The Initial Value Problem for the Navier-Stokes Equations with a Free Surface in L^q-Sobolev Spaces

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Abstract

We prove small-time existence of strong solutions of a free boundary value problem, which describes the motion of an incompressible viscous fluid occupying a semi-infinite domain bounded above by a free surface. This problem was studied by Beale [6] and others in L^2-Sobolev spaces. In contrast to the latter contribution we study solutions in L^q-Sobolev spaces for $q > n$ in space dimension $n \geq 2$. This approach has the advantage that the regularity assumptions can be reduced in comparison to [6]. In order to solve the linearized system, we use the instationary reduced Stokes equations with a mixed boundary conditions and the maximal regularity of the associated reduced Stokes operator.

Key words: Navier Stokes equations, free boundary value problems, maximal regularity
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1 Introduction

In the present contribution we consider the time-dependent flow of an incompressible, viscous fluid bounded above by a free surface and below by a fixed bottom under the influence of gravity; e.g. water in an infinite ocean. The effect of surface tension is neglected. This problem was studied by Beale [6] in the setting of anisotropic L^2-Sobolev spaces in $\mathbb{R}^3$. He proved short time existence of strong solutions for arbitrary large initial data. The large time existence was studied by Sylvester [17] and Tani and Tanaka [19]. Moreover, there are several works considering this problem with the effect of surface taken into account in three and two dimensions, cf. Beale [7], Beale and Nishiada [8], Tani [18], [19], and Allain [4]. As far as the author knows, there are no results in anisotropic L^q-Sobolev spaces for $q \neq 2$.

We show short-time existence of strong solutions of this free boundary value problem in the setting of anisotropic L^q-Sobolev spaces for $q > n$ in space dimension $n \geq 2$. The approach is similar to Solonnikov [16], where the motion of an isolated finite volume of viscous incompressible fluid is considered. It has the advantage that lower regularity of the data is needed since W^1_q is an algebra under pointwise multiplication if $q > n$ in contrast to W^m_2, where $m > \frac{n}{2}$ is needed.

The motion of the fluid is described by the solution of the Navier-Stokes equations

$$
\begin{aligned}
\partial_t v + (v \cdot \nabla) v - \Delta v + \nabla q &= -g_0 e_n && \text{with } x \in \Omega(t), t \in (0,T), && (1.1)\\
\operatorname{div} v &= 0 && \text{with } x \in \Omega(t), t \in (0,T), && (1.2)\\
T_1^+(v,q) &= -P_0 \nu && \text{with } x \in \partial\Omega^+(t), t \in (0,T), && (1.3)\\
v|_{\partial\Omega^-} &= 0 && \text{on } \partial\Omega(0)^- \times (0,T), && (1.4)\\
v|_{t=0} &= v_0 && \text{in } \Omega(0) && (1.5)
\end{aligned}
$$

in a *layer-like domain* $\Omega(t) = \{(x', x_n) \in \mathbb{R}^n : \gamma^-(x',t) < x_n < \gamma^+(x',t)\}$ with $\partial\Omega(t)^- = \partial\Omega(0)^-$, where $\partial\Omega^\pm(t) = \{(x', \gamma^\pm(x',t)) : x' \in \mathbb{R}^{n-1}\}$ and

$$
T_1^+(u,p) = (\nu \cdot Su - p\nu)|_{\partial\Omega^+}, \qquad Su = \nabla u + (\nabla u)^T.
$$

Here v_0 is a given initial velocity, $\Omega(0)$ is the initial domain filled by the fluid, $g_0 > 0$ is the constant due to the acceleration of gravity, e_n the n-th canonical unit vector, ν the exterior normal, and P_0 is the atmospheric pressure which is assumed to be constant. Moreover, the velocity field v and the domain $\Omega(t)$ have to satisfy a kinematic relation: Let $X(\xi,t)$, $t > 0$, be the trajectory of the mass particle, i.e., $X(\xi,t)$ solves

$$
\partial_t X(\xi,t) = v(X(\xi,t),t), \qquad \text{for } t > 0, \qquad X(\xi,0) = \xi.
$$

Then $X(\Omega(0),t) = \Omega(t)$ for $t > 0$.

Before we pass to Lagrangian coordinates, we replace the actual pressure q by the *hydrostatic pressure* $\tilde{q}(x,t) = q(x,t) - P_0 + g_0 x_n$. Then the term $-g_0 e_n$ disappears in (1.1) and (1.3) has to be replaced by

$$
T_1^+(v,\tilde{q}) = -g_0 x_n \nu \qquad \text{on } \partial\Omega(t)^+, t \in (0,T). \tag{1.6}
$$

Now let $u(\xi,t) = v(X(\xi,t),t)$, $p(\xi,t) = \tilde{q}(X(\xi,t),t)$ be the velocity and the pressure of the fluid in Lagrangian coordinates. Then

$$
X(\xi,t) = X_u(\xi,t) := \xi + \int_0^t u(\xi,\tau)\,d\tau
$$

and the system (1.1)-(1.5) is transformed to

$$
\begin{aligned}
\partial_t u - \Delta_u u + \nabla_u p &= 0 && \text{in } \Omega_T, && (1.7)\\
\operatorname{div}_u u &= 0 && \text{in } \Omega_T, && (1.8)\\
T_{1,u}^+(u,q) &= -g_0 X_{u,n} \nu_u && \text{on } \partial\Omega_T^+, && (1.9)\\
u|_{\partial\Omega^-} &= 0 && \text{on } \partial\Omega_T^-, && (1.10)\\
u|_{t=0} &= u_0 && \text{in } \Omega, && (1.11)
\end{aligned}
$$

where, using $A(u) = (D_\xi X_u)^{-T}(\xi, t)$,

$$\begin{aligned} \nabla_u &= A(u)\nabla, & \operatorname{div}_u v &= \nabla_u \cdot v = \operatorname{Tr}(A(u)\nabla v), \\ \Delta_u &= \operatorname{div}_u \nabla_u, & T^+_{1,u}(u,p) &= (\nu_u \cdot (S_u u - p\nu_u)|_{\partial\Omega^+}, \\ S_u v &= \nabla_u v + (\nabla_u v)^T, & \nu_u(\xi, t) &= \frac{A(u)\nu_\xi}{|(A(u)\nu_\xi|}, \end{aligned}$$

ν_ξ denotes the exterior norm at $\xi \in \partial\Omega^+$, and $\Omega = \Omega(0)$.

THEOREM 1.1 *Let $n \geq 2$, $n < q < \infty$, and let $\Omega(0) \subset \mathbb{R}^n$ be an asymptotically flat layer with $C^{1,1}$-boundary, cf. Definition 2.1 below, such that $\gamma^+ \in W_q^{1-\frac{1}{q}}(\mathbb{R}^{n-1})$. Then for every $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^n$ with $\operatorname{div} u_0 = 0$, $(Su_0)_\tau|_{\partial\Omega^+} = 0$ if $q > 3$, and*

$$\int_0^T \int_{\partial\Omega^+} \int_\Omega \frac{|u_0(y)|^q}{(|x' - y|^2 + t)^{1+n/2}} dy d\sigma(x') dt < \infty$$

if $q = 3$, there is a $T > 0$ such that (1.7)-(1.11) have a unique solution $(u, p) \in W_q^{2,1}(\Omega_T)^n \times W_q^{1,0}(\Omega_T)$ with $p|_{\partial\Omega^+} \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\partial\Omega^+)$.

Here τ denotes the tangential components.

The proof is done with the aid of the Banach contraction mapping principle using the unique solvability of the linearized system – the instationary *generalized* Stokes equations. In Section 3, the unique solvability of the latter system is reduced to the instationary reduced Stokes equations similarly to the approach in Grubb and Solonnikov [13] and other contributions by these authors. Then the unique solvability is proved using the maximal regularity of the associated reduced Stokes operator in asymptotically flat layers studied in [2, 3]. The maximal regularity itself is a consequence of the result of Dore and Venni [9] and the existence of a bounded H_∞-calculus, cf. McIntosh [14], of the reduced Stokes operator. Finally, the proof of Theorem 1.1 is given in Section 4.

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2 Preliminaries

2.1 Notation and Layer-Like Domains

For $s \in \mathbb{R}$ we denote by $[s]$ the largest integer $\leq s$, set $\{s\} := s - [s] \in [0, 1)$, and define $[s]_+ = \max\{s, 0\}$.

