Describing fields by implications between strong equations

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Abstract

In this note it will be shown that the class of all fields can be described by implications between strong equations. The signature is extended by the logical Craig projection to get more expressive power for the strong equations.

1 Existence equations and strong equations

For a signature Σ let $\mathsf{PAlg}(\Sigma)$ be the class of all partial algebras of Σ. The set of all terms over a set X of variables is denoted by $T_\Sigma(X)$. An existence equation (s,t) consists of two terms $s,t \in T_\Sigma(X)$ and is denoted by $s \stackrel{e}{=} t$. Formulas are defined recursively in the usual way (with $\neg, \vee, \wedge, \Rightarrow, \exists, \forall$), where the existence equations are the atomic formulas.

- A term existence statement is an existence equation of the form $s \stackrel{e}{=} s$ for $s \in T_\Sigma(X)$.
- An ECE-equation (existentially conditioned existence equation) is a formula of the form $\bigwedge\limits_{i=1}^{n} s_i \stackrel{e}{=} s_i \Rightarrow s \stackrel{e}{=} t$.
- A strong equation is a formula of the form $(s \stackrel{e}{=} s \Rightarrow s \stackrel{e}{=} t) \wedge (t \stackrel{e}{=} t \Rightarrow s \stackrel{e}{=} t)$ and it is denoted by $s \stackrel{s}{=} t$.
- A QE-equation (quasi-existence equation) is a formula of the form $\bigwedge\limits_{i=1}^{n} s_i \stackrel{e}{=} t_i \Rightarrow s \stackrel{e}{=} t$.
- A strong quasi-equation is a formula of the form $\bigwedge\limits_{i=1}^{n} s_i \stackrel{s}{=} t_i \Rightarrow s \stackrel{s}{=} t$.

In connection with strong equations the signature Σ is usually extended by binary operation symbols which are always interpreted as total first projection: For each pair (s,s') of sort symbols let $\epsilon_{s,s'} : s \times s' \to s$ be a new operation symbol. The extended signature is denoted by Σ_e. Every partial algebra $\underline{A} = ((A_s)_{s\in S}, (\phi^{\underline{A}})_{\phi\in\Omega}) \in$

$\mathsf{PAlg}(\Sigma)$ can also be seen as partial algebra of the extended signature Σ_e by defining $\epsilon^{\underline{A}}_{s,s'}(x,y) := x$ for all $x \in A_s, y \in A_{s'}$. With this extension the strong equations have the same expressive power like ECE-equations (when the empty algebra is excluded).[1]

For a formula α let $var(\alpha)$ be the set of all variables occuring in α and let $fvar(\alpha)$ be the set of all free variables in α. Let $\underline{A}$ be a partial algebra and $v : X \to A$ be a valuation of the variables. For $t \in T_\Sigma(X)$ let $t^{\underline{A}}(v)$ be the corresponding interpretation of t if it exists.[2] For a formula α with $fvar(\alpha) \subseteq X$ let $\alpha^{\underline{A}}(v) \in \{true, false\}$ be the interpretation of α, where the interpretation $(s \overset{e}{=} t)^{\underline{A}}(v)$ of an existence equation is true iff $s^{\underline{A}}(v)$ and $t^{\underline{A}}(v)$ exist and are equal. A formula α is valid in $\underline{A}$ if $\alpha^{\underline{A}}(v) = true$ for all mappings $v : fvar(\alpha) \to A$.

Classes which are definable by QE-equations[3] are closed with respect to isomorphic images, subalgebras and reduced products,[4] so these classes have many good properties: existence of inital objects, free objects, coproducts, universal solutions, etc. Classes which are definable by strong quasi-equations are much worse. These classes are in general not closed with respect to products, and an initial object need not exist (see next section for an example). The following theorem shows how a strong quasi-equation can be transformed into a set of implications between existence equations, where also disjunctions may appear in the conclusion.

