The Universal Central Extension of the Holomorphic Current Algebra

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Abstract

In the present paper we determine the universal central extension of the Lie algebra $\mathcal{O}(X,\mathfrak{k})$ of holomorphic functions of a complex manifold X which is a Riemannian domain over a Stein manifold with values in a finite-dimensional complex simple Lie algebra $\mathfrak{k}$. In view of the abstract description of the universal central extensions, this amounts to determine the universal differential module of the Fréchet algebra $\mathcal{O}(X)$ of holomorphic functions on X. We show that the de Rham differential into the space $\Omega^1(X)$ of holomorphic 1-forms on X is universal. Therefore the kernel of the universal central extension is the quotient space $\Omega^1(X)/d\mathcal{O}(X)$.

Introduction

Let A be a unital complex commutative Fréchet algebra, i.e., a complex Fréchet space together with a continuous bilinear commutative associative unital multiplication. Then an important invariant of A is its universal differential module $\Omega^1(A)$. This is a Fréchet space with a continuous A-module structure $A\times\Omega^1(A)\to\Omega^1(A)$ and a derivation $d\colon A\to\Omega^1(A)$ which is universal in the sense that for any other Fréchet module M of A and any derivation $D\colon A\to M$, there exists a unique continuous morphism $\alpha\colon\Omega^1(A)\to M$ of A-modules with $D=\alpha\circ d$ ([13]). If we consider the spectrum $\Gamma(A):=\mathrm{Hom}(A,\mathbb{C})$ of A, i.e., the space of all continuous algebra homomorphisms $A\to\mathbb{C}$, as a *topological invariant of* A, then the module $\Omega^1(A)$ is a *differential invariant of* A. The information contained in $\Omega^1(A)$ is in some sense finer than the information contained in $\Gamma(A)$, and this makes it often harder to determine $\Omega^1(A)$ in concrete terms for concrete algebras A.

If X is a complex manifold, then the algebra $\mathcal{O}(X)$ of holomorphic complex-valued functions on X is a commutative Fréchet algebra. The main result of the present paper is that for this algebra the de Rham differential

$$d\colon\mathcal{O}(X)\to\Omega^1(X)$$

into the Fréchet $\mathcal{O}(X)$-module $\Omega^1(X)$ of holomorphic 1-forms on X is universal whenever X is a Riemannian domain over a Stein manifold, hence in particular for any open subset of $\mathbb{C}^n$. Our result is that $\Omega^1(\mathcal{O}(X))$ is isomorphic to the space $\Omega^1(X)$ of holomorphic 1-forms on X, and that the differential

$$d\colon \mathcal{O}(X) \to \Omega^1(X), \quad f \mapsto df$$

is the universal differential. The methods we use are based on the theory of coherent sheaves on Stein manifolds. For the proof we proceed in two steps. First we prove the result for Stein manifolds, where we crucially apply the vanishing of the cohomology of coherent sheaves in degrees greater than one. As a second step, we extend the result to Riemannian domains over Stein manifolds by using results of H. Rossi, showing that if X is a Riemannian domain over a Stein manifold, then the spectrum $\widehat{X} := \Gamma(\mathcal{O}(X))$ of the algebra $\mathcal{O}(X)$ of holomorphic functions on X carries a natural complex manifold structure turning it into a Stein manifold. We thus obtain a natural open embedding $i_X : X \hookrightarrow \widehat{X}$, and each holomorphic function on X extends uniquely to $\widehat{X}$. In this sense $\widehat{X}$ is the *envelope of holomorphy of* X. As the algebras $\mathcal{O}(X)$ and $\mathcal{O}(\widehat{X})$ are naturally isomorphic, their universal differential modules are isomorphic, and we then derive that

$$\Omega^1(\mathcal{O}(X)) \cong \Omega^1(\mathcal{O}(\widehat{X})) \cong \Omega^1(\widehat{X}) \cong \Omega^1(X).$$

Similar results are well-known in several other contexts: In the algebraic context, where X is a non-singular affine complex variety, then the algebraic universal differential module of the algebra of regular functions on X can be identified with the space of regular 1-forms on X ([17]), result due to Bloch ([1]), Loday-Kassel ([12]). If M is a smooth finite-dimensional manifold, then the universal differential module of the Fréchet algebra $C^\infty(M, \mathbb{R})$ of smooth functions on M is the space $\Omega^1(M, \mathbb{R})$ of smooth 1-forms on M, result due (in some similar form to Connes ([2]), Pressley-Segal ([15]) and) Maier ([13]), who introduced Fréchet–Kähler differentials in this context. Leaving the context of Fréchet algebras and considering more general locally convex algebras, one can even show that for a non-compact manifold M the universal differential module of the algebra $C_c^\infty(M, \mathbb{R})$ of compactly supported smooth functions is the space of compactly supported smooth 1-forms ([13], [14]).

Our motivation to get precise information on the universal differential module $\Omega^1(X)$ of $\mathcal{O}(X)$ was to determine the universal central extension of the Lie algebras $\mathcal{O}(X, \mathfrak{k})$ of holomorphic maps $X \to \mathfrak{k}$, where $\mathfrak{k}$ is a simple finite-dimensional complex Lie algebra. In [13] P. Maier shows on an abstract level that for any Fréchet algebra A the Fréchet-Lie algebra $\mathfrak{g} := A \otimes \mathfrak{k}$ has a universal central extension $\widetilde{\mathfrak{g}}$ of the form

$$\mathbf{0} \to HC_1(A) \hookrightarrow \widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g} \to \mathbf{0},$$

where

$$HC_1(A) := \Omega^1(A)/\overline{dA}$$

is the quotient of $\Omega^1(A)$ modulo the closed subspace generated by all differentials.

For any complex manifold X the image of d, the space of exact 1-forms, is closed in $\Omega^1(X)$ because it consists of all closed 1-forms for which all integrals over loops in X vanish. Therefore our identification of $\Omega^1(\mathcal{O}(X))$ with $\Omega^1(X)$ implies that

$$HC_1(\mathcal{O}(X)) \cong \Omega^1(X)/d\mathcal{O}(X)$$

is the kernel of the universal central extension of $\mathcal{O}(X, \mathfrak{k})$.

For an open submanifold X of a Stein manifold, our identification of $\Omega^1(\mathcal{O}(X))$ with $\Omega^1(X)$ implies that the image of d, the space of exact 1-forms, is closed, so that

$$HC_1(\mathcal{O}(X)) \cong \Omega^1(X)/d\mathcal{O}(X)$$

is the kernel of the universal central extension of $\mathcal{O}(X, \mathfrak{k})$.

The universal central extension of the holomorphic current algebra takes an important intermediate place between Krichever-Novikov algebras, which are Lie algebras of meromorphic functions on a compact complex surface with fixed polar set and values in $\mathfrak{k}$, and algebras of smooth maps. For example, in the case of Riemann surfaces, the space of meromorphic functions with values in $\mathfrak{k}$ on the Riemann sphere $\mathbb{S}^2$ with poles only in $\{0, \infty\}$ is dense in the space of holomorphic functions on $\mathbb{S}^2 \setminus \{0, \infty\}$ which itself lies in the space of differentiable functions on the equator of $\mathbb{S}^2$ (all three with values in $\mathfrak{k}$). We project to explore the consequences of this situation in the representations theory of current algebras via coadjoint orbits (cf. [4]). Another direction for further research is the question under which conditions Lie groups correspond to the universal central extension of $\mathcal{O}(X, \mathfrak{k})$.

