ON TWODIMENSIONAL IMMERSIONS THAT ARE STABLE FOR PARAMETRIC FUNCTIONALS OF CONSTANT MEAN CURVATURE TYPE

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Abstract

We consider twodimensional immersions in Euclidean 3-space that are stable for parametric functionals of constant mean curvature type. We develop analytical and geometric concepts to give a perturbation result to estimate the principle curvatures of such mappings via uniformization.

1 Geometric and analytical basics

1.1 Introduction

Let $B := \{(u,v) \in \mathbb{R}^2 \,:\, u^2+v^2<1\}$ be the open unit disc, and let $\overline{B} \subset \mathbb{R}^2$ its topological closure. We consider twodimensional immersions $X = (x^1,x^2,x^3) \in \mathcal{C}(B,\mathbb{R}^3)$, where

$$\begin{aligned}\mathcal{C}(B,\mathbb{R}^3) \;:=\; \Big\{ X = X(u,v) \,:\, & X \in C^\omega(B,\mathbb{R}^3) \cap C^0(\overline{B},\mathbb{R}^3),\\ & |X_u(u,v) \wedge X_v(u,v)| > 0 \text{ for all } (u,v) \in B,\\ & \iint\limits_B |X_u(u,v) \wedge X_v(u,v)|\, dudv < +\infty \Big\}.\end{aligned}$$

Here, $\wedge$ means the vector product in $\mathbb{R}^3$, and the indices u and v indicate the partial derivatives of X. Furthermore, $X \in C^\omega(B,\mathbb{R}^3)$ means real analyticity in $B \subset \mathbb{R}^2$.

We investigate immersions $X \in \mathcal{C}(B,\mathbb{R}^3)$ that are stable for the parametric functional

$$\mathcal{H}[X] := \iint\limits_B \Big\{ F(X_u \wedge X_v) + \frac{2\gamma_0}{3}\, X \cdot (X_u \wedge X_v)^t \Big\}\, dudv \tag{1}$$

for real $\gamma_0 \in (0,+\infty)$. To be precise, there hold $\delta\mathcal{H}[X] = 0$ for its first variation, and $\delta^2\mathcal{H}[X;\varphi] \ge 0$ for all $\varphi \in C_0^\infty(B,\mathbb{R})$ for its second variation.

Furthermore, Z^t denotes the transposed vector of $Z \in \mathbb{R}^3$. The integrand $F \in C^\omega(\mathbb{R}^3 \setminus \{0\},\mathbb{R}) \cap C^0(\mathbb{R}^3,\mathbb{R})$ satisfies the homogeneity condition

$$F(\lambda Z) = \lambda F(Z) \quad \text{for all real } \lambda > 0,$$

which is necessary and sufficient for the parameter independence of $\mathcal{H}[X]$.

1.2 Some parametric functionals in differential geometry

Though we exclude $\gamma_0 = 0$, assume $\gamma_0 = 0$ for the moment. For $F(Z) = |Z|$ we have the area functional

$$\mathcal{A}[X] := \iint_B |X_u \wedge X_v| \, dudv.$$

Critical points of the associated variational problem $\mathcal{A}[X] \longrightarrow \min!$ are minimal surfaces, which are in fact real analytic in the interior. We refer the reader to [4] and [13].

Critial points for the functional

$$\mathcal{M}[X] := \iint_B \left\{ |X_u \wedge X_v| + \frac{2\gamma_0}{3} X \cdot (X_u \wedge X_v)^t \right\} dudv,$$

with real $\gamma_0 \in (0, +\infty)$, are surfaces of constant mean curvature $\gamma_0 > 0$. For our purpose we refer to [16].

Let us recall the differential system

$$\triangle X(u,v) = 2H(X) X_u(u,v) \wedge X_v(u,v) \quad \text{in } B$$

for a conformal parametrized immersion $X \in C^2(B, \mathbb{R}^3)$ of prescribed mean curvature $H = H(X)$. If the mean curvature is constant, then the immersion is real analytic in $B \subset \mathbb{R}^2$.

A more general version of a parametric functional is

$$\mathcal{F}[X] := \iint_B F(X_u \wedge X_v) \, dudv.$$

Its critical points $X = X(u,v)$ are immersions of minimal surface type (see e.g.[15], [19], [2], and [5]).

Finally, the most general case

$$\mathcal{S}[X] := \iint_B F(X, X_u \wedge X_v) \, dudv$$

is investigated in [18] and [10]. Critical points are immersions of mean curvature type. To the knowledge of the author, there are no a priori estimates of the derivatives of such immersions which are based on any differential system where these immersions are solutions of.

1.3 The arrangement

This paper is organized as follows:

Section 2: Following some notations, we derive the Euler-Lagrange equation of the variational problem

$$\mathcal{H}[X] \longrightarrow \min!$$

This system will be transformed into an equation of constant mean curvature (cmc-) type.

Section 3: For a minimizer $X \in \mathcal{C}(B, \mathbb{R}^3)$ we calculate $\delta^2 \mathcal{H}[X; \varphi]$ using weighted conformal parameters.

Section 4: We derive necessary geometric properties of surfaces of cmc-type.

Section 5: For X as well as for its unit normal vector N we establish a differential system with quadratic growth in the gradient.

Section 6: We investigate stable immersions X, that is $\delta^2 \mathcal{H}[X; \varphi] \geq 0$ for the second variation.

Section 7: We estimate the principle curvatures following the lines of [15].

1.4 Further notations

Let

$$F_Z(Z) := (F_{z^1}(Z), F_{z^2}(Z), F_{z^3}(Z)), \quad \mathbf{F}_{ZZ}(Z) := (F_{z^i z^j}(Z))_{i,j=1,2,3}.$$

From $F(\lambda Z) = \lambda F(Z)$ we derive

$$F_Z(Z) \cdot Z^t = F(Z), \quad \mathbf{F}_{ZZ}(Z) \circ Z^t = 0.$$

Thus, we consider $\mathbf{F}_{ZZ}(Z)$ as a mapping acting on the tangent space $\mathcal{T}_Z := \{Y \in \mathbb{R}^3 : Y \cdot Z^t = 0\}$.

Furthermore, we assume

$$\det \mathbf{F}_{ZZ}(Z) > 0,$$

and let $g_0 \in [0, +\infty)$ be a real constant such that

$$(1+g_0)^{-\frac{1}{2}}|\xi|^2 \le \xi \circ \mathbf{F}_{ZZ}(Z) \circ \xi^t \le (1+g_0)^{\frac{1}{2}}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3\,,\ \xi \cdot Z^t = 0.$$

Finally, by

$$N(u,v) := \frac{X_u(u,v) \wedge X_v(u,v)}{|X_u(u,v) \wedge X_v(u,v)|}\,, \quad (u,v) \in B,$$

we denote the unit normal vector of $X = X(u,v)$, and

$$W(u,v) := |X_u(u,v) \wedge X_v(u,v)| > 0 \quad \text{in } B$$

is its area element. For three vectors $X, Y, Z \in \mathbb{R}^3$ we set $[X,Y,Z] := X \cdot (Y \wedge Z)^t$. We will also use the notations $u^1 \equiv u$, $u^2 \equiv v$.