Theorem 1 *Let* $\delta = \bigwedge_{i=1}^{n-1} \alpha_i \Rightarrow \alpha_n$ *be a strong quasi-equation with* $\alpha_i = (\beta_{i,0} \Rightarrow \beta_{i,2}) \wedge (\beta_{i,1} \Rightarrow \beta_{i,2})$ *for* $i \leq n$*, where* $\beta_{i,0}, \beta_{i,1}$ *are term existence statements and* $\beta_{i,2}$ *is an existence equation. Let* $I = (\{0,1,\dots n-1\} \times \{0,1\}) \cup \{n\}$*. Then* δ *is semantically equivalent to the set*

$$\left\{ \left(\bigwedge_{(i,j)\in f^{-1}(1)} \beta_{i,2} \wedge \beta_{n,f(n)} \right) \Rightarrow \left(\bigvee_{(i,j)\in f^{-1}(0)} \beta_{i,j} \vee \beta_{n,2} \right) \;\middle|\; f \in \{0,1\}^I \right\}$$

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[1] See [Cr89] and [B95].

[2] If some operations of $\underline{A}$ are partial then it might happen that the term can not be interpreted for the valuation v, in this case $t^{\underline{A}}(v)$ is undefinded.

[3] Such classes are also called quasivarieties or QE-varieties.

[4] See [B86].

Proof. Some boolean transformations lead to the set of formulas:

$$\left(\bigwedge_{i=1}^{n-1} (\beta_{i,0} \Rightarrow \beta_{i,2}) \wedge (\beta_{i,1} \Rightarrow \beta_{i,2})\right) \Rightarrow ((\beta_{n,0} \Rightarrow \beta_{n,2}) \wedge (\beta_{n,1} \Rightarrow \beta_{n,2}))$$

$$\neg\left(\bigwedge_{i=1}^{n-1} (\beta_{i,0} \Rightarrow \beta_{i,2}) \wedge (\beta_{i,1} \Rightarrow \beta_{i,2})\right) \vee ((\beta_{n,0} \Rightarrow \beta_{n,2}) \wedge (\beta_{n,1} \Rightarrow \beta_{n,2}))$$

$$\left(\bigvee_{i=1}^{n-1} (\beta_{i,0} \wedge \neg\beta_{i,2}) \vee (\beta_{i,1} \wedge \neg\beta_{i,2})\right) \vee ((\neg\beta_{n,0} \vee \beta_{n,2}) \wedge (\neg\beta_{n,1} \vee \beta_{n,2}))$$

$$\bigwedge_{f\in\{0,1\}^I} \left(\left(\bigvee_{(i,j)\in f^{-1}(0)} \beta_{i,j}\right) \vee \left(\bigvee_{(i,j)\in f^{-1}(1)} \neg\beta_{i,2}\right) \vee \left(\neg\beta_{n,f(n)} \vee \beta_{n,2}\right)\right)$$

$$\left\{\left(\bigwedge_{(i,j)\in f^{-1}(1)} \beta_{i,2} \wedge \beta_{n,f(n)}\right) \Rightarrow \left(\bigvee_{(i,j)\in f^{-1}(0)} \beta_{i,j} \vee \beta_{n,2}\right) \mid f \in \{0,1\}^I\right\}$$

■

So each strong quasi-equation is equivalent to a set of implications, where each premisse is a conjunction of some existence equation, and the conclusion is a disjunction of some term existence statements and one existence equation. Note that all implications in the set have the same set of variables because of $var(\beta_{i,0}) \cup var(\beta_{i,1}) = var(\beta_{i,2})$.