If $M \subseteq \mathbb{R}^n$ is measurable, $L^q(M)$, $1 \leq q \leq \infty$ denotes the usual Lebesgue-space and $\|.\|_q$ its norm. Moreover, $L^q(M;X)$ denotes its vector-valued variant, where X is a Banach space. If $f \in L^q(M)$, $g \in L^{q'}(M)$, where $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$(f,g)_M := \int_M f(x)\overline{g(x)}dx.$$

The Banach space of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ that are k-times differentiable and have Lipschitz continuous k-th derivatives is denoted by $C^{k,1}(\mathbb{R}^n)$, $k \in \mathbb{N}_0$. Moreover, $C^\theta([0,T];X)$, $\theta \in (0,1)$, $T > 0$, denotes the space of Hölder continuous functions $f\colon [0,T] \to X$.

The dual of a topological vector space V is denoted by V'. If $v \in V$ and $v' \in V'$, then $\langle v, v'\rangle := v'(v)$ denotes the duality product. If $A\colon V \to W$ is a continuous linear operator, $A'\colon W' \to V'$ denotes its adjoint.

Finally, if $x \in \mathbb{R}^n$, $n \geq 2$, then we use the decomposition $x = (x', x_n)$, where x' denotes the first $n-1$ components.

2.2 Function spaces on Layer-Like Domains

Let $\Omega \subset \mathbb{R}^n$ be a domain. In the following $W^s_q(\Omega)$, $s \geq 0$, $1 \leq q < \infty$, denotes the usual Sobolev-Slobodeckij space normed by

$$\|u\|^q_{s,q} = \sum_{|\alpha|\leq s} \|D^\alpha u\|^q_q \qquad \text{if } s \in \mathbb{N}_0,$$

$$\|u\|^q_{s,q} = \sum_{|\alpha|\leq [s]} \|D^\alpha u\|^q_q + \sum_{|\alpha|=[s]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^q}{|x-y|^{n+q\{s\}}} dxdy \quad \text{if } s \notin \mathbb{N}_0.$$

Moreover, $W^s_q(\Omega;X)$ denotes its vector-valued variant, where X is a Banach space. Finally, $W^s_q(\partial\Omega)$ is defined in the same way as above with the Lebesgue measure replaced by the surface measure.

Definition 2.1 Let $n \geq 2$. If $\gamma = (\gamma^+, \gamma^-) \in C^{1,1}(\mathbb{R}^{n-1})^2$ with $\gamma^+(x') - \gamma^-(x') \geq c > 0$ for all $x' \in \mathbb{R}^{n-1}$, then

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : \gamma^-(x') < x_n < \gamma^+(x')\}$$

is called a *layer-like domain* with $C^{1,1}$-boundary. If additionally $\lim_{|x'|\to\infty} \gamma^\pm(x') = \gamma^\pm_\infty$ for some constants $\gamma^\pm_\infty \in \mathbb{R}$ and $\lim_{|x'|\to\infty} \nabla\gamma^\pm(x') = \lim_{|x'|\to\infty} \nabla^2\gamma^\pm(x') = 0$, then Ω_γ is called an *asymptotically flat layer*. Finally, $\partial\Omega^\pm(t) = \{(x', \gamma^\pm(x',t)) : x' \in \mathbb{R}^{n-1}\}$ denotes the upper and lower boundary, respectively.

If Ω is a layer-like domain with $C^{0,1}$-boundary, then

$${}^0W^1_q(\Omega) := \{u \in W^1_q(\Omega)\colon u|_{\partial\Omega^+} = 0\} \quad \text{and} \quad {}_0W^{-1}_q(\Omega) := ({}^0W^1_q(\Omega))'.$$

Now let $\Omega \subset \mathbb{R}^n$ be a domain and $T \in (0,\infty]$. Then $\Omega_T = \Omega \times (0,T)$ denotes the space-time cylinder and $\partial\Omega_T = \partial\Omega \times (0,T)$ its boundary in the spatial coordinates. The anisotropic Sobolev-Slobodeckij space is defined as $W_q^{2s,s}(\Omega_T) = L^q(0,T;W_q^{2s}(\Omega)) \cap W_q^s(0,T;L^q(\Omega))$, $s \geq 0$, normed by

$$\|u\|_{2s,s,q}^q = \|u\|_{L^q(0,T;W_q^{2s}(\Omega))}^q + \|u\|_{W_q^s(0,T;L^q(\Omega))}^q.$$

Moreover, we define $W_q^{m,0}(\Omega_T) = L^q(0,T;W_q^m(\Omega))$, $m \in \mathbb{N}$, and denote by $\|.\|_{m,0,q}$ the corresponding norm.

If $a \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\partial\Omega^+)$ and $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)$ with $q = 3$, we say that a and u_0 *coincide* at $\partial\Omega^+ \times \{0\}$ if

$$I[a,u_0] := \int_0^T \int_{\partial\Omega} \int_\Omega \frac{|a(x,t) - u_0(y)|^3}{(|x-y|^2 + t)^{1+n/2}} dy d\sigma(x) dt < \infty.$$

Note that, if $u_0 = 0$, then

$$I[a,0] \leq C \int_0^T \int_{\partial\Omega} |a(x,t)|^3 d\sigma(x) \frac{dt}{t}. \tag{2.1}$$

The following properties are well-known for bounded domains and are easily proved for layer-like domains.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a layer-like domain with $C^{1,1}$-boundary and let $q > \frac{3}{2}$. Then*

1. $\sup\limits_{0 \leq t \leq T} \|u(.,t)\|_{W_q^{2-\frac{2}{q}}(\Omega)} \leq C\|u\|_{2,1,q}$.

2. $\|\nabla u|_{\partial\Omega}\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q} \leq C\|u\|_{2,1,q}$.

3. *For every $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)$ with $u_0|_{\partial\Omega^-} = 0$ there is a $u \in W_q^{2,1}(\Omega_T)$ with $u|_{t=0} = u_0$, $u|_{\partial\Omega^-} = 0$, and $\|u\|_{2,1,q} \leq C\|u_0\|_{2-\frac{2}{q},q}$, where C can be chosen independently of $T \in (0,\infty]$.*

4. *For every $a^+ \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)^n$ with $a^+|_{t=0} = 0$ if $q > 3$ and $I[a^+,0] < \infty$ if $q = 3$ there is an $A \in W_q^{2,1}(\Omega_T)^n$ with $A|_{t=0} = 0$, $A|_{\partial\Omega^-} = 0$, and*

$$(\nu \cdot SA)_\tau|_{\partial\Omega^+} = a_\tau^+, \qquad \operatorname{div} A|_{\partial\Omega^+} = a_\nu^+,$$

where $SA = \nabla A + (\nabla A)^T$. Moreover, $\|A\|_{2,1,q} \leq C\|a^+\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$ if $q \neq 3$ and $\|A\|_{2,1,q} \leq C\left(\|a^+\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q} + I[a^+,0]^{\frac{1}{q}}\right)$ if $q = 3$, where C can be chosen independently of $T \in (0,\infty]$.

Proof: With the aid of a coordinate transformation $F\colon \Omega \to \mathbb{R}^{n-1} \times (-1,1)$ and simple cut-off functions the first three statements are easily reduced to case of an half-space $\mathbb{R}^n_+$, which are well-known, cf. Grubb [12, Appendix].

In order to prove 3., let $A \in W^{2,1}_q(\Omega_T)^n$ with $A|_{t=0}$, $A|_{\partial\Omega} = 0$, and $\partial_\nu A|_{\partial\Omega^+} = a$ such that $\|A\|_{2,1,q} \leq C\|a^+\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$ with an additional term $CI[a^+,0]^{\frac{1}{q}}$ if $q=3$. (As before, the existence A can be reduced to corresponding statement in $\mathbb{R}^n_+$.) Then

$$\begin{aligned}(\nu\cdot SA)_\tau|_{\partial\Omega^+} &= (\nabla_\tau A_\nu + \partial_\nu A_\tau)|_{\partial\Omega^+} = a^+_\tau,\\ \operatorname{div} A|_{\partial\Omega^+} &= (\operatorname{div}_\tau A_\tau + \partial_\nu A_\nu)|_{\partial\Omega^+} = a^+_\nu.\end{aligned}$$

The constant C can be chosen independently of T since we can extend a^+ to $\tilde{a}^+ \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\partial\Omega^+\times(0,\infty))^n$ with $\|\tilde{a}^+\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q} \leq C\|a^+\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$, where C does not depend on T, and restrict the corresponding $\widetilde{A} \in W^{2,1}_q(\Omega\times(0,\infty))^n$ to $(0,T)$ afterwards. The latter extension to $(0,\infty)$ can be done by first extending a^+ in an even way around $t=T$ to a function defined on $(0,2T)$ and then extending by zero, which yields an $\tilde{a}^+ \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\partial\Omega^+\times(0,\infty))^n$ since $\tilde{a}^+|_{t=2T} = a^+|_{t=0} = 0$ if $q>3$ and $\tilde{a}^+$ coincides with 0 at $\partial\Omega^+\times\{2T\}$ if $q=3$. ■

A simple but useful vector-valued variant of the Sobolev embedding theorem is given in the next lemma.

