Reduced and Generalized Stokes Resolvent Equations in Asymptotically Flat Layers, Part I: Unique Solvability

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Abstract

We study the generalized Stokes equations in asymptotically flat layers, which can be considered as compact perturbations of an infinite (flat) layer $\Omega_0 = \mathbb{R}^{n-1} \times (-1,1)$. Besides standard non-slip boundary conditions, we consider a mixture of slip and non-slip boundary conditions on the upper and lower boundary, respectively. In the first part we prove the unique solvability in L^q-Sobolev spaces, $1 < q < \infty$, by extending the known results in the case of an infinite layer Ω_0 via a perturbation argument to asymptotically flat layers which are sufficiently close to Ω_0. Combining this result with standard cut-off techniques and the parametrix constructed in the second part, we prove the unique solvability for an arbitrary asymptotically flat layer. Moreover, we show equivalence of unique solvability of the generalized and the reduced Stokes resolvent equations, which is essential for the second part of this contribution.

Key words: Stokes equations, free boundary value problems, boundary value problems for pseudodifferential operators, non-smooth pseudodifferential operators
AMS-Classification: 35 Q 30, 76 D 07, 35 R 35, 35 S 15

1 Introduction

We consider the generalized Stokes resolvent equations

$$
\begin{aligned}
(\lambda - \Delta)u + \nabla p &= f \quad && \text{in } \Omega_\gamma, && (1.1)\\
\operatorname{div} u &= g && \text{in } \Omega_\gamma, && (1.2)\\
T_j^+(u,p) &= a^+ && \text{on } \partial\Omega_\gamma^+, && (1.3)\\
u|_{\partial\Omega_\gamma^-} &= 0 && \text{on } \partial\Omega_\gamma^- && (1.4)
\end{aligned}
$$

with two kinds of boundary conditions, $j = 0$ or $j = 1$, where

$$
T_0^+(u,p) = u|_{\partial\Omega_\gamma^+}, \qquad T_1^+(u,p) = (\nu \cdot S(u) - \nu p)|_{\partial\Omega_\gamma^+}, \quad S(u) = \nabla u + (\nabla u)^T,
$$

and $\lambda \in \Sigma_\delta \cup \{0\}$. Here $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, is an asymptotically flat layer, which is a domain bounded by two surfaces $\partial\Omega_\gamma^+$ and $\partial\Omega_\gamma^-$, which get "close" to two parallel hyper-planes at infinity, see Definition 2.1 below.

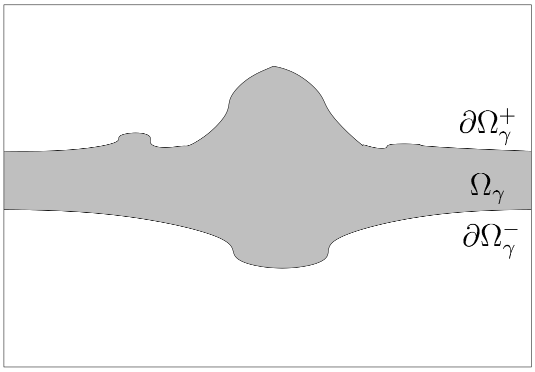

Figure 1: An asymptotically flat layer

The case $j = 0$ corresponds to standard non-slip boundary conditions. The mixed case $j = 1$ is important for application to free boundary value problems. Beale [7] and Sylvester [18] studied this case in a similar class of domains in L^2-Sobolev spaces. They applied their result to solve a free boundary value problem for the instationary Navier-Stokes equations, which describes the motion of an infinite ocean of water under force of gravity.

In the present contribution, we prove unique solvability of the system (1.1)-(1.4) in L^q-Sobolev spaces. The more general L^q-theory has many advantages for further applications to the non-linear Navier-Stokes equations. In the mixed case, the regularity assumptions on the data for the problem studied in [7, 18] can be reduced by using the embedding $W_q^1(\Omega) \hookrightarrow L^\infty(\Omega)$ for $q > n$ instead of $W_2^m(\Omega) \hookrightarrow L^\infty(\Omega)$ for $m > \frac{n}{2}$, cf. Solonnikov [17] for bounded domains and Abels [4]. Moreover, the following results in the case $j = 0$ imply that the Stokes operator generates a bounded analytic semi-group, which decays exponentially as $t \to \infty$. These properties can be used to construct strong solutions locally in time, cf. [14, 2].

Our main result is:

THEOREM 1.1 *Let $1 < q < \infty$, $j = 0, 1$, $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, and let $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, be an asymptotically flat $C^{1,1}$-layer. Then for every $(f, g, a^+) \in L^q(\Omega_\gamma)^n \times W^1_{q,\lambda}(\Omega_\gamma) \times W^{2-j-\frac{1}{q}}_{q,\lambda}(\partial\Omega_\gamma^+)^n$ with $g \in \dot{W}^{-1}_{q,0}(\Omega_\gamma)$ if $j = 0$ there is a unique solution $(u, p) \in W^2_{q,\lambda}(\Omega_\gamma)^n \times \dot{W}^1_q(\Omega_\gamma)$ of (1.1)-(1.4). Moreover,*

$$\begin{aligned} &(1 + |\lambda|)\|u\|_q + \|\nabla^2 u\|_q + \|\nabla p\|_q \\ &\leq \; C_\delta \left(\|f\|_q + \|g\|_{1,q,\lambda} + (1 + |\lambda|)\|g\|_{\cdot,-1,q,0} + \|a^+\|_{2-\frac{1}{q},q,\lambda} \right) \end{aligned} \tag{1.5}$$

if $j = 0$ *and*

$$\begin{aligned}(1+|\lambda|)\|u\|_q + \|\nabla^2 u\|_q + \|\nabla p\|_q + \|p|_{\partial\Omega_\gamma^+}\|_{1-\frac{1}{q},q,\lambda} \\ \leq \ C_\delta \left(\|f\|_q + \|g\|_{1,q,\lambda} + (1+|\lambda|)\|g\|_{{}_0W_q^{-1}} + \|a^+\|_{1-\frac{1}{q},q,\lambda}\right)\end{aligned} \tag{1.6}$$

if $j = 1$ *uniformly in* $\lambda \in \Sigma_\delta \cup \{0\}$. *If additionally* $(f, g, a^+) \in L^r(\Omega_\gamma)^n \times W_r^1(\Omega_\gamma) \times W_r^{2-j-\frac{1}{r}}(\partial\Omega_\gamma^+)^n$ *with* $g \in \dot{W}_{r,0}^{-1}(\Omega_\gamma)$ *if* $j = 0$, *then* $(u,p) \in W_r^2(\Omega_\gamma)^n \times \dot{W}_r^1(\Omega_\gamma)$.

The used function spaces are defined in Section 2 below.

In the case of an infinite layer $\Omega_0 = \mathbb{R}^{n-1} \times (-1,1)$, Theorem 1.1 was proved by Wiegner [19] in the case $j = 0$ and by Abels [3] in the case $j = 1$. Therefore we can use these results to obtain the unique solvability for asymptotically flat layers which are "sufficiently" close to an infinite layer. This will done by a similar perturbation argument as in Farwig and Sohr [10, Section 3]. Combining this result with cut-off techniques and the parametrix constructed in the second part, we prove Theorem 1.1 for arbitrary asymptotically flat layers.

The structure of the thesis is as follows:

In Section 2, we discuss some fundamental properties of L^q-Sobolev spaces on layer-like domains in parameter-dependent and homogeneous versions. In particular, the characterization of the homogeneous Sobolev space $\dot{W}_q^1(\Omega_\gamma)$ plays a central role. Then in Section 3 we prove equivalence of unique solvability of the generalized Stokes resolvent equations and a pseudodifferential boundary value problem – the reduced Stokes resolvent equations. This equivalence is fundamental for the parametrix construction in [5]. The main theorem is proved in Section 4 by the method described above. Finally, we prove the unique solvability of the weak Laplace resolvent equation with Neumann and mixed Neumann-Dirichlet boundary conditions in the appendix. These results are needed in Section 3. As a byproduct, we obtain the existence of the Helmholtz decomposition of $L^q(\Omega_\gamma)^n$ in the classical and a modified version with mixed boundary conditions.

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2 Preliminaries

2.1 Notation

In the following $\mathbb{N}$ denotes the set of natural numbers (without 0), $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

For $s \in \mathbb{R}$ we denote by $[s]$ the largest integer $\leq s$, set $\{s\} := s - [s] \in [0,1)$, and define $[s]_+ = \max\{s, 0\}$. Moreover, $\Sigma_\delta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \delta\}$, $\delta \in (0, \pi)$, denotes a sector with angle δ in $\mathbb{C}$, where $\arg z \in (-\pi, \pi]$, $z \neq 0$, denotes the angle of z in polar coordinates.

If $M \subseteq \mathbb{R}^n$ is measurable $L^q(M)$, $1 \leq q \leq \infty$ denotes the usual Lebesgue-space and $\|.\|_q$ its norm. If $\Omega \subset \mathbb{R}^n$ is an open set, then $L^q_{loc}(\overline{\Omega})$, $1 \leq q \leq \infty$, denotes the vector space of all measurable functions $f \colon \overline{\Omega} \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, such that $f \in L^q(B \cap \overline{\Omega})$ for all balls B with $B \cap \overline{\Omega} \neq \emptyset$. If $f \in L^q(M)$, $g \in L^{q'}(M)$, where $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$(f, g)_M := \int_M f(x)\overline{g(x)}dx.$$

The set of all smooth bounded functions $f \colon \mathbb{R}^n \to \mathbb{R}$ with bounded derivatives is denoted by $C^\infty(\mathbb{R}^n)$. If $\Omega \subseteq \mathbb{R}^n$ is a domain, $C_0^\infty(\Omega) = \mathcal{D}(\Omega)$ is the set of all functions $f \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} f \subset \Omega$ is compact. Furthermore,

$$C^\infty_{(0)}(\overline{\Omega}) \;\; := \;\; \{f \colon \overline{\Omega} \to \mathbb{R} : f = u|_{\overline{\Omega}}, \text{where } u \in C_0^\infty(\mathbb{R}^n)\}.$$

The Banach space of all functions $f \colon \mathbb{R}^n \to \mathbb{R}$ that are k-times differentiable and have Lipschitz continuous k-th derivatives is denoted by $C^{k,1}(\mathbb{R}^n)$, $k \in \mathbb{N}_0$.

The dual of a topological vector space V is denoted by V'. If $v \in V$ and $v' \in V'$, then $\langle v, v' \rangle := v'(v)$ denotes the duality product. If $A \colon V \to W$ is a continuous linear operator, $A' \colon W' \to V'$ denotes its adjoint.

Finally, if $x \in \mathbb{R}^n$, $n \geq 2$, then we use the decomposition $x = (x', x_n)$, where x' denotes the first $n-1$ components.