2 The first variation and the weighted CMC-system

2.1 The Euler-Lagrange equations

We start with

Lemma 2.1 *Let $X \in \mathcal{C}(B, \mathbb{R}^3)$ be a critical point of the functional $\mathcal{H}[X]$. Then it holds*

$$[N_u \circ \mathbf{F}_{ZZ}(N), N, X_v] + [N_v \circ \mathbf{F}_{ZZ}(N), X_u, N] = 2\gamma_0 W \quad \text{in } B.$$

Proof: For real $\varepsilon \in (-\varepsilon_0, +\varepsilon_0)$ and for any $\varphi \in C_0^\infty(B, \mathbb{R})$ we consider the normal variation

$$Y(u,v) := X(u,v) + \varepsilon\varphi(u,v)N(u,v), \quad (u,v) \in \overline{B}.$$

First, we have

$$Y_u \wedge Y_v = X_u \wedge X_v + \varepsilon\{X_u \wedge N_v + N_u \wedge X_v\}\varphi + \varepsilon\{X_u \wedge N\varphi_v + N \wedge X_v\varphi_u\} + o(\varepsilon).$$

We calculate

$$\begin{aligned} \delta\, X \cdot (X_u \wedge X_v)^t &= N \cdot (X_u \wedge X_v)^t\varphi + X \cdot (X_u \wedge N_v + N_u \wedge X_v)^t\varphi + X \cdot (X_u \wedge N\varphi_v + N \wedge X_v\varphi_u)^t \\ &= \operatorname{div}([X, N, X_v]\varphi, [X, X_u, N]\varphi) + 3W\varphi. \end{aligned}$$

Because $\varphi|_{\partial B} = 0$, the divergence theorem yields

$$\delta \iint\limits_B X \cdot (X_u \wedge X_v)^t\, dudv = 3 \iint\limits_B W\varphi\, dudv.$$

For the variation of $F = F(X_u \wedge X_v)$ we get

$$\begin{aligned} \delta\, F(X_u \wedge X_v) &= F_Z(N) \cdot (X_u \wedge N_v\varphi + N_u \wedge X_v\varphi + X_u \wedge N\varphi_v + N \wedge X_v\varphi_u)^t \\ &= \operatorname{div}([F_Z(N), N, X_v]\varphi, [F_Z(N), X_u, N]\varphi) \\ &\quad - [N_u \circ \mathbf{F}_{ZZ}(N), N, X_v]\varphi - [N_v \circ \mathbf{F}_{ZZ}(N), X_u, N]\varphi \end{aligned}$$

if we take $F_Z(X_u \wedge X_v) = F_Z(N)$ into account. Thus,

$$\delta \iint\limits_B F(X_u \wedge X_v)\, dudv = -\iint\limits_B \{[N_u \circ \mathbf{F}_{ZZ}(N), N, X_v] + [N_v \circ \mathbf{F}_{ZZ}(N), X_u, N]\}\varphi\, dudv.$$

Summarizing,

$$0 = -\iint\limits_B \{[N_u \circ \mathbf{F}_{ZZ}(N), N, X_v] + [N_v \circ \mathbf{F}_{ZZ}(N), X_u, N]\}\varphi\, dudv + 2\gamma_0 \iint\limits_B W\varphi\, dudv$$

holds true for all $\varphi \in C_0^\infty(B, \mathbb{R})$. The statement follows. □

2.2 The weight matrix

Let $S^2 := \{Z \in \mathbb{R}^3 \,:\, |Z| = 1\}$ denote the twodimensional unit sphere. To transform the above Euler-Lagrange equation into a system of constant mean curvature type, we introduce the weight matrix

$$\mathbf{G}(Z) := \left\{ \frac{1}{\sqrt{\det \mathbf{F}_{ZZ}(Z)}}\, \mathbf{F}_{ZZ}(Z) + (z^i z^j)_{i,j=1,2,3} \right\}^{-1}, \quad Z \in S^2\,,$$

as in [15]. It has the following properties (see section 1.4): For all $Z \in \mathbb{R}^3 \setminus \{0\}$, there hold

(G1) $\mathbf{G}(\lambda Z) = \mathbf{G}(Z)$ for all real $\lambda > 0$, where we continue the weight matrix along rays;

(G2) $\mathbf{G}(Z) \circ Z^t = Z^t$;

(G3) $(1+g_0)^{-1}|\xi|^2 \le \xi \circ \mathbf{G}(Z) \circ \xi^t \le (1+g_0)^2|\xi|^2$ for all $\xi \in \mathbb{R}^3$;

(G4) $\det \mathbf{G}(Z) = 1$.

2.3 The weighted fundamental forms

Let $X \in C^2(B, \mathbb{R}^3)$, and let $N = N(u,v)$ be its unit normal vector.

Definition 2.2 *As in [15], the weighted first, second, and third fundamental form are defined by*

$$\begin{aligned} I_G(X) &:= (X_{u^i} \circ \mathbf{G}(N) \circ X_{u^j}^t)_{i,j=1,2}\,, \\ II_G(X) &:= -\,(X_{u^i} \cdot N_{u^j}^t)_{i,j=1,2}\,, \\ III_G(X) &:= (N_{u^i} \circ \mathbf{G}(N)^{-1} \circ N_{u^j}^t)_{i,j=1,2}\,. \end{aligned}$$

We set

$$I_G(X) =: (h_{ij})_{i,j=1,2}\,, \quad II_G(X) =: -(L_{ij})_{i,j=1,2}\,, \quad III_G(X) := (e_{ij})_{i,j=1,2}\,.$$

Because $|X_u \wedge X_v| > 0$ in B, we can invert $I_G(X)$ to obtain $I_G(X)^{-1} =: (h^{ij})_{i,j=1,2}\,.$

2.4 The Weingarten equations

We use the summation convention. For the proof, compare [1], §55.

Lemma 2.3 *For $i = 1,2$, there hold*

$$N_{u^i} = -L_{ij} h^{jk} X_{u^k} \circ \mathbf{G}(N) \quad \textit{in } B.$$

Proof: From the ansatz $N_{u^i} = a_i^j X_{u^j} \circ \mathbf{G}(N)$ for $i = 1,2$ we find $-L_{ik} = a_i^j h_{jk}$ by multiplication with X_{u^k}, and, therefore, $-L_{ik} h^{kl} = a_i^j h_{jk} h^{kl} = a_i^l$. □

2.5 The weighted CMC-equation

From Lemma 2.1 and the definition of the weight matrix we conclude

$$[N, N_u \circ \mathbf{G}(N)^{-1}, X_v] + [N, X_u, N_v \circ \mathbf{G}(N)^{-1}] = -\frac{2\gamma_0}{\sqrt{\det \mathbf{F}_{ZZ}(N)}}\, W.$$

Using the Weingarten equations we calculate

$$\begin{aligned} \{N_u \circ \mathbf{G}(N)^{-1}\} \wedge X_v &= -(L_{11}h^{11} + L_{12}h^{21}) X_u \wedge X_v\,, \\ X_u \wedge \{N_v \circ \mathbf{G}(N)^{-1}\} &= -(L_{21}h^{12} + L_{22}h^{22}) X_u \wedge X_v\,, \end{aligned}$$

and, therefore,

$$L_{11}h^{11} + L_{12}h^{21} + L_{21}h^{12} + L_{22}h^{22} = \frac{2\gamma_0}{\sqrt{\det \mathbf{F}_{ZZ}(N)}}\,.$$

