

# Tensor products in the category of topological vector spaces are not associative

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**Abstract.** We show by example that the associative law does not hold for tensor products in the category of general (not necessarily locally convex) topological vector spaces. The same pathology occurs for tensor products of Hausdorff abelian topological groups.

## Introduction

Let  $\mathcal{A}$  be a class of (not necessarily Hausdorff) topological vector spaces over a non-discrete valued field  $\mathbb{K}$  (resp., of abelian topological groups), such that  $\mathcal{A}$  is closed under the formation of finite cartesian products. Given  $E_1, \dots, E_n \in \mathcal{A}$  (where  $2 \leq n \in \mathbb{N}$ ), we call an element  $T \in \mathcal{A}$ , together with an  $n$ -linear (resp.,  $n$ -additive) continuous map

$$\tau: E_1 \times \dots \times E_n \rightarrow T,$$

a *tensor product* of  $E_1, \dots, E_n$  in the class  $\mathcal{A}$  if for every  $E \in \mathcal{A}$  and  $n$ -linear (resp.,  $n$ -additive) continuous map  $f: E_1 \times \dots \times E_n \rightarrow E$ , there exists a unique continuous  $\mathbb{K}$ -linear map (resp., continuous homomorphism)  $\tilde{f}: T \rightarrow E$  such that  $\tilde{f} \circ \tau = f$ .

For example, the tensor products in the class of real Hausdorff locally convex spaces are the *projective tensor products*, going back to Grothendieck's memoir [8]. In this case, an explicit description of the locally convex topology (by means of suitable cross-seminorms) is available, and it is well-known that an associative law holds for iterated projective tensor products; this is important for applications in topological algebra. For locally convex spaces over spherically complete ultrametric fields, the situation is similar (cf. [14]).

Two-fold tensor products  $E \otimes F$  in the class of real topological vector spaces have been studied in [15], [17], [7], [10], [5] and the breakthrough papers [12], [13]. For two-fold tensor products in various classes of abelian topological groups, see [9], [6], [15], [1], and [11]. In none of these works, higher tensor products or iterated tensor products are discussed, and accordingly the question of associativity of tensor products has not been raised there.

The present paper intends to close this gap. Based on an explicit description of the topology on tensor products in the category of topological  $\mathbb{K}$ -vector spaces provided in Section 1 (Proposition 1), we first describe a sufficient condition ensuring that  $(E_1 \otimes E_2) \otimes E_3$  be canonically isomorphic to  $E_1 \otimes (E_2 \otimes E_3)$ : it suffices that the outer factors  $E_1$  and  $E_3$  be locally bounded (Proposition 2). We then establish the main result: For  $E := \mathbb{K}^{\mathbb{N}}$ , none of the tensor products  $(E \otimes E) \otimes E$ ,  $E \otimes (E \otimes E)$  and  $E \otimes E \otimes E$  (in the class of topological  $\mathbb{K}$ -vector spaces) are naturally isomorphic (Theorem 1). Likewise, the associative law fails for tensor products in the categories of Hausdorff real topological vector spaces and Hausdorff

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abelian topological groups (Remark 1). This failure of associativity had gone unnoticed so far even in the most interesting case of Hausdorff real topological vector spaces.

The present studies arose in connection with [2], where a framework of differential calculus over arbitrary non-discrete topological fields has been developed. In this context, it is of interest to know precisely which constructions of topological algebra work over general topological fields (or at least complete valued fields), in contrast to those which depend on specific properties of the real number field, or on local convexity.

## 1 Description of the topology on tensor products

In the following, valued fields are always assumed non-discrete. Topological  $\mathbb{K}$ -vector spaces are *not* presumed Hausdorff, nor are topological groups, unless saying otherwise explicitly.