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1 Preliminairies

Definition 1.1 If E and F are complete locally convex spaces, then we denote by $E \widehat{\otimes} F$ the completed projective tensor product of E and F. It has the universal property that the continuous bilinear maps $E \times F \to G$ into any complete locally convex space G are in one-to-one correspondence with the continuous linear maps $E \widehat{\otimes} F \to G$. The algebraic tensor product $E \otimes F$ is a dense subspace of $E \widehat{\otimes} G$.

Let A be a unital commutative associative complete locally convex algebra, i.e., a complete locally convex space with a continuous bilinear associative commutative unital multiplication. Then $A \widehat{\otimes} A$ also carries a natural algebra structure which is uniquely determined by

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb', \quad a, a', b, b' \in A,$$

and the multiplication map

$$\mu_A \colon A \widehat{\otimes} A \to A$$

is a morphism of locally convex algebras. It follows in particular that its kernel $I_A := \ker \mu_A$ is an ideal of $A \widehat{\otimes} A$.

We now define $\Omega^1(A)$ as the completion of the quotient $I_A/\overline{I_A^2}$, which carries a natural A-module structure inherited from the left multiplication action of $A \cong A \otimes \mathbf{1}$ on the tensor product $A \widehat{\otimes} A$. There is a continuous derivation

$$d_A : A \to \Omega^1(A), \quad a \mapsto [\mathbf{1} \otimes a - a \otimes \mathbf{1}],$$

where $[x]$ denotes the class of $x \in I_A$ in $I_A/\overline{I_A^2}$. The pair $(\Omega^1(A), d_A)$ is uniquely determined by the property that $\Omega^1(A)$ is a complete locally convex A-module and that the derivation d_A has the universal property that for each pair (V, D) of a complete locally convex A-module V and a continuous derivation $D : A \to V$, there is a unique continuous morphism of A-modules $\alpha : \Omega^1(A) \to V$ with $D = \alpha \circ d_A$.

Note that the uniqueness requirement in the universal property of d_A implies that the submodule of $\Omega^1(A)$ generated by $d_A(A)$ is dense, so that $A \cdot d_A(A)$ is dense in $\Omega^1(A)$.

Definition 1.2 Let X be a (second countable) complex manifold and $\mathcal{O}(X)$ the complex algebra of holomorphic functions on X. Then $\mathcal{O}(X)$ is a Fréchet space, i.e., a locally convex, metrizable complete topological vector space, with respect to the topology of uniform convergence on compact subsets of X. The natural algebra structure now turns $\mathcal{O}(X)$ into a unital commutative associative Fréchet algebra. We write $\Omega^1(X)$ for the space of holomorphic 1-forms on X. This space also has a natural Fréchet structure given by the uniform convergence on compact subsets of X, turning it into a Fréchet module of $\mathcal{O}(X)$. Moreover, the (de Rham) differential

$$d : \mathcal{O}(X) \to \Omega^1(X)$$

is a continuous derivation of $\mathcal{O}(X)$-modules, hence gives rise to a unique morphism of $\mathcal{O}(X)$-modules

$$\gamma_X : \Omega^1(\mathcal{O}(X)) \to \Omega^1(X) \quad \text{with} \quad \gamma_X \circ d_{\mathcal{O}(X)} = d.$$

In Section 3 we shall prove in our main theorem that γ_X is a homeomorphism if X is a Riemannian domain over a Stein manifold, which is equivalent to being an open subset of a Stein manifold.

Remark 1.3 A more explicit form of γ_X can be obtained as follows. Thanks to Grothendieck ([9], Ch. II, §3, no. 3, Ex. 2 after Theorem 13), the tensor product space $\mathcal{O}(X)\widehat{\otimes}\mathcal{O}(X)$ and the Fréchet space $\mathcal{O}(X, \mathcal{O}(X))$ of $\mathcal{O}(X)$-valued holomorphic functions on X are naturally isomorphic, and since the latter space can easily seen to be isomorphic to $\mathcal{O}(X \times X)$, we have an isomorphism of Fréchet algebras

$$\theta_X : \mathcal{O}(X)\widehat{\otimes}\mathcal{O}(X) \to \mathcal{O}(X \times X) \quad \text{with} \quad \theta_X(f \otimes g)(x, y) = f(x)g(y).$$

In this sense we identify the elements of the tensor product $\mathcal{O}(X)\widehat{\otimes}\mathcal{O}(X)$ with holomorphic functions on the product manifold $X \times X$.

We have a natural map

$$\widetilde{\gamma}_X : \mathcal{O}(X \times X) \to \Omega^1(X), \quad \widetilde{\gamma}_X(F)(x)(v) := dF(x, x)(0, v).$$

For $F = f \otimes g$ we then have

$$\widetilde{\gamma}_X(f \otimes g) = f \cdot dg,$$

and $\widetilde{\gamma}_X$ is a continuous morphism of $\mathcal{O}(X)$-modules, where $\mathcal{O}(X)$ acts on $\mathcal{O}(X \times X)$ by $(f.F)(x, y) := f(x)F(x, y)$. The restriction of $\widetilde{\gamma}_X$ to the ideal I_X satisfies

$$\widetilde{\gamma}_X(d_{\mathcal{O}(X)}(f)) = \widetilde{\gamma}_X(1 \otimes f - f \otimes 1) = df,$$

and therefore the uniqueness of γ_X implies that $\gamma_X = \widetilde{\gamma}_X|_{I_X}$, i.e.,

$$\gamma_X(F) = dF(x, x)(0, v).$$

Definition 1.4 For a sheaf $\mathcal{F}$ of $\mathcal{O}(X)$-modules, we will write $\mathcal{F}_x$ for the space of germs at $x \in X$. Given a morphism of sheaves $\zeta : \mathcal{F} \to \mathcal{G}$ on X, we shall for each open set $U \subset X$ write $\zeta_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for the corresponding map between spaces of sections over U. For ζ_X, we simply write ζ.

Let $\triangle : X \to X \times X$ be the diagonal map. It embeds X as a complex submanifold $\triangle(X)$ into $X \times X$. It induces a morphism between structure sheaves

$$\triangle^\sharp : \mathcal{O}_{X\times X} \to \triangle_*\mathcal{O}_X,$$

given on an open set $U \subset X \times X$ by

$$\triangle^\sharp_U(f) = f \circ \triangle, \quad \text{for} \quad f \in \mathcal{O}_{X\times X}(U).$$

Here $\triangle_*\mathcal{O}_X$ denotes the usual direct image sheaf, i.e., the sheaf of $\mathcal{O}_{X\times X}$-modules given on $U \subset X \times X$ by

$$(\triangle_*\mathcal{O}_X)(U) = \mathcal{O}_X(\Delta^{-1}(U)),$$

where the module structure is given by $f.v := (f \circ \Delta).v$. As $\triangle_*\mathcal{O}_X$ is a coherent analytic sheaf, i.e., a coherent sheaf of $\mathcal{O}_{X\times X}$-modules (cf. [8], p.20), the kernel $\mathcal{I}_X := \ker(\triangle^\sharp)$ is a coherent sheaf of $\mathcal{O}_{X\times X}$-modules (cf. [8], p.237). One has

$$\mathcal{I}_X(U) \;=\; \{f \in \mathcal{O}_{X\times X}(U) \,|\, f \circ \triangle|_{\Delta^{-1}(U)} = 0\},$$

because the sections on U of $\ker(\triangle^\sharp)$ are the kernel of $\triangle^\sharp_U : \mathcal{O}_{X\times X}(U) \to \triangle_*\mathcal{O}_X(U) = \mathcal{O}_X(\Delta^{-1}(U))$ by left-exactness of the global section functor.

