On elementary operators of length 1 and some order intervals in $\mathcal{B}_s(H)$

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1 Introduction

In the physical papers [1]–[4] there are some interesting propositions about operators acting in Hilbert spaces. For example in [3] there were considered all bijective mappings on $\mathcal{B}_s(H)$, which preserve the order in both directions. These mappings have been characterized with the help of elementary operators of length 1. In the present paper we will generalize these results.

It is well known that the order properties of operators in Hilbert spaces are very important in mathematical physics, because they are related to symmetry properties. If one considers $\mathcal{B}_s(H)$, the set of all bounded self-adjoint operators in a Hilbert space H, then $\mathcal{B}_s(H)$ is partially ordered by the relation $A \leq B \Leftrightarrow (Ax, x) \leq (Bx, x)$ for all $x \in H$. Also we can consider the set of all positive operators $0 \leq A \leq I$, also ordered by the same relation, with the additional mapping $A^{\perp} = I - A$ (the so-called effects). There is a question how to characterize the order relation in $\mathcal{B}_s(H)$. In [4] there is described the form of all bijective maps on $\mathcal{B}_s(H)$ which preserve the order $\leq$ in both directions. Also in [3] it is shown that for $\dim H = 2$ the $\perp$-order authomorphisms of the effects on H are induced by unitary or antiunitary operators, namely as $\Phi(A) = U^*AU$ (for $\dim H \leq 3$ this was shown by Wigner).

Another question to be considered is to characterize operators which map vectors to parallel or to orthogonal ones. For unitary or antiunitary operators this was solved in [2], where it is shown that for unitary operators they must be of the form $U = zI$, for antiunitary they must satisfy the equality $V^2 = -I$. In the present paper we would like to consider these properies for a wider class of operators. First, in Chapter 3 we characterize the elementary operators of length 1. In Chapter 4 for $T \in \mathcal{B}(H)$ we consider the maps of the form $\Phi(A) = T^*AT$ for all $A \in \mathcal{B}(H)$ (Φ is an elementary operator of length 1). We show that Φ preserves the order in both directions if and only if the range of T is dense in H, $\overline{T(H)} = H$. We also characterize commutativity of A and B by showing that $AB = BA$ is equivalent to $\Phi(A)\Phi(B) = \Phi(B) \cdot \Phi(A)$ if and only if $TT^* = \alpha I$ ($\alpha > 0$). We also show that $A\Phi(A) = \Phi(A)A$ if and only if $T = \alpha I$ ($\alpha \in \mathbb{C}$). Finally we show that if $T \in \mathcal{B}(H)$ maps vectors to orthogonal ones $(Tx, x) = 0$, or to parallel ones $Tx = \alpha x$ ($\alpha \in \mathbb{C}$), then $T = zI$. This is a characterization of identity operator, generalizing the main result of [3].

Finally in Chapter 5 we consider some order intervals of self adjoint operators in Hilbert space. We show that for $A, B \in \mathcal{B}_s(H)$ with $AB = BA$ and

$$C := \frac{1}{2}(A + B - |A - B|)$$

we have $[C, A] \cap [C, B] = \{C\}$. Here $[C, A]$ denotes the interval $\{X : C \leq X \leq A\}$ (similarly for $[C, B]$), and

$$|D| := \sqrt{D^*D} \quad \text{for } D \in \mathcal{B}_s(H).$$

In particular, this implies that for $A \in \mathcal{B}_s(H)$ $A^+ := \frac{1}{2}(A + |A|)$ is the only positive element in the order interval $[A, A^+]$. This generalizes the results of [1] to infinite dimensional case.

2 Notation and terminology

In the sequel let H denote a Hilbert space. By $\mathcal{B}(H)$ we denote the vector space of all bounded endomorphisms on H, and $\mathcal{B}_s(H)$ denotes the vector space of all bounded self–adjoint operators on H, equiped with the canonical partial order.

For $x, y \in H$, let $x \otimes y \in \mathcal{B}(H)$ be defined as

$$x \otimes y(z) = (z, x)y \text{ for all } z \in H,$$

where $(\cdot, \cdot)$ denotes the inner product in H.