Introducing the first Beltrami operator

$$\overline{\nabla}_{ds_g^2}(X,N) := h^{ij} X_{u^i} \cdot N_{u^j}^t = -h^{ij} L_{ij}$$

w.r.t. $ds_g^2 := h_{11}\, du^2 + 2h_{12}\, dudv + h_{22}\, dv^2$, and the weighted mean curvature

$$H_G(N) := \frac{\gamma_0}{\sqrt{\det \mathbf{F}_{ZZ}(N)}}\,,$$

we arrive at the

Proposition 2.4 *Let $X \in \mathcal{C}(B,\mathbb{R}^3)$ be a critical point of the functional $\mathcal{H}[X]$. Then*

$$\overline{\nabla}_{ds_g^2}(X,N) = -2H_G(N) \quad \textit{in } B.$$

Remark 2.5 *1. If $F(Z) = |Z|$, we have $H(X) \equiv \gamma_0$ for the mean curvature (see paragraph 1.2).*

2. Immersions of minimal surface type have vanishing weighted mean curvature w.r.t. the weight matrix $\mathbf{G} = \mathbf{G}(N)$ from paragraph 2.2 (see e.g. [5]).

3. Critical points of $\mathcal{S}[X]$ (see section 1.2) have a weighted mean curvature of the form $H_G = H_G(X,N)$.

3 The second variation

3.1 Weighted conformal parameters

According to [17], we introduce weighted conformal parameters $(u,v) \in B$, such that

$$X_u \circ \mathbf{G}(N) \circ X_u^t = W = X_v \circ \mathbf{G}(N) \circ X_v^t\,, \quad X_u \circ \mathbf{G}(N) \circ X_v^t = 0 \quad \text{in } B.$$

Remark 3.1 *We denote the coefficients of the first fundamental form of $X = X(u,v)$ by*

$$E := X_u \cdot X_u^t\,, \quad F := X_u \cdot X_v^t\,, \quad G := X_v \cdot X_v^t\,.$$

Then, it follows that $W = \sqrt{EG - F^2} > 0$. Because $\det \mathbf{G}(Z) = 1$ holds for all $Z \in \mathbb{R}^3 \setminus \{0\}$, we have

$$EG - F^2 = h_{11}h_{22} - h_{12}^2\,.$$

Remark 3.2 *In weighted conformal parameters, the triple $\{W^{-\frac{1}{2}}\,\mathbf{G}(N)^{\frac{1}{2}} \circ X_u^t, W^{-\frac{1}{2}}\,\mathbf{G}(N)^{\frac{1}{2}} \circ X_v^t, N^t\}$ forms an orthonormal moving frame for the immersion. Investing $\det \mathbf{G}(Z) = 1$ as well as*

$$(\mathbf{M} \circ X^t) \wedge (\mathbf{M} \circ Y^t) = (\det \mathbf{M})\, \mathbf{M}^{-1} \circ (X \wedge Y)^t$$

for any symmetric and regular matrix $\mathbf{M} \in \mathbb{R}^{3\times 3}$ (for details, cp. [15]), there follow

$$N \wedge X_u = X_v \circ \mathbf{G}(N), \quad X_v \wedge N = X_u \circ \mathbf{G}(N).$$

3.2 The curvature

The mean curvature H and the Gaussian curvature K of the immersion are

$$H := \frac{EL_{22} - 2FL_{12} + GL_{11}}{2(EG - F^2)}\,, \quad K := \frac{L_{11}L_{22} - L_{12}^2}{EG - F^2}\,.$$

Note, that the coefficients L_{ij}, $i,j = 1,2$, do not depend on the weight, and so the Gaussian curvature. For the weighted mean curvature we infer from paragraph 2.5

$$H_G(N) = \frac{h_{11}L_{22} - 2h_{12}L_{12} + h_{22}L_{11}}{2(h_{11}h_{22} - h_{12}^2)} = \frac{h_{11}L_{22} - 2h_{12}L_{12} + h_{22}L_{11}}{2(EG - F^2)}\,.$$

3.3 The second variation

Let $\varphi, \psi \in C^1(B,\mathbb{R})$. Using weighted conformal parameters, we get

$$\overline{\nabla}_{ds_g^2}(\varphi,\psi) = h^{ij}\varphi_{u^i}\psi_{u^j} = \frac{1}{W}\,\nabla\varphi\cdot(\nabla\psi)^t\,.$$

Let

$$\delta(N) := \det\mathbf{F}_{ZZ}(N), \quad \sigma(N) := \operatorname{trace}\mathbf{F}_{ZZ}(N).$$

Lemma 3.3 *Using weighted conformal parameters, for the second variation of $\mathcal{H}[X]$ we have*

$$\delta^2\mathcal{H}[X;\varphi] = \iint\limits_B \left\{\sqrt{\delta(N)}\,|\nabla\varphi|^2 + KW\sigma(N)\varphi^2\right\} dudv - 4\iint\limits_B H_G(N)H(u,v)\sqrt{\delta(N)}\,W\varphi^2\,dudv$$

for any $\varphi \in C_0^\infty(B,\mathbb{R})$.

Proof: Consider the normal variation $Y := X + \varepsilon\varphi N$ in B as in section 2.1. It follows that

$$\begin{aligned} Y_u\wedge Y_v \;&=\; X_u\wedge X_v + \varepsilon\{X_u\wedge N_v + N_u\wedge X_v\}\varphi + \varepsilon\{X_u\wedge N\varphi_v + N\wedge X_v\varphi_u\} \\ &\quad + \varepsilon^2(N_u\wedge N_v)\varphi^2 + \varepsilon^2\{N_u\wedge N\varphi_v + N\wedge N_v\varphi_u\}\varphi. \end{aligned}$$

1. In [5], Lemma 4, the second variation

   $$\delta^2F(X_u\wedge X_v) = \delta(X_u\wedge X_v)\circ\mathbf{F}_{ZZ}(X_u\wedge X_v)\circ\delta(X_u\wedge X_v)^t + F_Z(X_u\wedge X_v)\cdot\delta^2(X_u\wedge X_v)^t$$

   is evaluated to get

   $$\delta^2\iint\limits_B F(X_u\wedge X_v)\,dudv = \iint\limits_B\left\{\sqrt{\delta(N)}\,|\nabla\varphi|^2 + KW\sigma(N)\right\} dudv.$$

2. For $\delta^2X\cdot(X_u\wedge X_v)^t = 2N\cdot\delta(X_u\wedge X_v)^t\varphi + X\cdot\delta^2(X_u\wedge X_v)^t$ one calculates

   $$\delta^2\iint\limits_B X\cdot(X_u\wedge X_v)^t\,dudv = \iint\limits_B\left\{3N\cdot(X_u\wedge N_v + N_u\wedge X_v)^t\varphi^2\right\} dudv.$$