2.2 Layer-Like Domains

Definition 2.1 Let $k \in \mathbb{N}_0$ and $n \geq 2$. If $\gamma = (\gamma^+, \gamma^-) \in C^{k,1}(\mathbb{R}^{n-1})^2$ with $\gamma^+(x') - \gamma^-(x') \geq c > 0$ for all $x' \in \mathbb{R}^{n-1}$, then

$$\Omega_\gamma = \{(x', x_n) \in \mathbb{R}^n : \gamma^-(x') < x_n < \gamma^+(x')\}$$

is called a *layer-like domain* with $C^{k,1}$-boundary. If additionally $\lim_{|x'|\to\infty} \gamma^\pm(x') = \gamma^\pm_\infty$ for some constants $\gamma^\pm_\infty \in \mathbb{R}$ and $\lim_{|x'|\to\infty} D^\alpha_{x'}\gamma^\pm(x') = 0$ for all $|\alpha| \leq k+1$, then Ω_γ is called an *asymptotically flat layer*, cf. Figure 1. In the case that $\gamma^\pm(x') \equiv \gamma^\pm_\infty$ are constant, Ω_γ is called *infinite (flat) layer*.

We will assume w.l.o.g. that $\gamma^\pm_\infty = \pm 1$ and denote by $\Omega_0 = \mathbb{R}^{n-1} \times (-1, 1)$ the corresponding infinite layer.

If Ω_γ is a layer-like domain with $C^{k,1}$-boundary, then we will use the $C^{k,1}$-diffeomorphism $F : \Omega_0 \to \Omega_\gamma$ defined by

$$F(x) = \begin{pmatrix} x' \\ \frac{\gamma_+(x') - \gamma_-(x')}{2} x_n + \frac{\gamma_+(x') + \gamma_-(x')}{2} \end{pmatrix}, \quad x \in \Omega_0,$$

to reduce problems to the infinite layer Ω_0. If $u\colon \Omega_\gamma \to \mathbb{C}$, then $F^*(u)(x) = u(F(x))$ defines the pull-back of u to Ω_0. Similarly, if $v\colon \Omega_0 \to \mathbb{C}$, $F^{*,-1}(v) = v(F^{-1}(x))$ defines the push-forward of v.

We denote by $\partial\Omega_\gamma^\pm = \{(x', \gamma^\pm(x')) : x' \in \mathbb{R}^{n-1}\}$ the upper and lower boundary, respectively. In order to localize the domain around the upper and lower boundary, we choose a partition of unity $\varphi_0^\pm \in C^\infty([-1,1])$ with $\varphi_0^- \equiv 1$ on $[-1,-\frac{1}{2}]$ and $\varphi_0^+ \equiv 1$ on $[\frac{1}{2},1]$. Then $\varphi^\pm = F^{*,-1}(\varphi_0^\pm)$ is a partition of unity on $\overline{\Omega_\gamma}$ with $\varphi^\pm \equiv 1$ in a neighborhood of $\partial\Omega_\gamma^\pm$. Because of the construction, $\varphi^\pm \in C^{k,1}(\overline{\Omega_\gamma})$. By convolution with a suitable smooth function, we can achieve $\varphi^\pm \in C^\infty(\overline{\Omega_\gamma})$.

2.3 Non-Homogeneous Spaces

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Studying resolvent equations, it is natural to consider the usual Sobolev space with parameter-dependent norm

$$\begin{aligned} W^m_{q,\lambda}(\Omega) &= \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), |\alpha| \le m, \|u\|_{m,q,\lambda} < \infty\}, \\ \|u\|_{m,q,\lambda} &= \sum_{|\alpha|\le m} (1+|\lambda|)^{\frac{1}{2}(m-|\alpha|)} \|D^\alpha u\|_q, \end{aligned}$$

for $1 \le q \le \infty$, $m \in \mathbb{N}_0$, cf. [11, Section 1]. Moreover, $W^m_{q,0,\lambda}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^m_{q,\lambda}(\Omega)$ and

$$W^{-m}_{q,\lambda}(\Omega) \;:=\; \big(W^m_{q',0,\lambda}(\Omega)\big)', \qquad W^{-m}_{q,0,\lambda}(\Omega) := \big(W^m_{q',\lambda}(\Omega)\big)'.$$

Then $W^m_q(\Omega) := W^m_{q,\lambda}(\Omega)|_{\lambda=0}$ coincides with the usual parameter-independent Sobolev space.

Lemma 2.2 *Let $k \in \mathbb{N}_0$, $m \in \{0,1,\dots,k+1\}$, $1 < q < \infty$, and let $\Omega_\gamma \subset \mathbb{R}^n$ be a layer-like domain with $C^{k,1}$-boundary. Then F^* is a linear isomorphism in the following settings:*

$$F^*\colon W^m_{q,\lambda}(\Omega_\gamma) \to W^m_{q,\lambda}(\Omega_0), \quad F^*\colon L^q_{loc}(\overline{\Omega_\gamma}) \cap W^{-1}_{q,0}(\Omega_\gamma) \to L^q_{loc}(\overline{\Omega_0}) \cap W^{-1}_{q,0}(\Omega_0).$$

Here $L^q_{loc}(\overline{\Omega}) \cap W^{-1}_{q,0}(\Omega)$ is understood as the topological vector space of all functions $f \in L^q_{loc}(\overline{\Omega})$ that extend to functionals in $W^{-1}_{q,0}(\Omega)$ if the functions are identified with functionals in the canonical way.

Proof: The first part is a direct consequence of the chain rule, the transformation formula, and $0 < c \le \det DF = \frac{1}{2}(\gamma^+ - \gamma^-) \le C$. Since

$$\int_{\Omega_0} (F^*f)(x) g(x) dx \;=\; \int_{\Omega_\gamma} f(x) (F^{*,-1}g)(x) \det DF^{-1}(x) dx,$$

we get $|(F^*f, g)_{\Omega_0}| \le C\|f\|_{W^{-1}_{q,0}(\Omega_\gamma)} \|g\|_{W^1_{q'}(\Omega_0)}$ for all $g \in W^1_q(\Omega_0)$, which implies the second part. ■

The natural parameter-dependent Sobolev-Slobodeckij spaces are defined as

$$\begin{aligned}
W^s_{q,\lambda}(M) &= \{f \in L^q(M) : \|f\|_{s,q,\lambda} < \infty\},\\
\|f\|_{s,q,\lambda} &= (1+|\lambda|)^{\frac{s-[s]}{2}}\|f\|_{W^{[s]}_{q,\lambda}} + \sum_{|\alpha|=[s]}\left(\int_M\int_M \frac{|D^\alpha f(x)-D^\alpha f(y)|^q}{|x-y|^{n-1+q\{s\}}}d\sigma_x d\sigma_y\right)^{\frac{1}{q}},
\end{aligned}$$

for a given $(n-1)$-dimensional $C^{k,1}$-sub-manifold M of $\mathbb{R}^n$ without boundary and $0 < s \notin \mathbb{N}$ with $s < k+1$.

If Ω_γ is a layer-like domain with $C^{k,1}$-boundary, $\partial\Omega_\gamma$ is the disjoint union of two $C^{k,1}$-manifolds. Therefore we identify a function $a\colon \partial\Omega_\gamma \to \mathbb{C}$ with the tuple (a^+, a^-), where $a^\pm = a|_{\partial\Omega^\pm}$.

Let $F_0 = F|_{\partial\Omega_0}$ be the restriction of F to the boundary. Then

$$F_0^*\colon W^s_{q,\lambda}(\partial\Omega_\gamma) \to W^s_{q,\lambda}(\partial\Omega_0), \qquad F_0^{*,-1}\colon W^s_{q,\lambda}(\partial\Omega_0) \to W^s_{q,\lambda}(\partial\Omega_\gamma)$$

are continuous, if $0 \le s < k+1$, with operator norms uniformly bounded in $\lambda \in \mathbb{C}$.

We denote by $\gamma_0 u = u|_{\partial\Omega}$ the Dirichlet trace and by $\gamma_j u = (\gamma_j^+ u, \gamma_j^- u)$ the trace of the j-th normal derivative $\gamma_j u = \gamma_0(\partial_\nu)^j u$, where ν denotes the exterior normal vector on the boundary. Moreover, if $u\colon \Omega \to \mathbb{C}^n$ is a vector field, we also use the trace of the normal component $\gamma_\nu u = \nu \cdot \gamma_0 u$ and $\gamma_\nu^\pm u = \nu \cdot \gamma_0^\pm u$. Similarly, a_ν and a_τ denote the normal and the tangential components, resp., of a vector field $a\colon \partial\Omega \to \mathbb{C}^n$.

Lemma 2.3 *Let $1 < q < \infty$, $m \in \mathbb{N}$, and let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be a layer-like domain with $C^{m-1,1}$-boundary. Then:*

1. $(\gamma_0, \dots, \gamma_{m-1})\colon W^m_{q,\lambda}(\Omega) \to \prod_{j=0}^{m-1} W^{m-j-\frac{1}{q}}_{q,\lambda}(\partial\Omega)$ *is a surjective and continuous linear mapping.*
2. $W^m_{q,0,\lambda}(\Omega_\gamma) = \{f \in W^m_{q,\lambda}(\Omega_\gamma) : \gamma_j f = 0$ *for* $j = 0, \dots, m-1\}$

Proof: Using the coordinate transformation $F\colon \Omega_\gamma \to \Omega_0$, the statements are easily reduced to the case of an infinite layer Ω_0. Using a partition of unity on $\overline{\Omega_0}$, the statements are easily reduced to the corresponding statements for the half-spaces $\mathbb{R}^n_+$, cf. [11, Theorem 1.1]. ■

Since we will consider differential equations with mixed Dirichlet-Neumann boundary conditions, it is natural to consider

$$\begin{aligned}
{}^0W^m_q(\Omega_\gamma) &= \{u \in W^m_q(\Omega_\gamma) : \gamma_j^+ u = 0 \text{ for } j = 0, \dots, m-1\},\\
{}_0W^m_q(\Omega_\gamma) &= \{u \in W^m_q(\Omega_\gamma) : \gamma_j^- u = 0 \text{ for } j = 0, \dots, m-1\}.
\end{aligned}$$

Similarly to the notation above, we define

$${}^0W^{-m}_q(\Omega_\gamma) = \left({}_0W^m_{q'}(\Omega_\gamma)\right)', \qquad {}_0W^{-m}_q(\Omega_\gamma) = \left({}^0W^m_{q'}(\Omega_\gamma)\right)'$$

for $m \in \mathbb{N}$. Moreover, Lemma 2.2 holds if W^m_q and $W^{-1}_{q,0}$ are replaces by ${}_0W^m_q$ and ${}^0W^{-1}_q$, resp.

2.4 Homogeneous Sobolev Spaces

Although the usual (non-homogeneous) Sobolev, Besov, and Bessel potential spaces for a layer-like domain, defined in the last section, have the usual properties, the analysis of the homogeneous spaces causes more problems.

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let

$$\dot{W}_q^m(\Omega) = \{f \in L_{loc}^q(\overline{\Omega}) : D^\alpha f \in L^q(\Omega) \text{ for all } |\alpha| = m\}$$

be the usual homogeneous Sobolev space of order $m \in \mathbb{N}_0$. If we identify functions which differ by a polynomial of order $m-1$, then $\dot{W}_q^m(\Omega)$ is a Banach space. At first sight it is surprising that

$$\dot{W}_q^1(\Omega_\gamma) \neq r_{\Omega_\gamma} \dot{W}_q^1(\mathbb{R}^n) \tag{2.1}$$

if Ω_γ is a layer-like domain, where $r_{\Omega_\gamma} f = f|_{\Omega_\gamma}$. Of course $r_{\Omega_\gamma} : \dot{W}_q^1(\mathbb{R}^n) \to \dot{W}_q^1(\Omega_\gamma)$ is continuous. But there is no continuous extension operator $e : \dot{W}_q^1(\Omega_\gamma) \to \dot{W}_q^1(\mathbb{R}^n)$.