Lemma 1.5 *For any open subset $U \subset X$, let $\mu_U := \mu_{\mathcal{O}(U)} : \mathcal{O}(U)\widehat{\otimes}\mathcal{O}(U) \to \mathcal{O}(U)$ be the multiplication map of $\mathcal{O}(U)$. Then we have*

$$\mathcal{I}_X(U \times U) \;\cong I_{\mathcal{O}(U)} = \ker(\mu_U).$$

Proof. It is easy to see that for the isomorphism θ_U from Remark 1.3 the following diagramm commutes:

$$\begin{tikzcd} \mathcal{O}(U)\widehat{\otimes}\mathcal{O}(U) \arrow[r, "\theta_U"] \arrow[dr, "\mu_U"'] & \mathcal{O}(U \times U) \arrow[d, "\triangle^\sharp_{U\times U}"] \\ & \mathcal{O}(U) \end{tikzcd}$$

This implies that $\mathcal{I}_X(U \times U) \;= \ker\,(\triangle^\sharp_{U\times U}) \cong \ker \mu_U$. ■

Remark 1.6 The preceding lemma provides a sheaf version of the ideal $I_X := I_{\mathcal{O}(X)} = \ker \mu_X$ because it identifies for each open subset U of X the ideal I_U with the space of sections of the sheaf $\mathcal{I}_X$ over $U \times U$.

Lemma 1.7 *Let $n := \dim_{\mathbb{C}} X$ and $(x, y) \in X \times X$. Let $x_1, \ldots, x_n$ be coordinate functions of the first factor in $X \times X$ and $y_1, \ldots, y_n$ for the second factor around the point (x, y), whose coordinates are $(0, 0)$. Then the ideal $(\mathcal{I}_X)_{(x,y)} = \ker\,(\triangle^\sharp)_{(x,y)} \subset (\mathcal{O}_{X\times X})_{(x,y)}$ is generated by the germs of the functions*

$$(x_1 - y_1), \ldots, (x_n - y_n).$$

Proof. First we pass from the coordinate functions $x_1, \ldots, x_n, y_1, \ldots, y_n$ on $X \times X$ to coordinate functions $\widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{y}_1, \ldots, \widetilde{y}_n$ defined by $\widetilde{x}_i = x_i$ and $\widetilde{y}_i = x_i - y_i$ for $i = 1, \ldots, n$. Then the diagonal $\triangle(X)$ is the vanishing set of the ideal $\langle\widetilde{y}_1, \ldots, \widetilde{y}_n\rangle = \langle\widetilde{y}_1\rangle + \ldots + \langle\widetilde{y}_n\rangle$, generated by the functions $\widetilde{y}_1, \ldots, \widetilde{y}_n$. In our coordinate system the condition $F(\widetilde{x}, \widetilde{y}) \circ \triangle = 0$, i.e., $F(\widetilde{x}, \widetilde{y})|_{\triangle(X)} = 0$, on the level of germs in (x, y) is equivalent to $F(\widetilde{x}, \widetilde{y}) \in \langle\widetilde{y}_1, \ldots, \widetilde{y}_n\rangle$, i.e., $F(\widetilde{x}, \widetilde{y}) = \widetilde{y}_1 F_1(\widetilde{x}, \widetilde{y}) + \ldots + \widetilde{y}_n F_n(\widetilde{x}, \widetilde{y})$. ■

Definition 1.8 Let us now recall the definition of the *analytic inverse image functor* ([8], p. 18): Let $f : X \to Y$ be a holomorphic map and $\mathcal{F}$ a sheaf of $\mathcal{O}_Y$-modules. We define

$$\mathcal{F} \times_Y X := \{(a_y, x) \in \mathcal{F} \times X \,|\, y = f(x), a_y \in \mathcal{F}_y\}.$$

Here $\mathcal{F}_y$ is the space of germs or sections of $\mathcal{F}$ at $y \in Y$, and $\mathcal{F} \times_Y X$ is a sheaf on X. This *fibered product* is just the restriction $\mathcal{F}|_X$ of $\mathcal{F}$ to X in case $f : X \hookrightarrow Y$ is the inclusion of a submanifold. Continuing with the general construction, $\mathcal{F} \times_Y X$ is a sheaf of $\mathcal{O}_Y \times_Y X$-modules. Moreover, $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y \times_Y X$-modules by

$$(a_y, x) \cdot \varphi_x := (a_y \circ f)_x \cdot \varphi_x.$$

One then defines

$$f^*(\mathcal{F}) := (\mathcal{F} \times_Y X) \otimes_{\mathcal{O}_Y \times_Y X} \mathcal{O}_X.$$

This construction defines a right exact functor sending coherent sheaves of $\mathcal{O}_Y$-modules to coherent sheaves of $\mathcal{O}_X$-modules satisfying $f^*\mathcal{O}_Y = \mathcal{O}_X$ ([8], p. 18-19).

For $Y = X \times X$ and the embedding $f := \triangle : X \hookrightarrow X \times X$ we see in particular that $\triangle^*(\mathcal{I}_X)$ is a coherent analytic sheaf of $\mathcal{O}_X$-modules. The germs of this sheaf in $x \in X$ are the germs of holomorphic functions on $X \times X$ in $(x, x) \in X \times X$ vanishing on the diagonal of $X \times X$. In particular, we have

$$\triangle^*(\mathcal{I}_X) = (\mathcal{I}_X|_{\triangle(X)}) \otimes_{\mathcal{O}_{X \times X}|_{\triangle(X)}} \mathcal{O}_X.$$

Lemma 1.9 *Let X be a Stein manifold. Then the global section module of the sheaf $\triangle^*(\mathcal{I}_X)$ admits a finite presentation as an $\mathcal{O}_X(X)$-module.*

Proof. We recall that a coherent analytic sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is a sheaf such that for each point $x \in X$, there is an open set U, integers n_x and m_x and an exact sequence

$$(\mathcal{O}_X|_U)^{n_x} \to (\mathcal{O}_X|_U)^{m_x} \to \mathcal{F}|_U \to 0.$$

We want to show that the sheaf $\mathcal{F}$ posesses a finite presentation

$$(\mathcal{O}_X)^n \to (\mathcal{O}_X)^m \to \mathcal{F} \to 0.$$

Applying the global section functor, which is right exact for coherent analytic sheaves on Stein manifolds, the assertion of the lemma follows.

As $\triangle^*$ is a right exact functor (cf. [8], p. 19), a finite presentation of $\triangle^*(\mathcal{I}_X)$ can be obtained by a finite presentation of $\mathcal{I}_X$ as a sheaf of $\mathcal{O}_{X\times X}$-modules (recall that for a holomorphic map $f : X \to Y$, we have $\mathcal{O}_X \cong f^*(\mathcal{O}_Y)$ by [8], p. 19).