**Proposition 1** *Suppose that  $E_1, \dots, E_n$  are topological vector spaces over a valued field  $\mathbb{K}$ . Let  $T := E_1 \otimes_{\mathbb{K}} \dots \otimes_{\mathbb{K}} E_n$  be the (algebraic) tensor product of the  $\mathbb{K}$ -vector spaces  $E_1, \dots, E_n$ , and  $\tau: E_1 \times \dots \times E_n \rightarrow T$ ,  $\tau(x_1, \dots, x_n) := x_1 \otimes \dots \otimes x_n$ . Let  $\mathcal{B}$  be the set of all subsets  $U$  of  $T$  of the form*

$$U = \sum_{k \in \mathbb{N}} \tau(U_{k,1} \times \dots \times U_{k,n}) := \bigcup_{k \in \mathbb{N}} \sum_{\ell=1}^k \tau(U_{\ell,1} \times \dots \times U_{\ell,n}), \quad (1)$$

where  $(U_{k,j})_{k \in \mathbb{N}}$  is a sequence of balanced zero-neighbourhoods in  $E_j$ , for  $j = 1, \dots, n$ . Then the following holds:

- (a)  $\mathcal{B}$  is a basis for the filter of zero-neighbourhoods of some topology  $\mathcal{O}$  on  $T$  making  $T$  a topological  $\mathbb{K}$ -vector space.
- (b)  $\tau: E_1 \times \dots \times E_n \rightarrow (T, \mathcal{O})$  is continuous.
- (c)  $(T, \mathcal{O})$ , together with the continuous  $n$ -linear map  $\tau$ , is a tensor product of  $E_1, \dots, E_n$  in the category of topological  $\mathbb{K}$ -vector spaces.
- (d) If  $\mathbb{K} = \mathbb{R}$  and  $E_1, \dots, E_n$  are Hausdorff, then  $T$  is Hausdorff and hence is the tensor product of  $E_1, \dots, E_n$  in the category of Hausdorff real topological vector spaces.
- (e) If  $\mathbb{K} = \mathbb{R}$  and  $E_1, \dots, E_n$  are Hausdorff, then  $(T, \mathcal{O})$ , considered as an abelian topological group, together with the continuous  $n$ -additive map  $\tau$ , is a tensor product of the Hausdorff abelian topological groups  $(E_1, +), \dots, (E_n, +)$  in the category of Hausdorff abelian topological groups.

**Proof.** (a) It is obvious that every  $U \in \mathcal{B}$  is balanced, absorbing, and that  $zU \in \mathcal{B}$  for any  $z \in \mathbb{K}^\times$ ; hence conditions (EV<sub>I</sub>) and (EV<sub>II</sub>) of [3, I, §1, No. 5, Prop. 4] are satisfied. In order that  $\mathcal{B}$  be a basis of a vector topology, it remains to verify condition (EV<sub>III</sub>). To this end, let  $U$  be as in (1). We find balanced zero-neighbourhoods  $V_{k,j}$  in  $E_j$  for each

$k \in \mathbb{N}$  and  $j \in \{1, \dots, n\}$  such that  $V_{k,j} \subseteq U_{2k-1,j} \cap U_{2k,j}$ . Then, re-ordering terms and abbreviating  $V_{k,1} \otimes \dots \otimes V_{k,n} := \tau(V_{k,1} \times \dots \times V_{k,n})$ , we find that

$$\begin{aligned} & \left( \sum_{k \in \mathbb{N}} V_{k,1} \otimes \dots \otimes V_{k,n} \right) + \left( \sum_{k \in \mathbb{N}} V_{k,1} \otimes \dots \otimes V_{k,n} \right) \\ &= V_{1,1} \otimes \dots \otimes V_{1,n} + V_{1,1} \otimes \dots \otimes V_{1,n} + V_{2,1} \otimes \dots \otimes V_{2,n} + V_{2,1} \otimes \dots \otimes V_{2,n} + \dots \\ &\subseteq \sum_{k \in \mathbb{N}} U_{k,1} \otimes \dots \otimes U_{k,n}. \end{aligned}$$

Thus  $\mathcal{B}$  is a basis for the filter of zero-neighbourhoods of a vector topology  $\mathcal{O}$  on  $T$ .

(b) In order that the  $n$ -linear map  $\tau$  be continuous, we only need to show it is continuous at zero (see [3], Chap. I, §1, No. 6, Prop. 5). Now, given any zero neighbourhood  $U \in \mathcal{B}$  as in (1), we have  $\tau^{-1}(U) \supseteq U_{1,1} \times \dots \times U_{1,n}$ , which is a neighbourhood of  $(0, \dots, 0)$ .