This can easily be seen with the aid of the next important characterization.

Lemma 2.4 *Let $1 < q < \infty$ and let $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, be a Lipschitz layer-like domain. Then*

$$\dot{W}_q^1(\Omega_\gamma) = \{u \in L_{loc}^q(\overline{\Omega_\gamma}) : u(x) = u_1(x) + u_2(x'), u_1 \in W_q^1(\Omega_\gamma), u_2 \in \dot{W}_q^1(\mathbb{R}^{n-1})\},$$

where u_1 and u_2 in the decomposition of $u \in \dot{W}_q^1(\Omega_\gamma)$ can be chosen such that $\|u_1\|_{1,q}, \|\nabla' u_2\|_q \leq C\|\nabla u\|_q$. Finally, $C_{(0)}^\infty(\overline{\Omega_\gamma})$ is dense in $\dot{W}_q^1(\Omega_\gamma)$.

Proof: First of all, we note that $F^* \colon \dot{W}_q^1(\Omega_\gamma) \to \dot{W}_q^1(\Omega_0)$ since

$$\nabla F^*(u) = (D_x F^T) F^*(\nabla u),$$

where $D_x F \in L^\infty(\Omega_0)^{n\times n}$ if γ is Lipschitz. Now we set

$$u_2(x') = \frac{1}{\gamma^+ - \gamma^-}\int_{\gamma^-}^{\gamma^+} u(x', y_n) dy_n = \frac{1}{2}\int_{-1}^{1} u(F(x', y_n)) dy_n = F^*(u)_{[-1,1]}(x')$$

and $u_1 := u - u_2$. Here f_B denotes the mean-value of f on the set B. Then

$$\|\nabla' u_2\|_{L^q(\mathbb{R}^{n-1})} \leq \frac{1}{2}\int_{-1}^{1} \|\nabla' F^*(u)(.,y_n)\|_q dy_n \leq C\|\nabla u\|_{L^q(\Omega_\gamma)},$$

$$\|u_1\|_{L^q(\Omega_\gamma)}^q \leq C\int_{\mathbb{R}^{n-1}} \|F^*(u) - F^*(u)_{[-1,1]}\|_{L^q(-1,1)}^q dx' \leq C\|\nabla u\|_{L^q(\Omega_\gamma)}^q,$$

where we have used Poincaré's inequality on $(-1,1)$ for functions with vanishing mean-value. ∎

Corollary 2.5 *Let $1 < q < \infty$ and let $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, be a layer-like domain with $C^{0,1}$-boundary. Then F^* is a linear isomorphism in the following settings:*

$$F^*\colon \dot{W}^1_q(\Omega_\gamma) \to \dot{W}^1_q(\Omega_0), \quad F^*\colon L^q_{loc}(\overline{\Omega_\gamma}) \cap \dot{W}^{-1}_{q,0}(\Omega_\gamma) \to L^q_{loc}(\overline{\Omega_0}) \cap \dot{W}^{-1}_{q,0}(\Omega_0).$$

Here $L^q_{loc}(\overline{\Omega}) \cap \dot{W}^{-1}_{q,0}(\Omega)$ is understood as the topological vector space of all functions $f \in L^q_{loc}(\overline{\Omega})$ that extend to functionals in $\dot{W}^{-1}_{q,0}(\Omega)$ if the functions are identified with functionals in the canonical way.

Proof: The first mapping property was proved in the proof of Lemma 2.4.

Due to the characterization of $\dot{W}^1_{q'}(\Omega_\gamma)$, $g \in L^q_{loc}(\overline{\Omega_\gamma}) \cap \dot{W}^{-1}_{q,0}(\Omega_\gamma)$ iff $g \in L^q_{loc}(\overline{\Omega_\gamma}) \cap W^{-1}_{q,0}(\Omega_\gamma)$ and

$$\frac{1}{\gamma^+ - \gamma^-} \int_{\gamma^-}^{\gamma^+} g(x', x_n)dx_n \in L^q_{loc}(\mathbb{R}^{n-1}) \cap \dot{W}^{-1}_q(\mathbb{R}^{n-1}).$$

Replacing $\gamma^\pm$ with ± 1, we get the corresponding characterization of $L^q_{loc}(\overline{\Omega_0}) \cap \dot{W}^{-1}_{q,0}(\Omega_0)$.

Since $F^*\colon W^{-1}_q(\Omega_\gamma) \to W^{-1}_q(\Omega_0)$, it remains to show that

$$\frac{1}{2} \int_{-1}^{1} (F^*g)(x', x_n)dx_n \in L^q_{loc}(\mathbb{R}^{n-1}) \cap \dot{W}^{-1}_q(\mathbb{R}^{n-1}).$$

Using $\det DF^{-1}(x') = \frac{2}{(\gamma^+ - \gamma^-)(x')}$, we conclude

$$\int_{\mathbb{R}^{n-1}} \frac{1}{2} \int_{-1}^{1} (F^*g)(x', x_n)dx_n \varphi(x')dx' = \int_{\mathbb{R}^{n-1}} \frac{1}{\gamma^+ - \gamma^-} \int_{\gamma^-}^{\gamma^+} g(x', x_n)dx_n \varphi(x')dx'$$

for all $\varphi \in C^\infty_0(\mathbb{R}^{n-1}) \cap \dot{W}^1_{q'}(\mathbb{R}^{n-1})$. Hence $\frac{1}{2}\int_{-1}^{1}(F^*g)(x', x_n)dx_n \in \dot{W}^{-1}_q(\mathbb{R}^{n-1})$. ■

Remarks 2.6

1. To prove (2.1), let $u \in \dot{W}^1_q(\mathbb{R}^{n-1}) \setminus \dot{W}^{1-\frac{1}{q}}_q(\mathbb{R}^{n-1})$. Considering u as a function in $\dot{W}^1_q(\Omega_\gamma)$ independent of x_n, u does not have an extension $U \in \dot{W}^1_q(\mathbb{R}^n)$ since $U|_{x_n=0} = u \in \dot{W}^{1-\frac{1}{q}}_q(\mathbb{R}^{n-1})$ would contradict the assumption. Here $\dot{W}^{1-\frac{1}{q}}_q(\mathbb{R}^{n-1}) = \dot{B}^{1-\frac{1}{q}}_q(\mathbb{R}^{n-1})$ is defined as in [8, Section 6.3].

2. We can modify the decomposition Lemma 2.4 as follows: If $u_2(x') \in \dot{W}^1_q(\mathbb{R}^{n-1})$ as above, we split $u_2 = u_2' + u_2''$ with $u_2' = \mathcal{F}^{-1}_{\xi' \mapsto x'}[\varphi(\xi')\tilde{u}_2(\xi')]$ where $\varphi \in C^\infty_0(\mathbb{R}^{n-1})$, $\varphi \equiv 1$ on $B_1(0)$. Then $u_2' \in L^q_{loc}(\mathbb{R}^{n-1})$ with $\nabla' u_2' \in W^1_q(\mathbb{R}^{n-1})$, and $u_2'' \in W^1_q(\mathbb{R}^{n-1})$. Hence we get a decomposition of $u \in \dot{W}^1_q(\Omega_0)$ with

$$u(x) = u_1'(x) + u_2'(x'), \quad \|u_1'\|_{W^1_q(\Omega_0)} + \|\nabla' u_2'\|_{W^1_q(\mathbb{R}^{n-1})} \leq C\|\nabla u\|_{L^q(\Omega_0)}.$$

3. Although $\dot{W}^1_q(\Omega_\gamma)$ does not have the extension property to $\mathbb{R}^n$, there is a continuous extension operator $e\colon \dot{W}^1_q(\Omega_\gamma) \to \dot{W}^1_q(\Omega_{\gamma'})$ where $\Omega_{\gamma'} \supseteq \Omega_\gamma$ is a second layer-like domain.

As for the non-homogeneous spaces we set $\dot{W}^{-1}_{q,0}(\Omega) = (\dot{W}^1_{q'}(\Omega))'$. Note that $L^q(\Omega_\gamma)$ is not continuously embedded in $\dot{W}^{-1}_{q,0}(\Omega_\gamma)$, if we identify a function with a distribution in the usual sense. The norm

$$\|u\|_{\cdot,-1,q,0} \;:=\; \sup\left\{\left|\int_{\Omega_\gamma} uvdx\right| : v \in C^\infty_{(0)}(\overline{\Omega_\gamma}), \|\nabla v\|_{q'} = 1\right\}$$

does not have to be finite for $u \in L^q(\Omega_\gamma)$. If $u \in L^q_{loc}(\overline{\Omega_\gamma})$ and $\|u\|_{\cdot,-1,q,0} < \infty$, then we can extend $v \mapsto \int_{\Omega_\gamma} uvdx$, $v \in C^\infty_{(0)}(\overline{\Omega_\gamma})$, uniquely to a functional on $\dot{W}^1_{q'}(\Omega_\gamma)$. In this sense we understand $u \in L^q_{loc}(\overline{\Omega_\gamma}) \cap \dot{W}^{-1}_{q,0}(\Omega_\gamma)$. In particular, $W^1_q(\Omega_\gamma) \cap \dot{W}^1_{q,0}(\Omega_\gamma)$ has to be understood in this sense.

Remark 2.7 Using Lemma 2.4 again and $L^q(\mathbb{R}^{n-1}) \subset \dot{W}^{-1}_q(\mathbb{R}^{n-1}) \cap \dot{W}^1_q(\mathbb{R}^{n-1})$, it can be shown that $W^1_q(\Omega_\gamma) \cap \dot{W}^{-1}_{q,0}(\Omega_\gamma) = \dot{W}^1_q(\Omega_\gamma) \cap \dot{W}^{-1}_{q,0}(\Omega_\gamma)$ algebraically and topologically, cf. [6, Lemma 2.8].

The compactly supported elements of $L^q(\Omega) \cap \dot{W}^{-1}_{q,0}(\Omega)$ can be characterized as follows.

Lemma 2.8 *Let $n \geq 2$ and let $\Omega = \Omega_\gamma \subset \mathbb{R}^n$ be a layer-like Lipschitz domain or let $\Omega = \mathbb{R}^{n-1}$.*

1. *If $n = 2$ and $1 < q < \infty$ or $n \geq 3$ and $1 < q \leq \frac{n-1}{n-2}$, then every $g \in L^q(\Omega) \cap \dot{W}^{-1}_{q,0}(\Omega)$ with compact support satisfies*

$$\int_{\operatorname{supp} g} gdx = 0.$$

2. *If $n \geq 3$ and $q > \frac{n-1}{n-2}$, then $g \in L^q(\Omega)$ and $\operatorname{supp} g$ compact implies $g \in \dot{W}^{-1}_{q,0}(\Omega)$.*

Proof: First let $\Omega = \Omega_\gamma$ be a layer-like domain.