For U, $V \in \mathcal{B}(H)$ on the operator algebra $\mathcal{B}(H)$ there is defined the elementary operator $\Phi_{U,V}$ of length 1 as follows

$$\Phi_{U,V}(A) = UAV \text{ for all } A \in \mathcal{B}(H).$$

It is known that $\Phi_{U,V} \in \mathcal{B}(\mathcal{B}(H))$.

3 A characterization of the elementary operators of length 1

Let $U, V \in \mathcal{B}(H)$. By an easy calculation we can show that

$$\Phi_{U,V}(x \otimes y) = V^* x \otimes Uy$$

for all $x, y \in H$, where V^* is the adjoint of V.

Let $H \widehat{\otimes} H := \{x \otimes y : x, y \in H\}$. It is clear that

$$\Phi_{U,V}(H \widehat{\otimes} H) \subseteq H \widehat{\otimes} H.$$

However, this inclusion relation does not characterize the elementary operators of length 1, which is shown by the following example.

Let H be the Euclidean vector space $\mathbb{C}^2$. Then the algebra $\mathcal{B}(H)$ can be indentified with the algebra M_2 of all square matrices 2×2. Let the mapping $\Phi \in \mathcal{B}(\mathcal{B}(H))$ be defined as follows: For $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ let

$$\begin{aligned} \Phi(A) &= (a_{11} + 2a_{12} + 3a_{21} + 4a_{22} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= (a_{11} + 2a_{12} + 3a_{21} + 4a_{22}) e_1 \otimes e_1 \in H \widehat{\otimes} H. \end{aligned}$$

Then obviously $\Phi(H \widehat{\otimes} H) \subseteq H \widehat{\otimes} H$. Suppose that there would exist $R, S \in M_2$ with $\Phi(A) = RAS$ for all $A \in M_2$. Then we would have

$$(r_{11}a_{11} + r_{12}a_{21})s_{11} + (r_{11}a_{12} + r_{12}a_{22})s_{21} = a_{11} + 2a_{12} + 3a_{21} + 4a_{22}.$$

This would imply

$$r_{11}s_{11} = 1,\ r_{12} \cdot s_{11} = 3 \Rightarrow \frac{r_{11}}{r_{12}} = \frac{1}{3},\ r_{11} \cdot s_{21} = 2,\ r_{12} \cdot s_{21} = 4,\ \frac{r_{11}}{r_{12}} = \frac{2}{4} = \frac{1}{2},$$

which is a contradiction. Therefore, the mapping Φ is not an elementary operator of length 1.

With the help of $H \widehat{\otimes} H$ one can characterize the elementary operators of length 1 as follows.

Theorem 1. *Let $0 \neq \Phi \in \mathcal{B}(\mathcal{B}(H))$. Then there are $U, V \in \mathcal{B}(H)$ with $\Phi = \Phi_{U,V}$ if and only if the following conditions hold*

(i) Φ is continuous in the topology of pointwise convergence on $\mathcal{B}(H)$

(ii) There are mappings $\Phi_1,\ \Phi_2 : H \to H$ and $x_0, y_0 \in H$ with $\Phi_1(x_0) \neq 0$, $\Phi_2(y_0) \neq 0$ and

$$\Phi(x \otimes y) = \Phi_1(x) \otimes \Phi_2(y)$$

for all $x, y \in H$.

Proof. $\Rightarrow$ Let $\Phi = \Phi_{U,V}$ with $U, V \in \mathcal{B}(H)$. Moreover, let $S \in \mathcal{B}(H)$ and let (S_α) be a net in $\mathcal{B}(H)$ with $S = \lim_\alpha S_\alpha$ in the topology of pointwise convergence in $\mathcal{B}(H)$. Then for $x \in H$ we have

$$SVx = \lim_\alpha S_\alpha Vx \quad \text{ and } USVx = \lim_\alpha US_\alpha Vx.$$

Hence we also have $\Phi(S)x = \lim_\alpha \Phi(S_\alpha)x$.

Therefore Φ is continuous in the topology of pointwise convergence in $\mathcal{B}(H)$. Since $\Phi \neq 0$, we have $U \neq 0$ and $V \neq 0$. Hence there are $x_0, y_0 \in H$ with $Ux_0 \neq 0$ and $Vy_0 \neq 0$. Consequently, condition (ii) holds with $\Phi_1 = V^*$ and $\Phi_2 = U$.