3. The properties (G2) and (G4) of the weight matrix, the weighted Weingarten equations, and the calculus rules from Remark 3.2 yield

   $$\begin{aligned} N\wedge N_u \;&=\; -W^{-1}\,N\wedge\{L_{11}X_u\circ\mathbf{G}(N) + L_{12}X_v\circ\mathbf{G}(N)\} \\ &=\; -W^{-1}\{N\circ\mathbf{G}(N)\}\wedge\{L_{11}X_u\circ\mathbf{G}(N) + L_{12}X_v\circ\mathbf{G}(N)\} \\ &=\; -\frac{L_{11}}{W}\,X_v + \frac{L_{12}}{W}\,X_u\,, \end{aligned}$$

   and, analogously,

   $$N_v\wedge N = \frac{L_{12}}{W}\,X_v - \frac{L_{22}}{W}\,X_u\,.$$

   It follows, that

   $$\delta^2\iint\limits_B X\cdot(X_u\wedge X_v)^t\,dudv = -(2\cdot 3)\iint\limits_B HW\varphi^2$$

   with the mean curvature $H = H(u,v)$, and, therefore, according to paragraph 2.5

   $$\frac{2}{3}\iint\limits_B \gamma_0\delta^2X\cdot(X_u\wedge X_v)^t\,dudv = -4\iint\limits_B\gamma_0HW\varphi^2\,dudv = -4\iint\limits_B H_G(N)H\sqrt{\delta(N)}\,W\varphi^2\,dudv.$$

   The statement is proved. □

Remark 3.4 *For the functional $\mathcal{M} = \mathcal{M}[X]$ from paragraph 1.2 we have*

$$\delta^2\mathcal{M}[X] = \iint\limits_B\{|\nabla\varphi|^2 + 2KW\varphi^2\}\,dudv - 4\iint\limits_B H^2W\varphi^2\,dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$ with the mean curvature $H \equiv \gamma_0 \in (0,+\infty)$.

4 Some geometric properties

4.1 A linear connection between the weighted fundamental forms

Lemma 4.1 *Let $X \in C^2(B,\mathbb{R}^3)$. Then it holds*

$$I\!I\!I_G(X) - 2H_G(N)I\!I_G(X) + K(u,v)I_G(X) = \mathbf{0} \quad \textit{in } B.$$

Proof: Using weighted conformal parameters, the identity follows directly from the Weingarten equations. For $i = 1 = j$ we have

$$e_{11} = \left(\frac{L_{11}}{W}\,\mathbf{G}(N)\circ X_u^t + \frac{L_{12}}{W}\,\mathbf{G}(N)\circ X_v^t\right)^2 = \frac{L_{11}^2 + L_{12}^2}{W}\,,$$

while the stated identity reads as

$$e_{11} = \frac{L_{11}^2 + L_{11}L_{22} - L_{11}L_{22} + L_{12}^2}{W} = \frac{L_{11}^2 + L_{12}^2}{W}\,.$$

The remaining three equations are proved analogously. □

4.2 The weighted curvature relation

Lemma 4.2 *Let $X \in \mathcal{C}(B,\mathbb{R}^3)$ be a critical point of $\mathcal{H}[X]$. Then, there exist functions $\varrho_i = \varrho_i(X,N)$, $i = 1,2$, such that the curvature relation*

$$\varrho_1(X,N)\kappa_1(u,v) + \varrho_2(X,N)\kappa_2(u,v) = 2H_G(N) \quad \textit{in } B$$

holds true with the principle curvatures $\kappa_1 = \kappa_1(u,v)$ and $\kappa_2 = \kappa_2(u,v)$ of the surface.

Proof: By [12], Satz 5.4, as parameter lines for the surface we choose orthonormal curvature directions in any surface point. Then, $L_{11} = \kappa_1 E$ and $L_{22} = \kappa_2 G$, as well as $L_{12} = 0$, $F = 0$ in this point. We calculate

$$\begin{aligned} h^{11}L_{11} + h^{22}L_{22} &= \frac{X_v\circ\mathbf{G}(N)\circ X_v^t}{W^2}\,|X_u|^2\kappa_1 + \frac{X_u\circ\mathbf{G}(N)\circ X_u^t}{W^2}\,|X_v|^2\kappa_2 \\ &= \left(\frac{X_v}{|X_v|}\circ\mathbf{G}(N)\circ\frac{X_v^t}{|X_v|}\right)\kappa_1 + \left(\frac{X_u}{|X_u|}\circ\mathbf{G}(N)\circ\frac{X_u^t}{|X_u|}\right)\kappa_2 \end{aligned}$$

investing $W = |X_u||X_v|$. The vectors $|X_u|^{-1}X_u$, $|X_v|^{-1}X_v$ are orthonormal curvature directions. Setting $\varrho_1(X,N) := |X_v|^{-2}\,X_v\circ\mathbf{G}(N)\circ X_v^t$ and $\varrho_2(X,N) := |X_u|^{-2}\,X_u\circ\mathbf{G}(N)\circ X_u^t$, we have

$$\overline{\nabla}_{ds_g^2}(X,N) = -L_{ij}h^{ij} = -2H_G(N) = -\varrho_1(X,N)\kappa_1 - \varrho_2(X,N)\kappa_2\,.$$

This is the statement. □

Remark 4.3 *By (G3), there hold*

$$\frac{1}{1+g_0} \le \varrho_1(X,N),\varrho_2(X,N) \le 1 + g_0\,.$$

We point out the dependence on the position vector X, while the weight depends only on the normal direction.

5 An elliptic system

5.1 The Gauss equations

Let

$$\Gamma_{ij}^k := \frac{1}{2}\,h^{kl}\,(h_{li,j} + h_{jl,i} - h_{ij,l}), \quad i,j,k = 1,2,$$

denote the second Christoffel symbols, where $h_{ij,k}$ means the partial derivative of h_{ij} w.r.t. u^k. Furthermore,

$$\Omega_{ij}^k := -\frac{1}{2}\,h^{kl}(\omega_{lij} + \omega_{jli} - \omega_{ijl})\,, \quad \omega_{ijk} := X_{u^i}\circ\mathbf{G}(N)_{u^k}\circ X_{u^j}^t\,, \quad i,j,k = 1,2.$$

Lemma 5.1 *In B, there hold*

$$X_{u^iu^j}(u,v)=\Big\{\Gamma_{ij}^k(u,v)+\Omega_{ij}^k(u,v)\Big\}X_{u^k}(u,v)+L_{ij}(u,v)N(u,v)\quad \textit{for } i,j=1,2.$$

Proof: For the proof compare [1], § 57. We determine b_{ij}^k and c_{ij} of the ansatz

$$X_{u^iu^j}=b_{ij}^k\,X_{u^k}+c_{ij}\,N,\qquad i,j=1,2.$$

First, multiplication with the unit normal vector N yields $c_{ij}=-X_{u^i}\cdot N_{u^j}^t=L_{ij}$. Now, we multiply with $\mathbf{G}(N)\circ X_{u^l}^t$, $l=1,2$, to get

$$X_{u^iu^j}\circ\mathbf{G}(N)\circ X_{u^l}^t=b_{ij}^k\,X_{u^k}\circ\mathbf{G}(N)\circ X_{u^l}^t=b_{ij}^k\,h_{kl}=:b_{ilj}\,.$$