1. Let $M > 0$ be so large that $\operatorname{supp} g \subseteq \{x \in \Omega_\gamma : |x'| \leq M\} =: D_M$. Moreover, let $\varphi_R \in C^\infty_0(\mathbb{R}^{n-1})$ be such that $\varphi_R(x') \equiv 1$ on $B_M(0)$, $\operatorname{supp} \varphi_R \subseteq B_{M+R}(0)$, and $|\nabla'\varphi_R(x')| \leq CR^{-1}$. Then $\|\nabla'\varphi_R\|^{q'}_{q'} \leq CR^{n-1-q'}$, which is bounded as $R \to \infty$ if $q' \geq n-1$. If $n \geq 3$, the latter condition is equivalent to $q \leq \frac{n-1}{n-2}$. If $n = 2$, the condition is satisfied for all $1 < q < \infty$. Hence there is subsequence $\nabla'\varphi_{R_j}$, $j \in \mathbb{N}$, that converges weakly in $L^{q'}(\mathbb{R}^{n-1})$. Since $(\nabla'\varphi_R)_{R>0}$ converges pointwise to 0 as $R \to \infty$, the only possible weak accumulation point is 0. Hence $\nabla'\varphi_R \rightharpoonup 0$ as $R \to \infty$.

Because of the Lemma of Mazur, there is a sequence of convex combinations ψ_R of φ_R such that $\nabla'\psi_R \to 0$ in $L^{q'}(\mathbb{R}^{n-1})$ as $R\to\infty$. Therefore

$$\left|\int_{\operatorname{supp} g} g dx\right| = \left|\int_{\Omega_\gamma} g\psi_R dx\right| \le C\|\nabla'\psi_R\|_{q'} \to 0 \quad \text{as } R\to\infty.$$

2. Let $\varphi \in \dot W^1_{q'}(\Omega_\gamma)$ and let $\varphi = \varphi_1 + \varphi_2$, $\varphi_1 \in W^1_{q'}(\Omega_\gamma)$, $\varphi_2 \in \dot W^1_{q'}(\mathbb{R}^{n-1})$, be its decomposition due to Lemma 2.4. Using the Sobolev embedding $\dot W^1_{q'}(\mathbb{R}^{n-1}) \hookrightarrow L^r(\mathbb{R}^{n-1})$ with $\frac{1}{r} = \frac{1}{q'} - \frac{1}{n-1}$, we obtain $\varphi_2 \in L^r(\mathbb{R}^{n-1})$, where the condition $q > \frac{n-1}{n-2}$ is equivalent to $\frac{1}{r} = \frac{1}{q'} - \frac{1}{n-1} > 0$. Therefore

$$\begin{aligned} |(g,\varphi)_{\Omega_\gamma}| &\le \|g\|_q\|\varphi\|_{q',D_M} \le C\|g\|_q\left(\|\varphi_1\|_{q',D_M} + \|\varphi_2\|_{r,D_M}\right) \\ &\le C\|g\|_q\left(\|\varphi_1\|_{1,q'} + \|\nabla'\varphi_2\|_{q'}\right) \le C\|g\|_q\|\nabla\varphi\|_{q'} \end{aligned}$$

for arbitrary $\varphi \in \dot W^1_{q'}(\Omega_\gamma)$, where $B := \operatorname{supp} g$. Hence $g \in \dot W^{-1}_{q,0}(\Omega_\gamma)$.

The arguments for the proof in the case $\Omega = \mathbb{R}^{n-1}$ are contained in the proof for a layer-like domain. ■

If $f \in L^q(\Omega_\gamma)^n$ with $\operatorname{div} f \in L^q_{loc}(\overline{\Omega_\gamma}) \cap \dot W^{-1}_{q,0}(\Omega_\gamma)$, we can define a weak trace $\gamma_\nu f$ as

$$\langle \gamma_\nu f, v\rangle_{\partial\Omega_\gamma} := (f,\nabla V) + (\operatorname{div} f, V), \tag{2.2}$$

where $v \in \gamma_0(\dot W^1_{q'}(\Omega_\gamma))$ and $V \in \dot W^1_{q'}(\Omega_\gamma)$ is an arbitrary extension of v. If $f \in C^\infty_{(0)}(\overline{\Omega_\gamma})$, the definition coincides with the usual trace. Moreover, the definition does not depend on the choice of V. Hence

$$\gamma_\nu f \in (\gamma_0 \dot W^1_{q'}(\Omega_\gamma))' =: W^{-\frac{1}{q}}_{q,\nu}(\partial\Omega_\gamma)$$

and $\|\gamma_\nu f\|_{W^{-\frac{1}{q}}_{q,\nu}} \le C\left(\|f\|_{L^q} + \|\operatorname{div} f\|_{\dot W^{-1}_{q,0}}\right)$. In particular, $\gamma_\nu f$ is defined for $f \in L^q(\Omega_\gamma)^n$ with $\operatorname{div} f = 0$.

Remark 2.9 Using Lemma 2.4, we can characterize $\gamma_0(\dot W^1_{q'}(\Omega_\gamma))'$ as follows:

$$W^{-\frac{1}{q}}_{q,\nu}(\partial\Omega_\gamma) = \{a \in W_q^{-\frac{1}{q}}(\partial\Omega_\gamma) : (F_0^* a)^+\kappa^+ + (F_0^* a)^-\kappa^- \in \dot W_q^{-1}(\mathbb{R}^{n-1})\},$$

where $\kappa^\pm(x') = \sqrt{1+|\nabla'\gamma^\pm(x')|^2}$, cf. [6, Lemma 2.11] for details.

Similarly, if $u \in L^q(\Omega_\gamma)^n$ with $\operatorname{div} u \in L^q_{loc}(\overline{\Omega_\gamma}) \cap {}^0W_q^{-1}(\Omega_\gamma)$, then we define the trace $\gamma_\nu^+ u \in W_q^{-\frac{1}{q}}(\partial\Omega_\gamma^+)$ as

$$\langle \gamma_\nu^+ u, v\rangle = (u,\nabla V) + (\operatorname{div} u, V), \tag{2.3}$$

where $v \in W_{q'}^{1-\frac{1}{q'}}(\partial\Omega_\gamma^+)$ and $V \in {}_0W^1_{q'}(\Omega_\gamma)$ with $\gamma_0^+ V = v$. As in the case of $\gamma_\nu u$, the definition does not depend on the choice of V. Moreover,

$$\|\gamma_\nu^+ u\|_{-\frac{1}{q},q} \le C\left(\|u\|_q + \|\operatorname{div} u\|_{{}^0W_q^{-1}}\right).$$

In the same way we can define $\gamma_\nu^- u \in W_q^{-\frac{1}{q}}(\partial\Omega_\gamma^-)$.

3 Reduced Stokes Equations and Weak Laplace Resolvent Equations

The following reduction is a modification of the reduction introduced by Grubb and Solonnikov [12].

First we consider the Dirichlet case $j = 0$. Let $(u, p) \in W_q^2(\Omega_\gamma)^n \times \dot{W}_q^1(\Omega_\gamma)$ be a solution of (1.1)-(1.4) with

$$f \in L^q(\Omega_\gamma)^n, \ a^+ = 0, \ g \in W_q^1(\Omega_\gamma) \cap \dot{W}_{q,0}^{-1}(\Omega_\gamma). \tag{3.1}$$

Applying first $-\operatorname{div}$ to (1.1) then γ_ν, the pressure p is determined by

$$\begin{aligned} -\Delta p &= -\operatorname{div} f + (\lambda - \Delta) g && \text{in } \Omega_\gamma, \\ \gamma_1 p &= \gamma_\nu(f + \nabla g) + \gamma_\nu(\Delta - \nabla \operatorname{div}) u && \text{on } \partial\Omega_\gamma. \end{aligned} \tag{3.2}$$

Provided we have proved unique solvability of this weak Neumann problem, see Theorem 3.1 below, we can split $p = p_1 + p_2$ such that p_1 depends only on u and p_2 depends only on (f, g). Then we end up with the *reduced Stokes equations*

$$(\lambda - \Delta) u + G_0 u = f_r \quad \text{in } \Omega_\gamma, \tag{3.3}$$

$$\gamma_0 u = 0 \quad \text{on } \partial\Omega_\gamma \tag{3.4}$$

where $G_0 u = \nabla K_1 \gamma_\nu (\Delta - \nabla \operatorname{div}) u$ and $f_r = f - \nabla p_2$. Here K_1 denotes the Poisson operator for the Laplace equation.

The most important fact is that we can drop the equation $\operatorname{div} u = g$: If u solves the equations (3.3)-(3.4) with f_r defined as above, then

$$\begin{aligned} (\lambda - \Delta) \operatorname{div} u &= (\lambda - \Delta) g && \text{in } \Omega_\gamma, \\ \gamma_1 \operatorname{div} u &= \gamma_1 g && \text{on } \partial\Omega_\gamma \end{aligned}$$

because of the construction. Since these equations are uniquely solvable, we conclude $\operatorname{div} u = g$.

THEOREM 3.1 *Let $1 < q, r < \infty$, $\delta \in (0, \pi)$, and Ω_γ, $n \geq 2$, be an asymptotically flat layer with $C^{0,1}$-boundary. Then for every $f \in L^q(\Omega_\gamma)^n$ and $\lambda \in \Sigma_\delta \cup \{0\}$ there is a unique solution $u \in \dot{W}_q^1(\Omega_\gamma)$ with $\lambda u \in \dot{W}_{q,0}^{-1}(\Omega_\gamma)$ of the weak Laplace resolvent equation*

$$(\lambda - \Delta) u = -\operatorname{div} f \quad \textit{in } \Omega_\gamma, \tag{3.5}$$

$$\gamma_1 u = \gamma_\nu f \quad \textit{on } \partial\Omega_\gamma, \tag{3.6}$$

where (3.6) is understood as $\gamma_\nu(\nabla u - f) = 0$, cf. (2.2). Moreover, u is uniquely determined by

$$\lambda(u, v) + (\nabla u, \nabla v) = (f, \nabla v) \quad \textit{for all } v \in \dot{W}_{q'}^1(\Omega_\gamma) \tag{3.7}$$

and satisfies

$$|\lambda|\|u\|_{\dot{W}^{-1}_{q,0}} + |\lambda|^{\frac{1}{2}}\|u\|_q + \|\nabla u\|_q \leq C_{q,\delta,\varepsilon}\|f\|_q \tag{3.8}$$

uniformly in $\lambda \in \Sigma_\delta$ *with* $|\lambda| \geq \varepsilon > 0$ *and for* $\lambda = 0$. *If additionally* $f \in L^r(\Omega_\gamma)^n$, *then* $u \in \dot{W}^1_r(\Omega_\gamma)$ *and* $\lambda u \in \dot{W}^{-1}_{r,0}(\Omega_\gamma)$.

Hence the generalized Stokes equations (1.1)-(1.4) with right-hand side as in (3.1) are uniquely solvable if the reduced Stokes equations (3.3)-(3.4) are uniquely solvable for $f_r \in L^q(\Omega_\gamma)^n$.