Recall that the sheaf $\mathcal{I}_X$ is defined by

$$\mathcal{I}_X = \ker(\triangle^\sharp : \mathcal{O}_{X\times X} \to \triangle_*\mathcal{O}_X).$$

Furthermore, one can use Grauert's Embedding Theorem to embed $i : X \hookrightarrow \mathbb{C}^n$. Then $i^*\mathcal{O}_{\mathbb{C}^n} = \mathcal{O}_X$ implies

$$(i \times i)^*\mathcal{I}_{\mathbb{C}^n} = \mathcal{I}_X,$$

so that, in view of the right exactness of $(i \times i)^*$ ([8], p. 18), it suffices to obtain a finite presentation of $\mathcal{I}_{\mathbb{C}^n}$.

We may therefore assume that $X = \mathbb{C}^n$ and that $x = y = 0$. We then have to find a finite presentation of the sheaf $\mathcal{I}_{\mathbb{C}^n}$ which is the sheaf of ideals of the (complex analytic) diagonal subvariety $\mathbb{C}^n \hookrightarrow \mathbb{C}^n \times \mathbb{C}^n$. According to Lemma 1.7, it is generated by the functions $x_1 - y_1, \ldots, x_n - y_n$, where the x_i are the coordinate functions of the first factor in the product $\mathbb{C}^n \times \mathbb{C}^n$, and y_i are those on the second factor. Define a map

$$\zeta : \mathcal{O}^n_{\mathbb{C}^n\times\mathbb{C}^n} \to \mathcal{I}_{\mathbb{C}^n}, \quad \zeta(f_1, \ldots, f_n) := \sum_{i=1}^n (x_i - y_i) f_i,$$

where for an open subset $U \subset \mathbb{C}^n \times \mathbb{C}^n$, $f_1, \dots, f_n$ are elements of $\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n}(U) = \mathcal{O}(U)$, and write henceforth $z_k := x_k - y_k$ for $k = 1, \dots, n$. We want to determine generators and relations for $\mathcal{I}_{\mathbb{C}^n}$. Let $F = (f_1, \dots, f_n) \in \ker(\zeta)$, i.e., $z_1 f_1 + \dots + z_n f_n = 0$. Then f_1 vanishes on the common set of zeros of the functions $z_2, \dots, z_n$, hence is of the form $f_1 = \sum_{i=2}^n z_i f_{1i}$. Then the function

$$H \quad := \quad \sum_{i=2}^n (z_i f_{1i}, 0, \dots, 0, -z_1 f_{1i}, 0, \dots, 0) = \sum_{i=2}^n f_{1i}(z_i, 0, \dots, 0, -z_1, 0, \dots, 0)$$

is contained in $\mathcal{I}_{\mathbb{C}^n}$ and satisfies

$$\widetilde{F} := F - H = (0, \widetilde{f}_2, \dots, \widetilde{f}_n).$$

Now we use induction to see that the ideal $\ker(\zeta)$ is generated by the $\frac{n(n-1)}{2}$ functions

$$(0, \dots, 0, z_i, 0, \dots, 0, -z_j, 0, \dots, 0),$$

where z_i is at the j-th, and $-z_j$ is at the i-th position.

In conclusion, we obtain a finite presentation:

$$\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n}^{\frac{n(n-1)}{2}} \to \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n}^n \to \mathcal{I}_{\mathbb{C}^n} \to 0.$$

This completes the proof of the lemma. ∎

2 Universal differentials and holomorphic 1-forms

Theorem 2.1 *Let X be a Stein manifold. Then the map*

$$\gamma_X : \Omega^1(\mathcal{O}(X)) \to \Omega^1(X)$$

from Definition 1.2 *is an isomorphism of topological $\mathcal{O}(X)$-modules.*

The proof of Theorem 2.1 will be complete at the end of this section.

Remark 2.2 The Stein condition is essential in the theorem, but will be weakened in the next section. There exist counterexamples in the general case, for example $X = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. In this case $\mathcal{O}(X) \cong \mathbb{C}$, the constant functions, thus $\Omega^1(\mathcal{O}(X)) = 0$, but $\Omega^1(X) \cong \mathbb{C}$ is generated by the global holomorphic 1-form dz. Thus γ_X is not injective in this case.

The following lemma contains already a special case of our main theorem.

Lemma 2.3 *Let U be an open subset of $\mathbb{C}^n$ such that the restrictions of polynomials to U form a dense subspace of $\mathcal{O}(U)$. Then*

$$\gamma_U : \Omega^1(\mathcal{O}(U)) \to \Omega^1(U)$$

is an isomorphism of Fréchet $\mathcal{O}(U)$-modules.

Proof. For the proof we have to verify that the differential $d\colon \mathcal{O}(U) \to \Omega^1(U)$ has the universal property. Let V be a continuous $\mathcal{O}(U)$-module and $D\colon \mathcal{O}(U) \to V$ a continuous derivation.

Let $z_1, \ldots, z_n\colon U \to \mathbb{C}$ denote the coordinate functions. Since the tangent bundle of the open subset $U \subseteq \mathbb{C}^n$ is trivial, the $\mathcal{O}(U)$-module $\Omega^1(U)$ is free with basis $dz_1, \ldots, dz_n$. Hence there exists a unique continuous morphism of $\mathcal{O}(U)$-modules $\alpha\colon \Omega^1(U) \to V$ with $\alpha(dz_i) = D(z_i)$, namely

$$\alpha\Big(\sum_{j=1}^{n} f_j dz_j\Big) = \sum_{j=1}^{n} f_j.D(z_j).$$

We then have $\alpha \circ d = D$ on $z_1, \ldots, z_n \in \mathcal{O}(U)$, and since $\alpha \circ d$ and D are derivations $\mathcal{O}(U) \to V$, they also coincide on the subalgebra of $\mathcal{O}(U)$ generated by $z_1, \ldots, z_n$. By our assumption on U, this subalgebra is dense, so that the continuity of $\alpha \circ d$ and D entails that both maps coincide on the whole space $\mathcal{O}(U)$. This proves the existence of α, and since the relation $\alpha \circ d = D$ determines the image of dz_j under α, the module morphism α is uniquely determined by the condition $\alpha \circ d = D$. Hence $(\Omega^1(U), d)$ is the universal differential module of the Fréchet algebra $\mathcal{O}(U)$. ∎

Remark 2.4 For each point x in a complex manifold X there exists an open neighborhood U which is isomorphic to a polydisc. Since the polynomials are dense in $\mathcal{O}(U)$, Lemma 2.3 applies to U.