Lemma 2.3 *Let $1<q<\infty$, $0<T<\infty$, and X be a Banach space. Then $W^1_q(0,T;X) \hookrightarrow C^{\frac{1}{q'}}([0,T];X)$ is a continuous embedding.*

Proof: The lemma is a simple consequence of the fact that $u\colon (0,T)\to X$ is absolutely continuous, cf. Amann [5, Chapter III, Theorem 1.2.2], and the Hölder inequality. ■

2.3 Weak Mixed Dirichlet-Neumann Problem

If $u \in L^q(\Omega)^n$ with $\operatorname{div} u \in L^q_{loc}(\overline{\Omega}) \cap {}^0W^{-1}_q(\Omega)$, then we define the trace $\nu\cdot u|_{\partial\Omega^-} \in W_q^{-\frac{1}{q}}(\partial\Omega^-)$ as

$$\langle \nu\cdot u|_{\partial\Omega^-}, v\rangle \;=\; (u,\nabla V) + (\operatorname{div} u, V), \tag{2.2}$$

where $v \in W_{q'}^{1-\frac{1}{q'}}(\partial\Omega^-)$ and $V \in {}^0W^1_{q'}(\Omega_0)$ with $V|_{\partial\Omega^-} = v$. The definition does not depend on the choice of V. Moreover, $\|\gamma^-_\nu u\|_{-\frac{1}{q},q} \leq C\left(\|u\|_q + \|\operatorname{div} u\|_{{}^0W^{-1}_q(\Omega)}\right)$.

Now let $f \in L^q(\Omega)^n$. Then we consider the weak mixed Dirichlet-Neumann problem

$$\begin{aligned}\Delta u &= \operatorname{div} f && \text{in } \Omega, && (2.3)\\ u|_{\partial\Omega^+} &= 0 && \text{on } \partial\Omega^+, && (2.4)\\ \partial_\nu u|_{\partial\Omega^-} &= \nu\cdot f|_{\partial\Omega^-} && \text{on } \partial\Omega^-. && (2.5)\end{aligned}$$

Here $\partial_\nu u|_{\partial\Omega^-} = \nu\cdot f|_{\partial\Omega^-}$ is understood as $\nu\cdot(\nabla u - f)|_{\partial\Omega^-} = 0$.

Lemma 2.4 *Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then for every $f \in L^q(\Omega)^n$ there is a unique solution $u \in W^1_q(\Omega)$ of (2.3)-(2.5), which satisfies $\|u\|_{1,q} \leq C\|f\|_q$. Moreover, u is uniquely determined by*

$$(\nabla u, \nabla v) = (f, \nabla v) \qquad \text{for all } v \in {}^0W^1_{q'}(\Omega). \tag{2.6}$$

The lemma is a consequence of [2, Theorem 3.6]. Finally, we note that by the Hahn-Banach theorem every $F \in {}_0W^{-1}_q(\Omega)$ has a representation $\langle F, v\rangle = (f, \nabla v)$ for all $v \in {}^0W^1_{q'}(\Omega)$ with $f \in L^q(\Omega)^n$ and $\|f\|_q \leq C\|F\|_{{}_0W^{-1}_q}$.

3 Instationary Stokes Equations

In order to solve the free boundary value problem, we need the solvability of the linearized problem, i.e., the instationary generalized Stokes equations

$$\begin{aligned}
\partial_t u - \Delta u + \nabla p &= f && \text{in } \Omega_T, && (3.1)\\
\operatorname{div} u &= g && \text{in } \Omega_T, && (3.2)\\
T_1^+(u,p) &= a^+ && \text{on } \partial\Omega_T^+, && (3.3)\\
u|_{\partial\Omega^-} &= 0 && \text{on } \partial\Omega_T^-, && (3.4)\\
u|_{t=0} &= u_0 && \text{in } \Omega. && (3.5)
\end{aligned}$$

The solvability of this system is considered for data

$$f \in L^q(\Omega_T)^n, g \in W^{1,0}_q(\Omega_T), a^+ \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)^n, \text{ and } u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^n,$$

$n \geq 2$. We restrict ourselves to the case $q > \frac{3}{2}$, which is sufficient for our application. Moreover, we assume that the data satisfy the following *compatibility conditions:*

1. The vector-valued distributional derivative $\partial_t g$ in $\mathcal{D}'(0,T;{}_0W_q^{-1}(\Omega))$ belongs to $L^q(0,T;{}_0W_q^{-1}(\Omega))$. and $g|_{t=0} = \operatorname{div} u_0$ in ${}_0W_q^{-1}(\Omega)$.

2. $a^+_\tau|_{t=0} = (\nu\cdot Su_0)_\tau|_{\partial\Omega^+}$ if $q > 3$ and $I[a^+_\tau, (\nu\cdot Su_0)_\tau] < \infty$ if $q = 3$. (In the latter case $\nu \in C^{0,1}(\partial\Omega^+)$ has to be extended suitably to Ω.)

3. $u_0|_{\partial\Omega^-} = 0$.

Here $\mathcal{D}'(0,T;{}_0W_q^{-1}(\Omega)) := \mathcal{L}(\mathcal{D}(0,T), {}_0W_q^{-1}(\Omega))$, cf. [5, Section 4.1], and $\mathcal{D}(0,T)$ is the set of all smooth $f\colon (0,T) \to \mathbb{R}$ with $\operatorname{supp} f \subseteq (0,T)$.

In our application $g = \operatorname{div} R$ with $R \in W^{2,1}_q(\Omega_T)^n$, $\nu\cdot R|_{\partial\Omega^-} = 0$, $\operatorname{div} R|_{t=0} = 0$. Then

$$\int_0^T (\partial_t R(.,t), \nabla v)_\Omega \varphi(t)dt \quad = \quad \int_0^T (g(.,t), v)_\Omega \partial_t\varphi(t)dt$$

for all $\varphi \in \mathcal{D}(0,T)$, $v \in {}^0W^1_{q'}(\Omega)$. Hence $F(t) \in {}_0W_q^{-1}(\Omega)$, $t \in (0,T)$, defined by $\langle F(t), v\rangle = -(\partial_t R(.,t), \nabla v)_\Omega$ is the ${}_0W_q^{-1}(\Omega)$-valued distributional derivative $\partial_t g$. Thus $\partial_t g \in L^q(0,T;{}_0W_q^{-1}(\Omega))$ and $\|\partial_t g\|_{L^q(0,T;{}_0W_q^{-1}(\Omega))} \leq \|\partial_t R\|_{L^q(\Omega_T)}$. Conversely, the following lemma holds.

Lemma 3.1 *Let $1 < q < \infty$, $0 < T \leq \infty$ and let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an asymptotically flat layer with $C^{1,1}$-boundary. Then every $g \in W_q^{1,0}(\Omega_T)$ with $\partial_t g \in L^q(0,T;{}_0W_q^{-1}(\Omega))$ and $g|_{t=0} = 0$ possesses a representation $g = \operatorname{div} R$ where $R \in W^1_q(\Omega_T)^n$ with $R|_{t=0} = 0$, $\nu \cdot R|_{\partial\Omega^-} = 0$, and*

$$\|R\|_{W^1_q(\Omega_T)} \leq C\left(\|g\|_{1,0,q} + \|\partial_t g\|_{L^q(0,T;{}_0W_q^{-1}(\Omega))}\right),$$

where C does not depend on T.

