A, and since these cosets are uniformly invertible, we obtain the invertibility of the coset of A modulo the compact operators, i.e. the Fredholmness of A. ∎

Corollary 4.11 *Let $A \in \mathcal{A}^\$(L^2(\mathbb{R}^N))$. Then*

$$\|A\|_{ess} := \|A + K(L^2(\mathbb{R}^N))\| = \sup\{\|A_h\| : A_h \in \sigma_{op}(A)\}.$$

There is also a local version of the latter result. Given a radius $R > 0$, a direction $\eta \in S^{N-1}$ with S^{N-1} referring to the unit sphere in $\mathbb{R}^N$, and a neighborhood $U \subseteq S^{N-1}$ of η, define

$$W_{R,U} := \{z \in \mathbb{R}^N : |z| > R \text{ and } z/|z| \in U\}. \tag{27}$$

We call $W_{R,U}$ a *neighborhood at infinity of* η. If h is a sequence which tends to infinity, then we say that h tends into the direction of $\eta \in S^{N-1}$ if, for every neighborhood at infinity $W_{R,U}$ of η, there is an m_0 such that

$$h(m) \in W_{R,U} \quad \text{for all } m \geq m_0.$$

Finally, we call an operator $A \in L(L^2(\mathbb{R}^N))$ *locally invertible at the infinitely distant point* $\eta \in S^{N-1}$ if there exist a neighborhood at infinity W of η as well as operators R, $L \in L(L^2(\mathbb{R}^N))$ such that

$$LA\chi_W I = \chi_W AR = \chi_W I.$$

We denote by $\sigma_{op,\eta}(A)$ the set of all limit operators of $A \in B(L^2(\mathbb{R}^N))$ which are defined by sequences $h = (h_1, h_2) : \mathbb{N} \to \mathbb{Z}^N \times \mathbb{Z}^N$ for which h_1 tends to infinity into the direction of η.

Theorem 4.12 *Let $A \in \mathcal{W}^\$(L^2(\mathbb{R}^N))$. Then A is locally invertible at the infinitely distant point $\eta \in S^{N-1}$ if and only if all limit operators $A_h \in \sigma_{op,\eta}(A)$ are invertible.*

The proof is similar to the proof of Theorem 4.9. An analogous result (with the invertibility of all limit operators in the local operator spectrum replaced by their uniform invertibility) holds for operators in $\mathcal{A}^\$(L^2(\mathbb{R}^N))$.

Finally, we say that $\lambda \in \mathbb{C}$ belongs to the local spectrum $\sigma_\eta(A)$ of the operator A at η if $A - \lambda I$ is not locally invertible at the infinitely distant point $\eta \in S^{N-1}$. The following is a corollary of Theorem 4.12.

Theorem 4.13 *Let $A \in \mathcal{W}^\$(L^2(\mathbb{R}^N))$. Then*

$$\sigma_\eta(A) = \bigcup_{A_h \in \sigma_{op,\eta}(A)} \sigma(A_h).$$

4.3 Fredholmness of pseudodifferential operators in the class $OPS^0_{0,0}$

We have seen in Corollary 4.4, that every pseudodifferential operator with symbol in $S^0_{0,0}$ belongs to the Wiener algebra. Now we will show, moreover, that these pseudodifferential operators possess a rich operator spectrum. Thus, they become subject to Theorem 4.9.

Theorem 4.14 $OPS^0_{0,0} \subseteq \mathcal{W}^\$(L^2(\mathbb{R}^N))$.

Proof. Let $a \in S^0_{0,0}$ and $A := Op(a)$, and let $h \in \mathcal{H}$. For $k = (k_1, k_2) \in \mathbb{Z}^N \times \mathbb{Z}^N$, we consider the functions

$$a^{(k)} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \quad (x_1, x_2) \mapsto a(x_1 + k_1, x_2 + k_2).$$

Clearly, $U^*_{h(m)} A U_{h(m)} = Op(a^{(h(m))})$. The sequence $(a^{(h(m))})_{m \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ is bounded with respect to the supremum norm. Hence, by the Arzelà-Ascoli theorem, there exists a subsequence g of h such that the functions $a^{(g(m))}$ converge in the topology of $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ to a function a_g. It is easy to see that the limit function a_g belongs to $S^0_{0,0}$ and that

$$|a_g|_{k,l} \le |a|_{k,l} \qquad \text{for all } k,\, l \in \mathbb{N}.$$

We set $A_g := Op(a_g)$ and claim that A_g is the limit operator of A with respect to the sequence g, i.e., we claim that

$$\text{s-lim}_{m\to\infty} U^*_{g(m)} A U_{g(m)} = A_g \quad \text{and} \quad \text{s-lim}_{m\to\infty} U^*_{g(m)} A^* U_{g(m)} = A^*_g. \tag{28}$$

For the first assertion of (28), choose a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ which is equal to 1 in a neighborhood of the origin. Further, for $R > 0$, set $\varphi_R(x) := \varphi(x/R)$, and consider the cut-off functions $\psi_R(x, \xi) := \varphi_R(x)\varphi_R(\xi)$ on $\mathbb{R}^N \times \mathbb{R}^N$. Evidently,

$$\text{s-lim}_{R\to\infty} Op(\psi_R) = I. \tag{29}$$

The operator $Op(a)Op(\psi_R)$ is a pseudodifferential operator with symbol $c_R \in S^0_{0,0}$, given by the oscillatory integral

$$c_R(x,\,\xi) = os\,(2\pi)^{-N} \int\int_{\mathbb{R}^N} a(x,\,\xi+\eta)\psi_R(x+y,\,\xi)e^{-i\langle y,\,\eta\rangle}dy\,d\eta \tag{30}$$

(see, e.g. [16], Theorem 4.2.1). By means of the Lagrange formula, we write

$$\psi_R(x+y,\,\xi) = \psi_R(x,\,\xi) + q_R(x,\,y,\,\xi)$$

where $q_R(x,\,y,\,\xi) := \sum_{j=1}^N l_{j,R}(x,\,y,\,\xi)y_j$ and

$$l_{j,R}(x,\,y,\,\xi) := \int_0^1 (\partial_{x_j}\psi_R)(x+\theta y,\,\xi)\,d\theta.$$

Then we obtain (cf. [16], Corollary 2.2.2)

$$os\,(2\pi)^{-N} \int\int_{\mathbb{R}^N} a(x,\,\xi+\eta)e^{-i\langle y,\,\eta\rangle}dy\,d\eta = p(x,\,\xi),$$

such that (30) can be written as

$$c_R(x,\,\xi) = a(x,\,\xi)\psi_R(x,\,\xi) + t_R(x,\,\xi)$$

where

$$t_R(x,\,\xi) = (2\pi)^{-N}\sum_{j=1}^N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} l_{j,R}(x,\,y,\,\xi)(i\partial_{y_j})a(x,\,\xi+\eta)e^{-i\langle y,\,\eta\rangle}dy\,d\eta.$$

Simple manipulations yield the estimates

$$\left|\partial_x^\alpha\partial_\xi^\beta t_R(x+g_1(m),\,\xi+g_2(m))\right| \le C_{\alpha,\beta}\,|a|_{2k_1+|\alpha|,2k_2+|\beta|}\,(1+R)^{-1}$$

for all $2k_1 > N$ and $2k_2 > N$, and with a constant $C_{\alpha,\beta}$ independent of a. By the Calderon-Vaillancourt Theorem,

$$Op(t_R^{(g(m))}) \le C|a|_{N_1,N_2}(1+R)^{-1} \tag{31}$$

whenever N_1 and N_2 are sufficiently large. Here we used the convention

$$t_R^{(g(m))}(x,\,\xi) := t_R(x+g_1(m),\,\xi+g_2(m)).$$

Let now $u \in L^2(\mathbb{R}^N)$ and $\varepsilon > 0$. Due to (29) and (31), we can choose $R_0 > 0$ such that, for all $R > R_0$,

$$\|u - Op(\psi_R)u\| \le \frac{\varepsilon}{6\|u\|} \quad \text{and} \quad \sup_{m\in\mathbb{N}} \|Op(t_R^{(g(m))})\| \le \frac{\varepsilon}{3\|u\|}.$$