$\Leftarrow$ First we show that the mappings Φ_1 and Φ_2 are linear. Let $y_0 \in H$ with $\Phi_2(y_0) \neq 0$. Hence for $\lambda \in \mathbb{C}$ and $z_1, z_2 \in H$ we have:

$$\Phi((\lambda z_1) \otimes y_0) = \Phi_1(\lambda z_1) \otimes \Phi_2(y_0).$$

Since $(\lambda z_1) \otimes y_0 = \overline{\lambda}(z_1 \otimes y_0)$, we have also $\Phi((\lambda z_1) \otimes y_0) = \overline{\lambda}\Phi(z_1 \otimes y_0) = \overline{\lambda}(\Phi_1(z_1) \otimes \Phi_2(y_0))$, and we obtain for all $x \in H$

$$(x, \Phi_1(\lambda z_1))\Phi_2(y_0) = \overline{\lambda}(x, \Phi_1(z_1))\Phi_2(y_0) = (x, \lambda\Phi_1(z_1))\Phi_2(y_0).$$

Hence it follows that $\Phi_1(\lambda z_1) = \lambda\Phi_1(z_1)$. We have

$$\Phi(z_1 \otimes y_0) = \Phi_1(z_1) \otimes \Phi_2(y_0),$$
$$\Phi(z_2 \otimes y_0) = \Phi_1(z_2) \otimes \Phi_2(y_0)$$

and

$$\Phi((z_1 + z_2) \otimes y_0) = \Phi_1(z_1 + z_2) \otimes \Phi_2(y_0)$$

Since Φ is additive and

$$z_1 \otimes y_0 + z_2 \otimes y_0 = (z_1 + z_2) \otimes y_0,$$

we obtain

$$\Phi_1(z_1 + z_2) \otimes \Phi_2(y_0) = \Phi_1(z_1) \otimes \Phi_2(y_0) + \Phi_1(z_2) \otimes \Phi_2(y_0).$$

Hence it follows that

$$\Phi_1(z_1 + z_2) = \Phi_1(z_1) + \Phi_1(z_2).$$

The mapping Φ_1 is also linear. We can show analogously the linearity of Φ_2. The mappings Φ_1 and Φ_2 are bounded. Namely for $x, y \in H$ we have:

$$\begin{aligned} \|x \otimes y\| &= \sup\{\|(z, x)y\| : z \in H,\ \|z\| \leq 1\} \\ &= \sup\{|(z, x)| : z \in H,\ \|z\| \leq 1\} \cdot \|y\| = \|x\| \cdot \|y\| \end{aligned}$$

Therefore we obtain

$$\|\Phi_1(x)\|\ \|\Phi_2(y_0)\| = \|\Phi(x \otimes y_0)\| \leq \|\Phi\|\ \|x\| \cdot \|y_0\|.$$

Hence it follows that

$$\|\Phi_1(x)\| \leq \frac{\|\Phi\|\ \|y_0\|}{\|\Phi_2(y_0)\|}\|x\|$$

for all $x \in H$.

Hence Φ_1 is bounded. Analogously we show that Φ_2 is also bounded. Now we put $V := \Phi_1^*$ and $U := \Phi_2$. Clearly $U, V \in \mathcal{B}(H)$. Moreover, let

$$\widetilde{\Phi} : \mathcal{B}(H) \to \mathcal{B}(H)$$

be defined by

$$\widetilde{\Phi}(A) = UAV \text{ for all } A \in \mathcal{B}(H).$$

Then we have

$$\widetilde{\Phi}(x \otimes y) = V^*x \otimes Uy = \Phi_1(x) \otimes \Phi_2(y) = \Phi(x \otimes y)$$

for all $x, y \in H$.

It is known that, in the topology of pointwise convergence the tensor product $H \otimes H$, which is the linear span of $H\widehat{\otimes}H$, is dense in $\mathcal{B}(H)$ (see [6] p.108 and 109). Hence the mappings Φ and $\widetilde{\Phi}$ coincide on the dense subset $H \otimes H$. Since they are continuous in the topology of pointwise convergence, we have

$$\Phi = \widetilde{\Phi} = \Phi_{U,V}. \quad \square$$

4 Equivalence in both directions

Let $0 \neq T \in \mathcal{B}(H)$. In the sequel let Φ be the elementary operator on $\mathcal{B}(H)$ with $\Phi(A) = T^*AT$ for all $A \in \mathcal{B}(H)$. As it is easy to prove, we have

$$\Phi(\mathcal{B}_s(H)) \subseteq \mathcal{B}_s(H) \quad \text{and} \quad \Phi(A) \geq 0$$

for all positive $A \in \mathcal{B}_s(H)$.