Proof: We define $R = \nabla p$ where p is the solution of

$$\begin{aligned} \Delta p(.,t) &= g(.,t) && \text{in } \Omega, \\ p|_{\partial\Omega^+} &= 0 && \text{on } \partial\Omega^+, \\ \partial_\nu p|_{\partial\Omega^-} &= 0 && \text{on } \partial\Omega^-. \end{aligned}$$

Hence $\|p\|_{L^q(0,T;W^2_q(\Omega))} \leq C\|g\|_{1,0,q}$ because of [1, Lemma 3.3] which can be extended to arbitrary asymptotically flat layers with $C^{1,1}$-boundary using the same method as in [2, Appendix]. Because of the unique solvability of (2.3)-(2.5), $\partial_t g \in L^q(0,T,{}_0W_q^{-1}(\Omega))$ has a representation

$$\langle \partial_t g(.,t), v\rangle_\Omega = (\nabla\tilde{p}, \nabla v) \qquad \text{for all } v \in {}^0W^1_{q'}(\Omega)$$

for a.a. $t \in (0,T)$, where $\tilde{p} \in L^q(0,T;{}^0W^1_q(\Omega))$ with $\|\nabla\tilde{p}\|_{q,\Omega_T} \leq C\|\partial_t g\|_{L^q(0,T;{}_0W_q^{-1}(\Omega))}$. Therefore we conclude $\partial_t \nabla p \in L^q(\Omega_T)$ as follows: $g(.,t)$, $t \in (0,T)$, is an absolutely continuous ${}^0W_q^{-1}(\Omega)$-valued function, cf. [5, Theorem 1.2.2], and

$$(g(.,t), v)_\Omega = \int_0^t \langle \partial_t g(.,s), v\rangle_\Omega ds = (\nabla \int_0^t \tilde{p}(.,s)ds, \nabla v)$$

for all $v \in {}^0W^1_{q'}(\Omega)$. On the other hand,

$$(g(.,t), v)_\Omega = -(\nabla p, \nabla v) \qquad \text{for all } v \in {}^0W^1_{q'}(\Omega)$$

and a.a. $t \in (0,T)$. Because of Lemma 2.4, we conclude $\nabla p(.,t) = -\nabla \int_0^t \tilde{p}(.,s)ds$ for a.a. $t \in (0,T)$.

Thus $R = \nabla p \in W^1_q(\Omega_T)$ with $\nabla p|_{t=0} = 0$, $\partial_\nu p|_{\partial\Omega^-} = 0$, $\|\nabla p\|_{1,q,\Omega_T} \leq C\|g\|_{1,0,q}$, and $\|\partial_t \nabla p\|_{q,\Omega_T} \leq C\|\partial_t g\|_{L^q(0,T;{}_0W_q^{-1}(\Omega))}$. Finally, it is easy to check that all constants in the proof do not depend on T. ■

As in the case of the generalized Stokes resolvent equations, cf. [1], (3.1)-(3.5) can (at least formally) be reduced to the *instationary reduced Stokes equations*

$$\partial_t u - \Delta u + G_{10}u = f_r \qquad \text{in } \Omega_T, \tag{3.6}$$

$$T_1'^+ u = a_r \qquad \text{on } \partial\Omega_T^+, \tag{3.7}$$

$$u|_{\partial\Omega^-} = 0 \qquad \text{on } \partial\Omega_T^-, \tag{3.8}$$

$$u|_{t=0} = u_0 \qquad \text{in } \Omega, \tag{3.9}$$

where $(T_1'^+u)_\tau = (\nu \cdot Su)_\tau$ and $(T_1'^+u)_\nu = \operatorname{div} u|_{\partial\Omega^+}$ and

$$G_{10}u = \nabla K_{01}(2\partial_\nu u|_{\partial\Omega^+}, \nu \cdot (\Delta - \nabla \operatorname{div})u|_{\partial\Omega^-}).$$

Here K_{01} denotes the Poisson operator to (2.3)-(2.5), i.e., $\Delta K_{01} = 0$, $K_{01}(a^+, a^-)|_{\partial\Omega^+} = a^+$, and $\partial_\nu K_{01}(a^+, a^-)|_{\partial\Omega^-} = a^-$.

It is one of the main results of [3] that $A_{10} := -\Delta + G_{10}$ with domain

$$\mathcal{D}(A_{10}) = \left\{ u \in W_q^2(\Omega)^n : T_1'^+ u = 0, u|_{\partial\Omega^-} = 0 \right\}$$

admits a bounded H_∞-calculus in the sense of McIntosh [14]. In particular, A_{10} possesses bounded imaginary powers and the result of Dore and Venni [9, Theorem 3.2.] resp. its extension by Giga and Sohr [10, Theorem 2.1] implies that A_{10} has *maximal regularity*, i.e., for every $f \in L^q(0,T; L^q(\Omega)^n)$, $0 < T \leq \infty$, there is a unique solution

$$u \in W_q^1(0,T; L^q(\Omega)^n) \cap L^q(0,T; \mathcal{D}(A_{10}))$$

of the abstract ordinary differential equation

$$u'(t) + A_{10}u(t) = f(t), \quad t > 0, \quad u(0) = 0.$$

Combining this result with Lemma 2.2, we obtain:

THEOREM 3.2 *Let $\frac{3}{2} < q < \infty$, $0 < T \leq \infty$, and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an asymptotically flat layer with $C^{1,1}$-boundary. Moreover, let $(f_r, a_r^+, u_0) \in L^q(\Omega_T)^n \times W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Omega_T)^n \times W_q^{2-\frac{2}{q}}(\Omega)^n$ satisfy the compatibility conditions $u_0|_{\partial\Omega^-} = 0$, $T_1'^+u_0 = a^+|_{t=0}$ if $q > 3$, and $I[a_\tau^+, (\nu \cdot Su_0)_\tau], I[a_\nu^+, \operatorname{div} u_0] < \infty$ if $q = 3$. Then there is a unique solution $u \in W_q^{2,1}(\Omega_T)^n$ of (3.6)-(3.9), which satisfies*

$$\|u\|_{2,1,q} \leq C \left(\|f_r\|_q + \|a_r^+\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q} + \|u_0\|_{2-\frac{2}{q},q} \right)$$

with an additional $I[a_\tau^+, (\nu \cdot Su_0)_\tau]^{\frac{1}{q}} + I[a_\nu^+, \operatorname{div} u_0]^{\frac{1}{q}}$ on the right-hand side if $q = 3$. The constant C can be chosen independently of $T \in (0, \infty)$.

Using Theorem 3.2, we are able to prove the main result of this section.

THEOREM 3.3 *Let $n \geq 2$, $\frac{3}{2} < q < \infty$, and $0 < T < \infty$. Then for every $f \in L^q(\Omega_T)^n$, $g \in W_q^{1,0}(\Omega_T)$ with $\partial_t g \in L^q(0,T; {}_0W_q^{-1}(\Omega))$, $g|_{\partial\Omega^+} \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)^n$, $a^+ \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)^n$, and $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^n$ satisfying the compatibility condition above there is a unique solution $(u,p) \in W_q^{2,1}(\Omega_T)^n \times W_q^{1,0}(\Omega_T)$ of (3.1)-(3.5). Moreover,*

$$\begin{aligned} &\|u\|_{2,1,q} + \|p\|_{1,0,q} + \|p|_{\partial\Omega^+}\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q} \qquad (3.10)\\ &\leq\; C\left(\|(f,\nabla g)\|_q + \|\partial_t g\|_{L^q(0,T;_0W_q^{-1}(\Omega))} + \|(g|_{\partial\Omega^+},a^+)\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q} + \|u_0\|_{2-\frac{2}{q},q}\right), \end{aligned}$$

with an additional $I[a_\tau^+,(\nu\cdot Su_0)_\tau]^{\frac{1}{q}}$ on the right-hand side if $q=3$. The constant C can be chosen independently of $T \in (0,\infty)$.

Proof: For every $t \in (0,T)$ let $p_2(.,t) \in W_q^1(\Omega)$ be the solution of

$$(\nabla p_2(.,t),\nabla v) = (f + \nabla g - \partial_t R, \nabla v) \qquad \text{for all } v \in {}^0W_{q'}^1(\Omega)$$

with $p_2|_{\partial\Omega^+} = a_\nu^+$, cf. Lemma 2.4, where $\operatorname{div} R$ is a representation of g due to Lemma 3.1. Now we define $f_r = f - \nabla p_2$. Then

$$\|f_r\|_q \;\leq\; C\left(\|(f,\nabla g,\partial_t R)\|_q + \|a_\nu^+\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}\right)$$

with C independent of T. Moreover, let $(a_r^+)_\tau = a_\tau^+$ and $(a_r^+)_\nu = g|_{\partial\Omega^+}$.