Thus, for all $m \in \mathbb{N}$,

$$\begin{aligned} \|(U^*_{g(m)} A U_{g(m)} - A_g)u\| &\leq \|(U^*_{g(m)} A U_{g(m)} - A_g) Op(\psi_R) u\| + \varepsilon/3 \\ &\leq \|Op((a^{(g(m))} - a_g)\psi_R) u\| + 2\varepsilon/3. \end{aligned} \tag{32}$$

Since the functions $a^{(g(m))} - a_g$ tend to zero in the topology of $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, the sequence of the functions $(a^{(g(m))} - a_g)\psi_R$ tends uniformly to zero together with their derivatives. Hence, by the Calderon-Vaillancourt Theorem, there exist an m_0 such that, for all $m > m_0$

$$\|Op((a^{(g(m))} - a_g)\psi_R)\| \leq \frac{\varepsilon}{3\|u\|}. \tag{33}$$

Estimates (32) and (33) imply that, for arbitrary $u \in L^2(\mathbb{R}^N)$ and $\varepsilon > 0$, there exists an m_0 such that

$$\|(U^*_{g(m)} A U_{g(m)} - A_g)u\| < \varepsilon \quad \text{for all } m > m_0.$$

This settles the first assertion of (28). For the second one, notice that the symbol of the adjoint operator is given by the oscillatory integral

$$\mathrm{sym}_{A^*}(x, \xi) = os\, (2\pi)^{-N} \int\int_{\mathbb{R}^N} \bar{a}(x + y, \xi + \eta) e^{-i\langle y, \eta\rangle} dy\, d\eta$$

(Theorem 4.4.2 in [16]). Since $a^{(g(m))} \to a_g$ in the topology of $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, this implies that

$$\begin{aligned} &\mathrm{sym}_{A^*}(x + g_1(m), \xi + g_2(m)) \\ &\quad = os\, (2\pi)^{-N} \int\int_{\mathbb{R}^N} \bar{a}(x + g_1(m) + y, \xi + g_2(m) + \eta) e^{-i\langle y, \eta\rangle} dy\, d\eta \\ &\quad \to os\, (2\pi)^{-N} \int\int_{\mathbb{R}^N} \bar{a}_g(x + y, \xi + \eta) e^{-i\langle y, \eta\rangle} dy\, d\eta. \end{aligned}$$

Hence, the symbols $\mathrm{sym}_A^{(g(m))}$ converge to $\mathrm{sym}_{A^*_g}$ in the topology of $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ as $m \to \infty$. Repeating the above arguments, we obtain the second assertion of (28). ∎

Due to Theorem 4.14, the following results are straightforward consequences of Theorems 4.10 and 4.13 and of Corollary 4.11.

Theorem 4.15 *An operator $A \in OPS^0_{0,0}$ is Fredholm on $L^2(\mathbb{R}^N)$ if and only if all limit operators of A are invertible on $L^2(\mathbb{R}^N)$. Thus,*

$$\sigma_{ess}(A) := \sigma(A + K(L^2(\mathbb{R}^N))) = \cup_{A_h \in \sigma_{op}(A)} \sigma(A_h)$$

and, moreover,

$$\|A\|_{ess} := \|A + K(L^2(\mathbb{R}^N))\| = \inf_{K \in K(L^2(\mathbb{R}^N))} \|A - K\| = \sup_{A_h \in \sigma_{op}(A)} \|A_h\|. \tag{34}$$

Theorem 4.16 *An operator $A \in OPS^0_{0,0}$ is locally invertible at the infinitely distant point $\eta \in S^{N-1}$ if and only if all limit operators of A in $\sigma_{op,\eta}(A)$ are invertible. In particular,*

$$\sigma_\eta(A) = \cup_{A_h \in \sigma_{op,\eta}(A)} \sigma(A_h).$$

Remark. One also considers pseudodifferential operators with double symbols $a \in S^0_{0,0,0}$. The class $S^m_{0,0,0}$ consists of all functions $a \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N)$ such that

$$|a|_{r,\,s,\,t} := \sup_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} \sum_{|\alpha| \le r,\, |\beta| \le s,\, |\gamma| \le t} |\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,\, y,\, \xi)| \langle \xi \rangle^{-m} < \infty$$

for each choice of $r,\, s,\, t \in \mathbb{N}$. For each $a \in S^m_{0,0,0}$, the pseudodifferential operator $Op_d(a)$ with double symbol a is defined by

$$(Op_d(a)u)(x) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} a(x,\, y,\, \xi) u(y) e^{i\langle x-y,\, \xi \rangle} dy\, d\xi, \quad u \in S(\mathbb{R}^N).$$

The class of all operators $Op_d(a)$ with $a \in S^m_{0,0,0}$ is denoted by $OPS^m_{0,0,0}$. This class seems to be much larger than the class $OPS^m_{0,0}$, but actually, both classes coincide (Theorem 4.3.2 in [16]). Thus, the results of the previous theorems apply to pseudodifferential operators with double symbol $a \in S^0_{0,0,0}$, and what they yield is the following. For $k = (k_1,\, k_2) \in \mathbb{Z}^N \times \mathbb{Z}^N$, we set

$$a^{(k)}(x,\, y,\, \xi) := a(x + k_1,\, y + k_1,\, \xi + k_2).$$

Then $U^*_{h(m)} A U_{h(m)} = Op(a^{(h(m))})$, and the sequence h has a subsequence g such that the functions $a^{(g(m))}$ converge to a function a_g in the topology of $C^\infty(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N)$ as $m \to \infty$. The limit function a_g belongs to $S^0_{0,0,0}$, and the limit operator of A with respect to the sequence g exists and is equal to $Op(a_g)$. So, these operators possess a rich operator spectrum, and Theorems 4.15 and 4.16 remain valid without changes.

5 Applications

5.1 Operators with slowly oscillating symbols

A symbol $a \in S^0_{0,0}$ is called *slowly oscillating with respect to x* if

$$\lim_{x \to \infty} \sup_{\xi \in \mathbb{R}^N} |\partial_{x_j} a(x,\, \xi)| = 0 \quad \text{for all } j = 1,\, \ldots,\, N,$$

and a is *slowly oscillating with respect to ξ* if

$$\lim_{\xi \to \infty} \sup_{x \in \mathbb{R}^N} |\partial_{\xi_j} a(x,\, \xi)| = 0 \quad \text{for all } j = 1,\, \ldots,\, N.$$

Proposition 5.1 *Let the function $a \in S^0_{0,0}$ be slowly oscillating with respect to x. Then every limit operator of $A := Op(a)$, which is defined with respect to a sequence $h = (h_1, h_2) : \mathbb{N} \to \mathbb{Z}^N \times \mathbb{Z}^N$ with $h_1(m) \to \infty$ as $m \to \infty$, is a pseudodifferential operator $Op(a_h)$ with a symbol independent of x. In particular, $Op(a_h)$ is shift invariant and, thus, a convolution operator. Similarly, if a is slowly oscillating with respect to ξ, and if $h_2(m) \to \infty$ as $m \to \infty$, then the limit operator $Op(a_h)$ has a symbol independent of ξ and is, thus, a multiplication operator.*

Proof. We will prove the first assertion only. Let a be slowly oscillating with respect to x. As we have seen in the proof of Theorem 4.14, the symbol a_h of the limit operator is the $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$-limit of the functions

$$a^{(h(m))} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \quad (x, \xi) \mapsto a(x + h_1(m), \xi + h_2(m)).$$

Since, for fixed $x', x'' \in \mathbb{R}^N$,

$$\begin{aligned} &|a^{(h(m))}(x', \xi) - a^{(h(m))}(x'', \xi)| \\ &\quad \leq \sum_{j=1}^{N} |x'_j - x''_j| \int_0^1 |\partial_{x_j}(a^{(h(m))}((1-t)x' + tx'', \xi)| \, dt \to 0 \end{aligned}$$

as $m \to \infty$, the function a_h does not depend on x. ∎

The most simple (and, perhaps, most important) pseudodifferential operators with slowly oscillating symbols are those whose symbols are slowly oscillating with respect to both variables simultaneously. We denote this class of symbols by $SO^0_{0,0}$ and the corresponding set of pseudodifferential operators by $OPSO^0_{0,0}$. For operators in this class, all limit operators are operators of convolution or operators of multiplication (indeed, if the sequence $h = (h_1, h_2)$ tends to infinity, then at least one of the sequences h_1 and h_2 goes to infinity, too). For both kinds of limit operators, their invertibility can be easily checked.

