Fredholm indices of band-dominated operators

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Abstract

The Fredholmness of a band-dominated operator on $l^2(\mathbb{Z})$ is closely related with the invertibility of its limit operators: the operator is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded. The goal of the present note is to show how the Fredholm index of a Fredholm band-dominated operator can be determined in terms of its limit operators.

1 Introduction and results

We will work on the Hilbert space $l^2(\mathbb{Z})$ and write $L(l^2(\mathbb{Z}))$ for the C^*-algebra of all linear bounded operators on $l^2(\mathbb{Z})$. An operator $A \in L(l^2(\mathbb{Z}))$ with matrix representation (a_{ij}) with respect to the standard basis of $l^2(\mathbb{Z})$ is a *band operator* if there is an integer k such that $a_{ij} = 0$ whenever $|i-j| > k$. The closure in $L(l^2(\mathbb{Z}))$ of the set of all band operators is a C^*-subalgebra of $L(l^2(\mathbb{Z}))$ which we denote by $\mathcal{A}(\mathbb{Z})$. This is the same as the *rough algebra* of the coarse space $\mathbb{Z}$ which is discussed in [11]. The elements of $\mathcal{A}(\mathbb{Z})$ are called *band-dominated operators.*

Recall further that an operator $A \in L(l^2(\mathbb{Z}))$ is said to be a Fredholm operator if its kernel $\ker A := \{x \in l^2(\mathbb{Z}) : Ax = 0\}$ and its cokernel $\operatorname{coker} A := l^2(\mathbb{Z})/\operatorname{im} A$ are finite-dimensional linear spaces, and that in this case the number

$$\operatorname{ind} A := \dim \ker A - \dim \operatorname{coker} A$$

is called the Fredholm index of A.

In [10], a criterion for the Fredholmness of a band-dominated operator A in terms of the limit operators of A is derived. To restate this result, let $V_k \in L(l^2(\mathbb{Z}))$ stand for the operator of shift by $k \in \mathbb{Z}$,

$$(V_k x)(n) := x(n-k), \quad n \in \mathbb{Z}.$$

Further, let $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}$ which tend to infinity in the sense that, for each $R > 0$, there is an $N \in \mathbb{N}$ such that $|h(n)| \geq R$ for all $n \geq N$. An operator A_h is called a *limit operator* of $A \in L(l^2(\mathbb{Z}))$ with respect to the sequence $h \in \mathcal{H}$ if $V_{-h(n)}AV_{h(n)}$ tends *-strongly to A_h as $n \to \infty$. Clearly, every operator A can have at most one limit operator with respect to a given sequence $h \in \mathcal{H}$, which justifies this notation. The set $\sigma_{op}(A)$ of all limit operators of a given operator A is the *operator spectrum* of A. The operator spectrum splits into

$$\sigma_{op}(A) = \sigma_+(A) \cup \sigma_-(A)$$

where $\sigma_+(A)$ and $\sigma_-(A)$ stand for the sets of all limit operators of A which correspond to sequences tending to $+\infty$ and to $-\infty$, respectively. It is also clear that every limit operator of a compact operator is 0, and it is not hard to see that every limit operator of a Fredholm operator is invertible (see [10]). It is a basic result of [10] that the operator spectrum of a *band-dominated operator* is rich enough in order to guarantee the reverse implications.

Theorem 1.1 *Let A be a band-dominated operator. Then*

(a) every sequence $h \in \mathcal{H}$ possesses a subsequence g such that the limit operator A_g exists.

(b) the operator A is compact if and only if $\sigma_{op}(A) = \{0\}$.

(c) the operator A is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded.

The questions of whether and how the Fredholm index of a band-dominated Fredholm operator can be expressed in terms of its limit operators are left open in [10]. It is the goal of this note to answer these questions.

Let $\mathbb{Z}_+$ and $\mathbb{Z}_-$ stand for the sets of the non-negative and negative integers, and write P and Q for the orthogonal projections from $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z}_+)$ and $l^2(\mathbb{Z}_-)$, respectively. (We identify $l^2(\mathbb{Z}_+)$ and $l^2(\mathbb{Z}_-)$ with subspaces of $l^2(\mathbb{Z})$ in the obvious way.) If A is a band-dominated operator, then the operators PAQ and QAP are compact (they are of finite rank if A is a band operator). Hence, the operators $A - (PAP + Q)(P + QAQ)$ and $A - (P + QAQ)(PAP + Q)$ are compact for every band-dominated operator A, and this shows that a band-dominated operator A is Fredholm if and only if both operators $PAP + Q$ and $P + QAQ$ are Fredholm. In this case, we call $\mathrm{ind}_+ A := \mathrm{ind}\,(PAP + Q)$ and $\mathrm{ind}_- A := \mathrm{ind}\,(P + QAQ)$ the *plus-index* and the *minus-index* of A. Evidently,

$$\mathrm{ind}\, A = \mathrm{ind}_+ A + \mathrm{ind}_- A$$

for every Fredholm band-dominated operator A. The surprisingly simple answer to the index questions posed above is as follows.

Theorem 1.2 *Let A be a Fredholm band-dominated operator. Then*

(a) for all $B \in \sigma_\pm(A)$,

$$\mathrm{ind}_\pm(B) = \mathrm{ind}_\pm(A),$$

(b) all operators in $\sigma_+(A)$ have the same plus-index, and all operators in $\sigma_-(A)$ have the same minus-index.

(c) for arbitrarily chosen operators $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,

$$\text{ind}\, A = \text{ind}_+ B_+ + \text{ind}_- B_-. \tag{1}$$

So we can think of the plus- and the minus-index of A as local indices at $+\infty$ and $-\infty$.

To mention at least one example in which the identity (1) implies an explicit and effective formula for the computation of the Fredholm index, we consider band-dominated operators with slowly oscillating coefficients. These are the norm limits of band operators of the form $\sum_{n=-k}^{k} a_n V_n$ where the $a_n I$ are operators of multiplication by slowly oscillating functions. By definition, a function $a \in l^\infty(\mathbb{Z})$ is *slowly oscillating* if

$$\lim_{n \to \pm\infty} |a(n+1) - a(n)| = 0,$$

and the operator aI of multiplication by a is defined by $(ax)(n) := a(n)x(n)$. In [10], it has been shown that every limit operator A_h of a band-dominated operator A with slowly oscillating coefficients is shift invariant. Thus, there is a continuous function a_h on the unit circle $\mathbb{T}$ such that A_h is just the Laurent operator $L(a_h)$. Recall that every function $a \in C(\mathbb{T})$ induces a linear bounded Laurent operator $L(a)$ on $l^2(\mathbb{Z})$ by $(L(a)x)(n) := \sum_{k \in \mathbb{Z}} a_{n-k} x(n)$ where a_n refers to the nth Fourier coefficient of a. The Laurent operator $L(a)$ is invertible if and only if the function a is invertible in $C(\mathbb{T})$. Thus, Theorem 1.1 yields an effective criterion for the Fredholmness of band-dominated operators with slowly oscillating coefficients. Moreover, the compression $PL(a)P$ of the Laurent operator $L(a)$ onto $l^2(\mathbb{Z}_+)$ is the Toeplitz operator $T(a)$, which is Fredholm if and only if its generating function a is invertible in $C(\mathbb{T})$, and which has minus the winding number of a with respect to the origin as its index (see [2, 3, 6], for example). Thus, also the plus- and minus-index of Fredholm band-dominated operators with slowly oscillating coefficients can be effectively determined.