Let $u \in W_q^{2,1}(\Omega_T)^n$ is the solution of the reduced Stokes equations with right-hand side (f_r, a_r^+). Then (u,p) with $\nabla p = G_{10}u + \nabla p_2$ is a solution of the generalized Stokes equations, where it only remains to prove that $\operatorname{div} u = g$. Testing (3.6) with ∇v, $v \in L^{q'}(0,T;{}^0W_{q'}^1(\Omega))$, and using

$$(G_{10}u,\nabla v)_{\Omega_T} = \langle \nu\cdot(\Delta - \nabla\operatorname{div})u|_{\partial\Omega^-}, v|_{\partial\Omega^-}\rangle_{\partial\Omega_T} = ((\Delta-\nabla\operatorname{div})u,\nabla v)_{\Omega_T}$$

because of (2.2), we conclude

$$(\partial_t(u-R),\nabla v)_{\Omega_T} + (\nabla(\operatorname{div} u - \operatorname{div} R),\nabla v)_{\Omega_T} = 0 \tag{3.11}$$

for all $v \in L^{q'}(0,T;{}^0W_{q'}^1(\Omega))$. Moreover, $\operatorname{div}(u-R)|_{t=0} = 0$ because of the compatibility conditions.

Now let $\tilde f \in L^{q'}(\Omega_T)^n$ be arbitrary and let $\tilde u \in W_{q'}^{2,1}(\Omega_T)^n$ be a solution of the reduced Stokes equations with right-hand side $\tilde f$, $a^+ = 0$, and $u_0 = 0$. Then a similar calculation as above yields

$$(\partial_t\tilde u,\nabla w)_{\Omega_T} + (\nabla\operatorname{div}\tilde u,\nabla w)_{\Omega_T} = (\tilde f,\nabla w)_{\Omega_T} \quad \text{for all } w \in L^q(0,T;{}^0W_q^1(\Omega)).$$

If we set $v(x,t) = \operatorname{div}\tilde u(x,T-t)$ in (3.11), we get

$$\begin{aligned} 0 &= (\partial_t(u-R),\nabla\operatorname{div}\tilde u(x,T-.))_{\Omega_T} + (\nabla\operatorname{div}(u-R),\nabla\operatorname{div}\tilde u(x,T-.))_{\Omega_T}\\ &= (\nabla\operatorname{div}(u-R),(\partial_t\tilde u)(.,T-.))_{\Omega_T} + (\nabla\operatorname{div}(u-R),(\nabla\operatorname{div}\tilde u)(.,T-.))_{\Omega_T}\\ &= (\tilde f(.,T-.),\nabla\operatorname{div}(u-R))_{\Omega_T}, \end{aligned} \tag{3.12}$$

where we have used:

Proposition 3.4 *Let $1 < q < \infty$ and let $u \in W_q^{2,1}(\Omega_T)^n$, $v \in W_{q'}^{2,1}(\Omega_T)^n$ such that $\nu \cdot u|_{\partial\Omega_T^-} = \nu \cdot v|_{\partial\Omega_T^-} = 0$, $\operatorname{div} u|_{\partial\Omega_T^+} = \operatorname{div} v|_{\partial\Omega_T^+} = 0$, and $u|_{t=0} = v|_{t=T} = 0$. Then*

$$(\partial_t u, \nabla \operatorname{div} v)_{\Omega_T} = -(\nabla \operatorname{div} v, \partial_t v)_{\Omega_T}.$$

Proof: The lemma is simply proved by approximating $\partial_t u$ by difference quotients and using $u \in C^{\frac{1}{q'}}([0,T]; L^q(\Omega))^n$, $v \in C^{\frac{1}{q}}([0,T]; L^{q'}(\Omega))^n$ due to Lemma 2.3. ■

Since for every $F \in L^{q'}(0,T; {}_0W_{q'}^{-1}(\Omega)) = (L^q(0,T; {}^0W_q^1(\Omega))'$ there is an $\tilde{f} \in L^{q'}(\Omega_T)^n$ such that

$$\int_0^T \langle F(.,t), v(.,t)\rangle_{{}_0W_{q'}^{-1}(\Omega)} dt = (\tilde{f}(., T - .), \nabla v)_{\Omega_T} \qquad \text{for all } v \in L^q(0,T; {}^0W_q^1(\Omega)),$$

(3.12) and the Hahn-Banach theorem imply $\operatorname{div}(u - R) = 0$.

In order to prove uniqueness, let (u,p) be a solution with right-hand side 0. Then $\nabla p = G_{10} u$ since $\Delta p = 0$, $p|_{\partial\Omega^+} = 2\partial_\nu u_\nu$, and $\partial_\nu p|_{\partial\Omega^-} = \nu \cdot (\Delta - \nabla \operatorname{div}) u|_{\partial\Omega^-}$ by (3.1) and (3.3). Hence u solves (3.6)-(3.9) with $f_r = a_r^+ = u_0 = 0$ and therefore $u = \nabla p = 0$ due to Theorem 3.2. ■

4 Proof of Theorem 1.1

For simplicity we multipliy (1.9) by $|A(u)\nu|$ and replace it by

$$\widetilde{T}_{1,u}^+(u,p) = ((A(u)\nu) \cdot S_u u - pA(u)\nu)|_{\partial\Omega^+} = -g_0 X_{u,n} A(u)\nu \tag{4.1}$$

with $S_u v = \nabla_u v + (\nabla_u v)^T$.

We can formulate the initial boundary value problem as an abstract equation:

$$Lv \;=\; G(v) + h \quad \Leftrightarrow v = L^{-1}G(v) + L^{-1}h, \tag{4.2}$$

where $v = (u,p) \in X_T$, $h = (0, 0, -g_0\xi_n\nu, u_0) \in Y_T$, and

$$\begin{aligned}
X_T &= \{(u,p) : u \in W_q^{2,1}(\Omega_T)^n, u|_{\partial\Omega^-} = 0, \operatorname{div} u|_{t=0} = 0, p \in W_q^{1,0}(\Omega_T),\\
&\quad\; p|_{\partial\Omega^+} \in W_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(\partial\Omega_T^+), (\nu \cdot Su|_{t=0})_\tau|_{\partial\Omega^+} = 0\},\\
Y_T &= \{(f, g, a^+, u_0) : f \in L^q(\Omega_T)^n, g \in W_q^{1,0}(\Omega_T), \partial_t g \in L^q(0,T; {}_0W_q^{-1}(\Omega)),\\
&\quad\; g|_{\partial\Omega^+} \in W_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(\partial\Omega_T^+), g|_{t=0} = 0, a^+ \in W_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(\partial\Omega_T^+)^n, a_\tau^+|_{t=0} = 0\\
&\quad\; u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^n, \operatorname{div} u_0 = 0, u_0|_{\partial\Omega^-} = 0, (\nu \cdot Su_0)_\tau|_{\partial\Omega^+} = 0\},\\
L(u,p) &= ((\partial_t - \Delta)u + \nabla p, \operatorname{div} u, T_1^+(u,p), u|_{t=0}),\\
G(u,p) &= ((\Delta_u - \Delta)u + (\nabla - \nabla_u)p, (\operatorname{div} - \operatorname{div}_u)u, (T_1^+ - \widetilde{T}_{1,u}^+)(u,p) -\\
&\quad\; g_0(X_{u,n}A(u)\nu - \xi_n\nu), 0).
\end{aligned}$$

in the case $q > 3$. If $q = 3$, we replace the conditions $(\nu \cdot Su|_{t=0})_\tau|_{\partial\Omega^+} = 0$, $(\nu \cdot Su_0)_\tau|_{\partial\Omega^+}$, and $a_\tau^+|_{t=0} = 0$ by $I[0, (\nu \cdot Su|_{t=0})_\tau] < \infty$, $I[0, (\nu \cdot Su_0)_\tau] < \infty$, and $I[a_\tau^+, 0] < \infty$, respectively. In the case $q < 3$, we skip the latter conditions. The associated norms are defined in the obvious way with additional terms $I[0, (\nu \cdot Su|_{t=0})_\tau]^{\frac{1}{q}}$, $I[0, (\nu \cdot Su_0)_\tau]^{\frac{1}{q}}$, and $I[a_\tau^+, 0]^{\frac{1}{q}}$ if $q = 3$.