Theorem 5.2 *Let $a \in SO^0_{0,0}$. Then all limit operators of $Op(a)$ are invertible if and only if*

$$\lim_{R \to \infty} \inf_{|x|+|\xi| \geq R} |a(x, \xi)| > 0. \tag{35}$$

Proof. Let condition (35) be satisfied, and let $h = (h_1, h_2)$ be a sequence which defines a limit operator of $Op(a)$. Further assume for definiteness that the sequence h_1 tends to infinity (the case when $h_2 \to \infty$ can be treated similarly). Then, as we have seen in Proposition 5.1, the limit operator $Op(a)_h$ is shift invariant, i.e. there is a function a_h in $S^0_{0,0}$ which is independent of x such that $Op(a)_h = Op(a_h)$. Moreover, the functions

$$a^{(h(m))} : (x, \xi) \mapsto a(x + h_1(m), \xi + h_2(m))$$

converge to the function $(x, \xi) \mapsto a_h(\xi)$ in the topology of $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ as $m \to \infty$. Thus, for each $L > 0$,

$$\lim_{m\to\infty} \sup_{|x|+|\xi|\le L} |a(x + h_1(m), \xi + h_2(m)) - a_h(\xi)| = 0. \tag{36}$$

From (35) and (36) we conclude that $\inf_\xi |a_h(\xi)| > 0$, i.e. the limit operator $Op(a)_h$ is invertible.

To prove the reverse statement, suppose that all limit operators of $Op(a)$ are invertible, but that condition (35) is not fulfilled. Then there exists a sequence $h = (h_1, h_2) : \mathbb{N} \to \mathbb{Z}^N \times \mathbb{Z}^N$ which tends to infinity and for which $a(h_1(m), h_2(m)) \to 0$. Without loss we can assume that the limit operator of $Op(a)$ with respect to h exists (otherwise we choose a suitable subsequence of h). We further assume for definiteness that $h_1 \to \infty$ (the case when $h_2 \to \infty$ follows similarly). Then, as before, $Op(a)_h = Op(a_h)$ with a function a_h independent of x and such that the functions $a^{(h(m))}$ converge to a_h in $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. It follows from $a(h_1(m), h_2(m)) \to 0$ that $a_h(0) = 0$ which contradicts the invertibility of $Op(a_h)$. ■

Corollary 5.3 *An operator $Op(a) \in OPSO^0_{0,0}$ is Fredholm if and only if condition* (35) *holds. Moreover,*

$$\|Op(a)\|_{ess} = \lim_{R\to\infty} \sup_{|x|+|\xi|\ge R} |a(x, \xi)|.$$

The proof of the first assertion follows from the previous result and from Theorem 4.15. For the second assertion, recall Corollary 4.11. ■

These results admit generalizations to pseudodifferential operators with double symbols. For, we call the double symbol $a \in S^0_{0,0,0}$ *slowly oscillating* and write $a \in SO^0_{0,0,0}$ if, for arbitrary compact sets $K \subset \mathbb{R}^N$,

$$\lim_{x\to\infty} \sup_{(y,\xi)\in K\times\mathbb{R}^N} |\partial_{x_j} a(x, x + y, \xi)| = 0$$

and

$$\lim_{\xi\to\infty} \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} |\partial_{\xi_j} a(x, y, \xi)| = 0.$$

Proposition 5.4 (*a*) *Let $a \in SO^0_{0,0,0}$, and let $h = (h_1, h_2)$ be a sequence with $h_1 \to \infty$ for which the limit operator $Op_d(a)_h$ exists. Then this limit operator is of the form $Op(a_h)$ where a_h is the limit in the topology of $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ of the functions*

$$(x, \xi) \mapsto a(x + h_1(m), x + h_1(m), \xi + h_2(m))$$

as $m \to \infty$. The function a_h is independent of x in this case.

(*b*) *Let $a \in SO^0_{0,0,0}$, and let $h = (h_1, h_2)$ be a sequence with $h_2 \to \infty$ for which* *the limit operator $Op_d(a)_h$ exists. Then this limit operator is of the form $Op(a_h)$ where a_h is the limit in the topology of $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ of the functions*

$$(x, \xi) \mapsto a(x + h_1(m), x + h_1(m), \xi + h_2(m))$$

as $m \to \infty$. The function a_h is independent of ξ in this case.

Proof. We will check assertion (b) for example. The symbol a_h of the limit operator of $Op_d(a)$ with respect to h is defined as the limit as $m \to \infty$ of the oscillatory integral

$$os\,(2\pi)^{-N} \int\int_{\mathbb{R}^N} a(x + h_1(m), x + h_1(m) + y, \xi + h_2(m) + \eta) e^{-i\langle y, \eta\rangle}\, dy\, d\eta.$$

Thus,

$$a_h(x, x) = os\,(2\pi)^{-N} \int\int_{\mathbb{R}^N} a_h(x, x + y) e^{-i\langle y, \eta\rangle} dy\, d\eta$$

by Corollary 2.2.2 in [16]. ■

As in Theorem 5.2 and its Corollary 5.3, one can also prove that, if $a \in SO^0_{0,0,0}$, then all limit operators of $Op_d(a)$ are invertible if and only if

$$\lim_{R\to\infty} \sup_{|x|+|\xi|\geq R} |a(x, x, \xi)| > 0. \tag{37}$$

Hence, condition (37) is necessary and sufficient for Fredholmness of $Op_d(a)$, and

$$\|Op_d(a)\|_{ess} = \lim_{R\to\infty} \inf_{|x|+|\xi|\geq R} |a(x, x, \xi)|.$$

5.2 Operators with almost periodic symbols

A function a in $C_b(\mathbb{R}^N)$ (= the C^*-algebra of the bounded continuous functions on $\mathbb{R}^N$) is called *almost periodic* if the set $\{V_r a : r \in \mathbb{R}^N\}$ of all shifts of a is relatively compact in $C_b(\mathbb{R}^N)$, i.e. if every sequence in this set has a norm convergent subsequence. Here, $V_r a$ stands for the function $x \mapsto a(x - r)$. The class of all almost periodic functions will be denoted by $AP(\mathbb{R}^N)$. Note that $AP(\mathbb{R}^N)$ is a C^*-algebra with respect to the supremum norm. Nice references to this class are still [14, 15].

We set $AP^\infty(\mathbb{R}^N) := AP(\mathbb{R}^N) \cap C_b^\infty(\mathbb{R}^N)$ and denote by $\mathfrak{A}^0_{0,0}$ the closure in $S^0_{0,0}$ of the class of all functions of the form

$$a(x, \xi) = \sum_{j=1}^{J} c_j(x) b_j(\xi) \tag{38}$$

where $J \in \mathbb{N}$, $c_j \in AP^\infty(\mathbb{R}^N)$ and $b_j \in SO^0_{0,0}$. Pseudodifferential operators with symbols in this class possess limit operators with respect to the shifts V_k where the convergence is in the operator norm.