In the case $\lambda \neq 0$, the converse implication is also true: If $f_r \in L^q(\Omega_\gamma)^n$, then we get a solution u of the reduced Stokes equations (3.3)-(3.4) as follows: Let $(u,p) \in W^2_q(\Omega_\gamma) \times \dot{W}^1_q(\Omega_\gamma)$ be solution of the generalized Stokes equations (1.1)-(1.4) with right-hand side $(f_r, g, 0)$, where g is determined as solution of (3.5)-(3.6) with right-hand side f_r. Then $\nabla p = G_0 u$ since $-\Delta p = 0$ and $\gamma_1 p = \gamma_\nu(\Delta - \nabla \operatorname{div})u$. Hence u solves the reduced Stokes equations.

Therefore we have proved

Lemma 3.2 *Let* $1 < q < \infty$, $\lambda \in \Sigma_\delta$, $\delta \in (0,\pi)$, $j = 0$, *and let* $\Omega_\gamma \subseteq \mathbb{R}^n$, $n \geq 2$, *be an asymptotically flat layer with* $C^{0,1}$*-boundary. Then the generalized Stokes equations (1.1)-(1.4) are uniquely solvable for given data as in (3.1) iff the reduced Stokes equations (3.3)-(3.4) are uniquely solvable for every* $f_r \in L^q(\Omega_\gamma)^n$. *Moreover, the solutions of the generalized Stokes equations satisfy (1.5) for all* $\lambda \in \Sigma_\delta$ *with* $|\lambda| \geq \varepsilon > 0$ *iff the solutions of reduced Stokes equations satisfy*

$$|\lambda|\|u\|_q + |\lambda|^{\frac{1}{2}}\|\nabla u\|_q + \|\nabla^2 u\|_q \leq C_{q,\delta,\varepsilon}\|f_r\|_q \tag{3.9}$$

for all $\lambda \in \Sigma_\delta$ *with* $|\lambda| \geq \varepsilon > 0$.

Remark 3.3 We can consider $A_0 := -\Delta + G_0$ as unbounded operators with domain $\mathcal{D}(A_0) := \mathcal{D}(\Delta_D)^n = W^2_q(\Omega_\gamma)^n \cap W^1_{q,0}(\Omega_\gamma)^n$. We call this operator *reduced Stokes operator*. Then $A_0|_{J_{q,0}} = A_q$, where $A_q = -P_q\Delta_D$ is the usual Stokes operator and $P_q \colon L^q(\Omega_\gamma)^n \to J_{q,0}(\Omega_\gamma)$ is the Helmholtz projection, cf. Corollary A.3 below. This statement can be seen as follows: If $u \in \mathcal{D}(\Delta_D)^n \cap J_{q,0}(\Omega_\gamma)$, then $\operatorname{div}(-\Delta u + G_0 u) = 0$ and

$$\gamma_\nu(-\Delta u + G_0 u) = -\gamma_\nu \Delta u + \gamma_1 K_1 \gamma_\nu(\Delta - \nabla \operatorname{div})u = 0.$$

Hence $-\Delta u = (-\Delta + G_0)u - G_0 u$ is the Helmholtz decomposition of $-\Delta u$, i.e., $(-\Delta + G_0)u = P_q(-\Delta)u = A_q u$.

Moreover, if $f \in J_{q,0}(\Omega_\gamma)$ and $u = (\lambda - \Delta + G_0)^{-1} f$, then $\operatorname{div} u = 0$ and therefore $u \in J_{q,0}(\Omega_\gamma)$ since $(\lambda - \Delta)\operatorname{div} u = 0$ and $\gamma_1 \operatorname{div} u = 0$. Hence $(\lambda - \Delta + G_0)^{-1}|_{J_{q,0}} = (\lambda + A_q)^{-1}$ if $(\lambda - \Delta - G_0)^{-1}$ exists and $\lambda \notin (-\infty, 0)$.

Remark 3.4 Although the Stokes operator A_q is known to be invertible, cf. [19] and [1, 2], the reduced Stokes operator A_0 is not, which can be proved as follows: Let $f = \nabla p$ with $p \in \dot{W}^1_q(\Omega_\gamma) \setminus L^q(\Omega_\gamma)$. Then $p + c \notin \dot{W}^{-1}_{q,0}(\Omega_\gamma)$ for any $c \in \mathbb{R}$ because of

Remark 2.7. Thus there is no $u \in W_q^2(\Omega_\gamma)^n$ which solves (3.3)-(3.4) for $\lambda = 0$ since the reduced Stokes equations imply

$$\begin{aligned} -\Delta \operatorname{div} u &= -\Delta p && \text{in } \Omega_\gamma \\ \gamma_1 \operatorname{div} u &= \gamma_1 p && \text{on } \partial\Omega_\gamma \end{aligned}$$

and therefore $\operatorname{div} u = p + c \notin \dot{W}_{q,0}^{-1}(\Omega_\gamma)$ which contradicts the compatibility condition $g = \operatorname{div} u \in \dot{W}_{q,0}^{-1}(\Omega_\gamma) \cap W_q^1(\Omega_\gamma)$.

In the mixed case, $j = 1$, there is an analogous reduction of (1.1)-(1.4) to the system

$$(\lambda - \Delta)u + G_{10}u = f_r \quad \text{in } \Omega_0, \tag{3.10}$$

$$T_1'^{+}u = a_r^+ \quad \text{on } \partial\Omega_0^+, \tag{3.11}$$

$$\gamma_0^- u = 0 \quad \text{on } \partial\Omega_0^- \tag{3.12}$$

with

$$G_{10}u = \nabla K_{01} \begin{pmatrix} 2\gamma_1^+ u_\nu \\ \gamma_\nu^-(\Delta - \nabla \operatorname{div})u \end{pmatrix}, \quad (T_1'^{+}u)_\tau = (\gamma_\nu^+ S(u))_\tau, \quad (T_1'^{+}u)_\nu = \gamma_0^+ \operatorname{div} u,$$

$f_r = f - \nabla p_2$, $(a_r^+)_\tau = a_\tau^+$, and $a_\nu^+ = \gamma_0^+ g$, where p_2 solves (3.2) with boundary conditions $\gamma_0^+ p_2 = -a_\nu^+$ and $\gamma_1^- p_2 = \gamma_\nu^- f + \gamma_1^- g$.

Lemma 3.5 *Let $1 < q < \infty$, $\delta \in (0,\pi)$, $j = 1$, and let $\Omega_\gamma \subseteq \mathbb{R}^n$, $n \geq 2$, be an asymptotically flat layer with $C^{0,1}$-boundary. Then the generalized Stokes equations (1.1)-(1.4) are uniquely solvable for $(f, g, a^+) \in L^q(\Omega_\gamma)^n \times W_{q,\lambda}^1(\Omega_\gamma) \times W_{q,\lambda}^{1-\frac{1}{q}}(\partial\Omega_\gamma^+)^n$ iff the reduced Stokes equations (3.10)-(3.12) are uniquely solvable for every $f_r \in L^q(\Omega_\gamma)^n$ and $a_r^+ \in W_{q,\lambda}^{1-\frac{1}{q}}(\partial\Omega_\gamma^+)^n$. Moreover, the solutions of the generalized Stokes equations satisfy (1.6) for all $\lambda \in \Sigma_\delta \cup \{0\}$ iff the solutions of reduced Stokes equations satisfy*

$$(1 + |\lambda|)\|u\|_q + \|\nabla^2 u\|_q \leq C_{q,\delta}\left(\|f_r\|_q + \|a_r^+\|_{1-\frac{1}{q},q,\lambda}\right) \tag{3.13}$$

for all $\lambda \in \Sigma_\delta \cup \{0\}$.

Proof: The proof is done in the same way as in the Dirichlet case, $j = 0$, using Theorem 3.6 below instead of Theorem 3.1 and changing the boundary conditions in the obvious way. Details can be found in [3, Section 4.1], where Ω_0 has to be replaced by Ω_γ. ■

THEOREM 3.6 *Let $1 < q, r < \infty$, $\delta \in (0,\pi)$, and $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, be an asymptotically flat layer with $C^{0,1}$-boundary. Then for every $f \in L^q(\Omega_\gamma)^n$ and $\lambda \in \Sigma_\delta \cup \{0\}$ there is a unique solution $u \in {}^0W_q^1(\Omega_\gamma)$ of the weak Laplace resolvent equation*

$$(\lambda - \Delta)u = -\operatorname{div} f \quad \text{in } \Omega_\gamma, \tag{3.14}$$

$$\gamma_0^+ u = 0 \quad \text{on } \partial\Omega_\gamma^+, \tag{3.15}$$

$$\gamma_1^- u = \gamma_\nu^- f \quad \text{on } \partial\Omega_\gamma^-, \tag{3.16}$$

where (3.16) is understood as $\gamma_\nu^-(\nabla u - f) = 0$, cf. (2.3). Moreover, u is uniquely determined by

$$\lambda(u,v) + (\nabla u, \nabla v) = (f, \nabla v) \quad \text{for all } v \in {}_0W^1_{q'}(\Omega_\gamma). \tag{3.17}$$

and satisfies $(1+|\lambda|)\|u\|_{{}^0W_q^{-1}} + \|u\|_{1,q,\lambda} \le C_{q,\delta}\|f\|_q$ uniformly in $\lambda \in \Sigma_\delta \cup \{0\}$. If additionally $f \in L^r(\Omega_\gamma)^n$, then $u \in {}^0W^1_r(\Omega_\gamma)$.

4 Stokes Equations in Asymptotically Flat Layers

4.1 Small Global Perturbations

Let $\Omega_\gamma \subset \mathbb{R}^n$, $n \ge 2$, be an asymptotically flat layer with $C^{1,1}$-boundary, Ω_0 the infinite layer, and $F\colon \Omega_0 \to \Omega_\gamma$ be the diffeomorphism defined in Section 2.2. In the following we will denote by $\underline{\ }$ all variables, functions, and operators acting in Ω_0, e.g. $\underline{x}, \underline{u}(\underline{x}), \underline{\nabla}, \underline{\Delta}$. For the variables, functions, and operators on Ω_γ will use simple letters as $x, u(x), \nabla, \Delta$.

In order to apply a similar perturbation argument as in [10, Section 3], we observe

$$\begin{aligned}
\underline{\nabla} F^* u &= \begin{pmatrix} I' & \underline{\nabla}' a(\underline{x}) \\ 0 & \partial_n a(\underline{x}) \end{pmatrix} F^* \nabla u, \\
\underline{\Delta} F^* u &= F^*\Delta' u + \left((\underline{\partial}_n a)^2 + |\underline{\nabla}' a|^2\right) F^*(\partial_n^2 u) + 2\underline{\nabla}' a F^*(\partial_n \nabla' u) + \underline{\Delta}' a F^*(\partial_n u)
\end{aligned}$$

with $a(\underline{x}) = \frac{1}{2}(\gamma^+(\underline{x}') - \gamma^-(\underline{x}'))\underline{x}_n + \frac{1}{2}(\gamma^+(\underline{x}') + \gamma^-(\underline{x}'))$. Hence

$$F^{*,-1}\underline{\nabla} F^* u = \nabla u + R_1 u, \quad F^{*,-1}\underline{\operatorname{div}} F^* u = \operatorname{div} u + R_2 u, \tag{4.1}$$

$$F^{*,-1}\underline{\Delta} F^* u = \Delta u + R_3 u, \tag{4.2}$$

with

$$\|R_1 u\|_q \le C(\gamma)\|\nabla u\|_q, \quad \|R_2 u\|_{1,q,\lambda} \le C(\gamma)\|\nabla u\|_{2,q,\lambda}, \quad \|R_3 u\|_q \le C(\gamma)\|u\|_{2,q,\lambda},$$

where $C(\gamma) \to 0$ if $\|(\nabla'\gamma^\pm, \nabla'^2\gamma^\pm, \gamma^+ - \gamma^- - 2)\|_\infty \to 0$.