(c) Suppose that  $f: E_1 \times \dots \times E_n \rightarrow E$  is a continuous  $n$ -linear map to a topological  $\mathbb{K}$ -vector space  $E$ . Since  $(T, \tau)$  is the (algebraic) tensor product of the vector spaces  $E_1, \dots, E_n$ , there is a unique linear map  $\tilde{f}: T \rightarrow E$  such that  $\tilde{f} \circ \tau = f$ . It only remains to show that  $\tilde{f}$  is continuous. To this end, let  $W \subseteq E$  be a zero-neighbourhood. Then standard arguments provide a sequence of zero-neighbourhoods  $W_k \subseteq E$  such that  $\sum_{k \in \mathbb{N}} W_k \subseteq W$ . The  $n$ -linear map  $f$  being continuous, for each  $k \in \mathbb{N}$  we find balanced zero-neighbourhoods  $U_{k,j} \subseteq E_j$  for  $j = 1, \dots, n$  such that  $f(U_{k,1} \times \dots \times U_{k,n}) \subseteq W_k$ . Then  $U := \sum_{k \in \mathbb{N}} \tau(U_{k,1} \times \dots \times U_{k,n}) \in \mathcal{B}$  is a zero-neighbourhood in  $T$ , and

$$\tilde{f}(U) = \sum_{k \in \mathbb{N}} \tilde{f}(\tau(U_{k,1} \times \dots \times U_{k,n})) = \sum_{k \in \mathbb{N}} f(U_{k,1} \times \dots \times U_{k,n}) \subseteq \sum_{k \in \mathbb{N}} W_k \subseteq W.$$

Thus the linear map  $\tilde{f}$  is continuous at zero and thus continuous.

(d) The proof is by induction. For  $n = 2$ , the assertion is Turpin's famous result (see [12], [13]). Now, by induction  $E_1 \otimes \dots \otimes E_{n-1}$  is Hausdorff, and hence so is  $F := (E_1 \otimes \dots \otimes E_{n-1}) \otimes E_n$ . The continuous  $n$ -linear map  $f: E_1 \times \dots \times E_n \rightarrow F$ ,  $(x_1, \dots, x_n) \mapsto (x_1 \otimes \dots \otimes x_{n-1}) \otimes x_n$  induces a continuous linear map  $\tilde{f}: T = E_1 \otimes \dots \otimes E_n \rightarrow F$ , determined by  $\tilde{f} \circ \tau = f$ . It is known from abstract algebra that  $\tilde{f}$  is an isomorphism of vector spaces. Since  $F$  is Hausdorff and  $\tilde{f}$  is a continuous injection,  $T$  is Hausdorff.

(e) To outline the idea, let us assume that  $n = 2$  (the general case being analogous). Suppose that  $f: E_1 \times E_2 \rightarrow A$  is a continuous bi-additive map to an abelian topological group  $A$ . Using the given scalar multiplication also on the right, we consider  $E_1$  as a  $(\mathbb{Z}, \mathbb{R})$ -bimodule. Similarly,  $E_2$  is considered as a  $(\mathbb{R}, \mathbb{Z})$ -bimodule. Then  $f$  is  $\mathbb{R}$ -balanced in the sense of [4, p. 161]. In fact, for any  $x \in E_1$ ,  $y \in E_2$  and  $q \in \mathbb{Q}$ , say  $q = \frac{m}{n}$  with  $m \in \mathbb{Z}$  and  $n \in \mathbb{N} \setminus \{0\}$ , we have  $f(qx, y) = f(m \frac{1}{n}x, n \frac{1}{n}y) = f(n \frac{1}{n}x, m \frac{1}{n}y) = f(x, qy)$ , whence  $f(rx, y) = f(x, ry)$  for all  $r \in \mathbb{R}$ , by continuity. Now,  $f$  being  $\mathbb{R}$ -balanced, there is a uniquely determined homomorphism of groups  $\tilde{f}: T \rightarrow A$  such that  $\tilde{f} \circ \tau = f$  (cf. [4, pp. 161–162]). As in (c), we see that  $\tilde{f}$  is continuous.  $\square$