Proposition 5.5 *Let $A \in OP\mathfrak{A}^0_{0,0}$. Then each sequence $h : \mathbb{N} \to \mathbb{Z}^N$ which tends to infinity has a subsequence g such that there exists an operator $A_g \in OP\mathfrak{A}^0_{0,0}$ with*

$$\lim_{m\to\infty} \|V_{-g(m)}AV_{g(m)} - A_g\| = 0.$$

Proof. To start with, let $A = Op(a)$ where $a \in \mathfrak{A}^0_{0,0}$ is a symbol of the form (38), and let $h \in \mathcal{H}$. Since the functions c_k are almost periodic (and by a simple Cantor diagonal argument), there are a subsequence g of h as well as functions $c_{jg} \in AP(\mathbb{R}^N)$ such that

$$\lim_{m\to\infty} \sup_{x\in\mathbb{R}^N} |c_j(x + g(m)) - c_{jg}(x)| = 0 \tag{39}$$

for $1 \le j \le J$. Applying the inequality

$$\sup_{\mathbb{R}^N} \sum_{|\alpha|=1} |\partial^\alpha a(x)| \le C \left(\sup_{x\in\mathbb{R}^N} |a(x)| \left(\sup_{x\in\mathbb{R}^N} |a(x)| + \sup_{x\in\mathbb{R}^N} \sum_{|\alpha|=2} |\partial^\alpha a(x)| \right) \right)^{1/2}$$

(see, for instance, [22], p. 22), one obtains that the sequence of the shifted functions $V_{g(m)}c_j$ converges to c_{jg} in the topology of $C_b^\infty(\mathbb{R}^N)$, which implies that $c_{jg} \in AP^\infty(\mathbb{R}^N)$. Now set

$$A_g := Op(a_g) \quad \text{with} \quad a_g(x, \xi) := \sum_{j=1}^J c_{jg}(x)b_j(\xi).$$

Then it follows from (39) that indeed

$$\lim_{m\to\infty} \|V_{-g(m)}AV_{g(m)} - A_g\| = 0.$$

This settles the assertion for operators $A = Op(a)$ where a is of the form (38). The general case follows straightforwardly by a Cantor diagonalization procedure and standard continuity arguments. ■

One can also easily check that $A_g \in OP\mathfrak{A}^0_{0,0}$ again and that A_g is a limit operator of A defined by the sequence $h : m \mapsto (g(m), 0) \in \mathbb{Z}^N \times \mathbb{Z}^N$ and with respect to the shift operators $U_{h(m)}$ (cf. Section 3.3).

Theorem 5.6 *Let $A \in OP\mathfrak{A}^0_{0,0}$. Then the following assertions are equivalent:*

(a) A is a Fredholm operator.
(b) All limit operators of A are invertible.
(c) At least one limit operator of A is invertible.
(d) A is an invertible operator.

Proof. If A is Fredholm, then all limit operators of A are invertible. Let, conversely, A_h be an invertible limit operator of A. By Proposition 5.5, there is a subsequence g of h such that

$$\lim_{m\to\infty} \|V_{-g(m)} A V_{g(m)} - A_g\| = 0.$$

Then $A_h = A_g$ and, since the invertible operators form an open subset of $L(E)$, the operators $V_{-g(m)} A V_{g(m)}$ must be invertible for all sufficiently large m. Hence, A is invertible.

Similarly, if A is compact, then all limit operators of A are zero. Conversely, if 0 is a limit operator of A, then (again by Proposition 5.5) there is a subsequence g of h such that $\|V_{-g(m)} A V_{g(m)}\| \to 0$. Since the operators V_k are isometries, A must be the zero operator. ∎

Corollary 5.7 *The smallest C^*-subalgebra of $L(L^2(\mathbb{R}^N))$ which contains $OP\mathfrak{A}^0_{0,0}$ does not contain nonzero compact operators.*

We are now going to sketch briefly how these results specialize to symbols in a subclass of $\mathfrak{A}^0_{0,0}$, in which case the Fredholmness of the operator together with its uniform ellipticity and a certain index condition yields the invertibility of the operator.

We say that the function $a \in S^0_{0,0}$ belongs to $S^0_{1,0}$ if

$$|a|_l := \sum_{|\alpha|+|\beta|\le l} \sup_{(x,\,\xi)\in\mathbb{R}^N\times\mathbb{R}^N} |\partial_x^\beta \partial_\xi^\alpha a(x,\,\xi)| \langle \xi \rangle^{|\alpha|} < \infty$$

for all non-negative integers l. The semi-norms $|.|_l$ define the topology of $S^0_{1,0}$. Further, we consider the class $\mathfrak{A}^0_{1,0}$ which is the closure in $S^0_{1,0}$ of the set of all symbols of form (38) where the c_j satisfy the estimates

$$|\partial^\alpha c_j(\xi)| \le C_{\alpha,\,k} \langle \xi \rangle^{-|\alpha|}$$

for all multi-indices α. Finally, an operator $Op(a) \in OP\mathfrak{A}^0_{1,0}$ is called *uniformly elliptic* if

$$\lim_{R\to\infty} \inf_{x,\,\xi\in\mathbb{R}^N,\,|\xi|>R} |a(x,\,\xi)| > 0. \tag{40}$$

It is easy to see that an operator $Op(a) \in OP\mathfrak{A}^0_{1,0}$ is uniformly elliptic if and only if all limit operators of A defined by sequences $(g_1,\, g_2) : \mathbb{N} \to \mathbb{Z}^N \times \mathbb{Z}^N$ with $g_2 \to \infty$ are invertible. Thus, the uniform ellipticity is a *necessary* condition for the invertibility of $Op(a)$. An analogous result holds for almost periodic operators with matrix valued symbols, where one has to replace the value $a(x,\,\xi)$ in (40) by $\det a(x,\,\xi)$.

Let now $A \in OP\mathfrak{A}^0_{1,0}$ be a uniformly elliptic operator with $M \times M$-matrix-valued coefficients. Then the difference between its Fredholmness and its invertibility is measured by its *almost periodic index* $\kappa(A)$. This index has been introduced in [5] (see also [8]) by means of Breuer's Fredholm theory for II_∞ factors. In distinction to the usual (Fredholm) index, $\kappa(A)$ can be an arbitrary real number. We will not go into the details and restrict ourselves to rephrasing a few basic properties:

- If $A,\, B \in OP\mathfrak{A}^0_{1,0}$ are uniformly elliptic operators, then $\kappa(AB) = \kappa(A)\kappa(B)$.
- The almost periodic index is stable in the following sense. Given a uniformly elliptic operator $Op(a) \in OP\mathfrak{A}^0_{1,0}$, there exists an $\varepsilon > 0$ such that $\kappa(Op(b)) = \kappa(Op(a))$ for all operators $Op(b) \in OP\mathfrak{A}^0_{1,0}$ with
$$\lim_{R\to\infty} \sup_{x,\,\xi\in\mathbb{R}^N,\,|\xi|>R} \|a(x,\,\xi) - b(x,\,\xi)\|_{L(\mathbb{C}^M)} < \varepsilon.$$
- If $A \in OP\mathfrak{A}^0_{1,0}$ is invertible, then $\kappa(A) = 0$.
- Let $A \in OP\mathfrak{A}^0_{1,0}$ be uniformly elliptic and $\kappa(A) = 0$. Then A is invertible if and only if
$$\nu(A) := \inf_{\|\varphi\|\le 1} \|A\varphi\| > 0.$$
- Let $A \in OP\mathfrak{A}^0_{1,0}$ be a scalar uniformly elliptic operator, and let $N > 1$. Then $\kappa(A) = 0$. Thus, for such operators, the condition $\nu(A) > 0$ is necessary and sufficient for invertibility of A.

The condition $\nu(A) > 0$ is satisfied if and only if the operator A has a trivial kernel and a closed range, which holds, for example, if A is Fredholm. Hence, if $A \in OP\mathfrak{A}^0_{1,0}$ is a scalar uniformly elliptic and Fredholm operator with $\kappa(A) = 0$, then A is invertible.