THEOREM 4.1 *Let $1 < q, r < \infty$, $\lambda \in \Sigma_\delta \cup \{0\}$, $\delta \in (0,\pi)$, $n \ge 2$, and $j = 0$ or $j = 1$. Then there is a constant $K = K(q, r, \delta) > 0$ such that Theorem 1.1 holds under the additional assumption $\|(\nabla'\gamma^\pm, \nabla'^2\gamma^\pm, \gamma^+ + \gamma^- - 2)\|_\infty \le K$.*

Proof: First let $j = 0$. To apply the perturbation argument we introduce the operator

$$S_{0,\lambda}\colon X_0 \to Y_0, \quad S_{0,\lambda}(u,p) = (\lambda u - \Delta u + \nabla p, \operatorname{div} u)$$

on the Banach spaces $X_0 = \left(W^2_{q,\lambda}(\Omega_0)^n \cap W^1_{q,0,\lambda}(\Omega_0)^n\right) \times \dot{W}^1_q(\Omega_0)$ and $Y_0 = L^q(\Omega_0)^n \times (W^1_{q,\lambda}(\Omega_0) \cap \dot{W}^{-1}_{q,0}(\Omega_0))$, where $\|(u,p)\|_{X_0} := \|u\|_{2,q,\lambda} + \|\nabla p\|_q$ and $\|(f,g)\|_{Y_0} := \|f\|_q +$

$\|g\|_{1,q,\lambda} + (1+|\lambda|)\|g\|_{\dot{W}^{-1}_{q,0}(\Omega_0)}$. The corresponding operator and function spaces on Ω_γ are denoted by $S_{\gamma,\lambda}$, X_γ, and Y_γ.

Because of the unique solvability of the generalized Stokes equations in Ω_0 and the resolvent estimate, cf. [19], $S_{0,\lambda}\colon X_0 \to Y_0$ is an isomorphism with norms uniformly bounded in $\Sigma_\delta \cup \{0\}$. Due to Lemma 2.2 and Corollary 2.5, $F^{*,-1} S_{0,\lambda} F^*\colon X_\gamma \to Y_\gamma$ is an isomorphism. Using the identities (4.1)-(4.2), we get

$$F^{*,-1} S_{0,\lambda} F^* \;=\; S_{\gamma,\lambda} + R_{\gamma,\lambda},$$

where $R_{\gamma,\lambda}(u,p) = (-R_3 u + R_1 p, R_2 u)$ satisfying $\|R_{\gamma,\lambda}(u,p)\|_{\mathcal{L}(X_\gamma,Y_\gamma)} \le C(\gamma)$ with a constant $C(\gamma)$ which gets arbitrarily small if $K > 0$ in the assumption of the lemma is sufficiently small. Hence there is a $K > 0$ such that

$$\|R_{\gamma,\lambda}(u,p)\|_{Y_\gamma} \le \kappa \|F^{*,-1} S_{0,\lambda} F^*(u,p)\|_{Y_\gamma}$$

uniformly in $\lambda \in \Sigma_\delta \cup \{0\}$ with $\kappa < 1$. Thus $R_{\gamma,\lambda}$ is relatively bounded with respect to the isomorphism $F^{*,-1} S_{0,\lambda} F^*(u,p)$ and we can apply Kato's perturbation criterion, cf. [13, Chapter 4, Theorem 1.16]. Therefore $S_{\gamma,\lambda}\colon X_\gamma \to Y_\gamma$ is an isomorphism for all $\lambda \in \Sigma_\delta \cup \{0\}$ and $\|S_{\gamma,\lambda}\|_{\mathcal{L}(X_\gamma,Y_\gamma)} \le C_\delta$.

For the mixed case $j = 1$ we replace the spaces by

$$\begin{aligned}
X_{10} &= (W^2_{q,\lambda}(\Omega_0)^n \cap {}_0W^1_{q,\lambda}(\Omega_0)^n) \times \left\{ p \in \dot{W}^1_q(\Omega_0) : \gamma_0^+ p \in W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0^+) \right\}, \\
\|(u,p)\|_{X_{10}} &= \|u\|_{2,q,\lambda} + \|\nabla p\|_q + \|\gamma_0^+ p\|_{1-\frac{1}{q},q,\lambda}, \\
Y_{10} &= L^q(\Omega_0)^n \times W^1_{q,\lambda}(\Omega_0) \times W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_0^+)^n, \\
\|(f,g,a^+)\|_{Y_{10}} &= \|f\|_q + \|g\|_{1,q,\lambda} + (1+|\lambda|)\|g\|_{{}_0W^{-1}_q(\Omega_0)} + \|a^+\|_{1-\frac{1}{q},q,\lambda}
\end{aligned}$$

and the operator by

$$S_{10,\lambda}\colon X_{10} \to Y_{10}, \quad S_{10,\lambda}(u,p) = (\lambda u - \Delta u + \nabla p, \operatorname{div} u, T_1^+(u,p))$$

and carry on in the same way as in the Dirichlet case $j = 0$ with the additional perturbation term

$$R_4 u = \gamma_0^+(\nu_0^+ - \nu_\gamma^+) \cdot S(u) - \gamma_0^+(\nu_0^+ - \nu_\gamma^+) p + \gamma_0 \nu_\gamma^+ \cdot (R_1 u + (R_1 u)^T)$$

in the boundary condition on $\partial\Omega_\gamma^+$. Here $\nu_\gamma^+ = (-\nabla'\gamma^+(x'), 1)/\sqrt{1 + |\nabla'\gamma^+(x')|^2}$ denotes the exterior normal vector on $\partial\Omega_\gamma^+$ and $\nu_0^+ = e_n$ the exterior normal vector on $\partial\Omega_0^+$. This term can be estimated in the same way as before using additionally

$$\|\nu_\gamma^+ - \nu_0^+\|_{C^{0,1}(\mathbb{R}^{n-1})} \le C\|(\nabla'\gamma^+, \nabla'^2\gamma^+)\|_\infty.$$

■

4.2 Large Local Perturbations

Let $K > 0$ be the constant in Theorem 4.1 for given $1 < q < \infty$ and $r = 2$. If $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, is an arbitrary asymptotically flat layer with $C^{1,1}$-boundary, we can find an asymptotically flat domain Ω_{γ_0} such that $\|(\nabla\gamma_0^\pm, \nabla^2\gamma_0^\pm, \gamma_0^+ - \gamma_0^- - 2)\|_\infty \leq K$ and which coincides with Ω_γ on $\mathbb{R}^n \setminus B_R(0)$. Moreover, let Ω_+ and Ω_- be bounded $C^{1,1}$-domains with

$$\Omega_\gamma = (\Omega_{\gamma_0} \setminus B_{R+1}(0)) \cup \Omega_+ \cup \Omega_- \quad \text{and} \quad \operatorname{dist}(\Omega_\pm, \partial\Omega_\gamma^\mp) > 0.$$

Now let $\varphi_0, \varphi_\pm$ be a partition of unity on $\overline{\Omega_\gamma}$ and $\psi_0, \psi_\pm \in C^\infty_{(0)}(\overline{\Omega_\gamma})$ such that $\psi_* = 1$ on $\operatorname{supp}\varphi_*$, $* = 0, \pm$, $\operatorname{supp}\psi_0 \subset \mathbb{R}^n \setminus B_R(0)$, and $\operatorname{supp}\psi_\pm \subseteq \overline{\Omega_\pm}$.

Lemma 4.2 *Let $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. Then $S_{0,\lambda}$ and $S_{10,\lambda}$ defined as in the proof of Theorem 4.1 are Fredholm operators with trivial kernel.*

Proof: First let $j = 0$ and w.l.o.g. $a^+ = 0$. We set $g_0 = \psi_0 g - h_0$, where $h_0 \in C_0^\infty(\Omega_{\gamma_0} \cap \Omega_\gamma)$ with $\operatorname{supp} h_0 \cap \operatorname{supp}\psi_0 = \emptyset$ and $\|h_0\|_q \leq C\|g\|_q$ is chosen such that

$$\int_{\operatorname{supp}(1-\psi_0)\cap\Omega_\gamma} ((1 - \psi_0)g - h_0)dx = 0.$$

Then $g_0 \in \dot{W}^{-1}_{q,0}(\Omega_\gamma)$ with $\operatorname{supp} g_0 \subseteq \Omega_\gamma \cap \Omega_{\gamma_0}$. Hence $g_0 \in \dot{W}^{-1}_{q,0}(\Omega_{\gamma_0})$. Similarly, we choose $g_\pm = \psi_\pm g - h_\pm$, where $h_\pm \in C_0^\infty(\Omega_\pm)$ is chosen such that $\operatorname{supp} h_\pm \cap \operatorname{supp}\varphi_\pm = \emptyset$ and $\int_{\Omega_\pm} g_\pm dx = 0$. Moreover, let $f_* = \varphi_* f$, $* \in I := \{0, +, -\}$, and $D_\pm := \operatorname{supp}\varphi_\pm$. Now let (u_*, p_*), $* \in I$, be the solution of the generalized Stokes equations in $\Omega_{\gamma_0}, \Omega_\pm$, resp., with right-hand side (f_*, g_*) and non-slip boundary condition, where p_0 and $p_\pm$ are chosen such that

$$\int_{\Omega_{\gamma_0}\cap\Omega_\pm} p_0 dx = \int_{\Omega_{\gamma_0}\cap\Omega_\pm} p_\pm dx. \tag{4.3}$$

Then $(u, p) := \sum_{*\in I} \psi_*(u_*, p_*) \in W^2_{q,\lambda}(\Omega_\gamma)^n \times \dot{W}^1_q(\Omega_\gamma)$ depends continuously on (f, g) and solves

$$\begin{aligned} (\lambda - \Delta)u + \nabla p &= f + S_1 u + S_2 p && \text{in } \Omega_\gamma, \\ \operatorname{div} u &= g + S_3 u && \text{in } \Omega_\gamma, \\ \gamma_0 u &= 0 && \text{on } \partial\Omega_\gamma, \end{aligned}$$

where S_1 is a differential operator of order 1 with coefficients supported in $\operatorname{supp}\nabla\varphi_0 = \operatorname{supp}\nabla\varphi_+ \cup \operatorname{supp}\nabla\varphi_-$, $S_2 p = \nabla\varphi_0(p_0 - p_\pm)$ in $\Omega_\pm$, and $S_3 u = \nabla\varphi_0 \cdot (u_0 - u_\pm)$ in $\Omega_\pm$.

Since all error terms are supported in the bounded set $D_+ \cup D_-$ and possess higher regularity, the operator $K_\lambda(f, g) := (S_1 u + S_2 p, S_3 u)$ is compact from $L^q \times (W^1_{q,\lambda} \cap \dot{W}^{-1}_{q,0})$ to $L^q \times W^1_{q,\lambda}$.