Definition 2.5 We now define a relative of the map γ_X (cf. Definition 1.2) on sheaf level. Let $\mathcal{I}_X$ denote the kernel of the sheaf homomorphism $\triangle^\sharp$ (cf. Definition 1.4). Then the ideal sheaf $\mathcal{I}_X$ is a coherent sheaf of $\mathcal{O}_{X\times X}$-modules. Its coherent inverse image sheaf (constructed in Definition 1.8) $\triangle^*\mathcal{I}_X$ is the corresponding sheaf on X. We want to define

$$\underline{\varphi} \;:\; \triangle^*(\mathcal{I}_X) = \mathcal{I}_X|_{\triangle(X)} \otimes_{\mathcal{O}_{X\times X}|_{\triangle(X)}} \mathcal{O}_X \;\to\; \Omega^1_X.$$

The first step is to define $\underline{\varphi}_U$ for an open subset $U \subset X$. We first define

$$\underline{\widetilde{\varphi}}_U : \mathcal{I}_X|_{\triangle(X)}(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U) \to \Omega^1(U) \quad \text{by} \quad \underline{\widetilde{\varphi}}_U(F \otimes 1)(x, v) := dF(x,x)(0,v),$$

where F is a holomorphic function on $U \times U \subset X \times X$ vanishing on the diagonal, $x \in U$ and $v \in T_x(U)$ a tangent vector at x. Here it is understood that $\underline{\widetilde{\varphi}}_U$ is extended as a map of $\mathcal{O}_X(U)$-modules from $\mathcal{I}_X|_{\triangle(X)}(U) \otimes_{\mathbb{C}} 1$ to all of $\mathcal{I}_X|_{\triangle(X)}(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U)$, letting $\mathcal{O}_X(U)$ act on the right hand side of the tensor product. We then show that this definition factors to

$$\underline{\varphi}_U : \mathcal{I}_X|_{\triangle(X)}(U) \otimes_{\mathcal{O}_{X\times X}|_{\triangle(X)}(U)} \mathcal{O}_X(U) \to \Omega^1(U).$$

Lemma 2.6 *The map $\underline{\widetilde{\varphi}}_U : \mathcal{I}_X|_{\triangle(X)}(U) \otimes_{\mathbb{C}} \mathcal{O}_X(U) \to \Omega^1(U)$ factors through a map*

$$\underline{\varphi}_U\colon \triangle^*(\mathcal{I}_X)(U) = \big(\mathcal{I}_X|_{\triangle(X)}(U)\big) \otimes_{\mathcal{O}_{X\times X}|_{\triangle(X)}(U)} \mathcal{O}_X(U) \to \Omega^1(U).$$

As $\underline{\varphi}_U$ is obviously compatible with the restriction maps, it induces a sheaf map

$$\underline{\varphi} : \triangle^*(\mathcal{I}_X) \to \Omega^1_X.$$

On the level of germs, the kernel of $\underline{\varphi}$ in x equals $\triangle^*(\mathcal{I}_X)^2_x$.

Proof. The sheaf of algebras $\mathcal{O}_{X\times X}|_{\triangle(X)}$ acts on $\mathcal{O}_X$ by

$$(f_{(x,x)}, x)\cdot g_x \;=\; (f_{(x,x)}\circ\triangle)_x\cdot g_x.$$

Here $f_{(x,x)}$ denotes the germ of $f\in\mathcal{O}_{X\times X}(U\times U)$ at $(x,x)=\triangle(x)$, and $(f_{(x,x)}\circ\triangle)_x$ is the germ at x of $f_{(x,x)}\circ\triangle$ in $\mathcal{O}_X$. Thus, an element $(f_{(x,x)}, x)$ passes through the tensor product, acting on the left hand side by multiplication with $f_{(x,x)}$, and on the right hand side by multiplication with $(f_{(x,x)}\circ\triangle)_x$. Thanks to the formula

$$\underline{\widetilde{\varphi}}_U(FG) \;=\; \triangle_U^\sharp(F)\underline{\widetilde{\varphi}}_U(G)+\triangle_U^\sharp(G)\underline{\widetilde{\varphi}}_U(F),$$

for germs F, G, we have (using that $\underline{\widetilde{\varphi}}_U$ is an $\mathcal{O}_X(U)$-module homomorphism)

$$\begin{aligned}
\underline{\widetilde{\varphi}}_U(h_{(x,x)}f_{(x,x)}\otimes g_x) &= \underline{\widetilde{\varphi}}_U(h_{(x,x)}f_{(x,x)})g_x\\
&= (\triangle_U^\sharp(h_{(x,x)})\underline{\widetilde{\varphi}}_U(f_{(x,x)})+\triangle_U^\sharp(f_{(x,x)})\underline{\widetilde{\varphi}}_U(h_{(x,x)}))g_x\\
&= \triangle_U^\sharp(f_{(x,x)})\underline{\widetilde{\varphi}}_U(h_{(x,x)})g_x\\
&= \underline{\widetilde{\varphi}}_U(h_{(x,x)})(f_{(x,x)}\circ\triangle)_x g_x\\
&= \underline{\widetilde{\varphi}}_U(h_{(x,x)}\otimes(f_{(x,x)}\circ\triangle)_x g_x).
\end{aligned}$$

Thus $\underline{\varphi}_U$ is well-defined on the tensor product. The above mentioned formula shows $\triangle^*(\mathcal{I}_X)^2\subset\ker(\underline{\varphi})$. On the other hand, Lemma 1.7 shows that a germ F in $\ker(\triangle^\sharp)_x$ can be written

$$F \;=\; \sum_{k=1}^{n}(x_k-y_k)F_k$$

for the coordinate functions x_k and y_k on the two factors in $\mathbb{C}^n\times\mathbb{C}^n$, $k=1,\ldots,n$. Hence, application of $\underline{\varphi}$ gives

$$\underline{\varphi}(F)(x) = -\sum_{k=1}^{n}F_k(x,x)dx_k,$$

where we use $x_1,\ldots,x_n$ as coordinates on X, and now it is obvious that $F\in\ker(\underline{\varphi})$ means that the F_k vanish on the diagonal, hence $F\in\triangle^*(\mathcal{I}_X)^2_x$. ∎

To prove the injectivity of γ_X in Theorem 2.1, we shall need the following lemma.

Lemma 2.7 $\Gamma(X,\triangle^*(\mathcal{I}_X)^2)\cong\overline{I_X^2}=I_X^2$ *for* $I_X := I_{\mathcal{O}(X)}$.

Proof. In order to show the assertion of the preceding remark, we note that if $\mathcal{M}$ and $\mathcal{N}$ are two coherent sheaves of $\mathcal{O}_X$-modules such that their global section modules $M=\Gamma(X,\mathcal{M})$ and $N=\Gamma(X,\mathcal{N})$ are finitely presented $\mathcal{O}_X(X)$-modules, one has

$$M\otimes_{\mathcal{O}_X(X)}N\cong\Gamma(X,\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N})$$

(see e.g. [5], p. 403). Here it is not necessary to pass to the completions because the left hand side is complete owing to the finiteness of the presentation. Denote by ψ the multiplication map

$$\psi:\triangle^*(\mathcal{I}_X)\otimes\triangle^*(\mathcal{I}_X)\to\triangle^*(\mathcal{I}_X)^2 \;=\mathrm{im}(\psi),$$

and note that $\mathrm{im}(\psi)$ as an image sheaf, and $\ker(\psi)$ as a kernel sheaf are coherent ([7]). Then we consider the two short exact sequences:

$$0 \to \Gamma(X, \ker(\psi)) \to \Gamma(X, \Delta^*(\mathcal{I}_X) \otimes \Delta^*(\mathcal{I}_X)) \to \Gamma(X, \Delta^*(\mathcal{I}_X)^2) \to 0$$