For invertible operators T, the following theorem can be derived from [4], Th. 2:

Theorem 2. *Let $A, B \in \mathcal{B}_s(H)$. The following conditions are equivalent*

(i) $A \leq B \Leftrightarrow \Phi(A) \leq \Phi(B)$

(ii) The range of T is dense in H.

Proof. (i)$\Rightarrow$(ii). Assume that $\overline{T(H)} \neq H$. Then there is $y \in H$ with $\|y\| = 1$ and $y \perp \overline{T(H)}$. Let P be the orthogonal projection onto the linear span $\langle y \rangle$ of y, i.e. we have $P(x) = (x, y)y$ for all $x \in H$. Then there holds

$$0 = T^*(2P)T \leq T^*IT$$

and consequently $\Phi(2P) \leq \Phi(I)$, where I is the indentity on H.

Now we have

$$(2Py, y) = 2 \text{ and } (Iy, y) = 1$$

This means that $2P \not\leq I$, which is a contradiction with (i). Hence $\overline{T(H)} = H$.

(ii)$\rightarrow$ (i). Since Φ is positive, from $A \leq B$ it follows that $\Phi(A) \leq \Phi(B)$.

Now let $\Phi(A) \leq \Phi(B)$. Then for all $x \in H$ there holds

$$(T^*ATx, x) \leq (T^*BTx, x), \text{ and consequently } (ATx, Tx) \leq (BTx, Tx).$$

Let $y \in H$. Since we have

$$\overline{T(H)} = H,$$

there exists a sequence (x_n) in H such that $y = \lim_{n\to\infty} Tx_n$.

Hence we have

$$(ATx_n, Tx_n) \leq (BTx_n, Tx_n)$$

for all $n \in N$. This implies for $n \to \infty : (Ay, y) \leq (By, y)$. Hence $A \leq B$. □

The following theorem for unitary operators is known (see [4], Th 2, Cor 3).

Theorem 3. *Let $TT^* = \alpha I$ with $0 < \alpha \in \mathbb{R}$. Then for all $A, B \in \mathcal{B}(H)$ we have*

$$AB = BA \Leftrightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)$$

Proof. $\Rightarrow$ Let $AB = BA$. Then it follows that

$$\Phi(A)\Phi(B) = T^*ATT^*BT = \alpha T^*ABT$$

and

$$\Phi(B)\Phi(A) = T^*BTT^*AT = \alpha T^*BAT = \alpha T^*ABT$$

Hence we infer that

$$\Phi(A)\Phi(B) = \Phi(A)\Phi(B).$$

$\Leftarrow$ Let now $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)$. Then we have

$$T^*ATT^*BT = T^*BTT^*AT$$

Multiplying by T on the left and by T^* on the right be obtain

$$TT^*ATT^*BTT^* = TT^*BTT^*ATT^*$$

This implies

$$\alpha^3 AB = \alpha^3 BA \Rightarrow AB = BA. \quad \square$$

Corollary 4. *From $TT^* = \alpha I$ it follows that $T(H) = H$. Hence by Theorem 2 the operator Φ preserves in both directions the order relations.*

Example. Let H be the sequence space ℓ^2 and T the left–shift operator on H:

$$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

for all $(x_n) \in \ell^2$.

Then $T \in \mathcal{B}(H)$. As it is easy to prove, we have

$$T^*(x_1, x_2, \ldots) = (0, x_1, x_2 \ldots)$$

for all $(x_n) \in \ell^2$ and $TT^* = I$. But T is not invertible.

The next theorem generalizes a proposition about the unitary operators (see [2], Theorem 2.3).

Theorem 5. *Let $0 \neq S \in \mathcal{B}(H)$, and let S have the following property: For $0 \neq x \in H$ there holds $(Sx, x) = 0$ or $Sx = \alpha_x x$ with $0 \neq \alpha_x \in \mathbb{C}$. Then there exists $\alpha \in \mathbb{C}$ such that $S = \alpha I$.*

Proof. From $(Sx, x) = 0$ for all $x \in H$ it follows that $S = 0$. Hence there exists an eigenvalue $\alpha \neq 0$ and an eigenvector $x_0 \neq 0$ with $Sx_0 = \alpha x_0$.