To prove Theorem 1.2 we initially attempted to show that the unitary group of the C^*-algebra of the band-dominated operators on $l^2(\mathbb{Z}_+)$ is path connected. (Notice that this is definitely wrong for the unitary group of the band-dominated operators on $l^2(\mathbb{Z})$. Indeed, the plus-index of the unitary operator V_1 is -1, whereas the plus-index of the identity operator is 0. Since the plus-index is a continuous function on the set of the Fredholm band-dominated operators, the operators V_1 and I cannot be connected by a continuous path in that set.) Our attempt failed (and we do not know up to now whether this group is connected), and the final proof employs instead a K-theory argument which shows that this unitary group is at least 'stably' path connected. However, we obtained two partial results which might be of independent interest. For, we call an operator on $l^2(\mathbb{Z}_+)$ *elementary* if its matrix representation with respect to the standard basis of $l^2(\mathbb{Z}_+)$ is of the form

$$\operatorname{diag}(A_1, A_2, A_3, \ldots)$$

with blocks A_n of $k_n \times k_n$-matrices on the main diagonal.

Theorem 1.3 *(a) Every unitary tridiagonal operator on $l^2(\mathbb{Z}_+)$ is elementary with blocks of size 1×1 or 2×2.*

(b) Every unitary band operator on $l^2(\mathbb{Z}_+)$ is the product of two elementary unitary band operators.

Observe that these results imply that every unitary *band* operator on $l^2(\mathbb{Z}_+)$ can be connected with the identity operator by a continuous path running through the set of the unitary band operators. This is a simple consequence of the path connectedness of the unitary group of the algebra of all complex $k \times k$ matrices.

The following sections are devoted to the proofs of Theorems 1.2 and 1.3.

2 Proof of Theorem 1.2

Our strategy to prove Theorem 1.2 is as follows. Let $\mathcal{J}_+$ be the ideal of $\mathcal{A}(\mathbb{Z})$ generated by P. If A is a Fredholm band-dominated operator, then $PAP + Q$ is a Fredholm operator in the unitalization $\mathcal{A}_1$ of $\mathcal{J}_+$. We would like to show that

$$\operatorname{ind}(PAP + Q) = \operatorname{ind}(PA_hP + Q)$$

for every sequence h tending to $+\infty$ for which the limit operator A_h exists; and a simple reduction shows that it is enough to prove that the right-hand side vanishes if the left-hand side does. Suppose then that $PAP + Q$ has zero index; then it is a compact perturbation of an invertible in $\mathcal{A}_1$. If we knew that the group of invertibles in the C^*-algebra $\mathcal{A}_1$ was path connected, then we could produce a continuous path of Fredholm operators in $\mathcal{A}_1$ joining $PAP + Q$ to the identity. Taking limit operators (perhaps with respect to a suitable subsequence of h) produces a continuous path of Fredholm operators joining $PA_hP + Q$ to the identity, thus showing that the latter operator has index 0.

In fact, we do not know whether the group of invertibles of $\mathcal{A}_1$ is connected; but we can prove that the K-theory group $K_1(\mathcal{A}_1)$ vanishes. This implies that any invertible in $\mathcal{A}_1$ can be connected to the identity after 'stabilization' (taking the direct sum with the identity in a matrix algebra), and that is enough to carry out the argument sketched above.

This K-theory calculation uses techniques which are well known in the study of index theory on open manifolds and the coarse Baum–Connes conjecture. We first show that the algebra $\mathcal{A}(\mathbb{Z})$ can be identified with a crossed product of $l^\infty(\mathbb{Z})$ by the group $\mathbb{Z}$. The Pimsner-Voiculescu exact sequence allows us to compute the K_1-group of this crossed product. (This calculation is essentially due to Yu [13]; compare also [11], Lecture 4.) Then we plug in this result into a Mayer-Vietoris exact sequence to obtain that the K_1-group of $\mathcal{J}_+$ is $\{0\}$.

2.1 The algebra $\mathcal{A}(\mathbb{Z})$ as a crossed product

We start with recalling some facts on crossed products and reduced crossed products where we follow [1, 4, 8]. We will exclusively consider *C^*-dynamical systems* $(\mathcal{B}, G, \alpha)$ which consist of a C^*-algebra $\mathcal{A}$, a discrete group G, and a group homomorphism $\alpha : G \to \operatorname{Aut} \mathcal{B}$, $s \mapsto \alpha_s$. A pair (π, U) consisting of a *-representation $\pi : \mathcal{B} \to L(H)$ of $\mathcal{B}$ and a unitary representation $U : G \to L(H)$, $t \mapsto U_t$ of G on the same Hilbert space H, is called a *covariant representation* of the C^*-dynamical system $(\mathcal{B}, G, \alpha)$ if the covariance condition

$$U_t \pi(B) U_t^* = \pi(\alpha_t(B)) \quad \text{for all } B \in \mathcal{B} \text{ and } t \in G$$

is satisfied. A special class of covariant representations is obtained by taking the tensor product of a *-representation of $\mathcal{B}$ by the left regular representation of G which is defined as follows. Given a *-representation $\pi : \mathcal{B} \to L(H)$ of $\mathcal{B}$, let $l^2(G, H)$ refer to the Hilbert space of all square summable functions $x : G \to H$ with norm $\|x\|^2 := \sum_{t \in G} \|x(t)\|^2$. Then one has a covariant representation $(\tilde{\pi}, U)$ of $(\mathcal{B}, G, \alpha)$ which acts at $x \in l^2(G, H)$ by

$$(\tilde{\pi}(B)x)(s) := \pi(\alpha_s^{-1}(B))(x(s)) \quad \text{and} \quad (U_t x)(s) := x(t^{-1}s)$$

for $B \in \mathcal{B}$ and $s, t \in G$. If π is a faithful representation of $\mathcal{B}$, then the smallest C^*-subalgebra of $L(l^2(G, H))$ which contains all operators $\tilde{\pi}(B)$ with $B \in \mathcal{B}$ as well as all operators U_t with $t \in G$ is independent of the concrete choice of π. This algebra is called the *reduced crossed product of $\mathcal{B}$ by G* and is denoted by $\mathcal{B} \times_{\alpha r} G$ ([8], Theorem 7.7.5). Moreover, if the group G is amenable (for example, if G is commutative), then the reduced crossed product $\mathcal{B} \times_{\alpha r} G$ coincides with the crossed product $\mathcal{B} \times_\alpha G$ ([8], Theorem 7.7.7 and [4], Corollary VII.2.2).