(owing to the Stein property $H^1(X, \ker(\psi)) = 0$) and

$$0 \to \ker(\Gamma(X, \psi)) \to \Gamma(X, \Delta^*(\mathcal{I}_X)) \otimes_{\mathcal{O}(X)} \Gamma(X, \Delta^*(\mathcal{I}_X)) \to \Gamma(X, \Delta^*(\mathcal{I}_X))^2 \to 0$$

where once again $\Gamma(X, \Delta^*(\mathcal{I}_X))^2 = \mathrm{im}(\Gamma(X, \psi))$. By the left exactness of the global section functor Γ, the first terms of the two sequences coincide, and the second terms (and thus *à fortiori* the third terms) are isomorphic because the global section module of $\Delta^*(\mathcal{I}_X)$ is finitely presented by Lemma 1.9. We conclude with Lemma 2.6 that

$$\ker \varphi = \Gamma(X, \ker \underline{\varphi}) = \Gamma(X, \Delta^*(\mathcal{I}_X)^2) = \Gamma(X, \Delta^*(\mathcal{I}_X))^2 = I^2_{\mathcal{O}(X)}.$$

Therefore the assertion follows from the closedness of $\ker \varphi$ in $\mathcal{O}(X \times X)$. ■

Lemma 2.8 *If X is a Stein manifold, then the sheaf Ω^1_X of germs of holomorphic sections of $T^*(X)$ is coherent.*

Proof. The structure sheaf $\mathcal{O}_X$ of X is coherent because X is non-singular and coherence is a local property. Hence the same is true for $\mathcal{O}_X^{\oplus k}$ for each $k \in \mathbb{N}$. Since coherence is a local property and Ω^1_X is locally free, it is coherent. ■

Proposition 2.9 *For every Stein manifold X there exist finitely many functions $f_1, \ldots, f_N$ such that the $\mathcal{O}(X)$-module $\Omega^1(X)$ is generated by $df_1, \ldots, df_N$.*

Proof. We consider an embedding

$$F \colon X \to \mathbb{C}^N, \quad F = (f_1, \ldots, f_N)$$

of X as a closed submanifold of $\mathbb{C}^N$ ([11]). If $\dim X = n$ and $p \in X$, then there exist $i_1, \ldots, i_n$ such that $df_{i_j}(p)$, $j = 1, \ldots, n$, form a basis of $T_p(X)^*$. Let U denote an open neighborhood of p on which the 1-forms df_{i_j} are linearly independent. Then $T^*(X)|_U$ is a trivial bundle, hence $\Omega^1_X|_U$ is a free sheaf. It follows in particular that the map

$$\varphi \colon \Omega^1_{\mathbb{C}^N}|_X \to \Omega^1_X$$

is a surjective morphism of sheaves. As the sheaf $\Omega^1_{\mathbb{C}^N}|_X$ is free and Ω^1_X is coherent (Lemma 2.8), the sheaf $\ker \varphi$ is coherent ([7]). From Theorem B ([7]) we now derive that

$$H^1(X, \ker \varphi) = \{0\},$$

which means that the restriction map $\Omega^1(\mathbb{C}^N) \to \Omega^1(X)$ is surjective. If $z_1, \ldots, z_N$ are the canonical coordinate functions on $\mathbb{C}^N$, then $\Omega^1(\mathbb{C}^N)$ is generated by the differentials dz_j as a module of $\mathcal{O}(\mathbb{C}^N)$, and after restriction to X, we see that the restrictions $df_j = d(z_j|_X)$ generate $\Omega^1(X)$ as a module of $\mathcal{O}(X) = \mathcal{O}(\mathbb{C}^N)|_X$. Here we use that the restriction map $\mathcal{O}(\mathbb{C}^N) \to \mathcal{O}(X)$ is surjective ([7], Theorem 4, Ch. V, §4). ■

Now we complete the **proof of Theorem 2.1:** Left-exactness of the global section functor Γ implies that Γ and ker commute:

$$\Gamma(X, \ker(\underline{\varphi})) = \ker(\Gamma(X, \underline{\varphi})) = \ker(\varphi_X),$$

where

$$\varphi_X := \underline{\varphi}_X : \Gamma(X, \Delta^*(\mathcal{I}_X)) \to \Omega^1(X).$$

By Lemma 2.6, $\ker \underline{\varphi}$ and $\triangle^*(\mathcal{I})^2$ coincide as sheaves, so that Lemma 2.7 leads to

$$\ker(\varphi_X) \cong \Gamma(X, \triangle^*(\mathcal{I}_X)^2) = \overline{I_X^2}.$$

Since the map $\gamma_X \colon I_X/\overline{I_X^2} \to \Omega^1(X)$ coincides with the map induced by factorization of the map $\varphi_X : I_X \to \Omega^1(X)$, it follows that γ_X is injective.

In view of Proposition 2.9, the $\mathcal{O}(X)$-module $\Omega^1(X)$ is generated by $df_1, \dots df_N$ for some holomorphic functions $f_i \in \mathcal{O}(X)$. Therefore $\gamma_X(d_{\mathcal{O}(X)}u(f_i)) = df_i$ implies that γ_X is surjective, hence bijective. The map γ_X is continuous by construction, so that it is open by the Open Mapping Theorem. This completes the proof of Theorem 2.1. ∎

3 Extension to Riemannian domains over Stein manifolds

Definition 3.1 Let Y be a Stein manifold. A complex manifold X together with a holomorphic map $p : X \to Y$ which is everywhere regular will be called a *Riemannian domain over* Y.

One often considers arbitrary complex spaces X instead of manifolds in the preceding definition, but in order to have differential forms and therefore tangent spaces having everywhere the same dimension, we need manifolds.

The aim of this section is to generalize Theorem 2.1 to Riemannian domains over Stein manifolds.

Lemma 3.2 *If X is a Stein manifold, then its cotangent bundle T^*X is a Stein manifold.*

Proof. According to Grauert's Embedding Theorem, we may w.l.o.g. assume that X is a closed submanifold of some $\mathbb{C}^n$. Let $p \in X$ and $U \subseteq \mathbb{C}^n$ an open subset for which there exists a holomorphic function $F \colon U \to \mathbb{C}^k$ of constant rank k with

$$X \cap U = F^{-1}(0) \quad \text{and} \quad T_x(X) = \ker dF(x), \quad x \in X.$$

According to [3] (cf. also [16], Th. 4.1), the exact sequence of holomorphic vector bundles

$$0 \to TX \hookrightarrow T\mathbb{C}^n|_X \cong X \times \mathbb{C}^n \to N \to 0,$$

where N is the normal bundle of X in $\mathbb{C}^n$, splits. Let $\omega \colon N \to X \times \mathbb{C}^n$ be a holomorphic vector bundle map with

$$X \times \mathbb{C}^n \cong TX \oplus \omega(N).$$

Then we can identify T^*X with

$$\{(x, v) \in X \times \mathbb{C}^n : \langle v, \omega(N_x) \rangle = \{0\}\},$$

where $\langle z, w \rangle = \sum_{j=1}^n z_j w_j$. This implies that T^*X can be identified with a closed submanifold of $\mathbb{C}^n \times \mathbb{C}^n$, and thereore that T^*X is a Stein manifold. ∎

Theorem 3.3 *Let X be a Riemann domain over a Stein manifold. Then the map*

$$\gamma_X : \Omega^1(\mathcal{O}(X)) \to \Omega^1(X), \quad d_{\mathcal{O}(X)} f \mapsto df$$

is an isomorphism of Fréchet $\mathcal{O}(X)$-modules.