Next we show that α is the unique eigenvalue of S. Namely, let $Sy = \beta y$ with $0 \neq y \in H$ and $\beta \neq \alpha$. Then for all $n \in \mathbb{N}$ the vector $y + nx_0$ is not an eigenvector of S. In fact, assume that $S(y + mx_0) = \gamma(y + mx_0)$ for an $m \in \mathbb{N}$.

Then it follows that $\beta y + m\alpha x_0 = \gamma y + m\gamma x_0$, $(\beta - \gamma)y = m(\gamma - \alpha)x_0$. Since x_0 and y_0 are lineary independent, we infer that $\beta = \gamma$, $\alpha = \gamma$, hence $\alpha = \beta$, which is a contradiction. Since for all $n \in \mathbb{N}$ the vector $y + nx_0$ is not an eigenvector of S, there holds the assumption

$$\begin{aligned} (S(y + nx_0), y + nx_0) &= 0 \\ \beta y + n\alpha x_0, y + nx_0) &= 0 \\ \beta(y, y) + n\alpha(x_0, y) + \beta n(y, x_0) + n^2\alpha(x_0, x_0) &= 0 \end{aligned}$$

for all $n \in \mathbb{N}$. From the last equation it follows that $\alpha = 0$, which is a contradiction.

Let $M = \{x \in H : Sx = \alpha x\}$. Then we have $H = M \oplus M^\perp$. Let us assume that there exists $0 \neq u \in M^\perp$. Then we have for all $n \in \mathbb{N}$:

$$S(nu + x_0) \neq \alpha(nu + x_0).$$

Let us assume that there is $m \in \mathbb{N}$ such that

$$S(mu + x_0) = \alpha(mu + x_0).$$

Hence it follows that

$$mSu + \alpha x_0 = \alpha mu + \alpha x_0,$$

and consequently $u = 0$, which is a contradiction.

Since α is the unique eigenvalue, we have for all $n \in \mathbb{N}$

$$\begin{aligned} (S(nu + x_0), nu + x_0) &= 0, \quad \text{which implies} \\ (nSu + \alpha x_0, nu + x_0) &= 0, \quad \text{and consequently} \\ n^2(Su, u) + n\alpha(x_0, u) + n(Su, x_0) + \alpha(x_0, x_0) &= 0. \end{aligned}$$

Since the last equation holds for all $n \in \mathbb{N}$, it follows that $\alpha = 0$, which is a contradiction with $\alpha \neq 0$. Hence $M^\perp = \{0\}$, $M = H$, and $S = \alpha I$. □

From the previously proved theorems we can derive the following commutativity property of the operator Φ.

Theorem 6. *For the elementary operator Φ we have $A\Phi(A) = \Phi(A)A$ for all $A \in \mathcal{B}(H)$ if and only if $T = \alpha I$ with $\alpha \in \mathbb{C}$.*

Proof. $\Rightarrow$ Let $z \in H$, $\|z\| = 1$ and let P be the orthogonal projection from H onto the linear span $\langle z \rangle$ of z. Then the following equation holds:

$$(*) \qquad PT^*PT = T^*PTP.$$

We will show that T^* satisfies the assumptions of Theorem 5. We have to show that if $(T^*z, z) \neq 0$, there is $\alpha_z \in \mathbb{C}$ with $T^*z = \alpha_z z$. Let $(T^*z, z) \neq 0$.

Then we have

$$(PTz, z) = (Tz, Pz) = (Tz, z) = (z, T^*z) \neq 0.$$

This implies $PTz \neq 0$.

Hence $PTz = \beta z$ with $\beta \neq 0$. Therefore it follows that

$$PTPz = \beta z, \ (T^*PTP)(z) = T^*(\beta z) = \beta T^*z.$$

According to $(*)$ we have

$$\beta T^*z = \gamma z.$$

This means that $T^*z = \frac{\gamma}{\beta} z$.

According to the theorem we have $T^* = \alpha I$. Hence $T = \overline{\alpha} I$.