Now we consider the special dynamical system $(l^\infty(\mathbb{Z}), \mathbb{Z}, \alpha)$ where $\alpha_k = \alpha(k)$, $k \in \mathbb{Z}$, acts on $a \in l^\infty(\mathbb{Z})$ by

$$(\alpha_k(a))(n) = a(n-k), \quad n \in \mathbb{Z}. \tag{2}$$

Proposition 2.1 *For the dynamical system $(l^\infty(\mathbb{Z}), \mathbb{Z}, \alpha)$ with α specified by (2), one has*

$$l^\infty(\mathbb{Z}) \times_\alpha \mathbb{Z} = l^\infty(\mathbb{Z}) \times_{\alpha r} \mathbb{Z} \cong \mathcal{A}(\mathbb{Z}).$$

Proof. We have already mentioned that the first identity holds in general for products by amenable groups. So we are left with showing that the algebra $\mathcal{A}(\mathbb{Z})$ is *-isomorphic to the reduced crossed product $l^\infty(\mathbb{Z}) \times_{\alpha r} \mathbb{Z}$.

The mapping π which associates with every sequence $a \in l^\infty(\mathbb{Z})$ the operator $aI \in L(l^2(\mathbb{Z}))$ of multiplication by a represents the C^*-algebra $l^\infty(\mathbb{Z})$ faithfully. This representation induces a covariant representation of the dynamical system $(l^\infty(\mathbb{Z}), \mathbb{Z}, \alpha)$ on the Hilbert space $H = l^2(\mathbb{Z}, l^2(\mathbb{Z}))$ via

$$(\tilde{\pi}(a)x)(s) := \pi(\alpha_s^{-1}(a))(x(s)) \quad \text{and} \quad (U_t x)(s) := x(s-t)$$

where $a \in l^\infty(\mathbb{Z})$ and $t \in \mathbb{Z}$. We identify $l^2(\mathbb{Z}, l^2(\mathbb{Z}))$ with $l^2(\mathbb{Z} \times \mathbb{Z})$ via $x(s, n) := (x(s))(n)$. Then we can identify $\tilde{\pi}(a)$ and U_t with the operators

$$(\tilde{\pi}(a)x)(s, n) := a(n+s)x(s, n) \quad \text{and} \quad (U_t x)(s, n) := x(s-t, n). \tag{3}$$

Let $\mathcal{C}$ refer to the smallest C^*-subalgebra of $L(l^2(\mathbb{Z} \times \mathbb{Z}))$ which contains all operators $\tilde{\pi}(a)$ and U_t with $a \in l^\infty(\mathbb{Z})$ and $t \in \mathbb{Z}$, given by (3). This algebra is *-isomorphic to the reduced crossed product $l^\infty(\mathbb{Z}) \times_{\alpha r} \mathbb{Z}$ as quoted above, and we claim that it is also *-isomorphic to the algebra $\mathcal{A}(\mathbb{Z})$ of the band-dominated operators on $l^2(\mathbb{Z})$. For $n \in \mathbb{Z}$, let

$$H_n := \{x \in l^2(\mathbb{Z} \times \mathbb{Z}) : x(s, m) = 0 \text{ whenever } m \neq n\}.$$

We identify $l^2(\mathbb{Z} \times \mathbb{Z})$ with the orthogonal sum $\oplus_{n \in \mathbb{Z}} H_n$ such that $x \in l^2(\mathbb{Z} \times \mathbb{Z})$ is identified with $\oplus h_n \in \oplus H_n$ if $x(s, n) = h_n(s)$. From (3) we conclude that each space H_n is invariant with respect to each operator $C \in \mathcal{C}$. Hence, each operator $C \in \mathcal{C}$ corresponds to a diagonal matrix operator $\operatorname{diag}(\ldots, C_n, C_{n+1}, \ldots)$ with respect to the decomposition of $l^2(\mathbb{Z} \times \mathbb{Z})$ into the orthogonal sum of its subspaces H_n. In particular, C_n is nothing but the restriction of C onto H_n. Let $\mathcal{C}_n$ denote the C^*-algebra of all restrictions of operators in $\mathcal{C}$ onto H_n.

It is clear that each of the spaces H_n is isometric to $l^2(\mathbb{Z})$ with the isometry given by

$$J_n : H_n \to l^2(\mathbb{Z}), \quad (J_n x)(s) := x(s, n).$$

Thus, $J_n \mathcal{C}_n J_n^{-1}$ is a C^*-subalgebra of $L(l^2(\mathbb{Z}))$ which we denote by $\mathcal{B}_n$. Clearly, for $a \in l^\infty(\mathbb{Z})$, the operator $J_n \tilde{\pi}(a) J_n^{-1}$ is just the operator $\pi(\alpha_n(a))$, whereas $J_n U_t J_n^{-1}$ is the shift operator V_t. Since $\pi(\alpha_{-n}(a)) = V_n \pi(a) V_n^*$ and $V_t = V_n V_t V_n^*$, the mapping $B \mapsto V_n B V_{-n}$ is a *-isomorphism from $\mathcal{B}_n$ onto $\mathcal{A}(\mathbb{Z})$. Consequently, the mapping

$$\mathcal{A}(\mathbb{Z}) \to \mathcal{C}, \quad A \mapsto \operatorname{diag}(\ldots, J_n^{-1} V_n^* A V_n J_n, \ldots)$$

is a *-isomorphism. ∎

2.2 The K_1-group of $\mathcal{A}(\mathbb{Z})$.

To compute the K_1-group of the algebra $\mathcal{A}(\mathbb{Z})$ we will make use of the fact that $\mathcal{A}(\mathbb{Z})$ is *-isomorphic to the crossed product $l^\infty(\mathbb{Z}) \times_\alpha \mathbb{Z}$ by Proposition 2.1. The K-theory of crossed products by $\mathbb{Z}$ is dominated by the Pimsner-Voiculescu exact sequence ([9], see also [4], Theorem VIII.5.1) which we restate below. Recall in this connection that every automorphism α of a C^*-algebra $\mathcal{B}$ induces a group homomorphism from $\mathbb{Z}$ into $\operatorname{Aut} \mathcal{B}$ by $n \mapsto \alpha^n$ which we denote by α again.

Theorem 2.2 (The Pimsner-Voiculescu exact sequence.) *Let α be an automorphism of the C^*-algebra $\mathcal{B}$. Then there is a cyclic six term exact sequence*

$$\begin{array}{ccccc} K_0(\mathcal{B}) & \xrightarrow{\mathrm{id}_*-\alpha_*} & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{B}\times_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{B}\times_\alpha \mathbb{Z}) & \longleftarrow & K_1(\mathcal{B}) & \xleftarrow{\mathrm{id}_*-\alpha_*} & K_1(\mathcal{B}) \end{array} \tag{4}$$

We wish to apply this exact sequence to the algebra $\mathcal{A}(\mathbb{Z}) = l^\infty(\mathbb{Z})\times_\alpha\mathbb{Z}$, i. e. with $\mathcal{B} = l^\infty(\mathbb{Z})$. Since $l^\infty(\mathbb{Z})$ is a von Neumann algebra, one has $K_1(l^\infty(\mathbb{Z})) = \{0\}$ ([12], Exercise 8.14). Thus, (4) becomes

$$\begin{array}{ccccc} K_0(l^\infty(\mathbb{Z})) & \xrightarrow{\mathrm{id}_*-\alpha_*} & K_0(l^\infty(\mathbb{Z})) & \longrightarrow & K_0(\mathcal{A}(\mathbb{Z})) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A}(\mathbb{Z})) & \longleftarrow & \{0\} & \longleftarrow & \{0\} \end{array} \tag{5}$$

The K_0-group of $l^\infty(\mathbb{Z})$. The K_0-group of the algebra $l^\infty(\mathbb{Z})$ coincides with the group of all bounded functions from $\mathbb{Z}$ into $\mathbb{Z}$ which we denote by $\mathbb{Z}_b^\mathbb{Z}$. Since we have not found an explicit reference of this result, and for the reader's convenience, we include its proof here. Again we start with recalling the basic steps in the definition of the K_0-group of a C^*-algebra, where we follow [12], Chapter 3.