In order to apply the Banach contraction mapping principle to (4.2), we have to estimate $G(v) - G(w)$ for all $v, w \in \overline{B_R(0)}$ for some suitable $R > 0$.

First of all, we note that

$$\|f\|^q_{W^s_q(0,T;X)} = \|f\|^q_{L^q(0,T;X)} + \int_0^T \int_0^t \frac{\|\Delta_h f(t)\|^q_X}{h^{1+qs}} dh dt,$$

where $\Delta_h f(t) = f(t) - f(t-h)$.

Lemma 4.1 *Let $n \geq 2$, $q > n$, and $R > 0$. Moreover, let $F(u) = X_u$ and $Z = W^2_q(\Omega)^n$ or $F(u) = A(u)$ and $Z = W^1_q(\Omega)^{n\times n}$. Then there is a $T_0 = T_0(R) > 0$ and a constant $C > 0$ such that for all $0 < T \leq T_0$*

$$\begin{aligned}
\sup_{0\leq t\leq T} \|F(u) - F(v)\|_Z &\leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q}, &(4.3)\\
\sup_{0\leq t\leq T} \left(\int_0^t \frac{\|\Delta_h(F(u) - F(v))(.,t)\|^q_Z}{h^{1+\frac{q}{2q'}}} dh\right)^{\frac{1}{q}} &\leq CT^{\frac{1}{2q'}} \|u - v\|_{2,1,q}, &(4.4)\\
\left(\int_0^T \int_0^t \frac{\|\Delta_h(F(u) - F(v))(.,t)\|^q_Z}{h^{1+\frac{q}{2q'}}} dh dt\right)^{\frac{1}{q}} &\leq CT^{\frac{1}{q}+\frac{1}{2q'}} \|u - v\|_{2,1,q} &(4.5)
\end{aligned}$$

for all $u, v \in W^{2,1}_q(\Omega_T)^n$ with $\|u\|_{2,1,q}, \|v\|_{2,1,q} \leq R$.

Proof: First of all, we observe that

$$\|F(u) - F(v)\|_{W^1_q(0,T;Z)} \leq C\|u - v\|_{2,1,q} \tag{4.6}$$

for all $0 < T \leq T_0$ if $T_0 > 0$ is chosen sufficiently small.

If $F(u) = X_u$, (4.6) is a consequence of

$$X_u(\xi, t) - X_v(\xi, t) = \int_0^t (u(\xi, \tau) - v(\xi, \tau)) d\tau \in W^1_q(0, T; W^2_q(\Omega)^n).$$

Since $\int_0^t \nabla u(., \tau) d\tau \in W^1_q(0, T; W^1_q(\Omega)^{n\times n}) \hookrightarrow C^{\frac{1}{q'}}(0, T; W^1_q(\Omega)^{n\times n})$, we have

$$\sup_{0\leq t\leq T} \left\|\int_0^t \nabla u(., \tau) d\tau\right\|_\infty \leq C \sup_{0\leq t\leq T} \left\|\int_0^t \nabla u(., \tau) d\tau\right\|_{1,q} \leq CT^{\frac{1}{q'}} R$$

for all $u \in \overline{B_R(0)} \subset W^{2,1}_q(\Omega_T)$. Thus we can find a $T_0 > 0$ sufficiently small such that $A(u) = \left(I + \int_0^t \nabla u(x, \tau) d\tau\right)^{-T}$ exists for all $u \in \overline{B_R(0)}$. Since matrix inversion is a smooth mapping on $\mathrm{GL}(n)$, $A(u)$ depends smoothly on $M = \int_0^t \nabla u(x,\tau)d\tau$. Therefore $A(u) - A(v) \in W_q^1(0,T;W_q^1(\Omega)^n)$ and (4.6) holds for $F(u) = A(u)$.

Now (4.3) is a consequence of Lemma 2.3, (4.6), and $(F(u) - F(v))|_{t=0} = 0$. Moreover,

$$\|\Delta_h(F(u) - F(v))(.,t)\|_Z \leq C h^{\frac{1}{q'}} \|u - v\|_{2,1,q}$$

and therefore

$$\begin{aligned} &\sup_{0<t\leq T} \left(\int_0^t \frac{\|\Delta_h(F(u) - F(v))(.,t)\|_Z^q}{h^{1+\frac{q}{2q'}}} dh \right)^{\frac{1}{q}} \\ \leq\ & C \sup_{0<t\leq T} \left(\int_0^t h^{\frac{q}{q'}-1-\frac{q}{2q'}} dh \right)^{\frac{1}{q}} \|u-v\|_{2,1,q} \leq C T^{\frac{1}{2q'}} \|u-v\|_{2,1,q}. \end{aligned}$$

Because of $\|f\|_{L^q(0,T)} \leq T^{\frac{1}{q}} \|f\|_{L^\infty(0,T)}$, (4.5) is a consequence of (4.4). ■

Lemma 4.2 *Let $1 < q < \infty$, $T_0 > 0$, and $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a layer-like domain with $C^{1,1}$-boundary. Then*

$$\|\nabla u\|_{W_q^{\frac{1}{2}-\varepsilon}(0,T;L^q(\Omega))} \leq C_{\varepsilon,T_0} \|u\|_{2,1,q}$$

for every $u \in W_q^{2,1}(\Omega_T)$, $\varepsilon \in (0,\frac{1}{2})$, and $0 < T \leq T_0$.

Proof: Because of $\|\nabla u(.,t)\|_{q,\Omega} \leq C \|u(.,t)\|_{q,\Omega}^{\frac{1}{2}} \|u(.,t)\|_{2,q,\Omega}^{\frac{1}{2}}$,

$$\begin{aligned} &\left(\int_0^T \int_0^t \frac{\|(\Delta_h \nabla u)(.,t)\|_{q,\Omega}^q}{h^{1+(\frac{1}{2}-\varepsilon)q}} dh dt \right)^{\frac{1}{q}} \\ \leq\ & \left(\int_0^T \int_0^t \frac{\|\Delta_h u(.,t)\|_{q,\Omega}^q}{h^{1+(1-\varepsilon)q}} dh dt \right)^{\frac{1}{2q}} \left(\int_0^T \int_0^t \frac{\|\Delta_h u(.,t)\|_{2,q,\Omega}^q}{h^{1-\varepsilon q}} dh dt \right)^{\frac{1}{2q}}. \end{aligned}$$

Since $\|\Delta_h u(.,t)\|_{2,q,\Omega} \leq 2\|u(.,t)\|_{2,q,\Omega}$, the second term can be estimated by $C_{\varepsilon,T_0} \|u\|_{2,1,q}^{\frac{1}{2}}$. For the first term we use $\|\Delta_h u(.,t)\|_q^q \leq h^{\frac{q}{q'}} \int_{t-h}^t \|\partial_t u(.,\tau)\|_q^q d\tau$, and obtain

$$\begin{aligned} \left(\int_0^T \int_0^t \frac{\|\Delta_h u(.,t)\|_q^q}{h^{1+(1-\varepsilon)q}} dh dt \right)^{\frac{1}{q}} &\leq \left(\int_0^T \int_h^T \int_{t-h}^t \|\partial_t u(.,\tau)\|_q^q h^{-2+\varepsilon q} d\tau dt dh \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^T \int_0^T \int_{\max\{h,\tau\}}^{\min\{\tau+h,T\}} dt \|\partial_t u(.,\tau)\|_q^q h^{-2+\varepsilon q} d\tau dh \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^T h^{-1+\varepsilon q} dh \right)^{\frac{1}{q}} \|\partial_t u\|_{q,\Omega_T} \leq C_{\varepsilon,T_0} \|u\|_{2,1,q}. \end{aligned}$$

■

Lemma 4.3 *Let $n \geq 2$, $q > n$, and $\kappa > 0$ and let Ω be an asymptotically flat layer with $C^{1,1}$-boundary such that $\gamma^+ \in W_q^{1-\frac{1}{q}}(\mathbb{R}^{n-1})$, and let $G\colon X_T \to Y_T$, $T > 0$, be defined as above. Then for every $R > 0$ there is a $T_0 > 0$ such that*

$$\|G(u) - G(v)\|_{Y_T} \leq \kappa \|u - v\|_{X_T} \tag{4.7}$$

for all $u, v \in \overline{B_R(0)} \subset X_T$.