$\Leftarrow$ Let $T = \alpha I$. Then we have

$$\Phi(A) = T^*AT = \overline{\alpha} I A \alpha I = |\alpha|^2 A \Rightarrow \Phi = |\alpha| I$$

for all $A \in \mathcal{B}(H)$. Hence $\Phi(A)$ commutes with A. □

With the help of Theorem 5 we can prove the converse of Theorem 3.

Theorem 7. *Let the operator Φ have the property:*
For A, $B \in \mathcal{B}(H)$ there holds $AB = BA \Leftrightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)$.
Then there exists $\alpha > 0$ such that

$$TT^* = \alpha I.$$

Proof. First we show that T^* is injective, which is necessary for the validitity of the theorem. Let us suppose that $F = \{x \in H : T^*x = 0\} \neq \{0\}$. Let P be the orthogonal projection from H onto F. Then we have $P \neq 0$,

$$\Phi(P) = T^*PT = 0 \qquad \Phi(P)\Phi(A) = \Phi(A)\Phi(P)$$

and consequently $PA = AP$ for all $A \in \mathcal{B}(H)$.

Hence, in particular, the orthogonal projection P commutes with all orthogonal projections on H. But this implies $P = I$. Hence $F = H$, $T^* = 0$, and consequently $T = 0$. But this is a contradiction. Hence T^* is injective.

According to [5], 12.10, there holds $T(H)^{\perp} = \{0\}$ and consequently $\overline{T(H)} = H$.

For all $0 \neq x \in H$ the operators $x \otimes x$ and I commute. Hence $\Phi(I)$ commutes with all $\Phi(x \otimes x)$ and consequently T^*T commutes with all $T^*x \otimes T^*x := A_x$. Hence there holds for all $z \in H$:

$$A_x(z) = (Tz, x)T^*x,$$

$$A_xT^*T(z) = (TT^*Tz, x)T^*x$$

and

$$T^*TA_x(z) = (Tz, x)T^*TT^*x$$

Hence we obtain

$$(TT^*Tz, x)T^*x = (Tz, x)T^*TT^*x,$$

$$T^*\{(TT^*Tz, x)x - (Tz, x)TT^*x\} = 0$$

Since T^* is injective, it follows that

$$(Tz, x)TT^*x = (T^*Tz, T^*x)x.$$

Since $\overline{T(H)} = H$, there is $z_0 \in H$ with $(Tz_0, x) \neq 0$.

Hence

$$TT^*x = \frac{(T^*Tz_0, T^*x)}{(Tz_0, x)}x$$

According to Theorem 5, there exists $\alpha \in \mathbb{C}$ with $TT^* = \alpha I$. Since $TT^* > 0$, we also have $\alpha > 0$. □

5 On the order intervals of self–adjoint operators

Here $\mathcal{B}_s(H)$ denotes the set of all bounded self–adjoint operators on H. In the paper [1] the following fact on the order intervals in $\mathcal{B}_s(H)$ (in the finite–dimensional case) has been proved:

Let H be a Hilbert space with $\dim H < \infty$.

Let $A, B \in \mathcal{B}_s(H)$ with $A \geq 0$, $B \geq 0$ and $AB = BA$. Moreover, let

$$C := \frac{1}{2}(A + B - |A - B|),$$

where $|D| = \sqrt{D^*D}$ for $D \in \mathcal{B}_s(H)$. Then there holds in $\mathcal{B}_s(H)$

$$[C, A] \cap [C, B] = \{C\}$$

As usual, $[C, A]$ denotes the order interval

$$[C, A] := \{X \in \mathcal{B}_s(H)| : C \leq X \leq A\}$$

and similarly $[C, B]$.

This theorem can be generalized for infinite–dimensional spaces as follows.