Proof. The spectrum $\widehat{X} := \Gamma(\mathcal{O}(X)) := \mathrm{Hom}(\mathcal{O}(X), \mathbb{C})$ of the algebra $\mathcal{O}(X)$ of holomorphic functions on X carries a natural complex manifold structure turning it into a Riemannian domain over X which is a Stein manifold ([16], Th. 4.6). The restriction map

$$R \colon \mathcal{O}(\widehat{X}) \to \mathcal{O}(X)$$

is an isomorphism of Fréchet algebras, which in turn induces a natural isomorphism

$$\alpha \colon \Omega^1(\mathcal{O}(\widehat{X})) \to \Omega^1(\mathcal{O}(X)).$$

On the other hand, we have a continuous restriction map $\beta \colon \Omega^1(\widehat{X}) \to \Omega^1(X)$, which leads to a commutative diagram

$$\begin{tikzcd} \Omega^1(\mathcal{O}(\widehat{X})) \arrow[r, "\alpha"] \arrow[d, "\gamma_{\widehat{X}}"] & \Omega^1(\mathcal{O}(X)) \arrow[d, "\gamma_X"] \\ \Omega^1(\widehat{X}) \arrow[r, "\beta"] & \Omega^1(X) \end{tikzcd}$$

We know from Theorem 2.1 that $\gamma_{\widehat{X}}$ is an isomorphism. It therefore suffices to show that β is an isomorphism to see that γ_X is an isomorphism, too. Since β is a continuous linear map between Fréchet spaces, it suffices to show that it is surjective, and then apply the Open Mapping Theorem. This means that we have to show that each holomorphic 1-form $\omega \in \Omega^1(X)$ is the restriction of a holomorphic 1-form $\widehat{\omega}$ on $\widehat{X}$.

The envelope of holomorphy $\widehat{X}$ has the property that all holomorphic functions on X with values in an arbitrary Stein manifold S extend uniquely to holomorphic functions from $\widehat{X}$ to S. In fact, we may embed S as a closed submanifold of $\mathbb{C}^n$, and then extend the function $f \colon X \to S$ to a function $F \colon \widehat{X} \to \mathbb{C}^n$. Since F maps the open subset X into S, it follows by analytic continuation that $F(\widehat{X}) \subseteq S$.

Let $\omega \in \Omega^1(X)$ be a holomorphic 1-form. We consider ω as a section of the cotangent bundle: $\omega : X \to T^*X$. We may identify $T^*X \cong T^*\widehat{X}|_X$ as an open submanifold of the Stein manifold $T^*\widehat{X}$ (Lemma 3.2). Therefore ω admits a unique extension to a holomorphic function $\widehat{\omega} \colon \widehat{X} \to T^*\widehat{X}$. If $\pi \colon T^*\widehat{X} \to \widehat{X}$ is the bundle projection, then $\pi \circ \omega = \mathrm{id}_X$, so that $\pi \circ \widehat{\omega}|_X = \mathrm{id}_X$, and therefore the uniqueness of the extension implies that $\pi \circ \widehat{\omega} = \mathrm{id}_{\widehat{X}}$. This means that $\widehat{\omega}$ is a section of the cotangent bundle of $\widehat{X}$, i.e., a holomorphic 1-form. ■

Remark 3.4 The condition that a complex manifold X is an open subset of a Stein manifold is equivalent to the condition that it is a Riemannian domain over a Stein manifold. In fact, each open subset is trivially a Riemannian domain. Conversely, each Riemannian domain X over a Stein manifold embeds as an open subset $X \hookrightarrow \widehat{X}$, where $\widehat{X}$ is its envelope of holomorphy, which is a Stein manifold ([16], Th. 4.6).

Corollary 3.5 *If X is a Riemannian domain over a Stein manifold, then the image of the universal differential $d \colon \mathcal{O}(X) \to \Omega^1(\mathcal{O}(X))$ is closed, and*

$$HC_1(\mathcal{O}(X)) \cong \Omega^1(X)/d\mathcal{O}(X).$$

Proof. In view of Theorem 3.3, we may identify $\Omega^1(\mathcal{O}(X))$ with the Fréchet space $\Omega^1(X)$ of holomorphic 1-forms on X and d with the de Rham differential. Its image consists of the exact 1-forms. That the space of exact 1-forms is closed follows from the fact that it is defined by the equations $\int_\gamma \omega = 0$, where γ is a piecewise smooth closed path in X, and the integration maps

$$\Omega^1(X) \to \mathbb{C}, \quad \omega \mapsto \int_\gamma \omega$$

are continuous linear functionals. ■

The following result was our initial motivation for the present work:

Theorem 3.6 *Let $\mathfrak{k}$ be a simple complex finite-dimensional Lie algebra and X a Riemann domain over a Stein manifold. Then $\mathfrak{g} := \mathcal{O}(X, \mathfrak{k}) \cong \mathcal{O}(X) \otimes \mathfrak{k}$ is a Fréchet–Lie algebra with respect to the pointwise bracket:*

$$[f \otimes x, g \otimes y] := fg \otimes [x, y].$$

If κ is the Cartan–Killing form of $\mathfrak{k}$, then

$$c(f \otimes x, g \otimes y) := \kappa(x, y)(f \cdot dg) \mod d(\mathcal{O}(X))$$

defines a continuous Lie algebra cocycle

$$c \colon \mathfrak{g} \times \mathfrak{g} \to HC_1(\mathcal{O}(X)) \cong \Omega^1(X)/d(\mathcal{O}(X)),$$

and the corresponding central extension

$$HC_1(\mathcal{O}(X)) \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$$

is universal.

Proof. It is shown in [13] that for each Fréchet algebra A the universal central extension of $\mathfrak{g} := A \otimes \mathfrak{k}$ is given by the cocycle

$$c(a \otimes x, b \otimes y) := \kappa(x, y)[a \cdot d_A b] \in HC_1(A).$$

Therefore the assertion follows from our identification of $HC_1(\mathcal{O}(X))$ in Theorem 3.3. ■

4 The algebra of germs of holomorphic functions

In this section we consider a compact subset $K \subseteq \mathbb{C}^n$. We write $\mathcal{O}(K)$ for the complex algebra of all germs of holomorphic functions on K, which is the direct limit of all algebras $\mathcal{O}(U)$, where U runs through the set of all open neighborhoods of K. Let $i_U \colon \mathcal{O}(U) \to \mathcal{O}(K), f \mapsto [f]$ denote the map assigning to $f \in \mathcal{O}(U)$ its germ on K and, for $U \subseteq V$, write

$$i_{U,V} \colon \mathcal{O}(V) \to \mathcal{O}(U), f \mapsto f|_U$$

for the restriction map.

Each of the algebras $\mathcal{O}(U)$ is a Fréchet algebra, and we consider on $\mathcal{O}(K)$ the locally convex direct limit topology, which is defined by all seminorms p on $\mathcal{O}(K)$ for which all compositions $p \circ i_U$ are continuous seminorms on $\mathcal{O}(U)$. Then all the maps

$$i_U : \mathcal{O}(U) \to \mathcal{O}(K)$$

are continuous and $\mathcal{O}(K)$ has the universal property of the direct limit: a linear map $\varphi : \mathcal{O}(K) \to V$ to a locally convex space V is continuous if and only if all compositions $\varphi \circ i_U$ are continuous. That the topology on $\mathcal{O}(K)$ is Hausdorff and the group multiplication is continuous is shown by H. Glöckner in Theorem 6.1 of [6].