5.3 Operators with semi-almost periodic symbols

The class $\mathfrak{B}^0_{1,0}$ of the *semi-almost periodic symbols with respect to* x is defined as the closure in the topology of $S^0_{1,0}$ of the set of all functions of the form
$$a(x,\,\xi) = \sum_{j=1}^{J} c_j(x) b_j(x,\,\xi)$$
where $J \in \mathbb{N}$, $c_j \in AP^\infty(\mathbb{R}^N)$ and $b_j \in SO^0_{1,0} := SO^0_{0,0} \cap S^0_{1,0}$.

Theorem 5.8 *Let $N > 1$, and let $a \in \mathfrak{B}^0_{1,0}$. Then the operator $A := Op(a)$ is a Fredholm operator if and only if the following conditions are satisfied:*
(a) A is uniformly elliptic, that is
$$\lim_{R\to\infty} \inf_{x,\,\xi\in\mathbb{R}^N,\,|\xi|>R} |a(x,\,\xi)| > 0.$$
(b) For each limit operator A_g of A which is defined by a sequence $g = (g_1,\, g_2) : \mathbb{N} \to \mathbb{Z}^N \times \mathbb{Z}^N$ with $g_2 \to \infty$, one has $\nu(A_g) > 0$.

Proof. Let conditions (a) and (b) be satisfied. In the same way as in the proof of Theorem 5.2, we obtain that condition (a) implies the invertibility of all limit operators of A which correspond to sequences $g = (g_1, g_2)$ with $g_2 \to \infty$. Let now $g = (g_1, g_2)$ be a sequence with $g_1 \to \infty$ for which the limit operator A_g exists. Then, by the definition of the class $\mathfrak{B}^0_{1,0}$, this limit operator belongs to $OP\mathfrak{A}^0_{1,0}$ and, due to condition (a), the operator A_g is uniformly elliptic with $\kappa(A_g) = 0$ (since A is an operator with scalar-valued symbol). It follows from the last remark in the preceding subsection that A_g is invertible if the lower norm $\nu(A_g)$ is positive. Thus, conditions (a) and (b) provide us with the invertibility of all limit operators of A. By Theorem 4.15, A is a Fredholm operator.

Let, conversely, A be a Fredholm operator. Then, by Theorem 4.15 again, all limit operators of A are invertible. The invertibility of all limit operators with respect to sequences $g = (g_1, g_2)$ with $g_2 \to \infty$ yields the uniform ellipticity of A, that is condition (a), whereas the invertibility of all limit operators corresponding to sequences $g = (g_1, g_2)$ with $g_1 \to \infty$ evidently implies condition (b). ■

5.4 Pseudodifferential operators of nonzero order

Let $a \in S^m_{0,0}$. Then the pseudodifferential operator $A := Op(a)$ acts as a linear bounded operator from $H^{s+m}(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$ for every $s \in \mathbb{R}$ (which is a simple consequence of the Calderon-Vaillancourt theorem). We are going to study the Fredholm properties of that operator by reducing it in a standard way to a pseudodifferential operator acting on $H^0(\mathbb{R}^N) = L^2(\mathbb{R}^N)$. For, let $\langle D\rangle^r$ refer to the pseudodifferential operator with symbol $(x, \xi) \mapsto (1 + |\xi|_2^2)^{r/2}$. The operator $\langle D\rangle^r$ is an isometry from $H^{s+r}(\mathbb{R}^N)$ onto $H^s(\mathbb{R}^N)$ for each real s. Thus,

$$A : H^{s+m}(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$$

is a Fredholm operator if and only if

$$B := \langle D\rangle^s A \langle D\rangle^{-s-m} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

is a Fredholm operator. The operator B is a pseudodifferential operator in the class to $OPS^0_{0,0}$. Hence, Theorem 4.15 implies the following.

Theorem 5.9 *Let $a \in S^m_{0,0}$. Then the operator $A = Op(a) : H^{s+m}(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$ is Fredholm operator if and only if all limit operators of the operator $B := \langle D\rangle^s A \langle D\rangle^{-s-m} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ are invertible. In particular,*

$$\sigma_{ess}(A) = \cup_{B_h \in \sigma_{op}(B)} \sigma(B_h).$$

These conditions can be made more explicit for symbols which are slowly oscillating in the following sense. We say that the function a is in the class $SO^m_{0,0}$ with $m \in \mathbb{N}$ if the function $(x, \xi) \mapsto a(x, \xi)\langle\xi\rangle^{-m}$ belongs to $SO^0_{0,0}$. Analogously, the double symbol a is said to be in $SO^m_{0,0,0}$ if the function $(x, y, \xi) \mapsto a(x, y, \xi)\langle\xi\rangle^{-m}$ belongs to $SO^0_{0,0,0}$.

Proposition 5.10 *(a) Let* $A := Op(a) \in OPSO^{m_1}_{0,0}$ *and* $B := Op(b) \in OPSO^{m_2}_{0,0}$*. Then* $AB \in OPSO^{m_1+m_2}_{0,0}$*, and the symbol of* AB *is of the form* $\mathrm{sym}_{AB} = ab + t$ *with* t *satisfying*

$$\lim_{(x,\xi)\to\infty} t(x,\, \xi)\langle\xi\rangle^{-m_1-m_2} = 0. \tag{41}$$

(b) Let $A := Op_d(a) \in OPSO^m_{0,0,0}$*. Then* $A \in OPS^m_{0,0}$*, and the formal symbol of that operator is given by* $\mathrm{sym}_A(x,\, \xi) := a(x,\, x,\, \xi) + t(x,\, \xi)$ *where* t *is such that*

$$\lim_{(x,\xi)\to\infty} t(x,\, \xi)\langle\xi\rangle^{-m} = 0.$$

Proof. (a) By Theorem 4.2.1 in [16], the operator AB belongs to $OPSO^{m_1+m_2}_{0,0}$, and its symbol is given by

$$\mathrm{sym}_{AB}(x,\, \xi) = os\, (2\pi)^{-N} \int\int_{\mathbb{R}^N} a(x,\, \xi+\eta) b(x+y,\, \xi) e^{-i\langle y,\, \eta\rangle} dy\, d\eta.$$

By Lagrange's formula, we have

$$a(x,\, \xi+\eta) = a(x,\, \xi) + \sum_{j=1}^N \eta_j \int_0^1 \partial_{\xi_j} a(x,\, \xi+\theta\eta)\, d\theta,$$

whence via Corollary 2.2.2 in [16],

$$\mathrm{sym}_{AB}(x,\, \xi) = a(x,\, \xi)b(x,\, \xi) + t(x,\, \xi),$$

with

$$t(x,\, \xi) = \sum_{j=1}^N \int_0^1 L_j(x,\, \xi,\, \theta)\, d\theta$$

and

$$\begin{aligned} & L_j(x,\, \xi,\, \theta) \\ &\quad = os\, (2\pi)^{-N} \int\int_{\mathbb{R}^N} \partial_{\xi_j} a(x,\, \xi+\theta\eta)(-i\partial_{x_j}) b(x+y,\, \xi) e^{-i\langle y,\, \eta\rangle} dy\, d\eta \\ &\quad = os\, (2\pi)^{-N} \int\int_{\mathbb{R}^N} \langle\eta\rangle^{-2k_2} \langle D_y\rangle^{2k_2} \times \\ &\qquad \times \left\{ \langle y\rangle^{-2k_1} \langle D_\eta\rangle^{2k_1} \partial_{\xi_j} a(x,\, \xi+\theta\eta)(-i\partial_{x_j}) b(x+y,\, \xi)) \right\} e^{-i\langle y,\, \xi\rangle} dy\, d\eta \end{aligned}$$

for all k_1, k_2 with $2k_1 > N$ and $2k_2 > N + |m_1|$. Taking into account the elementary inequality

$$\langle\xi+\eta\rangle^l \le 2^{l/2}\langle\eta\rangle^{|l|}\langle\xi\rangle^l \quad \text{for } l \in \mathbb{R},$$

we obtain

$$L_j(x,\, \xi,\, \theta) \le C\langle\xi\rangle^{m_1+m_2} K_j(x,\, \xi,\, \theta)$$

where

$$K_j(x,\,\xi,\,\theta)$$
$$= \; os\,(2\pi)^{-N}\int\int_{\mathbb{R}^N}\langle\xi+\theta\eta\rangle^{-m_1}\langle\eta\rangle^{-2k_2+|m_1|}\times$$
$$\times\left|\langle D_y\rangle^{2k_2}\langle y\rangle^{-2k_1}\langle D_\eta\rangle^{2k_1}\partial_{\xi_j}a(x,\,\xi+\theta\eta)(-i\partial_{x_j})b(x+y,\,\xi)\langle\xi\rangle^{-m_2})\right|\,dy\,d\eta.$$