We see the symmetry conditions $b_{ilj}=b_{jli}$ for $i,j,l=1,2$. Furthermore, we have

$$\begin{aligned} b_{ilj} &= \{X_{u^i}\circ\mathbf{G}(N)\circ X_{u^l}^t\}_{u^j}-X_{u^i}\circ\mathbf{G}(N)_{u^j}\circ X_{u^l}^t-X_{u^lu^j}\circ\mathbf{G}(N)\circ X_{u^i}^t\\ &= h_{il,j}-b_{lij}-X_{u^i}\circ\mathbf{G}(N)_{u^j}\circ X_{u^l}^t\,, \end{aligned}$$

and it follows that $h_{il,j}=b_{ilj}+b_{lij}+\omega_{ilj}$. Using the above symmetry conditions we have

$$h_{jl,i}+h_{li,j}-h_{ij,l}=2b_{ilj}+\omega_{jli}+\omega_{lij}-\omega_{ijl}\,.$$

Rearranging for b_{ij}^k, multiplication with $\frac{1}{2}\,(h^{lm})_{l,m=1,2}$, and summation proves the statement. □

5.2 The differential system

Proposition 5.2 *Let $X\in C^3(B,\mathbb{R}^3)$. In weighted conformal parameters, there hold*

$$\triangle X=(\Omega_{11}^1+\Omega_{22}^1)X_u+(\Omega_{11}^2+\Omega_{22}^2)X_v+2H_G(N)WN$$

as well as

$$\begin{aligned} \triangle N^t \;=\; &-2\Big\{H_G(N)_u+H_G(N)(\Omega_{11}^1+\Omega_{22}^1)\Big\}\,\mathbf{G}(N)\circ X_u^t\\ &-2\Big\{H_G(N)_v+H_G(N)(\Omega_{11}^2+\Omega_{22}^2)\Big\}\,\mathbf{G}(N)\circ X_v^t\\ &-2\Big\{2H_G(N)^2-K\Big\}\,WN^t\\ &+\mathbf{G}(N)_u\circ\mathbf{G}(N)^{-1}\circ N_u^t+\mathbf{G}(N)_v\circ\mathbf{G}(N)^{-1}\circ N_v^t\,. \end{aligned}$$

Proof:

1. The first system follows from the Gauss equations, namely,

$$\begin{aligned} \triangle X &= (\Gamma_{11}^1+\Gamma_{22}^1+\Omega_{11}^1+\Omega_{22}^1)X_u+(\Gamma_{11}^2+\Gamma_{22}^2+\Omega_{11}^2+\Omega_{22}^2)X_v+(L_{11}+L_{22})N\\ &= (\Omega_{11}^1+\Omega_{22}^1)X_u+(\Omega_{11}^2+\Omega_{22}^2)X_v+2H_G(N)WN. \end{aligned}$$

2. We proceed as in [14]: From section 3.3 we recall

$$N\wedge N_u=-\frac{L_{11}}{W}\,X_v+\frac{L_{12}}{W}\,X_u\,,\quad N\wedge N_v=-\frac{L_{12}}{W}\,X_v+\frac{L_{22}}{W}\,X_u\,.$$

   Investing the Weingarten equations, there follow

$$\begin{aligned} N\wedge N_u &= \frac{L_{12}}{W}\,X_u+\frac{L_{22}}{W}\,X_v-\frac{L_{11}+L_{22}}{W}\,X_v \;=\; -N_v\circ\mathbf{G}(N)^{-1}-2H_G(N)\,X_v\,,\\ N\wedge N_v &= -\frac{L_{12}}{W}\,X_v-\frac{L_{11}}{W}\,X_u+\frac{L_{11}+L_{22}}{W}\,X_u \;=\; N_u\circ\mathbf{G}(N)^{-1}+2H_G(N)\,X_u\,. \end{aligned}$$

We conclude

$$\begin{aligned} N_u^t &= \mathbf{G}(N)\circ(N\wedge N_v)^t-2H_G(N)\,\mathbf{G}(N)\circ X_u^t\,,\\ N_v^t &= -\,\mathbf{G}(N)\circ(N\wedge N_u)^t-2H_G(N)\,\mathbf{G}(N)\circ X_v^t\,. \end{aligned}$$

Differentiation yields

$$\begin{aligned} N_{uu}^t &= \mathbf{G}(N)_u\circ(N\wedge N_v)^t+\mathbf{G}(N)\circ(N_u\wedge N_v)^t+\mathbf{G}(N)\circ(N\wedge N_{uv})^t\\ &\quad -2H_G(N)_u\,\mathbf{G}(N)\circ X_u^t-2H_G(N)\,\mathbf{G}(N)_u\circ X_u^t-2H_G(N)\,\mathbf{G}(N)\circ X_{uu}^t\,,\\ N_{vv}^t &= -\,\mathbf{G}(N)_v\circ(N\wedge N_u)^t-\mathbf{G}(N)\circ(N_v\wedge N_u)^t-\mathbf{G}(N)\circ(N\wedge N_{uv})^t\\ &\quad -2H_G(N)_v\;\mathbf{G}(N)\circ X_v^t-2H_G(N)\,\mathbf{G}(N)_v\circ X_v^t-2H_G(N)\,\mathbf{G}(N)\circ X_{vv}^t\,. \end{aligned}$$

Taking (G2) into account, we arrive at

$$\begin{aligned} \triangle N^t &= 2(N_u\wedge N_v)^t-2H_G(N)_u\,\mathbf{G}(N)\circ X_u^t-2H_G(N)_v\,\mathbf{G}(N)\circ X_v^t\\ &\quad +\mathbf{G}(N)_u\circ(N\wedge N_v)^t-2H_G(N)\,\mathbf{G}(N)_u\circ X_u^t\\ &\quad +\mathbf{G}(N)_v\circ(N_u\wedge N)^t-2H_G(N)\,\mathbf{G}(N)_v\circ X_v^t\\ &\quad -2H_G(N)\,\mathbf{G}(N)\circ(X_{uu}^t+X_{vv}^t)\\ &= 2KWN^t-2H_G(N)_u\,\mathbf{G}(N)\circ X_u^t-2H_G(N)_v\,\mathbf{G}(N)\circ X_v^t\\ &\quad +\mathbf{G}(N)_u\,\mathbf{G}(N)^{-1}\circ N_u^t+\mathbf{G}(N)_v\,\mathbf{G}(N)^{-1}\circ N_v^t\\ &\quad -2H_G(N)(\Omega_{11}^1+\Omega_{22}^1)\,\mathbf{G}(N)\circ X_u^t-2H_G(N)(\Omega_{11}^2+\Omega_{22}^2)\,\mathbf{G}(N)\circ X_v^t\\ &\quad -4H_G(N)^2WN^t\,. \end{aligned}$$

Collecting all the terms proves the statement. □

5.3 Arrangement of the a priori data

For simplicity we refer to a set $\{g_0,g_1,\gamma_0,\delta_0\}$ of weight data in the following form: In addition to (G3), let $g_1\in[0,+\infty)$ be a real constant such that

$$\sqrt{\sum_{i,j,k=1}^3 g_{ij,z^k}(N)^2}\le g_1\,.$$

Let us recall the weighted mean curvature $H_G(N)=\delta(N)^{-\frac12}\gamma_0$. This means

$$\frac{\gamma_0}{\sqrt{1+g_0}}\le H_G(N)\le\sqrt{1+g_0}\,\gamma_0\in(0,+\infty).$$