## 2 A criterion for associativity of tensor products

Given a valued field  $\mathbb{K}$  and topological  $\mathbb{K}$ -vector spaces  $E_1, E_2, E_3$ , there is a continuous linear map  $\phi: E_1 \otimes E_2 \otimes E_3 \rightarrow (E_1 \otimes E_2) \otimes E_3$ , uniquely determined by  $\phi(x \otimes y \otimes z) = (x \otimes y) \otimes z$ , and  $\phi$  is an isomorphism of vector spaces (cf. proof of Proposition 1 (d)). Likewise, there is a unique continuous linear map (and isomorphism of vector spaces)  $\psi: E_1 \otimes E_2 \otimes E_3 \rightarrow E_1 \otimes (E_2 \otimes E_3)$ , determined by  $\psi(x \otimes y \otimes z) = x \otimes (y \otimes z)$ .

The following proposition describes criteria ensuring that  $\phi, \psi$  and  $\theta := \psi^{-1} \circ \phi$  be isomorphisms of topological vector spaces. Since  $\phi$  and  $\psi$  are isomorphisms of vector spaces, we can always identify  $(E_1 \otimes E_2) \otimes E_3, E_1 \otimes (E_2 \otimes E_3)$  and  $E_1 \otimes E_2 \otimes E_3$  as vector spaces for simplicity of notation; only the topologies may differ.

**Proposition 2** *In the preceding situation, we have:*

- (a) *If  $E_3$  is locally bounded, then  $\phi$  is an isomorphism of topological vector spaces.*
- (b) *If  $E_1$  and  $E_3$  are locally bounded, then the natural isomorphism of vector spaces*

$$\theta: (E_1 \otimes E_2) \otimes E_3 \rightarrow E_1 \otimes (E_2 \otimes E_3)$$

*taking  $(x \otimes y) \otimes z$  to  $x \otimes (y \otimes z)$  is an isomorphism of topological vector spaces.*

**Proof.** (a) Let  $B$  be a bounded, balanced zero-neighbourhood in  $E_3$ . Then  $\{tB: t \in \mathbb{K}^\times\}$  is a basis of zero-neighbourhoods in  $E_3$ . Hence a basis of zero-neighbourhoods of  $E_1 \otimes E_2 \otimes E_3$  is given by the sets of the form

$$\sum_{n \in \mathbb{N}} U_n \otimes V_n \otimes (t_n B) = \sum_{n \in \mathbb{N}} (t_n U_n) \otimes V_n \otimes B,$$

where  $U_n$  and  $V_n$  are open zero-neighbourhoods in  $E_1$  and  $E_2$ , respectively, and  $t_n \in \mathbb{K}^\times$ . Replacing  $U_n$  with  $t_n U_n$ , we see that it suffices to take  $t_n = 1$  for all  $n \in \mathbb{N}$  here: the sets

$$W := \sum_{n \in \mathbb{N}} U_n \otimes V_n \otimes B$$

form a basis of zero-neighbourhoods. Suppose such a  $W$  is given. We choose a bijection  $\lambda: \mathbb{N} \rightarrow \mathbb{N}^2$  and define  $P_{\lambda(n)} := U_n$  and  $Q_{\lambda(n)} := V_n$  for  $n \in \mathbb{N}$ . Then

$$W = \sum_{n \in \mathbb{N}} P_{\lambda(n)} \otimes Q_{\lambda(n)} \otimes B = \sum_{(j,k) \in \mathbb{N}^2} P_{(j,k)} \otimes Q_{(j,k)} \otimes B \supseteq \sum_{j \in \mathbb{N}} \left( \sum_{k \in \mathbb{N}} P_{(j,k)} \otimes Q_{(j,k)} \right) \otimes B,$$

where the right hand side is a zero-neighbourhood in  $(E_1 \otimes E_2) \otimes E_3$ . Thus  $\phi$  is open and thus  $\phi$  is an isomorphism of topological vector spaces.