The latter integral converges uniformly with respect to $x,\,\xi\in\mathbb{R}^N$ and $\theta\in[0,\,1]$. Hence, we can pass to the limit as $(x,\,\xi)\to\infty$ under this integral, which yields

$$\lim_{(x,\,\xi)\to\infty}\sup_{\theta\in[0,\,1]}K_j(x,\,\xi,\,\theta)=0.$$

This implies (41). Assertion (b) can be checked in the same way. ■

A consequence of this proposition is that, if $A=Op(a)\in OPSO^0_{0,0}$, then

$$B:=\langle D\rangle^sA\langle D\rangle^{-(s+m)}=Op(a_m)+Op(t)$$

where
$$a_m(x,\,\xi):=a(x,\,\xi)\langle\xi\rangle^{-m}\quad\text{and}\quad\lim_{(x,\,\xi)\to\infty}t(x,\,\xi)=0.$$

Thus, all limit operators B_g of B depend on the main part a_m of the symbol of B only. Moreover, these limit operators are pseudodifferential operators $B_g=Op(b_g)$ which are invariant with respect to shifts (i.e. their symbols b_g depend on ξ only), or they are operators of multiplication (i.e. their symbols are only dependent on x). So we arrive at the following theorem.

Theorem 5.11 *(a) Consider the operator $A=Op(a)\in OPSO^m_{0,0}$ as acting from $H^{s+m}(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$. Then all limit operators of $B:=\langle D\rangle^sA\langle D\rangle^{-(s+m)}:L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$ invertible if and only if*

$$\lim_{R\to\infty}\inf_{|x|+|\xi|\geq R}|a(x,\,\xi)|\langle\xi\rangle^{-m}>0. \tag{42}$$

The condition (42) is necessary and sufficient for the Fredholmness of A.
(b) Consider the operator $A=Op_d(a)\in OPSO^m_{0,0,0}$ as acting from $H^{s+m}(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$. Then all limit operators of $B:=\langle D\rangle^sA\langle D\rangle^{-(s+m)}:L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$ are invertible if and only if

$$\lim_{R\to\infty}\inf_{|x|+|\xi|\geq R}|a(x,\,x,\,\xi)|\langle\xi\rangle^{-m}>0. \tag{43}$$

Condition (43) is necessary and sufficient for the Fredholmness of A.

5.5 Differential operators

The results of the previous section apply to study the Fredholmness of differential operators on $\mathbb{R}^N$ by means of their limit operators. Let

$$P = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

be a differential operator of order m with coefficients $a_\alpha \in C_b^\infty(\mathbb{R}^N)$. We consider this operator as acting from $H^{s+m}(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$. The function

$$p_m : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \quad (x, \xi) \mapsto \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

is called the *main symbol* of P, and the operator P is called *uniformly elliptic* if

$$\inf_{x \in \mathbb{R}^N} |p_m(x, \omega)| > 0 \quad \text{for all } \omega \in S^{N-1}.$$

Let $h : \mathbb{N} \to \mathbb{Z}^N$ be a sequence which tends to infinity. Then there exist a subsequence g of h and functions $a_\alpha^g \in C_b^\infty(\mathbb{R}^N)$ such that the functions $x \mapsto a_\alpha(x + g(k))$ converge to a_α^g in the topology of $C_b^\infty(\mathbb{R}^N)$ for every α. We set

$$P_g := \sum_{|\alpha| \le m} a_\alpha^g D^\alpha,$$

consider P_g as an operator from $H^{s+m}(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$ again, and denote by $\sigma_{op}^1(P)$ the set of all operators which arise in this way.

Theorem 5.12 *The differential operator $P : H^{s+m}(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$ is Fredholm if and only if the following conditions are satisfied:*
(a) All operators $P_g \in \sigma_{op}^1(P)$ are invertible.
(b) The operator P is uniformly elliptic.

Proof. It follows from Theorem 5.9 that P is a Fredholm operator if and only if all limit operators of $\langle D \rangle^s P \langle D \rangle^{-s-m}$ are invertible on $L^2(\mathbb{R}^N)$.

Let $h = (h_1, h_2) : \mathbb{N} \to \mathbb{Z}^N \times \mathbb{Z}^N$ be a sequence such that $h_1 \to \infty$ but h_2 is bounded. Then there exists a subsequence $g = (g_1, g_2)$ of h such that, for every α, the functions $x \mapsto a_\alpha(x + g_1(k))$ converge to certain functions $a_\alpha^{g_1}$ in the topology of $C_b^\infty(\mathbb{R}^N)$ and that the sequence g_2 is constant, say $g_2(k) = \gamma_2 \in \mathbb{Z}^N$ for all k. In this case, it is easy to see that

$$\text{s-lim}_{k \to \infty} U_{g(k)}^* \langle D \rangle^s P \langle D \rangle^{-s-m} U_{g(k)} = E_{\gamma_2}^* \langle D \rangle^s P_{g_1} \langle D \rangle^{-s-m} E_{\gamma_2}$$

with $(E_\gamma u)(x) := e^{i\langle \gamma, x \rangle} u(x)$. Thus, the limit operators of $\langle D \rangle^s P \langle D \rangle^{-s-m}$ which are defined be sequences of this kind are invertible if and only if condition (a) holds.

Now consider limit operators of $\langle D\rangle^s P\langle D\rangle^{-s-m}$ which are defined by sequences $g = (g_1, g_2)$ such that $g_2 \to \infty$ and g_1 is constant, say $g_1(k) = \gamma_1 \in \mathbb{Z}^N$. Suppose for definiteness that g_2 tends to infinity into the direction of the infinitely distant point $\omega \in S^{N-1}$. Then

$$\text{s-lim}_{k\to\infty} E^*_{g_2(k)}\langle D\rangle^s P\langle D\rangle^{-s-m} E_{g_2(k)} = p_m(\,.,\, \omega)I$$

whence

$$\text{s-lim}_{k\to\infty} U^*_{g(k)}\langle D\rangle^s P\langle D\rangle^{-s-m} U_{g(k)} = \sum_{|\alpha|=m} a(\,.\, - \gamma_1)\omega^\alpha I.$$

Hence, all limit operators defined by these sequences are operators of multiplication by the functions

$$p_{m,\,g} : (x,\, \omega) \mapsto \sum_{|\alpha|=m} a(.\, - \gamma_1)\omega^\alpha.$$

Finally, if both g_1 and g_2 go to infinity, and if g_1 and g_2 are chosen such that the functions $x \mapsto a_\alpha(x + g_1(k))$ converge to certain functions $a_\alpha^{g_1}$ in the topology of $C^\infty(\mathbb{R}^N)$ and that g_2 tends to infinity into the direction of the infinitely distant point $\omega \in S^{N-1}$, then

$$\text{s-lim}_{k\to\infty} U^*_{g(k)}\langle D\rangle^s P\langle D\rangle^{-s-m} U_{g(k)} = \sum_{|\alpha|=m} a_\alpha^{g_1}\omega^\alpha I.$$

Thus, we get multiplication operators again, this time by the functions

$$p_{m,\,g} : (x,\, \omega) \mapsto \sum_{|\alpha|=m} a_\alpha^{g_1}(x)\omega^\alpha.$$