Finally, we introduce a real constant $\delta_0^*\in[0,+\infty)$ estimating the gradient of the inverse of $\sqrt{\det\delta(N)}$, that is, for the weighted mean curvature

$$|H_{G,Z}(N)|\le\delta_0^*\gamma_0=\delta_0^*\gamma_0\cdot1\le\delta_0^*\sqrt{1+g_0}\,H_G(N)=:\delta_0H_G(N),$$

where

$$\delta_0:=\sqrt{1+g_0}\,\delta_0^*\,.$$

Lemma 5.3 *There hold*

$$\|\mathbf{G}(N)_{u^m}\|\le g_1|N_{u^m}(u,v)|,\quad m=1,2.$$

Proof: Let $\xi=(\xi^1,\xi^2,\xi^3)\in\mathbb{R}^3$ be given with $|\xi|=1$. Then, we calculate

$$\begin{aligned} |\mathbf{G}(N)_{u^m}\circ\xi^t|^2 &= \sum_{i=1}^3\Big\{g_{ij,z^k}(N)N_{u^m}^k\xi^j\Big\}^2\le\sum_{i,j=1}^3\Big\{g_{ij,z^k}(N)N_{u^m}^k\Big\}^2\\ &\le |N_{u^m}|^2\sum_{i,j,k=1}^3 g_{ij,z^k}(N)^2\le g_1^2|N_{u^m}|^2 \end{aligned}$$

for $m=1,2$. The statement follows. □

5.4 Quadratic growth in the gradient

The next result enables us to derive a modulus of continuity for the spherical mapping $N = N(u,v)$.

Proposition 5.4 *Let $X \in C^3(B,\mathbb{R}^3)$ be given in weighted conformal parameters. Then, there hold*

$$|\triangle X| \le (1+g_0)g_1(|X_u|+|X_v|)(|N_u|+|N_v|) + \sqrt{2}\,(1+g_0)|\nabla X||\nabla N|.$$

as well as

$$|\triangle N| \le 2\{H_G(N)^2 - K\}W + 2(1+g_0)^2\{2\delta_0 + g_1 + 4(1+g_0)g_1\}\{2H_G(N)^2 - K\}W.$$

Proof:

1. First, we infer from the Weingarten equations

$$\begin{aligned}\mathbf{G}(N)^{-\frac{1}{2}} \circ N_u^t &= -\frac{L_{11}}{W}\,\mathbf{G}(N)^{\frac{1}{2}} \circ X_u^t - \frac{L_{12}}{W}\,\mathbf{G}(N)^{\frac{1}{2}} \circ X_v^t\,,\\ \mathbf{G}(N)^{-\frac{1}{2}} \circ N_v^t &= -\frac{L_{12}}{W}\,\mathbf{G}(N)^{\frac{1}{2}} \circ X_u^t - \frac{L_{22}}{W}\,\mathbf{G}(N)^{\frac{1}{2}} \circ X_v^t\end{aligned}$$

   the estimates

$$\begin{aligned}|\nabla N|^2 &\le (1+g_0)\,\frac{L_{11}^2+2L_{12}^2+L_{22}^2}{W} = (1+g_0)\,\frac{(L_{11}+L_{22})^2 - 2(L_{11}L_{22}-L_{12}^2)}{W}\\ &= 2(1+g_0)\{2H_G(N)^2-K\}W\end{aligned}$$

   as well as

$$|\nabla N|^2 \ge \frac{1}{1+g_0}\,\frac{L_{11}^2+2L_{12}^2+L_{22}^2}{W} = \frac{2}{1+g_0}\{2H_G(N)^2-K\}W.$$

   Furthermore, we remark $K \le H_G(N)^2$ in B, because from the Hopf function $\mathcal{H} := L_{11} - L_{22} - 2iL_{12}$ we infer

$$|\mathcal{H}|^2 = 4\{H_G(N)^2 - K\}W^2 \ge 0.$$

   Likewise, we have $K \le H^2$ B. By $H_G(N) > 0$, we get

$$H_G(N)\sqrt{2H_G(N)^2-K} \le \sqrt{2H_G(N)^2-K}\sqrt{2H_G(N)^2-K} = 2H_G(N)^2-K.$$

2. We have

$$|\triangle X| \le (|\Omega_{11}^1| + |\Omega_{22}^1|)|X_u| + (|\Omega_{11}^2|+|\Omega_{22}^2|)|X_v| + 2H_G(N)W.$$

   Explicitely, we estimate

$$|\Omega_{11}^1| \le \frac{1}{2W}\,|X_u \circ \mathbf{G}(N)_u \circ X_u^t| \le \frac{|X_u|^2}{2W}\,\|\mathbf{G}(N)_u\| \le \frac{(1+g_0)g_1}{2}\,|N_u|.$$

   Here, we take

$$|X_u|^2 \le (1+g_0)\,X_u \circ \mathbf{G}(N) \circ X_u^t = (1+g_0)W$$

   into account; analogously we estimate for $|X_v|$. In the same manner, we have

$$\begin{aligned}|\Omega_{11}^2| &\le (1+g_0)g_1|N_u| + \frac{(1+g_0)g_1}{2}\,|N_v|, \quad |\Omega_{22}^1| \le \frac{(1+g_0)g_1}{2}\,|N_u| + (1+g_0)g_1|N_v|,\\ |\Omega_{22}^2| &\le \frac{(1+g_0)g_1}{2}\,|N_v|.\end{aligned}$$

   It follows

$$|\triangle X| \le (1+g_0)g_1(|X_u|+|X_v|)(|N_u|+|N_v|) + 2H_G(N)W.$$

   Investing

$$\begin{aligned}2H_G(N)W &\le 2H_G(N)|X_u||X_v| \le 2\sqrt{1+g_0}\,H_G(N)|\nabla X|\sqrt{W}\\ &\le 2\sqrt{1+g_0}\,\sqrt{2H_G(N)^2-K}\,\sqrt{W}\,|\nabla X| \le \sqrt{2}\,(1+g_0)|\nabla X||\nabla N|,\end{aligned}$$

   the first estimate is proved.