Theorem 8. *Let H be a Hilbert space and $A, B \in \mathcal{B}_s(H)$ with $AB = BA$. Moreover, let*

$$C := \frac{1}{2}(A + B - |A - B|)$$

Then there holds

$$[C, A] \cap [C, B] = \{C\}.$$

Proof. First we consider the case $A \geq 0$, $B \geq 0$ and $C = 0$. Let $T \in \mathcal{B}_s(H)$ with $0 \leq T \leq A$ and $0 \leq T \leq B$. We show that $\ker A \subseteq \ker T$ and $\ker B \subseteq \ker T$. Let $x \in H$ and $Ax = 0$. Then there holds

$$0 \leq (Tx, x) \leq (Ax, x) = 0.$$

Then we have

$$0 = (Tx, x) = ((\sqrt{T})^2 x, x) = (\sqrt{T}x, \sqrt{T}x) \quad \text{and } \sqrt{T}x = 0.$$

Since $\ker T = \ker \sqrt{T}$ (see [5],12.28) we infer that

$$\ker A \subseteq \ker T$$

Analogously we show that

$$\ker B \subseteq \ker T.$$

Let now $\mathcal{B}^*$ be the smallest closed subalgebra of $\mathcal{B}(H)$, which contains A, B and the identity I. Hence $\mathcal{B}^*$ is a commutative B^*–algebra. Let M be the space of maximal ideals of $\mathcal{B}^*$. Then there exists a unique decomposition of the identity E on the σ–algebra of the Borel sets of M and there holds (see [5], 12.22):

(1) $T = \int_M \widehat{T} dE$ for all $T \in \mathcal{B}^*$, where $\widehat{T}$ is the Gelfand transform of T.

(2) $E(\omega)A = AE(\omega)$ for all Borel sets ω in M and $A \in \mathcal{B}^*$.

$C = 0$ means that in $C(M)$ $\min(\widehat{A}, \widehat{B}) = 0$.

Let $\omega_1 := \{t \in M : \widehat{A}(t) = 0\}$ and $\omega_2 = \{t \in M : \widehat{B}(t) = 0\}$. Then we obtain

$$M = \omega_1 \cup \omega_2 = \omega_1 \cup (\omega_2 \setminus \omega_1) \quad \text{and } I = E(\omega_1) + E(\omega_2 \setminus \omega_1).$$

Furthermore there holds

$$A \cdot E(\omega_1) = \int_M \widehat{A} \cdot \chi_{\omega_1} dE = 0$$

and

$$B \cdot E(\omega_2 \setminus \omega_1) = \int_M \widehat{B} \chi_{\omega_2 \setminus \omega_1} dE = 0$$

where χ_{ω_1} and $\chi_{\omega_2 \setminus \omega_1}$are the characteristic functions of ω_1 and $\omega_2 \setminus \omega_1$.

Hence there follows

$$E(\omega_1)(H) \subseteq \ker A$$

and

$$E(\omega_2 \setminus \omega_1)(H) \subseteq \ker B.$$

For $x \in H$ we have

$$x = E(\omega_1)x + E(\omega_2 \setminus \omega_1)x$$

$$Tx = T(E(\omega_1)x) + T(E(\omega_2 \setminus \omega_1)x) = 0 + 0 = 0$$

Hence $T = 0$, which was to be proved.

By translation we obtain the statement of the theorem.

Let

$$T \in \mathcal{B}_s(H) \quad \text{with } C \leq T \leq A \quad \text{and } C \leq T \leq B.$$

We put $\widetilde{T} := T - C$, $\widetilde{A} := A - C$ and $\widetilde{B} := B - C$. Then we obtain $0 \leq \widetilde{T} \leq \widetilde{A}$, $0 \leq \widetilde{T} \leq \widetilde{B}$, $\widetilde{A}\widetilde{B} = \widetilde{B}\widetilde{A}$ and $\frac{1}{2}(\widetilde{A} + \widetilde{B} - |\widetilde{A} - \widetilde{B}|) = 0$. We have proved previously that $\widetilde{T} = 0$, hence $T = C$. Also there holds

$$[C, A] \cap [C, B] = \{C\}.$$

Corollary 9. *For $A \in \mathcal{B}_s(H)$ let $A^+ := \frac{1}{2}(A + |A|)$. Then A^+ is the unique positive element in the order interval $[A, A^+]$.*

Proof. There holds

$$-A^+ = \frac{1}{2}(-A + 0 - | - A - 0|)$$

and consequently

$$[-A^+, -A] \cap [-A^+, 0] = \{-A^+\}.$$

Hence we obtain

$$[A, A^+] \cap [0, A^+] = \{A^+\}. \quad \square$$

The conclusion of the theorem holds in every vector lattice. It is astonishing that it also holds in the antilattice $\mathcal{B}_s(H)$.

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