Evidently, if the operator is uniformly elliptic, then in all cases

$$\inf_{x\in\mathbb{R}^N} |p_{m,\,g}(x,\, \omega)| > 0.$$

Hence, the limit operators $Op(p_{m,\,g})I$ are invertible on $L^2(\mathbb{R}^N)$, and condition (b) implies the invertibility of all limit operators defined by sequences g with $g_2 \to \infty$. Conversely, choosing sequences $g = (g_1,\, g_2)$ with $g_1(k) = 0$ for all k and with g_2 tending to infinity into the direction of $\omega \in S^{N-1}$, we obtain that the invertibility of all associated limit operators implies condition (b). ∎

Corollary 5.13 *Let $P : H^m(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be a uniformly elliptic differential operator of order m. Then*

$$\sigma_{ess}(P) = \cup_{P_g\in\sigma^1_{op}(P)}\sigma(P_g).$$

Proof. By Theorem 5.9, the operator $P - \lambda I : H^m(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is Fredholm if and only if all limit operators

$$P_g - \lambda I : H^m(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \qquad P_g \in \sigma^1_{op}(P),$$

are invertible and if $P - \lambda I$ is uniformly elliptic. Since the uniform ellipticity of a differential operator depends on its main symbol only, the uniform ellipticity of $P - \lambda I$ follows from the conditions of the corollary. ∎

We denote by $SO^\infty(\mathbb{R}^N)$ the class of the smooth slowly oscillating functions on $\mathbb{R}^N$, that is the class of all functions $a \in C_b^\infty(\mathbb{R}^N)$ with

$$\lim_{x\to\infty} \partial_{x_j} a(x) = 0 \quad \text{for all } j = 1, \ldots, N.$$

Let the coefficients a_α of the differential operator P belong to $SO^\infty(\mathbb{R}^N)$. Then all limit operators $P_g \in \sigma^1_{op}(P)$ are of the form

$$P_g = Op(p_g) = \sum_{|\alpha|\le m} a_\alpha^{g_1} D^\alpha$$

with *constant* coefficients $a_\alpha^{g_1}$. The operator P_g is invertible if and only if

$$\inf_{\xi\in\mathbb{R}^N} |p_g(\xi)| \langle\xi\rangle^{-m} = \inf_{\xi\in\mathbb{R}^N} \left| \sum_{|\alpha|\le m} a_\alpha^{g_1} \xi^\alpha \right| \langle\xi\rangle^{-m} > 0.$$

Hence, if P is a differential operator with smooth slowly oscillating coefficients, then

$$\sigma_{ess}(P) = \bigcup_{P_g\in\sigma^1_{op}(P)} \overline{\{p_g(\xi) : \xi \in \mathbb{R}^N\}}.$$

Remark. A differential operator P of order m can be considered as an unbounded operator on the Hilbert space $L^2(\mathbb{R}^N)$ with domain $H^m(\mathbb{R}^N)$. If P is uniformly elliptic, then P is a closed operator. An unbounded operator P is called a *Fredholm operator* if its range is closed in $L^2(\mathbb{R}^N)$ and if $\ker A$ and $\ker A^*$ are finite dimensional spaces, and the *essential spectrum* $\sigma_{ess}(A)$ *of* A consists of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not a Fredholm operator.

It is well known that, if P is uniformly elliptic, then P is a Fredholm operator in this sense (i.e. as an unbounded operator) if and only if $P : H^m(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a Fredholm operator in the common sense (i.e. as a bounded operator). Hence, if P is a uniformly elliptic differential operator, then

$$\sigma_{ess}(P) = \cup_{P_g\in\sigma^1_{op}(P)} \sigma(P_g),$$

where now both the essential spectrum on the left hand side and the spectra on the right hand side are understood in the unbounded operator sense. ∎

5.6 Schrödinger operators

Here we are going to specialize the results of the previous section to operators of the form

$$\mathbf{H} = \sum_{l,m=1}^{N} (i\partial_{x_l} + a_l I) g^{lm} (i\partial_{x_m} + a_m I) + wI$$

where g^{lm}, a_l and w are real-valued functions in $C_b^\infty(\mathbb{R}^N)$. This operator can be viewed of as the electro-magnetic Schrödinger operator on the Riemann space $\mathbb{R}^N$ provided with the metric tensor $(g_{lm})_{l,m=1}^N$ which is the tensor inverse of $(g^{lm})_{l,m=1}^N$. Schrödinger operators of this form arise in multi-particle problems after separating the mass center of the system (see, for instance, [6], pp. 29 – 33 and [11], pp. 172 – 176). Throughout this section, we will suppose that

$$\inf_{x\in\mathbb{R}^N,\, \eta\in S^{N-1}} \sum_{l,m=1}^{N} g_{lm}(x)\eta_l\eta_m > 0.$$

Let $h : \mathbb{N} \to \mathbb{Z}^N$ be a sequence which tends to infinity. Then there exists a subsequence k of h such that the functions

$$x \mapsto g^{lm}(x + k(n)), \quad x \mapsto a_l(x + k(n)) \quad \text{and} \quad x \mapsto w(x + k(n))$$

converge in the topology of $C_b^\infty(\mathbb{R}^N)$ to certain functions g_k^{lm}, a_l^k and w^k, respectively. In particular, these limit functions belong to $C_b^\infty(\mathbb{R}^N)$ again. If k is chosen in this way, then the limit operator $\mathbf{H}_k$ of $\mathbf{H}$ with respect to k exists, and

$$\mathbf{H}_k = \sum_{l,m=1}^{N} (i\partial_{x_l} + a_l^k I) g_k^{lm} (i\partial_{x_m} + a_m^k I) + w^k I.$$

We consider $\mathbf{H}$ as an unbounded operator on $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$. Note that $\lambda \in \mathbb{C}$ is a point in the discrete spectrum of the unbounded operator $\mathbf{H}$ if and only if λ belongs to the discrete spectrum of the bounded operator $\mathbf{H} : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. Hence, the essential spectrum of $\mathbf{H}$, considered as an unbounded operator, coincides with the essential spectrum of the bounded operator $\mathbf{H} : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. With Corollary 5.13, we find

$$\sigma_{ess}(\mathbf{H}) = \bigcup_{\mathbf{H}_k \in \sigma_{op}^1(\mathbf{H})} \sigma(\mathbf{H}_k). \tag{44}$$

Here are a few instances where the structure of the limit operators is sufficiently simple such that their invertibility can be effectively checked.

Example A. Let the functions g^{lm}, a_l and w be in $SO^\infty(\mathbb{R}^N)$. Then each limit operator of $\mathbf{H}$ is a differential operator with constant coefficients, i.e.

$$\mathbf{H}_k = \sum_{l,m=1}^{N} (i\partial_{x_l} + a_l^k I) g_k^{lm} (i\partial_{x_m} + a_m^k I) + w^k I.$$

with real numbers g_k^{lm}, a_l^k and w^k. Set $a^k := (a_1^k, \ldots, a_N^k)$ and $(E_\alpha u)(x) := e^{i\langle \alpha, x\rangle} u(x)$ for $\alpha \in \mathbb{R}^N$. Then

$$E_{a^k} \mathbf{H}_k E_{a^k}^{-1} = - \sum_{l,m=1}^{N} g_k^{lm} \partial_{x_l} \partial_{x_m} + w^k I.$$

Thus,

$$\sigma(\mathbf{H}_k) = \left\{ \sum_{l,m=1}^{N} g_k^{lm} \xi_l \xi_m + w^k : (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \right\} = [w^k, +\infty],$$

and the essential spectrum of $\mathbf{H}$ is

$$\sigma_{ess}(\mathbf{H}) = \bigcup [w^k, +\infty] = [m_w, +\infty]$$

where $m_w := \inf w^k = \liminf_{x \in \mathbb{R}^N} w(x)$. ∎

Example B. We let v_1, v_2 and v_{12} be C^∞-functions on $\mathbb{R}^3$ with

$$\lim_{y\to\infty} v_1(y) = \lim_{y\to\infty} v_2(y) = \lim_{y\to\infty} v_{12}(y) = 0,$$

define functions w_1, w_2, w_{12} on $\mathbb{R}^3 \times \mathbb{R}^3$ by

$$w_1(x) := v_1(x^{(1)}), \quad w_2(x) := v_2(x^{(2)}), \quad w_{12}(x) := v_{12}(x^{(1)} - x^{(2)})$$

where $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^3 \times \mathbb{R}^3$, and consider the Hamiltonian on $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$\mathbf{H} := -\Delta_{x^{(1)}} - \Delta_{x^{(2)}} - w_1 I - w_2 I - w_{12} I.$$

Hamiltonians of this special structure arise in nuclear physics (but, usually, with non-smooth functions v_1, v_2 and v_{12}, which moreover will have singularities at 0; see, for instance, [7], p. 163, and [6], p. 29).