For n a positive integer and $\mathcal{B}$ a unital C^*-algebra, let $\mathcal{P}_n(\mathcal{B})$ stand for the set of all projections (i.e. self-adjoint idempotents) in the algebra $\mathcal{B}_{n\times n}$ of all $n \times n$ matrices with entries in $\mathcal{B}$, and set $\mathcal{P}_\infty(\mathcal{B}) := \cup_n \mathcal{P}_n(\mathcal{B})$. One defines a binary operation $\oplus$ and a relation $\sim$ on $\mathcal{P}_\infty(\mathcal{B})$ as follows. For $p \in \mathcal{P}_n(\mathcal{B})$ and $q \in \mathcal{P}_m(\mathcal{B})$, one sets

$$p \oplus q := \mathrm{diag}\,(p,\, q) \in \mathcal{P}_{n+m}(\mathcal{B}),$$

and one writes $p \sim q$ if there is an element $v \in \mathcal{B}_{m\times n}$ such that $p = v^*v$ and $q = vv^*$. Thus, if both p and q belong to $\mathcal{P}_n(\mathcal{B})$ for some n, then $p \sim q$ if and only if p and q are Murray - von Neumann equivalent. The following is Proposition 2.3.2 in [12].

Proposition 2.3 *Let $p, q, r, p', q' \in \mathcal{P}_\infty(\mathcal{B})$ for some unital C^*-algebra $\mathcal{B}$. Then*
(a) $p \sim p \oplus 0_{n\times n}$.
(b) If $p \sim p'$ and $q \sim q'$, then $p \oplus q \sim p' \oplus q'$.
(c) $p \oplus q \sim q \oplus p$.
(d) If $p, q \in \mathcal{P}_n(\mathcal{B})$ and $pq = 0$, then $p + q \in \mathcal{P}_n(\mathcal{B})$ and $p + q \sim p \oplus q$.
(e) $(p \oplus q) \oplus r \sim p \oplus (q \oplus r)$.

Let $D(\mathcal{B}) := \mathcal{P}_\infty(\mathcal{B})/\sim$, write $[p]_\sim$ for the equivalence class of $p \in \mathcal{P}_\infty(\mathcal{B})$ in $D(\mathcal{B})$, and define an operation $+$ on $D(\mathcal{B})$ by $[p]_\sim + [q]_\sim := [p \oplus q]_\sim$. Then $D(\mathcal{B})$

becomes an abelian semigroup, and the Grothendieck group of $D(\mathcal{B})$ is called the K_0-group of $\mathcal{B}$.

Now we specify $\mathcal{B} = l^\infty(\mathbb{Z})$ and let $P \in \mathcal{P}_\infty(\mathcal{B})$. Then $P \in \mathcal{P}_k(l^\infty(\mathbb{Z})) = \mathcal{P}(l^\infty(\mathbb{Z})_{k\times k})$ for some k. Since $l^\infty(\mathbb{Z})_{k\times k} = l^\infty(\mathbb{Z}, \mathbb{C}_{k\times k})$, we can think of P as a sequence of projections in $\mathbb{C}_{k\times k}$. Conversely, each sequence of projections in $\mathbb{C}_{k\times k}$ determines an element of $\mathcal{P}_k(l^\infty(\mathbb{Z}))$.

For $P \in \mathcal{P}(l^\infty(\mathbb{Z}, \mathbb{C}_{k\times k}))$, let $\operatorname{rank} P$ be the sequence

$$\mathbb{Z} \to \mathbb{Z}_+, \quad n \mapsto \operatorname{rank} P(n).$$

Clearly, this sequence is bounded by k and, conversely, every bounded sequence from $\mathbb{Z}$ into $\mathbb{Z}_+$ is the rank of a certain projection in $\mathcal{P}_\infty(l^\infty(\mathbb{Z}))$.

We claim that, if $P, Q \in \mathcal{P}_\infty(l^\infty(\mathbb{Z}))$, then

$$P \sim Q \quad \Longleftrightarrow \quad \operatorname{rank} P = \operatorname{rank} Q. \tag{6}$$

Since $\sim$ is an equivalence relation and by Proposition 2.3 (a), we can assume without loss of generality that $P, Q \in \mathcal{P}_k(l^\infty(\mathbb{Z}))$ with some positive integer k. Then the implication $\Leftarrow$ in (6) can be seen as follows. If the matrices $P(n), Q(n) \in \mathcal{P}(\mathbb{C}_{k\times k})$ have rank $l \le k$, then there are unitary operators U_n and V_n such that

$$U_n^* P(n) U_n = \operatorname{diag}(\underbrace{1, \ldots, 1}_{l}, 0, \ldots, 0) = V_n^* Q(n) V_n.$$

Define $W \in l^\infty(\mathbb{Z}, \mathbb{C}_{k\times k})$ by $W(n) := V_n U_n^*$. Then W is a unitary element in $l^\infty(\mathbb{Z}, \mathbb{C}_{k\times k})$, and $P = W^* Q W$. Hence, the projections P and Q are unitarily equivalent, which implies their Murray - von Neumann equivalence ([12], Proposition 2.2.2).

For the reverse implication in (6), let $P, Q \in \mathcal{P}(l^\infty(\mathbb{Z}, \mathbb{C}_{k\times k}))$ and $P \sim Q$. Then $P(n) \sim Q(n)$ for every $n \in \mathbb{Z}$. By elementary linear algebra, this implies that $\operatorname{rank} P(n) = \operatorname{rank} Q(n)$ and, hence, $\operatorname{rank} P = \operatorname{rank} Q$ ([12], Exercise 2.9).

This proves (6), and from the definition of the addition $\oplus$ in $\mathcal{P}_\infty(\mathcal{B})$ we conclude that $D(l^\infty(\mathbb{Z}))$ is isomorphic to the semigroup of all bounded sequences from $\mathbb{Z}$ into $\mathbb{Z}_+$, provided with the operation of pointwise addition. Passing to the Grothendieck group of this semigroup, we get

$$K_0(l^\infty(\mathbb{Z})) \cong \mathbb{Z}_b^{\mathbb{Z}}. \tag{7}$$

The mapping $\mathrm{id}_* - \alpha_*$ and its kernel. K-theory is functorial, i.e. given C^*-algebras $\mathcal{B}$ and $\mathcal{C}$ and a *-homomorphism $\varphi : \mathcal{B} \to \mathcal{C}$, there is a unique group homomorphism $\varphi_* : K_0(\mathcal{B}) \to K_0(\mathcal{C})$ such that

$$\varphi_* : [p]_\sim \mapsto [\varphi(p)]_\sim \quad \text{for } p \in \mathcal{P}_\infty.$$

Here, $\varphi(p)$ is defined as follows: the mapping φ extends to a *-homomorphism from $\mathcal{B}_{k\times k}$ into $\mathcal{C}_{k\times k}$ by

$$\varphi : (b_{ij})_{i,j=1}^{k} \mapsto (\varphi(b_{ij}))_{i,j=1}^{k},$$

and since φ maps projections to projections, it maps $\mathcal{P}_\infty(\mathcal{B})$ into $\mathcal{P}_\infty(\mathcal{C})$.