Proof: The operator $G\colon X_T \to Y_T$ can be written as $G(u,p) = H(u)(u,p) - H(0)(u,p) + F(u) - F(0)$, where

$$\begin{aligned} H(u)(f,p) &= (\Delta_u f - \nabla_u p, -\operatorname{div}_u f, -\tilde{T}^+_{1,u}(f,p), 0), \\ F(u) &= (0, 0, g_0 X_{u,n} A(u)^T \nu|_{\partial\Omega^+}, 0), \end{aligned}$$

and $H(u)(f,p)$ is a linear operator in (f,p) depending on u. Hence

$$\begin{aligned} G(u,p_1) - G(v,p_2) &= (H(u) - H(v))(u,p_1) + (H(v) - H(0))(u - v, p_1 - p_2) \\ &\quad + F(u) - F(v). \end{aligned}$$

Therefore it is sufficient to prove

$$\|(H(u) - H(v))(f,p)\|_{Y_T} \leq C\varphi(T)\|u - v\|_{2,1,q}\|(f,p)\|_{X_T}, \tag{4.8}$$

$$\|F(u) - F(v)\|_{Y_T} \leq C\varphi(T)\|u - v\|_{2,1,q} \tag{4.9}$$

for a function $\varphi(T)$ such that $\lim_{T\to 0}\varphi(T) = 0$ and $(u,0), (v,0), (f,p) \in X_T$ with $\|u,v\|_{2,1,q}, \|(f,p)\|_{X_T} \leq R$.

Using (4.3) and $W^1_q(\Omega) \hookrightarrow L^\infty(\Omega)$, it is easy to prove that

$$\begin{aligned} \|(\nabla_u - \nabla_v)p\|_{q,\Omega_T} &\leq CT^{\frac{1}{q'}}\|u - v\|_{2,1,q}\|p\|_{1,0,q}, \\ \|(\operatorname{div}_u - \operatorname{div}_v)f\|_{1,0,q} &\leq CT^{\frac{1}{q'}}\|u - v\|_{2,1,q}\|f\|_{2,0,q}, \\ \|(\Delta_u - \Delta_v)f\|_{q,\Omega_T} &\leq CT^{\frac{1}{q'}}(T^{\frac{1}{q'}}R + 1)\|u - v\|_{2,1,q}\|f\|_{2,0,q}, \\ \|(S_u - S_v)f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} &\leq CT^{\frac{1}{q'}}\|u - v\|_{2,1,q}\|f\|_{2,0,q}. \end{aligned}$$

Since $\nu^+ \in C^{0,1}(\partial\Omega^+)$ and since $W_q^{1-\frac{1}{q}}(\partial\Omega^+)$ is an algebra under pointwise multiplication,

$$\begin{aligned} &\|(A(u)^T\nu)\cdot S_u f|_{\partial\Omega^+} - (A(v)^T\nu)\cdot S_v f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} \\ &\leq C\left(\sup_{0\leq t\leq T}\|(A(u) - A(v))^T\nu|_{\partial\Omega^+}\|_{1-\frac{1}{q},q}\right)\|S_u f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} \\ &\quad + C\left(\sup_{0\leq t\leq T}\|A(v)^T\nu|_{\partial\Omega^+}\|_{1-\frac{1}{q},q}\right)\|(S_u - S_v)f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} \\ &\leq CT^{\frac{1}{q'}}(\|(u,v)\|_{2,1,q} + 1)\|u - v\|_{2,1,q}\|f\|_{2,0,q}. \end{aligned}$$

Furthermore,

$$\begin{aligned}\|(A(u)-A(v))^T\nu p|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} &\le CT^{\frac{1}{q'}}\|u-v\|_{2,1,q}\|p\|_{1,0,q},\\ \|(\mathrm{div}_u-\mathrm{div}_v)f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} &\le CT^{\frac{1}{q'}}\|u-v\|_{2,1,q}\|f\|_{2,0,q}.\end{aligned}$$

Hence

$$\|(\widetilde{T}_u^+-\widetilde{T}_v^+)(f,p)\|_{1-\frac{1}{q},0,q} \le CT^{\frac{1}{q'}}\|u-v\|_{2,1,q}(\|f\|_{2,0,q}+\|p\|_{1,0,q}).$$

In the same way we get

$$\begin{aligned}&\|\gamma_0^+(X_{u,n}A(u)^T-X_{v,n}A(v)^T)\nu\|_{1-\frac{1}{q},0,q}\\ &\le CT^{\frac{1}{q'}}\|u-v\|_{2,1,q}+C\|X_{u,n}-X_{v,n}\|_{1,0,q}\le CT^{\frac{1}{q'}}\|u-v\|_{2,1,q}\end{aligned}$$

because of (4.3). But it remains to estimate the semi-norm

$$\|a\|_{\cdot,0,\frac{1}{2q'},q} := \left(\int_0^T\int_0^t \|(\Delta_h a)(.,t)\|^q_{q,\partial\Omega^+}h^{-1-\frac{q}{2q'}}dhdt\right)^{\frac{1}{q}}$$

for the boundary terms. Because of (4.4) and

$$\Delta_h(fg)(t)=(\Delta_h f)(t)g(t-h)+f(t)(\Delta_h g)(t),$$

we conclude

$$\begin{aligned}&\|(A(u)-A(v))|_{\partial\Omega^+}a^+\|_{\cdot,0,\frac{1}{2q'},q}\\ &\le C\left(\sup_{0\le t\le T}\|A(u)-A(v)\|_{\infty,\Omega}\|a^+\|_{\cdot,0,\frac{1}{2q'},q}\right.\\ &\qquad\left.+\sup_{0\le t\le T}\left(\int_0^t\|\Delta_h A(u-v)\|^q_{\infty,\Omega}h^{-1-\frac{q}{2q'}}dh\right)^{\frac{1}{q}}\|a^+\|_{q,\partial\Omega_T^+}\right)\\ &\le CT^{\frac{1}{2q'}}\|u-v\|_{2,1,q}\|a^+\|_{W_q^{\frac{1}{2q'}}(0,T;L^q(\partial\Omega^+))}.\end{aligned}\tag{4.10}$$

This implies

$$\begin{aligned}\|(\widetilde{T}_u^+-\widetilde{T}_v^+)(f,p)\|_{\cdot,0,\frac{1}{2q'},q} &\le CT^{\frac{1}{2q'}}\|u-v\|_{2,1,q}(\|f\|_{2,1,q}+\|p|_{\partial\Omega^+}\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}),\\ \|(\mathrm{div}_u-\mathrm{div}_v)f|_{\partial\Omega^+}\|_{\cdot,0,\frac{1}{2q'},q} &\le CT^{\frac{1}{2q'}}\|u-v\|_{2,1,q}\|f\|_{2,1,q}\end{aligned}$$

due to Lemma 2.2.2. Because of (4.5) and $\|X_u-X_v\|_{q,\partial\Omega_T^+}\le T\|(u-v)|_{\partial\Omega^+}\|_{q,\partial\Omega_T^+}\le CT\|u-v\|_{2,1,q}$, we have

$$\|(X_{u,n}-X_{v,n})|_{\partial\Omega^+}\|_{B_q^{\frac{1}{2q'}}(0,T;L^q(\partial\Omega^+))}\le CT^{\frac{1}{q}+\frac{1}{2q'}}\|u-v\|_{2,1,q}.$$

Thus (4.10) and the latter estimate imply

$$\|(X_{u,n}-\xi_n)A(u)^T\nu|_{\partial\Omega^+}-(X_{v,n}-\xi_n)A(v)^T\nu|_{\partial\Omega^+}\|_{\cdot,0,\frac{1}{2q'},q} \;\le\; CT^{\frac{1}{2q'}}\|u-v\|_{2,1,q}.$$

Since $\xi_n|_{\partial\Omega^+}=\gamma^+\in W_q^{1-\frac{1}{q}}$, (4.10) yields

$$\|(A(u)^T-A(v)^T)\nu\xi_n|_{\partial\Omega^+}\|_{\cdot,0,\frac{1}{2q'},q}\le CT^{\frac{1}{2q'}}\|u-v\|_{2,1,q},$$

which implies

$$\|X_{u,n}A(u)^T\nu|_{\partial\Omega^+}-X_{v,n}A(v)^T\nu|_{\partial\Omega^+}\|_{\cdot,0,\frac{1}{2q'},q} \;\le\; CT^{\frac{1}{2q'}}\|u-v\|_{2,1,q}.$$