Now $\mathcal{O}(K)$ is a locally convex unital algebra to which we can associate the universal differential module $\Omega^1(\mathcal{O}(K))$ together with the universal differential $d_{\mathcal{O}(K)} : \mathcal{O}(K) \to \Omega^1(\mathcal{O}(K))$. We write

$$\Omega^1(K) := \varinjlim \Omega^1(U)$$

for the space of germs of holomorphic 1-forms on K.

The main result of this section is the following:

Theorem 4.1 *The de Rham differential*

$$d : \mathcal{O}(K) \to \Omega^1(K)$$

is universal, when $\Omega^1(K)$ is endowed with the locally convex direct limit topology of the spaces $\Omega^1(U)$, U a neighborhood of K. In particular $\Omega^1(K) \cong \Omega^1(\mathcal{O}(K))$.

Proof. Since the cotangent bundle of any open subset $U \subseteq \mathbb{C}^n$ is trivial, it follows that

$$\Omega^1(K) \cong \bigoplus_{j=1}^{n} \mathcal{O}(K).[dz_j],$$

where $[\alpha]$ denotes the germ of the holomorphic 1-form α which is defined on a neighborhood of K. In particular $\Omega^1(K)$ is a free $\mathcal{O}(K)$-module of rank n, and we see that the module structure

$$\mathcal{O}(K) \times \Omega^1(K) \to \Omega^1(K), \quad ([f],[\alpha]) \mapsto [f\alpha]$$

is continuous because the multiplication in the algebra $\mathcal{O}(K)$ is continuous.

To prove the universality of d, let M be a continuous $\mathcal{O}(K)$-module and $D : \mathcal{O}(K) \to M$ a continuous derivation. Since the maps i_U are continuous, M inherits a natural structure of an $\mathcal{O}(U)$-module via

$$f.m := i_U(f).m, \quad f \in \mathcal{O}(U), m \in M.$$

For each open neighborhood $U \subseteq K$ the map

$$D_U : = D \circ i_U : \mathcal{O}(U) \to M$$

is a continuous derivation, and our main Theorem 3.3 therefore implies the existence of a unique continuous linear map

$$\alpha_U : \Omega^1(U) \to M \quad \text{with} \quad \alpha_U \circ d_U = D_U,$$

where $d_U\colon \mathcal{O}(U) \to \Omega^1(U)$ is the de Rham differential.

Let $j_U\colon \Omega^1(U) \to \Omega^1(K), \alpha \mapsto [\alpha]$ be the natural map, and for $U \subseteq V$, we write

$$j_{U,V}\colon \Omega^1(V) \to \Omega^1(U), \quad \alpha \mapsto \alpha|_U$$

for the restriction map. We clearly have $d_U \circ i_{U,V} = j_{U,V} \circ d_V$. For $U \subseteq V$ we now have $i_V = i_U \circ i_{U,V}$, so that $D_V = D_U \circ i_{U,V}$. Therefore

$$\alpha_U \circ j_{U,V} \circ d_V = \alpha_U \circ d_U \circ i_{U,V} = D_U \circ i_{U,V} = D \circ i_U \circ i_{U,V} = D \circ i_V = D_V,$$

so that the uniqueness of α_V leads to

$$\alpha_V = \alpha_U \circ j_{U,V}.$$

Hence the universal property of $\Omega^1(K)$ as a locally convex direct limit space implies the existence of a continuous linear map

$$\alpha\colon \Omega^1(K) \to M$$

with $\alpha \circ j_U = \alpha_U$ for each open neighborhood U of K.

We claim that α is a morphism of $\mathcal{O}(K)$-modules. For each $f \in \mathcal{O}(V)$ we choose $U \subseteq V$ and obtain for $\beta \in \Omega^1(U)$:

$$f.\alpha([\beta]) = f.\alpha_U(\beta) = i_{U,V}(f).\alpha_U(\beta) = \alpha_U(i_{U,V}(f)\beta) = \alpha \circ j_U(i_{U,V}(f)\beta) = \alpha([f].[\beta]).$$

The uniqueness of α follows from the fact that $\Omega^1(K)$ is generated as a $\mathcal{O}(K)$-module by the image of d, which contains the classes $[dz_1], \ldots, [dz_n]$. ■

Remark 4.2 In view of the preceding theorem, Theorem 3.6 generalizes in the obvious way to the Lie algebra $\mathcal{O}(K, \mathfrak{k})$ of germs of $\mathfrak{k}$-valued holomorphic functions on K.

5 Further remarks

Suppose the Stein manifold X is the complex analytic manifold corresponding to a complex smooth affine algebraic variety X^{aff} such that $X^{\text{aff}} \subset P$ for a complex smooth projective algebraic variety P and $D := P \setminus X^{\text{aff}}$ is an ample divisor on P. In this situation, some density theorem applies to show that (usual) Kähler 1-forms $\Omega^1_{\text{alg}}(X^{\text{aff}})$ on X^{aff} are dense in (usual holomorphic) 1-forms $\Omega^1(X)$ on X in the subspace topology (cf. [19]). Recall that for usual Kähler 1-forms, one has

$$\Omega^1_{\text{alg}}(X^{\text{aff}}) = \Omega^1(\text{Reg}(X)) = J^2/J,$$

where

$$J := \ker(\mu : \text{Reg}(X^{\text{aff}}) \otimes \text{Reg}(X^{\text{aff}}) \to \text{Reg}(X^{\text{aff}}))$$

the kernel of the multiplication map on the space $\text{Reg}(X^{\text{aff}})$ of *regular functions* on the affine variety X^{aff} (cf. [17]). Here, the density of $\Omega^1_{\text{alg}}(X^{\text{aff}})$ in the space of 1-forms $\Omega^1(X)$ can be independently deduced from Theorem 2.1, because of the following lemma.

Lemma 5.1 $\Omega^1_{\text{alg}}(X^{\text{aff}})$ *is a dense subspace of* $\Omega^1(\mathcal{O}(X))$.

Proof. It is well-known (cf. for example [19]) that $B := \mathrm{Reg}(X^{\mathrm{aff}}) \subset \mathcal{O}(X)$ is a dense subspace. We consider $J = \ker(\mu_X \cap (B \otimes B))$ as a subspace of the ideal $I_X = \ker \mu_X \subseteq \mathcal{O}(X \times X)$. Then the map

$$f \otimes g \mapsto f \otimes g - fg \otimes \mathbf{1}, \quad F \mapsto F - \mu_X(F) \otimes \mathbf{1}$$

is a continuous surjection $\mathcal{O}(X \times X) \to I_X$ which maps B onto J (cf. [13], Lemma 5), so that the density of B in $\mathcal{O}(X)$ implies the density of J in I_X. The map

$$\varphi_X : I_X \to \Omega^1(X)$$

whose kernel is I_X^2 maps J onto $\Omega^1_{\mathrm{alg}}(X)$, which is isomorphic to J/J^2. Therefore the density of J in I_X implies the density of $\Omega^1_{\mathrm{alg}}(X)$ in $\Omega^1(X)$. ■

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