Because of $g, \operatorname{div} u \in \dot{W}^{-1}_{q,0}(\Omega_\gamma)$, also $S_3 u \in \dot{W}^{-1}_{q,0}(\Omega_\gamma)$. Since $\operatorname{supp} S_3 u \subseteq D_+ \cup D_-$, this is equivalent to

$$\int_{D_+ \cup D_-} S_3 u dx = 0$$

if $n = 2$ or $n \geq 3$, $1 < q \leq \frac{n-1}{n-2}$, due to Lemma 2.8. If $n \geq 3$ and $q > \frac{n-1}{n-2}$, every compactly supported L^q function is in $\dot{W}^{-1}_{q,0}(\Omega_\gamma)$. Hence $\|S_3 u\|_{\cdot,-1,q,0} \leq C\|S_3 u\|_q$ and K_λ is a compact mapping on $L^q \times (W^1_q \cap \dot{W}^{-1}_{q,0})$. Thus $S_{0,\lambda}$ is a semi-Fredholm operator with finite co-dimension of $\mathcal{R}(S_{0,\lambda})$.

Finally, let $(u, p) \in W^2_{q,\lambda}(\Omega_\gamma)^n \times \dot{W}^1_q(\Omega_\gamma)$ be a zero-solution. If $q = 2$, testing the equation (1.1) with u and integrating by parts immediately implies that $u = 0$ and therefore $\nabla p = 0$. If $q \neq 2$, then $(v, \tilde{p}) = (\varphi^0 u, \varphi^0 p)$ solves the generalized Stokes equations in Ω_{γ_0} with compactly supported right-hand side in $W^1_q(\Omega_{\gamma_0})^n \times W^2_q(\Omega_{\gamma_0})$. Therefore, if $q > 2$, $(v, \tilde{p}) \in W^2_2(\Omega_{\gamma_0})^n \times \dot{W}^1_2(\Omega_{\gamma_0})$ because of the regularity assertion of Theorem 4.1 for $r = 2$. Thus $(u, p) \in W^2_2(\Omega_\gamma)^n \times \dot{W}^1_2(\Omega_\gamma)$, which implies $(u, \nabla p) = 0$. In the case $1 < q < 2$, we use Sobolev's embedding $W^1_q(\Omega_{\gamma_0}) \hookrightarrow L^r(\Omega_{\gamma_0})$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{n}$, to obtain $(v, \tilde{p}) \in W^2_r(\Omega_{\gamma_0})^n \times \dot{W}^1_r(\Omega_{\gamma_0})$. Repeating this argument finitely many times and choosing $K > 0$ a priori sufficiently small, we obtain $(u, p) \in W^2_s(\Omega_\gamma)^n \times \dot{W}^1_s(\Omega_\gamma)$ for some $s > 2$, which implies $(u, \nabla p) = 0$.

Now we consider the case $j = 1$. In this case we replace $\Omega_\pm$ by bounded domains $\Omega_1, \ldots, \Omega_N, \Omega_-$ with $C^{1,1}$-boundary such that

$$\Omega_\gamma = (\Omega_{\gamma_0} \setminus B_{R+1}(0)) \cup \bigcup_{k=1}^N \Omega_k \cup \Omega_-, \quad \operatorname{dist}(\bigcup_{k=1}^N \Omega_k, \partial\Omega^-_\gamma), \operatorname{dist}(\Omega_-, \partial\Omega^+_\gamma) > 0.$$

Moreover, we assume that Ω_k, $k = 1, \ldots, N$, are chosen such that there are asymptotically flat layers $\Omega_{\gamma_k} \supset \Omega_k$, $k = 1, \ldots, N$, (possibly rotated) satisfying the assumptions of Theorem 4.1 and $\partial\Omega^+_\gamma \cap \partial\Omega_k = \partial\Omega^+_\gamma \cap \partial\Omega^+_{\gamma_k}$. Moreover, let $\varphi_0, \ldots, \varphi_N, \varphi_-$ and $\psi_0, \ldots, \psi_N, \psi_-$ be associated cut-off functions chosen such that $\operatorname{supp} \psi_k \subseteq \overline{\Omega_k}$, $k = 0, \ldots, N$, and $\operatorname{supp} \psi_- \subseteq \overline{\Omega_-}$.

As before we set $f_* = \varphi_* f$, $* \in I := \{0, \ldots, N, -\}$. Moreover, let $g_* = \varphi_* g$ if $* = 0, \ldots, N$ and let g_- be defined as before. Finally, we set $a^+_k = \varphi_k a^+$, $k = 0, \ldots, N$. Now let (u_k, p_k), $k = 0, \ldots, N$ be the solution of (1.1)-(1.4) in Ω_{γ_k}, $k = 0, \ldots, N$, with right-hand side (f_k, g_k, a^+_k) and $j = 1$. Moreover, let (u_-, p_-) be the solution of the Stokes resolvent equations in the bounded domain Ω_- with non-slip boundary condition and right-hand side (f_-, g_-) where p_- is chosen as in (4.3).

Similarly to the Dirichlet case, we set $(u, p) := \sum_{* \in I} \psi_*(u_*, p_*)$. As before we get a solution of the generalized Stokes equations with compactly supported perturbations of higher order. Then we proceed in the same way as before, showing injectivity. ■

Lemma 4.3 *Let $1 < q < \infty$, $\delta \in (0, \pi)$, and $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, be an asymptotically flat layer with $C^{1,1}$-boundary. Then there is an $R = R(q, \delta, \gamma) > 0$ such that Theorem 1.1 holds for all $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R > 0$ and $q = r$.*

Proof: The lemma is a consequence of Lemma 5.18 in the second part of this contribution [5], which is proved independently of the result obtained in the present first part.

We can assume w.l.o.g. $a^+ = 0$. Firstly, let $j = 0$. Then [5, Lemma 5.18] implies that there is an approximate solution operator $R_{0,\lambda}$ such that

$$\begin{aligned}(\lambda - \Delta + G_0)R_{0,\lambda} f &= (I + S_{1,\lambda})f \quad &&\text{in } \Omega_\gamma,\\ \gamma_0 R_{0,\lambda} f &= 0 &&\text{on } \partial\Omega_\gamma\end{aligned}$$

for $f \in L^q(\Omega_\gamma)^n$ with $\|S_{1,\lambda} f\|_q \leq C_\delta(1 + |\lambda|)^{-\varepsilon}\|f\|_q$, $\varepsilon > 0$, $\lambda \in \Sigma_\delta \cup \{0\}$. Hence there is an $R > 0$ such that $I + S_{1,\lambda}$ is invertible for all $\lambda \in \Sigma_\delta$, $|\lambda| \geq R$. Thus the reduced Stokes equations are uniquely solvable for $\lambda \in \Sigma_\delta$, $|\lambda| \geq R$. Because of Lemma 3.2, the same is true for the generalized Stokes equations. Moreover, (1.5) is a consequence of $\|R_{0,\lambda} f\|_{2,q,\lambda} \leq C_{q,\delta}\|f\|_q$ and $\|(I + S_{1,\lambda})^{-1} f\|_q \leq \|f\|_q$ for $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R$.

In the case $j = 1$, there is an $R_{10,\lambda}$ such that

$$\begin{aligned}(\lambda - \Delta + G_{10})R_{10,\lambda} f &= (I + S_{2,\lambda})f \quad &&\text{in } \Omega_\gamma,\\ T_1'^{+} R_{10,\lambda} f &= S_{3,\lambda} f &&\text{on } \partial\Omega_\gamma^+,\\ \gamma_0^- R_{10,\lambda} f &= 0 &&\text{on } \partial\Omega_\gamma^-\end{aligned}$$

for all $f \in L^q(\Omega_\gamma)^n$ and

$$\|S_{2,\lambda}\|_{\mathcal{L}(L^q(\Omega_\gamma))} + \|S_{3,\lambda}\|_{\mathcal{L}(L^q(\Omega_\gamma), W^{1-\frac{1}{q}}_{q,\lambda}(\partial\Omega_\gamma))} \leq C_{q,\delta}(1 + |\lambda|)^{-\varepsilon},$$

cf. [5, Lemma 5.18]. Choosing $v \in W^2_{q,\lambda}(\Omega_\gamma)^n$ with $T_1'^{+} v = S_{3,\lambda} f$ and $\|v\|_{2,q,\lambda} \leq C\|S_{3,\lambda} f\|_{1-\frac{1}{q},q,\lambda}$, we can modify $R_{10,\lambda}$ such that $T_1'^{+} R_{10,\lambda} f = 0$ and the estimate of $S_{2,\lambda}$ is preserved. The rest of the proof is done in the same way as before. ■

Proof of Theorem 1.1: Because of Lemma 4.2, the unique solvability of the generalized Stokes equations (1.1)-(1.4) is a consequence of Lemma 4.3. This implies that $S_{0,\lambda}$ and $S_{10,\lambda}$ have index zero for all $\lambda \in \mathbb{C}\setminus(-\infty, 0)$ because of the homotopy invariance of the Fredholm index. Moreover, it is sufficient to prove the a priori estimates (1.5) and (1.6) for large λ, which is also a consequence of Lemma 4.3 below.

Alternatively, one can prove that the range of $S_{0,\lambda}$ and $S_{10,\lambda}$ are dense, which can be done analogously to the proof of Lemma 4.1 (iii) in [10]. Moreover, the a priori estimate can be proved similarly to [10, Lemma 4.2].

Finally, the regularity assertion is proved by considering $(\varphi_0 u, \varphi_0 p)$ in Ω_{γ_0} and applying similar arguments as in Lemma 4.2. ■

A Proof of Theorem 3.1 and Theorem 3.6

Lemma A.1 *Theorem 3.1 holds if $\Omega_\gamma = \Omega_0$ is an infinite layer.*

Proof: If $\lambda = 0$, the unique solvability of the weak Neumann problem is equivalent to the existence of the classical Helmholtz decomposition of $L^q(\Omega_0)^n$, which was proved by Miyakawa [15] and by Farwig [9]. For $\lambda \in \Sigma_\delta$ with $|\lambda| \geq \varepsilon > 0$, the proof can be done in the same way as [3, Lemma 3.1] using the Poisson operator $K_{1,\lambda} = \mathrm{OP}'(k_{1,\lambda}(\xi', D_n))$ with

$$k_{1,\lambda}(\xi', D_n)\tilde{a} \;:=\; \frac{\cosh(\zeta_\lambda x_n)}{\zeta_\lambda \sinh \zeta_\lambda} \frac{\tilde{a}^+ + \tilde{a}^-}{2} + \frac{\sinh(\zeta_\lambda x_n)}{\zeta_\lambda \cosh \zeta_\lambda} \frac{\tilde{a}^+ - \tilde{a}^-}{2},$$

$\zeta_\lambda = (\lambda + |\xi'|^2)^{\frac{1}{2}}$, instead of $K_{10,\lambda}$. Details can be found in [6, Section 4]. Finally, the regularity assertion holds since the solution operators for different q and r coincide on the dense subset $C^\infty_{(0)}(\overline{\Omega_0})$. ■

Lemma A.2 *Let $1 < q, r < \infty$ and $\delta \in (0, \pi)$. Then there is a $K = K(q, r, \delta)$ such that Theorem 3.1 and Theorem 3.6 hold for all layer-like domains $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, such that $\|(\nabla'\gamma^\pm, \nabla'^2\gamma^\pm, \gamma^+ + \gamma^- - 2)\|_\infty \leq K$.*