In order to estimate $I[\widetilde{T}_u(f,p)-\widetilde{T}_v(f,p),0]^{\frac{1}{q}}$ in the case $q=3$ and $n=2$, we use (2.1). Then

$$\begin{aligned}&\int_0^T\int_{\partial\Omega}|\widetilde{T}_u(f,p)-\widetilde{T}_v(f,p)|^3t^{-1}d\sigma dt\\ &\quad\le\; C\left(\|f\|_{2,1,3}+\|p\|_{1,0,3}\right)^3\sup_{0\le t\le T}t^{-1}\|(A(u)-A(v))(.,t)\|_{1,3}^3,\end{aligned}$$

where $\|(A(u)-A(v))(.,t)\|_{1,3}\le Ct^{\frac{2}{3}}\|u-v\|_{2,1,q}$ because of (4.3). Hence

$$I[\widetilde{T}_u(f,p)-\widetilde{T}_v(f,p),0]^{\frac{1}{q}}\le CT^{\frac{1}{3}}\left(\|f\|_{2,1,3}+\|p\|_{1,0,3}\right)\|u-v\|_{2,1,q}.$$

Similarly, using

$$(X_{u,n}A(u)^T-X_{v,n}A(v)^T)|_{\partial\Omega^+}\in W_q^1(0,T;W_q^{1-\frac{1}{q}}(\partial\Omega^+))\hookrightarrow C^{\frac{2}{3}}(0,T;W_q^{1-\frac{1}{q}}(\partial\Omega^+))$$

$(X_{u,n}A(u)^T-X_{v,n}A(v)^T)|_{t=0}=0$, and (2.1), we conclude

$$I[(X_{u,n}A(u)^T-X_{v,n}A(v)^T)|_{\partial\Omega^+},0]^{\frac{1}{q}}\le CT^{\frac{1}{3}}\|u-v\|_{2,1,q}.$$

Finally, we estimate the $L^q(0,T;{}_0W_q^{-1}(\Omega))$-norm of $\partial_t(\mathrm{div}_u-\mathrm{div}_v)f$. Because of the definition of the ${}_0W_q^{-1}(\Omega)$-valued distributional derivate ∂_t,

$$\begin{aligned}&-\langle\langle\partial_t(\mathrm{div}_u-\mathrm{div}_v)f,\varphi\rangle_{(0,T)},w\rangle_\Omega\\ &= \int_{\Omega_T}(\mathrm{div}_u-\mathrm{div}_v)f(x,t)\partial_t\varphi(t)w(x)d(x,t)\\ &= -\int_{\Omega_T}f(x,t)\partial_t\varphi(t)\nabla\left[(A(u)-A(v))^Tw(x)\right]d(x,t)\\ &= \int_{\Omega_T}\partial_tf\varphi\nabla\left[(A(u)-A(v))^Tw\right]d(x,t)-\int_{\Omega_T}\mathrm{div}\,f\varphi\partial_t(A(u)^T-A^T(v))wd(x,t)\end{aligned}$$

for all $\varphi \in \mathcal{D}(0,T)$, $w \in {}^0W^1_{q'}(\Omega)$, and $u, v \in W^{2,1}_q(\Omega_T)$ such that $(u,0), (v,0), (f,0) \in X_T$. Firstly,

$$\begin{aligned} &\left| \int_\Omega \int_0^T \partial_t f \varphi \nabla \left[(A(u) - A(v))^T w \right] dt dx \right| \\ &\quad \leq \quad \|\partial_t f\|_{q,\Omega_T} \|\varphi\|_{q',(0,T)} \sup_{0\leq t\leq T} \left\| \nabla[(A(u) - A(v))^T(t,.)w] \right\|_{q',\Omega}. \end{aligned}$$

Using Sobolev inequalities, we have $\|w\|_r \leq \|w\|_{1,q'}$ for all $\frac{1}{q'} \geq \frac{1}{r} \geq \frac{1}{q'} - \frac{1}{n}$. Therefore $\|vw\|_{q'} \leq \|v\|_q \|w\|_r \leq C\|v\|_q\|w\|_{1,q'}$ with $\frac{1}{r} = \frac{1}{q'} - \frac{1}{q} > \frac{1}{q'} - \frac{1}{n}$. Hence

$$\begin{aligned} &\sup_{0\leq t\leq T} \left\| \nabla \left[(A(u) - A(v))^T(.,t)w \right] \right\|_{q',\Omega} \\ &\quad \leq \sup_{0\leq t\leq T} \left[\|\nabla (A(u) - A(v))\|_{q,\Omega} \|w\|_{r,\Omega} + \|A(u) - A(v)\|_{\infty,\Omega} \|\nabla w\|_{q',\Omega} \right] \\ &\quad \leq C \sup_{0\leq t\leq T} \|(A(u) - A(v))(.,t)\|_{1,q,\Omega} \|w\|_{1,q',\Omega} \leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} \|w\|_{1,q',\Omega} \end{aligned}$$

by (4.3). Secondly,

$$\begin{aligned} &\left| \int_\Omega \int_0^T \operatorname{div} f \partial_t (A(u) - A(v))^T \varphi w dt dx \right| \\ &\quad \leq \sup_{0\leq t\leq T} \|\operatorname{div} f(.,t)\|_{q,\Omega} \|\varphi\|_{q',(0,T)} \|\partial_t (A(u) - A(v))(.,t)\|_{L^q(0,T;L^\infty(\Omega))} \|w\|_{L^{q'}(\Omega)} \\ &\quad \leq C_\varepsilon T^{\frac{1}{2} - \frac{1}{q} - \varepsilon} \|f\|_{2,1,q} \|\varphi\|_{q',(0,T)} \|w\|_{1,q',\Omega} \|u - v\|_{2,1,q}. \end{aligned}$$

Here we have used (4.6) and $\sup_{0\leq t\leq T} \|\operatorname{div} f(.,t)\|_{q,\Omega} \leq C_\varepsilon T^{\frac{1}{2} - \frac{1}{q} - \varepsilon} \|f\|_{2,1,q}$ for $\varepsilon \in (0, \frac{1}{2} - \frac{1}{q})$ since $\operatorname{div} f|_{t=0} = 0$ and

$$\operatorname{div} f \in B_q^{\frac{1}{2} - \varepsilon}(0,T; L^q(\Omega)) \hookrightarrow C^{\frac{1}{2} - \frac{1}{q} - \varepsilon}(0,T; L^q(\Omega))$$

because of Lemma 4.2 and Simon [15, Corollary 26]. ■

Proof of Theorem 1.1: Let $R > 0$ be chosen so large that $\|L^{-1}h\|_{X_T} \leq \frac{R}{2}$ for $h = (0, 0, g_0\xi_n\nu, u_0)$ for some $T > 0$. Then, because of Lemma 4.3, there is a $0 < T_0 \leq T$ such that $L^{-1}G\colon \overline{B_R(0)}|_{X_{T_0}} \to \overline{B_R(0)}|_{X_{T_0}}$ is a contraction with Lipschitz constant $\kappa = \frac{1}{2}$. Now let $F(v) := L^{-1}G(v) + L^{-1}h$, $v \in X_{T_0}$. Then $\|F(v)\|_{X_{T_0}} \leq R$ for $v \in \overline{B_R(0)}|_{X_{T_0}}$ since $\|L^{-1}h\|_{X_{T_0}} \leq \|L^{-1}h\|_{X_T} \leq \frac{R}{2}$ and $\|L^{-1}G(v)\|_{X_{T_0}} = \|L^{-1}G(v) - L^{-1}G(0)\|_{X_{T_0}} \leq \frac{R}{2}$. Hence the Banach contraction mapping principle implies the existence of a unique solution in $\overline{B_R(0)}$.

Finally, if $v \in X_T$ is an arbitrary solution, we choose $R > 0$ in the construction above a priori so large that $R \geq \|v\|_{X_T}$. Then v coincides with the solution u obtained by the Banach fixed point theorem for some $0 < T_0 \leq T$. Since this argument can be repeated with initial data $v|_{t=T_0}$, the solution is unique as long as the solution exists. ■

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