2 Axioms for fields

In this section the class of all fields (together with all terminal algebras, i.e. one-element total algebras in the signature of the fields) are described by strong quasi-equations. Let Σ be the signature which consists of one sort symbol, two binary operation symbols $+$ and $\cdot$, two unary operation symbols $-$ and $^{-1}$ and two constants 0 and 1. Let E be the following system of axioms (in the extended signature Σ_e):

1. $\epsilon(x, y) \stackrel{s}{=} x$
2. $\epsilon(x, x + y) \stackrel{s}{=} x$
3. $\epsilon(x, x \cdot y) \stackrel{s}{=} x$
4. $x + (y + z) \stackrel{s}{=} (x + y) + z$
5. $x + 0 \stackrel{s}{=} x$

6. $x + (-x) \stackrel{s}{=} 0$

7. $x + y \stackrel{s}{=} y + x$

8. $x \cdot (y \cdot z) \stackrel{s}{=} (x \cdot y) \cdot z$

9. $x \cdot 1 \stackrel{s}{=} x$

10. $x \cdot y \stackrel{s}{=} y \cdot x$

11. $x \cdot (y + z) \stackrel{s}{=} x \cdot y + x \cdot z$

12. $\epsilon(x, x^{-1}) \stackrel{s}{=} x \Rightarrow x \cdot x^{-1} \stackrel{s}{=} 1$

13. $\epsilon(0, x^{-1}) \stackrel{s}{=} \epsilon(1, x^{-1}) \Rightarrow \epsilon(x, 1^{-1}) \stackrel{s}{=} 0$

Theorem 2 *A partial algebra $\underline{A} \in \mathsf{PAlg}(\Sigma_e)$ is a model of E iff one of the following two conditions hold:*

(1) The reduct $\underline{A}|_\Sigma$ is a field and $\epsilon^{\underline{A}}$ is the total first projection

(2) $\underline{A}$ is a total algebra with $|A| = 1$.

Proof.
"$\Leftarrow$":
If (1) is satisfied then the axioms 1-12 are trivially satisfied so we only have to check the last axiom: Let $a \in A$ such that the strong equation $\epsilon(0, x^{-1}) \stackrel{s}{=} \epsilon(1, x^{-1})$ is satisfied for the valuation $v(x) := a$. Then $a^{-1^{\underline{A}}}$ does not exist, because $0^{\underline{A}} \neq 1^{\underline{A}}$ follows from (1). So we get $a = 0^{\underline{A}}$ and the strong equation $\epsilon(x, 1^{-1}) \stackrel{s}{=} 0$ is satisfied for the valuation v. If (2) is satisfied then all strong equations hold in $\underline{A}$, so all axioms are valid in $\underline{A}$.
"$\Rightarrow$":
The first axiom implies that $\epsilon^{\underline{A}}$ is the total first projection. The axioms 5 and 9 imply that $0^{\underline{A}}$ and $1^{\underline{A}}$ exist. The axioms 2, 3 and 6 imply that $+^{\underline{A}}$, $\cdot^{\underline{A}}$ and $-^{\underline{A}}$ are total operations. The axioms 4–11 imply that $(A, +, -, 0, \cdot, 1)$ is a commutative unitary ring. If $|A| = 1$ then the last axiom implies that $(1^{\underline{A}})^{-1^{\underline{A}}}$ exists, so (2) holds. Now assume $|A| > 1$. Let $a \in A$ with $a \neq 0^{\underline{A}}$. If $a^{-1^{\underline{A}}}$ does not exist, then the last axiom implies $a = 0^{\underline{A}}$, which is a contradiction. So $a^{-1^{\underline{A}}}$ exists and with axiom 12 we get $a \cdot a^{-1^{\underline{A}}} = 1^{\underline{A}}$, and the reduct $\underline{A}|_\Sigma$ is a field. ■

So the fields (together with the terminal algebras) can be characterized by strong quasi-equations. But the fields are not a quasivariety, because they are not closed with respect to products. If we use existence equations as atomic formulas then the quasivarieties of partial algebras behave in a similar way like the quasivarieties of total algebras. If we use the strong equations as atomic formulas then the theory becomes completly different. It is not known which algebraic operator (like ISP_r for quasi-existence equations[5]) correspond to strong quasi-equations. For algebraic characterizations of strong equations and mixed strong and existence equations without the Craig operator ϵ see [StSt94].

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[5]See [B86].

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