We will describe the essential spectrum of $\mathbf{H}$ by means of its limit operators. Let the sequence $h := (h_1, h_2) : \mathbb{N} \to \mathbb{Z}^3 \times \mathbb{Z}^3$ tend to infinity. After passing to suitable subsequences of h, if necessary, we have to distinguish between four cases.

[A] We have $h_1 \to \infty$, and h_2 is a constant sequence, say $h_2(k) = \gamma_2 \in \mathbb{Z}^N$ for all k. Then the limit operator of $\mathbf{H}$ with respect to h exists, and

$$(\mathbf{H}_h u)(x) = -(\Delta_{x^{(1)}} u)(x) - (\Delta_{x^{(2)}} u)(x) - w_2(x^{(2)} + \gamma_2) u(x).$$

The operator $\mathbf{H}_h$ is unitarily equivalent to the operator

$$\mathbf{H}^1 := -\Delta_{x^{(1)}} - \Delta_{x^{(2)}} - w_2 I.$$

[B] If $h_2 \to \infty$, and if $h_1(k) = \gamma_1 \in \mathbb{Z}^N$ for all k, then the limit operator of $\mathbf{H}$ with respect to h exists, and it is unitarily equivalent to the operator

$$\mathbf{H}^2 := -\Delta_{x^{(1)}} - \Delta_{x^{(2)}} - w_1 I.$$

[C] If both h_1 and h_2 tend to infinity, and if also $h_1 - h_2 \to \infty$, then the limit operator of $\mathbf{H}$ is equal to the Laplacian

$$\mathbf{H}^3 := -\Delta_{x^{(1)}} - \Delta_{x^{(2)}}.$$

[D] If, finally, h_1 and h_2 tend to infinity, and if the difference $h_1 - h_2$ is a constant sequence, then the limit operator of $\mathbf{H}$ with respect to h exists, and it is unitarily equivalent to the operator

$$\mathbf{H}^4 := -\Delta_{x^{(1)}} - \Delta_{x^{(2)}} - w_{12} I.$$

Let $j = 1, 2$. Applying the Fourier transform with respect to $x^{(j)}$, we obtain that the operator $\mathbf{H}^j$ is unitarily equivalent to the operator of multiplication by the operator-valued function

$$\widehat{\mathbf{H}}^j : \mathbb{R}^3 \to L(L^2(\mathbb{R}^3 \times \mathbb{R}^3)), \quad \xi \mapsto |\xi|^2 - \Delta_{x^{(3-j)}} - w_{3-j} I.$$

It is well-known that the essential spectrum of the operator $A_j := -\Delta_{x^{(3-j)}} - w_{3-j} I$ is the interval $[0, \infty)$ and that its discrete spectrum consists of finitely many points in $(-\infty, 0)$. Let $\lambda_{\min}^{(j)} < 0$ be the minimal eigenvalue of A_j. Then, since $|\xi|^2$ varies over $[0, \infty)$, the spectrum of $\mathbf{H}^j$ is the interval $[\lambda_{\min}^{(j)}, \infty)$.

Now consider the operator $\mathbf{H}^4$. After a change of variables

$$y^{(1)} := x^{(1)} + x^{(2)}, \quad y^{(2)} := x^{(1)} - x^{(2)},$$

the operator $\mathbf{H}^4$ becomes

$$-2(\Delta_{y^{(1)}} + \Delta_{y^{(2)}}) - \hat{w}_{12} I$$

with $\hat{w}_{12}(y) := v_{12}(y^{(2)})$. The spectrum of this operator is the interval $[\lambda_{\min}^{(12)}, \infty)$ where $\lambda_{\min}^{(12)} < 0$ is the minimal eigenvalue of $-2\Delta_{y^{(2)}} - \hat{w}_{12} I$.

Summarizing, we get

$$\sigma_{ess}(\mathbf{H}) = [\lambda_{\min}, \infty) \quad \text{where} \quad \lambda_{\min} := \min\{\lambda_{\min}^{(1)}, \lambda_{\min}^{(2)}, \lambda_{\min}^{(12)}\}.$$

5.7 Partial differential-difference operators

Finally, we consider differential-difference operators of the form

$$P := \sum_{|\alpha|\le m,\, j\le N} a_{\alpha j} D^\alpha V_{\beta_{\alpha j}}$$

where $(V_\beta u)(x) = u(x-\beta)$ for $\beta \in \mathbb{R}^N$ and where the coefficients $a_{\alpha j}$ belong to $SO^\infty(\mathbb{R}^N)$. The operator P is a pseudodifferential operator in the class $OPS^m_{0,0}$ with symbol

$$p(x,\, \xi) := \sum_{|\alpha|\le m,\, j\le N} a_{\alpha j}(x) \xi^\alpha e^{i\langle \beta_{\alpha j},\, \xi\rangle}.$$

Hence, $P : H^m(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a Fredholm operator if and only if all limit operators of the operator $R := P\langle D\rangle^{-m} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ are invertible.

Let $h = (h_1,\, h_2) : \mathbb{N} \to \mathbb{Z}^N \times \mathbb{Z}^N$ be a sequence tending to infinity which defines a limit operator of R. We distinguish between three cases for the sequence h.

[A] Let $h_1 \to \infty$, and let h_2 tend to infinity into the direction of the infinitely distant point $\eta \in S^{N-1}$. Then the limit operator of R is a difference operator with constant coefficients the form

$$R_h := \sum_{|\alpha|=m,\, j\le N} a^h_{\alpha j} \eta^\alpha V_{\beta_{\alpha j}},$$

i.e. with numbers $a^h_{\alpha j} \in \mathbb{C}$. It is evident that R_h is invariant with respect to shifts, and this operator is invertible if and only if

$$\inf_{\xi\in\mathbb{R}^N} \left| \sum_{|\alpha|=m,\, j\le N} a^h_{\alpha j} \eta^\alpha e^{i\langle \beta_{\alpha j},\, \xi\rangle} \right| > 0.$$

[B] Let $h_1 \to \infty$, and let h_2 be a constant sequence. Then the limit operator R_h is unitarily equivalent to the pseudodifferential operator with symbol

$$r_h : \xi \mapsto \sum_{|\alpha|\le m,\, j\le N} a^h_{\alpha j} \frac{\xi^\alpha}{\langle \xi\rangle^m} e^{i\langle \beta_{\alpha j},\, \xi\rangle}.$$

Clearly, this operator is invertible if and only if

$$\inf\{|r_h(\xi)| : \xi \in \mathbb{R}^N\} > 0.$$

[C] Finally, let h_2 tend to infinity into the direction of the infinitely distant point $\eta \in S^{N-1}$, and let h_1 be a constant sequence. Then the limit operator R_h is unitarily equivalent to the difference operator with variable coefficients,

$$\sum_{|\alpha|=m,\, j\le N} a_{\alpha j} \eta^\alpha V_{\beta_{\alpha j}}.$$

Effective sufficient conditions for the invertibility of difference operators with variable coefficients can be found in the monographs [1, 2, 3].

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