3. From Proposition 5.2 and the settings of section 5.3 we infer the estimate

$$\begin{aligned}|\triangle N| &\le 2\{2H_G(N)^2-K\}W+2(1+g_0)\delta_0(|X_u||N_u|+|X_v||N_v|)H_G(N)+(1+g_0)g_1|\nabla N|^2\\ &\quad +2(1+g_0)H_G(N)(|\Omega_{11}^1|+|\Omega_{22}^1|)|X_u|+2(1+g_0)H_G(N)(|\Omega_{11}^2|+|\Omega_{22}^2|)|X_v|\\ &\le 2\{2H_G(N)^2-K\}W+2(1+g_0)\delta_0(|X_u||N_u|+|X_v||N_v|)H_G(N)+(1+g_0)g_1|\nabla N|^2\\ &\quad +2(1+g_0)^2g_1H_G(N)(|X_u|+|X_v|)(|N_u|+|N_v|)\\ &\le 2\{2H_G(N)^2-K\}W+2\sqrt{2}\,(1+g_0)^{\frac{3}{2}}\delta_0H_G(N)|\nabla N|\sqrt{W}+(1+g_0)g_1|\nabla N|^2\\ &\quad +4\sqrt{2}\,(1+g_0)^{\frac{5}{2}}g_1H_G(N)|\nabla N|\sqrt{W}\,.\end{aligned}$$

We invest the results from the first point of the proof and arrive at

$$|\triangle N|\le 2\{H_G(N)^2-K\}W+2(1+g_0)^2\{2\delta_0+g_1+4(1+g_0)g_1\}\{2H_G(N)^2-K\}W.$$

This proves the Proposition. □

Corollary 5.5 *The estimate*

$$|\triangle N|\le(1+g_0)\left\{1+(1+g_0)^2[2\delta_0+g_1+4(1+g_0)g_1]\right\}|\nabla N|^2\quad\textit{in } B$$

holds true.

Proof: From Proposition 5.4 together with $\{2H_G(N)^2-K\}W\le\frac{1+g_0}{2}\,|\nabla N|^2$ we deduce the statement. □

6 Properties of stable immersion of CMC-type

Assume that the stable immersion $X=X(u,v)$ represents a geodesic disc $\mathfrak{B}_r(X_0)$ of radius $r>0$ with the center $X_0:=X(0,0)$. In geodesic polar coordinates we get a mapping $Z=Z(\varrho,\varphi)\,:\,[0,r]\times[0,2\pi]\longrightarrow\mathbb{R}^3$ with the metrical element $ds_P^2=d\varrho^2+P(\varrho,\varphi)\,d\varphi^2$ (cp. [1], § 79).

Lemma 6.1 *Let $X\in\mathcal{C}(B,\mathbb{R})$ be a stable critical point of $\mathcal{H}[X]$, given in weighted conformal parameters. Let X represent a geodesic disc $\mathfrak{B}_r(X_0)$ of radius $r>0$ centred at $X_0:=X(0,0)$. Finally, assume*

$$\mu:=\frac{2-4g_0-6g_0^2-2g_0^3}{(1+g_0)^2}>\frac{1+g_0}{2}\,.$$

Then, for the area of the geodesic disc we have the estimate

$$\mathcal{A}(Z):=\int_0^r\int_0^{2\pi}\sqrt{P(\varrho,\varphi)}\,d\varrho d\varphi\le\frac{2\pi\mu}{2\mu-(1+g_0)}\,r^2\,.$$

Proof:

1. From the stability condition (we omit $dudv$),

$$\iint_B|\nabla\varphi|^2\sqrt{\delta(N)}\ge\iint_B\{4H_G(N)H\sqrt{\delta(N)}-K\sigma(N)\}W\varphi^2$$

for all $\varphi\in C_0^\infty(B,\mathbb{R})$, we derive the μ-stability condition

$$\iint_B|\nabla\varphi|^2\ge\mu\iint_B(2H^2-K)W\varphi^2$$

with the above $\mu>0$.

The first inequality yields

$$\begin{aligned}\iint\limits_B |\nabla\varphi|^2\sqrt{\delta(N)} \;&\geq\; \iint\limits_B \{\varrho_1\kappa_1^2+\varrho_2\kappa_2^2+(\varrho_1+\varrho_2)\kappa_1\kappa_2\}\sqrt{\delta(N)}\,W\varphi^2+\iint\limits_B(-K)W\sigma(N)\varphi^2\\ &\geq\; \frac{2}{1+g_0}\iint\limits_B(2H^2-K)W\sqrt{\delta(N)}\,\varphi^2+\iint\limits_B\{(\varrho_1+\varrho_2)\sqrt{\delta(N)}-\sigma(N)\}KW\varphi^2\,.\end{aligned}$$

We estimate

$$(\varrho_1+\varrho_2)\sqrt{\delta(N)}-\sigma(N)\leq 2(1+g_0)^{\frac{3}{2}}-\frac{2}{\sqrt{1+g_0}}\leq 2\,\frac{(1+g_0)^2-1}{\sqrt{1+g_0}}=\frac{4g_0+2g_0^2}{\sqrt{1+g_0}}\,,$$

and by $K\leq H^2$ we arrive at

$$\iint\limits_B|\nabla\varphi|^2\sqrt{\delta(N)}\geq\frac{2}{1+g_0}\iint\limits_B(2H^2-K)\sqrt{\delta(N)}\,W\varphi^2-\frac{4g_0+2g_0^2}{\sqrt{1+g_0}}\iint\limits_B(2H^2-K)W\varphi^2\,.$$

The stated inequality follows.

2. The above μ-stability condition can be rewritten in the form

$$\iint\limits_B\overline{\nabla}_{ds^2}(\varphi,\varphi)W\geq\frac{\mu}{1+g_0}\iint\limits_B\{2H^2-K\}\varphi^2W$$

with the metrical element $ds^2:=E\,du^2+2F\,dudv+G\,dv^2$. Now, we follow exactly the lines of [5], Lemma 1, by inserting the special test function $\Phi(\varrho):=1-r^{-1}\varrho$, $0\leq\varrho\leq r$, followed by partial integration. □

Lemma 6.2 *Let $X\in\mathcal{C}(B,\mathbb{R}^3)$ be a stable critical point of the functional $\mathcal{H}[X]$, given in weighted conformal parameters $(u,v)\in B$. Let $\nu\in(0,1)$ be a real number. Then the estimate*

$$\iint\limits_{|w|\leq 1-\nu}|\nabla N(u,v)|^2\,dudv\leq\frac{8\pi(1+g_0)}{\mu^*\nu^2}$$

holds true, where

$$0<\mu^*<\frac{2-2(1+g_0)^3+2(1+g_0)}{(1+g_0)^2+(1+g_0)^4-1}\,,$$

and $g_0\geq 0$ is chosen sufficiently small.

Proof:

1. For $\tilde{\mu}>0$ still to be computed we derive the integral inequality (we omit $dudv$)

$$\iint\limits_B\{(\varrho_1\kappa_1^2+\varrho_2\kappa_2^2)\sqrt{\delta(N)}+[(\varrho_1+\varrho_2)\sqrt{\delta(N)}-\sigma(N)]\kappa_1\kappa_2\}\,W\varphi^2\geq\tilde{\mu}\iint\limits_B\{2H_G(N)^2-K\}W\varphi^2\,.$$

For this we have to ensure the positivity

$$\left(\varrho_1\sqrt{\delta}-\frac{\tilde{\mu}\varrho_1^2}{2}\right)\kappa_1^2+\frac{2}{2}\{(\varrho_1+\varrho_2)\sqrt{\delta}-\sigma-\tilde{\mu}\varrho_1\varrho_2+\tilde{\mu}\}\kappa_1\kappa_2+\left(\varrho_2\sqrt{\delta}-\frac{\tilde{\mu}\varrho_2^2}{2}\right)\kappa_2^2\geq 0.$$

For any quadratic form $ax^2+2bxy+cy^2$ with $a,b>0$, we know $ax^2+2bxy+cy^2\geq(a-b)x^2+(c-b)y^2$, and, therefore, it is positive if $a>b$ and $c>b$. Applied to our quadratic form we get the condition

$$\tilde{\mu}<\frac{2\sqrt{1+g_0}-2(1+g_0)^{\frac{7}{2}}+2(1+g_0)^{\frac{3}{2}}}{(1+g_0)^2+(1+g_0)^4-1}\,,$$

where $g_0 \geq 0$ is chosen small enough. Finally, we have the μ-stability condition

$$\iint_B |\nabla\varphi|^2 \geq \mu^* \iint_B \{2H_G(N)^2 - K)\}W\varphi^2$$

with $\mu^* := \tilde{\mu}(1+g_0)^{-\frac{1}{2}}$, and $\tilde{\mu} > 0$ chosen w.r.t. the above restriction.