Thus, in our concrete setting, the mapping

$$\mathrm{id}_* : K_0(l^\infty(\mathbb{Z})) \to K_0(l^\infty(\mathbb{Z}))$$

which is induced by the identical mapping on $l^\infty(\mathbb{Z})$ is just the identical mapping on the associated K_0-groups. It is also clear that, still under the identification of $l^\infty(\mathbb{Z})_{k\times k}$ with $l^\infty(\mathbb{Z}, \mathbb{C}_{k\times k})$, the mapping

$$\alpha : \mathcal{P}_\infty(l^\infty(\mathbb{Z})) \to \mathcal{P}_\infty(l^\infty(\mathbb{Z})) \tag{8}$$

acts as the shift operator. Moreover, the equivalence (6) implies that, for $P,\, Q \in \mathcal{P}_\infty(l^\infty(\mathbb{Z}))$,

$$P \sim Q \quad \Longleftrightarrow \quad \alpha(P) \sim \alpha(Q).$$

Thus, the mapping (8) is compatible with the relation $\sim$, which shows that α induces the shift operator on $D(l^\infty(\mathbb{Z})) \cong \mathbb{Z}_b^{\mathbb{N}}$. This finally implies that α_* acts as the shift operator on $K_0(l^\infty(\mathbb{Z})) \cong \mathbb{Z}_b^{\mathbb{Z}}$.

Consequently, the kernel of the group homomorphism $\mathrm{id}_* - \alpha_*$ consists of all shift invariant sequences in $\mathbb{Z}_b^{\mathbb{Z}}$, i.e. of all constant sequences. The subgroup of $\mathbb{Z}_b^{\mathbb{Z}}$ of all constant sequences is isomorphic to $\mathbb{Z}$; so what we get is

$$\ker(\mathrm{id}_* - \alpha_*) \cong \mathbb{Z}. \tag{9}$$

Identification of $K_1(\mathcal{A}(\mathbb{Z}))$. The picture we have obtained so far is

$$\begin{array}{ccccc} \mathbb{Z}_b^{\mathbb{Z}} & \xrightarrow{\mathrm{id}_* - \alpha_*} & \mathbb{Z}_b^{\mathbb{Z}} & \longrightarrow & K_0(\mathcal{A}(\mathbb{Z})) \\ \uparrow{\scriptstyle\beta} & & & & \downarrow \\ K_1(\mathcal{A}(\mathbb{Z})) & \xleftarrow{\iota} & \{0\} & \longleftarrow & \{0\}. \end{array} \tag{10}$$

Since group homomorphisms map the zero element to the zero element, we have $\mathrm{im}\,\iota = \{0\}$, which implies that $\ker\beta = \{0\}$ due to the exactness of (10) at $K_1(\mathcal{A}(\mathbb{Z}))$. Further, by (9) and since (10) is exact at its left upper corner, we have $\mathrm{im}\,\beta \cong \mathbb{Z}$. Hence, β is a injective group homomorphism on $K_1(\mathcal{A}(\mathbb{Z}))$ with range $\mathbb{Z}$. Summing up, we find that

$$K_1(\mathcal{A}(\mathbb{Z})) \cong \mathbb{Z}. \tag{11}$$

2.3 The K_1-group of $\mathcal{A}_\pm$.

Following [7] we now split the algebra $\mathcal{A}(\mathbb{Z})$ into two subalgebras $\mathcal{A}_\pm$ which essentially contain the band-dominated operators on $\mathbb{Z}_\pm$, and we compute their respective K_1-groups. The basic device for this computation is the Mayer-Vietoris exact sequence which can be found in the following form in [7], Section 3, lemma 1 (for instance).

Theorem 2.4 (The Mayer-Vietoris exact sequence.) *Let $\mathcal{B}$ be a C^*-algebra and let $\mathcal{I}$ and $\mathcal{J}$ be closed ideals of $\mathcal{B}$ such that $\mathcal{I}+\mathcal{J}=\mathcal{B}$. Then there is a cyclic six term exact sequence*

$$\begin{array}{ccccc} K_0(\mathcal{I}\cap\mathcal{J}) & \longrightarrow & K_0(\mathcal{I})\oplus K_0(\mathcal{J}) & \longrightarrow & K_0(\mathcal{B}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{B}) & \longleftarrow & K_1(\mathcal{I})\oplus K_1(\mathcal{J}) & \longleftarrow & K_1(\mathcal{I}\cap\mathcal{J}). \end{array} \tag{12}$$

Let $\mathcal{J}_+$ denote the smallest closed subalgebra of the algebra $\mathcal{A}(\mathbb{Z})$ of all band-dominated operators on $\mathbb{Z}$ which contains the algebra $P\mathcal{A}(\mathbb{Z})P$ and the ideal $\mathcal{K}$ of the compact operators on $l^2(\mathbb{Z})$. We further define $\mathcal{J}_-$ by replacing P by Q in that definition. Then $\mathcal{J}_+$ and $\mathcal{J}_-$ are closed ideals of $\mathcal{A}(\mathbb{Z})$ which satisfy $\mathcal{J}_+ + \mathcal{J}_- = \mathcal{A}(\mathbb{Z})$ and $\mathcal{J}_+ \cap \mathcal{J}_- = \mathcal{K}$. Indeed, the inclusion $\mathcal{K} \subseteq \mathcal{J}_+ \cap \mathcal{J}_-$ follows from the definitions. Conversely, if $K \in \mathcal{J}_+ \cap \mathcal{J}_-$, then PKQ, QKP and QKQ are compact since $K \in \mathcal{J}_+$, and PKP is compact since $K \in \mathcal{J}_-$. Hence, $K = PKP + PKQ + QKP + QKQ$ is compact.

Let further $\mathcal{A}_+ := \mathcal{J}_- + \mathbb{C}Q$ and $\mathcal{A}_- := \mathcal{J}_- + \mathbb{C}P$. Then $\mathcal{A}_+$ and $\mathcal{A}_-$ are C^*-subalgebras of $\mathcal{A}(\mathbb{Z})$ which are *-isomorphic to the (minimal) unitizations of the ideals $\mathcal{J}_+$ and $\mathcal{J}_-$, respectively. For, one easily checks that every operator $A \in \mathcal{A}_+$ can be written as $A = PAP + K + \alpha Q$ where $PAP + K \in P\mathcal{A}(\mathbb{Z})P + \mathcal{K}$ and $\alpha \in \mathbb{C}$ are uniquely determined, and that

$$\mathcal{A}_+ \to \mathcal{J}_+ \times \mathbb{C}, \quad PAP + K + \alpha Q \mapsto (PAP + K - \alpha P,\, \alpha)$$

is a *-isomorphism from $\mathcal{A}_+$ onto the unitization $\mathcal{J}_+ \times \mathbb{C}$ of the ideal $\mathcal{J}_+$.