Proof: The proof is done with the same perturbation argument used in the proof of Theorem 4.1. ■

Proof of Theorem 3.6: Let Ω_{γ_0} be an asymptotically flat layer with Lipschitz-boundary chosen in the same way as in Section 4.2. Moreover, let $\Omega_{1,\pm}, \ldots, \Omega_{N,\pm}$, Ω_b be bounded Lipschitz-domains such that

$$\Omega_\gamma = (\Omega_{\gamma_0} \setminus B_{R+1}(0)) \cup \bigcup_{k=1}^N (\Omega_{k,+} \cup \Omega_{k,-}) \cup \Omega_b, \quad \mathrm{dist}(\bigcup_{k=1}^N \Omega_{k,+}, \partial\Omega_\gamma^-) > 0$$

$$\mathrm{dist}(\bigcup_{k=1}^N \Omega_{k,-}, \partial\Omega_\gamma^+) > 0, \quad \mathrm{dist}(\Omega_b, \partial\Omega_\gamma) > 0.$$

Moreover, we assume that $\Omega_{k,\pm}$, $k = 1, \ldots, N$, are chosen such that there are asymptotically flat layers $\Omega_{\gamma_{k,\pm}} \supset \Omega_{k,\pm}$, $k = 1, \ldots, N$, (possibly rotated) satisfying the assumptions of Lemma A.2 and $\partial\Omega_\gamma^\pm \cap \partial\Omega_{k,\pm} = \partial\Omega_\gamma^\pm \cap \partial\Omega^\pm_{\gamma_{k,\pm}}$. Furthermore, let Ω_{γ_b} be an infinite layer containing Ω_b. Finally, let φ_*, ψ_*, $* \in I := \{0, (k, \pm), b : k = 1, \ldots, N\}$, be chosen in the same way as before.

Then, because of Lemma A.2 there are u_*, $* \in I$, which solve

$$\lambda(u_*, v)_{\Omega_{\gamma_*}} + (\nabla u_*, \nabla v)_{\Omega_{\gamma_*}} = (f, \nabla(\varphi_* v))_{\Omega_{\gamma_*}} \text{ for all } v \in {}_0W^1_{q'}(\Omega_{\gamma_*}). \tag{A.1}$$

Now we define $R_\lambda f = \sum_{*\in I} \psi_* u_*$. Because of the construction, $R_\lambda f \in {}_0W^1_{q,\lambda}(\Omega_\gamma)$ and $\|R_\lambda f\|_{1,q,\lambda} \leq C_{\delta,q}\|f\|_q$. Moreover,

$$(R_\lambda f, v)_{\Omega_\gamma} \;=\; \sum_{*\in I} (u_*, \psi_* v)_{\Omega_{\gamma_*}}, \tag{A.2}$$

where $\psi_* v \in {}_0W^1_{q'}(\Omega_{\gamma *})$, $* \in I$. Hence by Lemma A.2, $(1+|\lambda|)\|Rf\|_{{}^0W^{-1}_q(\Omega_\gamma)} \le C_{q,\delta}\|f\|_q$. Furthermore,

$$(\nabla R_\lambda f, \nabla v)_{\Omega_\gamma} = \sum_{*\in I} (\nabla u_*, \nabla(\psi_* v))_{\Omega_{\gamma *}} + \langle S_\lambda f, v\rangle, \tag{A.3}$$

where $\langle S_\lambda f, v\rangle$ is the sum of the terms

$$\langle S_{\lambda,1,*} f, v\rangle := ((\nabla\psi_*)u_*, \nabla v)_{\Omega_{\gamma *}}, \quad \langle S_{\lambda,2,*} f, v\rangle = -(\nabla u_*, (\nabla\psi_*)v)_{\Omega_{\gamma *}}, \quad * \in I.$$

Since the mapping $A\colon L^q(\Omega_\gamma) \to {}^0W^{-1}_q(\Omega_\gamma)$ with $\langle Au_*, v\rangle = ((\nabla\psi_*)u_*, \nabla v)$, $v \in {}_0W^1_{q'}(\Omega_\gamma)$, is continuous, $\operatorname{supp}\psi_*$ is compact, and since $u_* \in {}_0W^1_q(\Omega_{\gamma_*})$ depends continuously on $f \in L^q(\Omega_\gamma)^n$, we conclude that $S_{\lambda,1,*}\colon L^q(\Omega_\gamma)^n \to {}^0W^{-1}_q(\Omega_\gamma)$ is a compact mapping. The same is true for $S_{\lambda,j,*}$ by duality. Moreover,

$$|\langle S_{\lambda,j,*} f, v\rangle| \le C_\delta(1+|\lambda|)^{-\frac{1}{2}}\|f\|_q\|v\|_{1,q',\lambda}, \qquad j = 1, 2,$$

since $\|u_*\|_{1,q,\lambda} \le C_\delta\|f\|_q$.

Representing the functional $S_\lambda f \in {}^0W^{-1}_q(\Omega_\gamma)$ by

$$\langle S_\lambda f, v\rangle = (g, \nabla v)_{\Omega_\gamma} \qquad \text{for all } v \in {}_0W^1_{q'}(\Omega_\gamma),$$

where $g \in L^q(\Omega_\gamma)^n$ and $\|g\|_q \le C\|S_\lambda f\|_{{}^0W^{-1}_{q,\lambda}(\Omega_\gamma)} \le C_\delta(1+|\lambda|)^{\frac{1}{2}}\|f\|_q$, we can consider S_λ as a compact operator on $L^q(\Omega_\gamma)^n$ with $S_\lambda = O(|\lambda|^{-\frac{1}{2}})$ as $|\lambda| \to \infty$ in Σ_δ.

Combining (A.1), (A.2), and (A.3) and using $\varphi_*\psi_* = \varphi_*$, we get

$$\lambda(R_\lambda f, v)_{\Omega_\gamma} + (\nabla R_\lambda f, \nabla v)_{\Omega_\gamma} = \sum_{*\in I}(f, \nabla(\varphi_*\psi_* v))_{\Omega_{\gamma *}} + \langle S_\lambda f, v\rangle = (f, \nabla v)_{\Omega_\gamma} + \langle S_\lambda, v\rangle$$

for all $v \in {}_0W^1_{q'}(\Omega_\gamma)$. Hence R_λ is a solution operator modulo a compact operator $S_\lambda \in \mathcal{L}(L^q(\Omega_\gamma))$, which tends to zero as $|\lambda| \to \infty$. Therefore (3.14)-(3.16) are uniquely solvable for all $\lambda \in \Sigma_\delta$, $|\lambda| \ge R$ and Fredholm solvable for all $\lambda \in \Sigma_\delta \cup \{0\}$ with index 0.

Next we show that the kernel is trivial. Let $u \in W^1_q(\Omega_\gamma)$ be a zero-solution. If $q = 2$, we immediately get $u = 0$ by (3.17). In the case $q \ne 2$ we consider $\varphi_0 u$ and conclude in the same way as in the proof of Lemma 4.2 that $u \in W^1_2(\Omega_\gamma)$.

Finally, the a priori estimate is a consequence of

$$\|u\|_{1,q,\lambda} = \|R_\lambda(1+S_\lambda)^{-1}f\|_{1,q,\lambda} \le C_{q,\delta}\|(1+S_\lambda)^{-1}f\|_q \le C_{q,\delta}\|f\|_q$$

for all $\lambda \in \Sigma_\delta$, $|\lambda| \ge R$, $R > 0$. ∎

Proof of Theorem 3.1: The proof is done in the same way as in the mixed case. But in the Neumann case we have to deal with homogeneous Sobolev spaces, which cause more problems. Nevertheless, this case can be proved in the same way since all error terms are supported in a ball $B_R(0)$ for an $R > 0$ and we can choose $v \in \dot{W}^1_q(\Omega_{\gamma_*})$ such that $\|v\|_{L^q(B_R(0)\cap\Omega_\gamma)} \le C\|\nabla v\|_q$.

The main problem is that the solution for $\lambda = 0$ is an element of $\dot{W}^1_q(\Omega_\gamma)$ but the solution for $\lambda \neq 0$ is in the set $\dot{W}^1_q(\Omega) \cap \dot{W}^{-1}_{q,0}(\Omega)$. Therefore we cannot use the argument that the Fredholm index is locally constant for $\lambda = 0$ since the space changes. Hence we have to treat this case separately. Nevertheless, the equation is still Fredholm solvable since the parametrix construction also holds in the case $\lambda = 0$.

If $q = 2$, it is well-known that the weak Neumann equation is uniquely solvable for any domain Ω. If $q > 2$, we can show that the kernel is trivial by the same localization technique as in the mixed case. Since the range of the equation is closed, it is sufficient to prove that the range is dense in $L^q(\Omega_\gamma)$ in the case $q > 2$. Then by duality the same is true for $1 < q < 2$.

If $f \in L^2(\Omega_\gamma)^n \cap L^q(\Omega_\gamma)^n$, $q > 2$, there is a unique solution $u \in \dot{W}^1_2(\Omega_\gamma)$. Then $\varphi_0 u$ solves a weak Neumann problem in Ω_{γ_0} with right-hand side in $L^q(\Omega_{\gamma_0}) \cap L^2(\Omega_{\gamma_0})$. Hence $\varphi_0 u \in \dot{W}^1_q(\Omega_{\gamma_0})$ and therefore $u \in \dot{W}^1_q(\Omega_\gamma)$. ■

Corollary A.3 (Helmholtz decomposition)
Let $1 < q < \infty$, $n \geq 2$, and $\Omega_\gamma \subset \mathbb{R}^n$ be an asymptotically flat layer with $C^{0,1}$-boundary. Then there are continuous projections $P_q, {}^0P_q \colon L^q(\Omega_\gamma)^n \to L^q(\Omega_\gamma)^n$ with

$$\begin{aligned}
\mathcal{R}(P_q) &= J_{q,0}(\Omega_\gamma) := \{u \in L^q(\Omega)^n : \operatorname{div} u = 0, \gamma_\nu u = 0\},\\
\mathcal{N}(P_q) &= G_q(\Omega_\gamma) \ := \{\nabla p : p \in \dot{W}^1_q(\Omega)\},\\
\mathcal{R}({}^0P_q) &= {}_0J_q(\Omega_\gamma) \ := \{u \in L^q(\Omega_\gamma)^n : \operatorname{div} u = 0, \gamma_\nu^- u = 0\},\\
\mathcal{N}({}^0P_q) &= {}^0G_q(\Omega_\gamma) := \{\nabla p : p \in {}^0W^1_q(\Omega_\gamma)\}.
\end{aligned}$$

Proof: It is well-known that the existence of the (classical) Helmholtz projection P_q is equivalent to the unique solvability of (3.5)-(3.6) for $\lambda = 0$ and $f \in L^q(\Omega_\gamma)^n$, cf. e.g. [16]. The proof for the mixed case ${}_0P_q$ is an easy modification. If $f \in L^q(\Omega_\gamma)^n$, then ${}^0P_q f := f - \nabla p$, where p solves (3.14)-(3.16) for $\lambda = 0$. ■

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