(b) If also  $E_1$  is locally bounded, we see in the same way that  $\psi$  is an isomorphism of topological vector spaces, whence so is  $\theta = \psi \circ \phi^{-1}$ .  $\square$

### 3 Examples where associativity fails

Let  $\mathbb{K}$  be a valued field. We show by example that tensor products in the category of topological  $\mathbb{K}$ -vector spaces need not be associative.

**Theorem 1** *Let  $E := \mathbb{K}^{\mathbb{N}}$  with the product topology. Then the following holds:*

(a) *The canonical isomorphism of vector spaces*

$$\theta: (E \otimes E) \otimes E \rightarrow E \otimes (E \otimes E)$$

*is not continuous (and hence not an isomorphism of topological vector spaces).*

(b) *The canonical isomorphisms of vector spaces*

$$\phi: E \otimes E \otimes E \rightarrow (E \otimes E) \otimes E \quad \text{and} \quad \psi: E \otimes E \otimes E \rightarrow E \otimes (E \otimes E)$$

*are continuous but not open.*

**Proof.** (a) Given  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , the set  $U_{n,\varepsilon} := \{z \in \mathbb{K} : |z| < \varepsilon\}^n \times \mathbb{K}^{\{n+1, n+2, \dots\}}$  is a zero-neighbourhood in  $E$ , whence

$$U := \sum_{n \in \mathbb{N}} U_{2n, 2^{-n}} \otimes \left( \sum_{k \in \mathbb{N}} (U_{2n, 2^{-k}} \otimes U_{2n, 1}) \right)$$

is a zero-neighbourhood in  $E \otimes (E \otimes E)$ . Then  $U$  is not a zero-neighbourhood in  $(E \otimes E) \otimes E$ . In fact, otherwise we find zero-neighbourhoods  $A_{n,k}$ ,  $B_{n,k}$  and  $C_n$  in  $E$  such that

$$V := \sum_{n \in \mathbb{N}} \left( \sum_{k \in \mathbb{N}} A_{n,k} \otimes B_{n,k} \right) \otimes C_n \subseteq U.$$

There is  $N \in 2\mathbb{N}$  such that  $\mathbb{K} e_N \subseteq C_1$ , where  $e_N := \delta_{N, \bullet} : \mathbb{N} \rightarrow \{0, 1\} \subseteq \mathbb{K}$  is defined using Kronecker's delta. The sets  $A_{1,k}$  and  $B_{1,k}$  being absorbing, we then have

$$\left( \sum_{k \in \mathbb{N}} E \otimes E \right) \otimes \mathbb{K} e_N \subseteq V \subseteq U. \quad (2)$$

Let  $p: E \otimes E \otimes E \rightarrow \mathbb{K}^N \otimes \mathbb{K}^N$  be the linear map uniquely determined by

$$p(x \otimes y \otimes z) = z(N) \cdot (x|_{\{1, \dots, N\}}) \otimes (y|_{\{1, \dots, N\}}).$$

We now identify  $\mathbb{K}^N$  with  $\mathbb{K}^N \times \{0\} \subseteq \mathbb{K}^{\mathbb{N}}$ . Given  $v \in \mathbb{K}^N \otimes \mathbb{K}^N$  and  $R \in \mathbb{K}^\times$ , we have  $(Rv) \otimes e_N \in U$  by (2), whence  $(Rv) \otimes e_N = \sum_{k, n \in \mathbb{N}} x_n \otimes y_{n,k} \otimes z_{n,k}$  for suitable  $x_n \in U_{2n, 2^{-n}}$ ,  $y_{n,k} \in U_{2n, 2^{-k}}$ , and  $z_{n,k} \in U_{2n, 1}$  (depending on  $R$ ), almost all of which are zero. Hence

$$v = p(v \otimes e_N) = \sum_{k, n \in \mathbb{N}} \frac{c_{n,k}}{R} \cdot u_n \otimes v_{n,k} = \sum_{n \in \mathbb{N}} u_n \otimes \underbrace{\left( \sum_{k \in \mathbb{N}} \frac{c_{n,k}}{R} \cdot v_{n,k} \right)}_{=: w_n}$$