Thus, we can apply the Mayer-Vietoris exact sequence with $\mathcal{A}(\mathbb{Z})$, $\mathcal{J}_+$, $\mathcal{J}_-$ and $\mathcal{K}$ in place of $\mathcal{B}, \mathcal{I}$, $\mathcal{J}$ and $\mathcal{I}\cap\mathcal{J}$. The K-theory of $\mathcal{K}$ is well known,

$$K_0(\mathcal{K}) \cong \mathbb{Z} \quad \text{and} \quad K_1(\mathcal{K}) = \{0\}$$

(Corollary 6.4.2 and Example 8.2.9 in [12]). Thus, and by (11), the general exact sequence (12) specifies to

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(\mathcal{J}_+)\oplus K_0(\mathcal{J}_-) & \longrightarrow & K_0(\mathcal{A}(\mathbb{Z})) \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \xleftarrow{\beta} & K_1(\mathcal{J}_+)\oplus K_1(\mathcal{J}_-) & \xleftarrow{\gamma} & \{0\} \end{array}$$

with certain group homomorphisms β and γ. From $\operatorname{im}\gamma = \{0\}$ we conclude that β is injective. Hence, $K_1(\mathcal{J}_+) \oplus K_1(\mathcal{J}_-)$ is isomorphic to a subgroup of $\mathbb{Z}$. But each subgroup of $\mathbb{Z}$ is either isomorphic to $\mathbb{Z}$ or equal to $\{0\}$. Suppose for a moment that $K_1(\mathcal{J}_+) \oplus K_1(\mathcal{J}_-) \cong \mathbb{Z}$. Since the ideals $\mathcal{J}_+$ and $\mathcal{J}_-$ are *-isomorphic (a *-isomorphism is given by $K \mapsto JKJ$ where $J : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is given by $(Jx)(n) := x(-n-1)$), their K_1-groups are isomorphic, too:

$$K_1(\mathcal{J}_+) \cong K_1(\mathcal{J}_-) =: \Gamma.$$

Thus, $\mathbb{Z}$ is isomorphic to $\Gamma \oplus \Gamma$, the direct sum of two copies of Γ. But $\mathbb{Z}$ is singly generated (by 1, for example), whereas $\Gamma \oplus \Gamma$ cannot be generated by a single element. This contradiction shows that

$$K_1(\mathcal{J}_+) \oplus K_1(\mathcal{J}_-) \cong \Gamma \oplus \Gamma = \{0\},$$

whence

$$K_1(\mathcal{J}_+) = K_1(\mathcal{J}_-) = \{0\}.$$

Finally, the K_1-groups of a C^*-algebra and of its unitization coincide (Proposition 8.1.6 and Equality (8.4) in [12]) which implies that

$$K_1(\mathcal{A}_+) \cong K_1(\mathcal{A}_-) = \{0\}. \tag{13}$$

2.4 Indices of band-dominated operators

In this section, we will prove assertion (a) of Theorem 1.2, which has assertions (b) and (c) as its corollaries. In the course of the proof, we will make use of some of the following elementary properties of the plus- and minus-indices of Fredholm band-dominated operators.

Proposition 2.5 *Let A and B be Fredholm operators in $\mathcal{A}(\mathbb{Z})$. Then*
(a) $\operatorname{ind}_\pm A$ *is invariant with respect to small perturbations.*
(b) $\operatorname{ind}_\pm A$ *is invariant with respect to compact perturbations.*
(c) $\operatorname{ind}_\pm A^* = -\operatorname{ind}_\pm A$.
(d) $\operatorname{ind}_\pm AB = \operatorname{ind}_\pm A + \operatorname{ind}_\pm A$.

The latter property follows from

$$PABP + Q = (PAP + Q)(PBP + Q) + \mathit{compact}.$$

Further we need the following continuity property of limit operators which is proved in [10].

Proposition 2.6 *Let C_n, $C \in L(l^2(\mathbb{Z}))$ be operators with $\|C_n - C\| \to 0$, and let the limit operators $(C_n)_g$ exist with respect to a given sequence g and for all n. Then the limit operator C_g exists, too, and $\|(C_n)_g - C_g\| \to 0$.*

Proof of Theorem 1.2 (a)**.** We abbreviate the C^*-algebra of all $k \times k$ matrices with entries in $\mathcal{A}(\mathbb{Z})_+$ to $\mathcal{A}_k$ and write P_k and Q_k for the operators

$$\mathrm{diag}\,(P, \ldots, P), \quad \mathrm{diag}\,(Q, \ldots, Q) : \mathcal{A}_k \to \mathcal{A}_k.$$

It is clearly sufficient to prove the theorem for the plus-case where it reads as follows:

$$\mathrm{ind}_+ A = \mathrm{ind}_+ A_h \quad \text{for all } \ A_h \in \sigma_+(A). \tag{14}$$

It is further sufficient to prove (14) only in the case when $\mathrm{ind}_+ A = 0$. Indeed, for the shift operator V_1 one has $\mathrm{ind}_+ V_1 = -1$ and $\mathrm{ind}_- V_1 = 1$. Thus, if $A \in \mathcal{A}(\mathbb{Z})$ is a Fredholm operator with plus-index r, then AV_1^r is a Fredholm operator with plus-index 0. If the identity (14) holds for all Fredholm operators with vanishing plus-index, then this implies that

$$\mathrm{ind}_+(AV_1^r)_h = 0 \quad \text{for every limit operator of } AV_1^r.$$

But, evidently, every limit operator of AV_1^r is of the form $A_h V_1^r$ since V_1 is shift invariant. Thus,

$$\mathrm{ind}_+(A_h V_1^r) = 0 \quad \text{for every } A_h \in \sigma_+(A),$$

whence, by Proposition 2.5 (d),

$$0 = \mathrm{ind}_+(A_h V_1^r) = \mathrm{ind}_+ A_h + \mathrm{ind}_+ V_1^r = \mathrm{ind}_+ A_h - r$$

and, finally, $\mathrm{ind}_+ A_h = r$ for every limit operator of A in $\sigma_+(A)$.

So, what we really have to check is that, for all Fredholm band-dominated operators A,

$$\mathrm{ind}_+ A = 0 \quad \Longrightarrow \quad \mathrm{ind}_+ A_h = 0 \quad \text{for all } A_h \in \sigma_+(A). \tag{15}$$

Let $\mathrm{ind}_+ A = 0$, i.e. $\mathrm{ind}\,(PAP + Q) = 0$. Let further K be a compact operator such that $B := PAP + Q + K \in \mathcal{A}_+$ is invertible, and let $B = UR$ be the polar decomposition of B, i.e. U is a unitary operator in $\mathcal{A}_1$, and R is a positive definite operator in $\mathcal{A}_1$. A consequence of the vanishing of the K_1-group of $\mathcal{A}_1$ (according to (13)) is that U is stably path connected with the identity operator (see the Definition 8.1.3 of the K_1-group in [12]). Thus, there is a positive integer k such that

$$\begin{pmatrix} U & 0 \\ 0 & I_{k-1} \end{pmatrix} \sim_h \begin{pmatrix} I & 0 \\ 0 & I_{k-1} \end{pmatrix}$$

in the set of the unitary operators of $\mathcal{A}_k$. Here, $\sim_h$ denotes homotopy equivalence, and I_{k-1} refers to the identity operator in $\mathcal{A}_{k-1}$.