with  $u_n := x_n|_{\{1, \dots, N\}}$ ,  $v_{n,k} := y_{n,k}|_{\{1, \dots, N\}}$  and  $c_{n,k} := z_{n,k}(N)$ . Thus

$$v = \sum_{n=1}^{N/2} u_n \otimes w_n + \rho \quad (3)$$

with  $\rho := \sum_{n>N/2} u_n \otimes w_n = \sum_{i,j=1}^N \left( \sum_{n>N/2} \sum_{k \in \mathbb{N}} \frac{c_{n,k}}{R} u_n(i) v_{n,k}(j) \right) e_i \otimes e_j$ . Note that, as  $2n > N \geq i, j$ , we have  $|u_n(i)| < 2^{-n}$ ,  $|v_{n,k}| < 2^{-k}$  and  $|c_{n,k}| < 1$ . Thus

$$\sum_{n>N/2} \sum_{k \in \mathbb{N}} \left| \frac{c_{n,k}}{R} u_n(i) v_{n,k}(j) \right| \leq \frac{1}{|R|} \sum_{n,k=1}^{\infty} 2^{-n} 2^{-k} = \frac{1}{|R|},$$

which can be made arbitrarily small for large  $|R|$ . Thus (3) shows that  $v$  is contained in the closure of the set  $S$  of  $N/2$ -fold sums of elementary tensors, with respect to the canonical Hausdorff vector topology ( $\cong \mathbb{K}^{N^2}$ ) on  $\mathbb{K}^N \otimes \mathbb{K}^N$ . Hence  $S$  is dense in  $\mathbb{K}^N \otimes \mathbb{K}^N$ . However, under the usual isomorphism  $\mathbb{K}^N \otimes \mathbb{K}^N \cong M_N(\mathbb{K})$  the set  $S$  corresponds to the set of matrices of rank  $\leq N/2$ , which is not dense in  $M_N(\mathbb{K})$  as its closure does not meet the open set  $\text{GL}_N(\mathbb{K})$  of invertible matrices. We have reached a contradiction.

(b) If one of  $\phi$  and  $\psi$  was an isomorphism of topological vector spaces, then also the other, because  $\psi = \alpha \circ \phi \circ \beta$ , where the linear maps  $\beta : E \otimes E \otimes E \rightarrow E \otimes E \otimes E$  and  $\alpha : (E \otimes E) \otimes E \rightarrow E \otimes (E \otimes E)$  determined by  $\beta(x \otimes y \otimes z) := y \otimes z \otimes x$  and  $\alpha((x \otimes y) \otimes z) = z \otimes (x \otimes y)$  are isomorphisms of topological vector spaces. Thus  $\theta = \psi \circ \phi^{-1}$  would be an isomorphism of topological vector spaces, contradicting (a).  $\square$

**Remark 1** If  $\mathbb{K} = \mathbb{R}$ , then  $E \otimes E$ ,  $(E \otimes E) \otimes E$  and  $E \otimes (E \otimes E)$  are also tensor products in the category of Hausdorff real topological vector spaces (Proposition 1 (d)), whence Theorem 1 (a) entails that tensoring is not associative in this category. Nor is tensoring associative in the category of complete Hausdorff real topological vector spaces, since  $E \otimes E$ ,  $(E \otimes E) \otimes E$  and  $E \otimes (E \otimes E)$  are complete (see [13]). Likewise, considering  $E \otimes E$ ,  $(E \otimes E) \otimes E$  and  $E \otimes (E \otimes E)$  as tensor products in the category of Hausdorff abelian topological groups (Proposition 1 (d)), we deduce that tensoring is not associative in the category of Hausdorff abelian topological groups.

*Open problems.* Does Proposition 1 (d) carry over to Hausdorff topological vector spaces over complete valued fields  $\mathbb{K}$  other than  $\mathbb{R}$  (or  $\mathbb{C}$ )? Are tensor products of complete Hausdorff topological  $\mathbb{K}$ -vector spaces in the class of Hausdorff topological  $\mathbb{K}$ -vector spaces always complete, as in in the real case ([13])?

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