Choose a continuous unitary-valued function

$$f_1 : [0, 1] \to \mathcal{A}_k \quad \text{with} \quad f_1(0) = \begin{pmatrix} U & 0 \\ 0 & I_{k-1} \end{pmatrix}, \quad f_1(1) = \begin{pmatrix} I & 0 \\ 0 & I_{k-1} \end{pmatrix}.$$

Further, let

$$f_2 : [0, 1] \to \mathcal{A}_k, \quad t \mapsto (1-t) \begin{pmatrix} R & 0 \\ 0 & I_{k-1} \end{pmatrix} + t \begin{pmatrix} I & 0 \\ 0 & I_{k-1} \end{pmatrix}$$

and

$$f_3 : [0, 1] \to \mathcal{A}_k, \quad t \mapsto (1-t) \begin{pmatrix} K & 0 \\ 0 & 0_{k-1} \end{pmatrix}.$$

Then f_2 is a continuous function having only positive definite operators as its values, and f_3 is a continuous function with compact values. Hence,

$$f := f_1 f_2 - f_3 : [0, 1] \to \mathcal{A}_k$$

is a continuous function with

$$f(0) = \begin{pmatrix} U & 0 \\ 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & I_{k-1} \end{pmatrix} - \begin{pmatrix} K & 0 \\ 0 & 0_{k-1} \end{pmatrix} = \begin{pmatrix} PAP + Q & 0 \\ 0 & I_{k-1} \end{pmatrix}$$

and

$$f(1) = \begin{pmatrix} I & 0 \\ 0 & I_{k-1} \end{pmatrix},$$

and all values of that function are Fredholm operators (with index 0).

Let now $h : \mathbb{N} \to \mathbb{Z}$ be a sequence which tends to $+\infty$ and for which the limit operator A_h exists. Then, obviously, the limit operator of P with respect to h exists, and $P_h = I$. Hence, the limit operator of $f(0)$ with respect to h exists, and

$$f(0)_h = \begin{pmatrix} A_h & 0 \\ 0 & I_{k-1} \end{pmatrix}.$$

We use a Cantor diagonal argument in order to produce a subsequence g of h such that the limit operator $f(q)_g$ exists for every rational number q in $[0, 1]$. For, let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Then one can find a subsequence g_1 of h such that $f(q_1)_{g_1}$ exists (recall Theorem 1.1 (a)), further a subsequence g_2 of g_1 such that $f(q_2)_{g_2}$ exists, etc. The sequence defined by $g(n) := g_n(n)$ has the desired property.

Since $\mathbb{Q} \cap [0, 1]$ is dense in $[0, 1]$, we conclude from Proposition 2.6 that the limit operator $f(t)_g$ exists for every $t \in [0, 1]$ and that

$$[0, 1] \to \mathcal{A}_k, \quad t \mapsto f(t)_g \tag{16}$$

is a continuous function with

$$f(0)_g = \begin{pmatrix} A_h & 0 \\ 0 & I_{k-1} \end{pmatrix} \quad \text{and} \quad f(1)_g = \begin{pmatrix} I & 0 \\ 0 & I_{k-1} \end{pmatrix}.$$

Moreover, all values of the function (16) are invertible operators (because limit operators of Fredholm operators are invertible). Thus,

$$F : [0, 1] \to \mathcal{A}_k, \quad t \mapsto P_k f(t)_g P_k + Q_k$$

is a continuous function with

$$F(0) = \begin{pmatrix} PA_hP + Q & 0 \\ 0 & I_{k-1} \end{pmatrix} \quad \text{and} \quad F(1) = \begin{pmatrix} I & 0 \\ 0 & I_{k-1} \end{pmatrix}$$

all values of which are Fredholm operators (recall that P_kBQ_k and Q_kBP_k are compact for all band-dominated operators B). From the continuity of the index we finally conclude that

$$\operatorname{ind} F(0) = \operatorname{ind} \begin{pmatrix} PA_hP + Q & 0 \\ 0 & I_{k-1} \end{pmatrix} = \operatorname{ind} \begin{pmatrix} I & 0 \\ 0 & I_{k-1} \end{pmatrix} = \operatorname{ind} F(1),$$

whence $\operatorname{ind}(PA_hP + Q) = \operatorname{ind}_+ A_h = 0$. ■

Remark: The argument of this section can also be expressed in K-theoretic terms. Namely, consider the quotient algebra $\mathcal{A}_1/\mathcal{K}$. Since $K_1(\mathcal{A}_1) = 0$, the six term exact sequence of K-theory shows that $K_1(\mathcal{A}_1/\mathcal{K}) = \mathbb{Z}$, with the isomorphism being implemented by the Fredholm index. The continuity of the limit operation expressed by Proposition 2.6 shows that the assignment

$$U \mapsto \text{plus-index of a plus-limit operator of } U$$

gives a homomorphism $K_1(\mathcal{A}_1/\mathcal{K}) \to \mathbb{Z}$, and to check that it agrees with the Fredholm index it suffices to check one example, the generator of $K_1(\mathcal{A}_1/\mathcal{K})$ given by $[V_1]$.

3 Proof of Theorem 1.3

Assertion (a)**.** Let A be a tridiagonal unitary operator on $l^2(\mathbb{Z}_+)$ with matrix representation

$$A = \begin{pmatrix} a_0 & b_1 & & & \\ c_1 & a_1 & b_2 & & \\ & c_2 & a_2 & b_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with respect to the standard basis of $l^2(\mathbb{Z}_+)$.

We choose unimodular numbers u_n and v_n such that $u_1a_0v_1$ as well as all numbers $u_{n+1}c_nv_n$ and $u_nb_nv_{n+1}$ are non-negative, and we set $U := \operatorname{diag}(u_1, u_2, \ldots)$ and $V := \operatorname{diag}(v_1, v_2, \ldots)$. Then U and V are unitary operators, and $T := UAV$ is a unitary tridiagonal operator

$$T = \begin{pmatrix} \alpha_0 & \beta_1 & & & \\ \gamma_1 & \alpha_1 & \beta_2 & & \\ & \gamma_2 & \alpha_2 & \beta_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with α_0, β_n, $\gamma_n \in \mathbb{R}_+$ for all positive integers n.

Consider the entries of the main diagonals of $TT^* = I$ and $T^*T = I$. The first of these entries are equal to

$$\alpha_0^2 + \beta_1^2 = 1 = \alpha_0^2 + \gamma_1^2,$$

whence $\beta_1 = \gamma_1$ due to the non-negativity of β_1 and γ_1. The second pair of these entries is

$$\gamma_1^2 + |\alpha_1|^2 + \beta_2^2 = 1 = \beta_1^2 + |\alpha_1|^2 + \gamma_2^2,$$

whence $\beta_2 = \gamma_2$. Proceeding in this way we see that T is necessarily of the form

$$T = \begin{pmatrix} \alpha_0 & \beta_1 & & & \\ \beta_1 & \alpha_1 & \beta_2 & & \\ & \beta_2 & \alpha_2 & \beta_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We claim that, if $\beta_1 \neq 0$, then $\beta_2 = 0$. Indeed, the 12-entry of $TT^* = I$ equals $\alpha_0\beta_1 + \beta_1\overline{\alpha_1} = 0$, whence $\overline{\alpha_1} = -\alpha_0$. Thus, the first and the second entry on the main diagonal of $TT^* = I$ are actually given by $|\alpha_0|^2 + \beta_1^2 = 1$ and $\beta_1^2 + |\alpha_1|^2 + \beta_2^2 = \beta_1^2 + |\alpha_0|^2 + \beta_2^2 = 1$, respectively. These equalities imply that $\beta_2 = 0$.

Consequently, there is either a unitary 1×1-block (if $\beta_1 = 0$) or a unitary 2×2-block (if $\beta_1 \neq 0$ and hence $\beta_2 = 0$) in the upper left corner of T. Applying the same arguments to the remaining part of T (which evidently also can be identified with a unitary tridiagonal operator on $l^2(\mathbb{Z}_+)$), we obtain assertion (a) of Theorem 1.3.

Assertion (b)**.** Let A be a unitary band operator on $l^2(\mathbb{Z}_+)$ with matrix representation

$$A = \begin{pmatrix} A_0 & B_1 & & & \\ C_1 & A_1 & B_2 & & \\ & C_2 & A_2 & B_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with respect to the standard basis of $l^2(\mathbb{Z}_+)$ where the A_n, B_n and C_n are $k \times k$-blocks of the same block size k.

We choose unitary $k \times k$ matrices U_n and V_n such that the matrix $U_1A_0V_1$ and all matrices $U_{n+1}C_nV_n$ and $U_nB_nV_{n+1}$ with $n \geq 1$ become non-negative (choose $U_1 := I$ and use the polar decomposition to define successively V_1, U_2, V_2, U_3, $\ldots$) and set $U_{(1)} := \operatorname{diag}(U_1, U_2, \ldots)$ and $V_{(1)} := \operatorname{diag}(V_1, V_2, \ldots)$. Then $U_{(1)}$ and $V_{(1)}$ are unitary operators, and $T_1 := U_{(1)}AV_{(1)}$ is a unitary tridiagonal operator, the

$k \times k$ block entries of which we denote by A_n, B_n and C_n again, i.e.

$$T_1 = \begin{pmatrix} A_0 & B_1 & & & \\ C_1 & A_1 & B_2 & & \\ & C_2 & A_2 & B_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where now A_0, B_n and C_n are self-adjoint and non-negative. The upper left $N \times N$ corner of $T_1 T_1^* = I$ is $A_0^2 + B_1^2 = I$. Hence, the matrix A_0 is a contraction, B_1 is equal to $S_0 := (I - A_0^2)^{1/2}$, and the operator

$$W_1 := \begin{pmatrix} A_0 & S_0 & 0 & \\ S_0 & -A_0 & 0 & \\ 0 & 0 & I & \\ & & & \ddots \end{pmatrix}$$

is unitary. Further we get as in the proof of part (a) that $C_1 = B_1$. Thus, we have

$$\begin{aligned} A^{(1)} := W_1 T_1 &= \begin{pmatrix} A_0 & S_0 & 0 & \\ S_0 & -A_0 & 0 & \\ 0 & 0 & I & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} A_0 & S_0 & & & \\ S_0 & A_1 & B_2 & & \\ & C_2 & A_2 & B_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \\ &= \begin{pmatrix} I & A_0 S_0 + S_0 A_1 & S_0 B_2 & 0 & \\ 0 & I - A_0^2 - A_0 A_1 & -A_0 B_2 & 0 & \\ 0 & C_2 & A_2 & B_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \end{aligned}$$

Being the product of unitary operators, the operator $A^{(1)}$ is unitary, too. Thus, multiplying the first row of this operator by the first column of its adjoint, we get

$$I + (A_0 S_0 + S_0 A_1)(A_0 S_0 + S_0 A_1)^* + (S_0 B_2)(S_0 B_2)^* = I$$

whence

$$A_0 S_0 + S_0 A_1 = 0 \quad \text{and} \quad S_0 B_2 = 0.$$

Thus, $A^{(1)}$ is actually a unitary operator of the form

$$\begin{pmatrix} I & 0 & 0 & 0 & \\ 0 & A_1' & B_2' & 0 & \\ 0 & C_2 & A_2 & B_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with $A_1' := I - A_0^2 - A_0 A_1$ and $B_2' := -A_0 B_2$, and the operator A can be written as

$$A = U_{(1)}^* W_1^* A^{(1)} V_{(1)}^*.$$

Now we repeat the same arguments to the second block column of the unitary operator $A^{(1)}$. That is, we choose unitary $k \times k$ block operators $U_{(2)}$ and $V_{(2)}$ such that

$$T_2 := U_{(2)} A^{(1)} V_{(2)} = \begin{pmatrix} I & 0 & 0 & 0 & & \\ 0 & A_1' & B_2' & 0 & & \\ 0 & C_2 & A_2 & B_3 & & \\ & & \ddots & \ddots & \ddots & \end{pmatrix}$$

with non-negative matrices A_1', B_2', B_n and C_n, and we set $S_1 := (I - (A_1')^2)^{1/2}$ and

$$W_2 := \begin{pmatrix} A_1' & S_1 & 0 & \\ S_1 & -A_1' & 0 & \\ 0 & 0 & I & \\ & & & \ddots \end{pmatrix}.$$

Then W_2 is an elementary unitary operator, $A^{(2)} := T_2 W_2$ is a unitary operator of the form

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & \\ 0 & I & 0 & 0 & 0 & 0 & \\ 0 & 0 & A_2' & B_3 & 0 & 0 & \\ 0 & 0 & C_3' & A_3 & B_4 & 0 & \\ & & & \ddots & \ddots & \ddots & \end{pmatrix},$$

and

$$A = U_{(1)}^* W_1^* U_{(2)}^* A^{(2)} W_2^* V_{(2)}^* V_{(1)}^*.$$

Now we deal with the third row of $A^{(2)}$ (by operating from the left hand side again), after this with its forth column (from the right hand side) etc. What we finally get is that $A = \tilde{U}\tilde{V}$ where $\tilde{U}$ and $\tilde{V}$ are diagonal operators

$$\tilde{U} := \operatorname{diag}(\tilde{U}_1, \tilde{U}_2, \ldots) \quad \text{and} \quad \tilde{V} := \operatorname{diag}(\tilde{V}_1, \tilde{V}_2, \ldots)$$

with a unitary $k \times k$ matrix $\tilde{V}_1$ and with unitary $2k \times 2k$ matrices $\tilde{U}_n$ $(n \geq 1)$ and $\tilde{V}_n$ $(n \geq 2)$. Thus, $\tilde{U}$ and $\tilde{V}$ are elementary unitary operators on $l^2(\mathbb{Z}_+)$. ∎

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