2. Recall $|\nabla N|^2 \leq 2(1+g_0)\{2H_G(N)^2 - K\}W$. For real $\nu \in (0,1)$ we choose $\varphi \in C_0^\infty(B,\mathbb{R})$ such that

$$\begin{aligned} \varphi(u,v) &= 1 \quad \text{in } B_{1-\nu}(0,0) := \{(u,v) \in B \,:\, u^2+v^2 < (1-\nu)^2\}, \\ |\nabla\varphi(u,v)| &\leq \frac{2}{\nu} \quad \text{in } B. \end{aligned}$$

Investing the stability condition in the above form we find

$$\begin{aligned} \iint_{|w|\leq 1-\nu} |\nabla N|^2\,dudv \;\leq\; 2(1+g_0) \iint_{|w|\leq 1-\nu} \{2H_G(N)^2 - K\}W \;\leq 2(1+g_0) \iint_B \{2H_G(N)^2 - K\}W\varphi^2 \\ \leq \frac{2(1+g_0)}{\mu^*} \iint_B |\nabla\varphi|^2 \;\leq\; \frac{8\pi(1+g_0)}{\mu^*\nu^2}\,. \end{aligned}$$

The statement follows. □

Motivated by [14], Hilfssatz 6, we prove the

Proposition 6.3 *Let $X \in \mathcal{C}(B,\mathbb{R}^3)$ be a stable critical point of the functional $\mathcal{H}[X]$, given in weighted conformal parameters $(u,v) \in B$. For $w_0 \in B$ and real $\nu \in (0, 1-|w_0|)$ we assume*

$$\psi^*(u,v) := N(u,v)\cdot(0,0,1)^t > \omega \quad \textit{for all } (u,v) \in \partial B_\nu(w_0),$$

where

$$1 > \omega \geq \frac{2(1+g_0)^2\{2\delta_0 + g_1 + 4(1+g_0)g_1\} + 2}{\mu^*} - 1 \geq 0$$

with $\mu^* > 0$ from Lemma 6.2 and g_0, g_1, δ_0 sufficiently small. Then the inequality

$$N(u,v)\cdot(0,0,1)^t \geq \omega \quad \textit{for all } (u,v) \in \overline{B}_\nu(w_0)$$

holds true. In particular, $X|_{\overline{B}_\nu(w_0)}$ represents a graph over the plane perpendicular to the vector $(0,0,1)$.

Proof:

1. Let $\psi := \psi^* - \omega$. Setting $q := \{2H_G(N)^2 - K\}W$, $q > 0$, from Proposition 5.2 we get

$$\triangle N = -2qN + R,$$

where the remaining term R can be seen from the representation given in Proposition 5.2; we will discuss it later in detail. There follow

$$\triangle\psi^* + 2q\psi^* = R\cdot(0,0,1)^t$$

as well as

$$\triangle\psi^* = -2q\psi^* + R\cdot(0,0,1)^t = -2q\psi - 2q\omega + R\cdot(0,0,1)^t = \triangle\psi.$$

2. We define the cut off function

$$\psi^-(u,v) := \min\left(\psi(u,v), 0\right) \in H_1^2(\overline{B}_\nu(w_0),\mathbb{R}) \cap C^0(\overline{B}_\nu(w_0),\mathbb{R}).$$

It remains to prove $\psi^- \equiv 0$. Using a generalized Gauß theorem we calculate (we set $B^* := B_\nu(w_0)$, and omit $dudv$)

$$\iint_{B^*} |\nabla\psi^-|^2 = -\iint_{B^*} \psi^-\triangle\psi = \mu^* \iint_{B^*} q|\psi^-|^2 + [2-\mu^*] \iint_{B^*} q|\psi^-|^2 + 2\omega \iint_{B^*} q\psi^- - \iint_{B^*} \psi^- R\cdot(0,0,1)^t\,.$$

3. For the admissable test function

$$\varphi(u,v) := \psi^-(u,v) + \varepsilon\chi(u,v), \quad \chi \in C_0^\infty(B^*,\mathbb{R}),\ \varepsilon \in \mathbb{R},$$

the stability condition in the form of Lemma 6.2 implies

$$\iint_{B^*} |\nabla\psi^-|^2 + 2\varepsilon \iint_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint_{B^*} |\nabla\chi|^2$$

$$\geq \mu^* \iint_{B^*} q|\psi^-|^2 + 2\mu^*\varepsilon \iint_{B^*} q\psi^-\chi + \mu^*\varepsilon^2 \iint_{B^*} q\chi^2 \,.$$

It follows the inequality

$$[2-\mu^*] \iint_{B^*} q|\psi^-|^2 + 2\omega \iint_{B^*} q\psi^- - \iint_{B^*} \psi^- R \cdot (0,0,1)^t + 2\varepsilon \iint_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint_{B^*} |\nabla\chi|^2$$

$$\geq 2\mu^*\varepsilon \iint_{B^*} q\psi^-\chi + \mu^*\varepsilon^2 \iint_{B^*} q\chi^2 \,.$$

4. Rearranging the terms yields

$$2\varepsilon \iint_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint_{B^*} |\nabla\chi|^2$$

$$\geq -[2-\mu^*] \iint_{B^*} q|\psi^-|^2 - 2\omega \iint_{B^*} q\psi^- + \iint_{B^*} \psi^- R \cdot (0,0,1)^t + 2\mu^*\varepsilon \iint_{B^*} q\psi^-\chi + \mu^*\varepsilon^2 \iint_{B^*} q\chi^2$$

$$\geq -[2-\mu^*] \iint_{B^*} q|\psi^-||\psi^-| + 2\omega \iint_{B^*} q|\psi^-| - \iint_{B^*} |\psi^-||R \cdot (0,0,1)^t|$$

$$+ 2\mu^*\varepsilon \iint_{B^*} q\psi^-\chi + \mu^*\varepsilon^2 \iint_{B^*} q\chi^2$$

$$\geq \{-[2-\mu^*](1+\omega) + 2\omega - \Lambda\} \iint_{B^*} q|\psi^-| + 2\mu^*\varepsilon \iint_{B^*} q\psi^-\chi + \mu^*\varepsilon^2 \iint_{B^*} q\chi^2$$

investing $-1-\omega \leq \psi^- \leq 0$, $\mu^* < 2$, and $\Lambda := 2(1+g_0)^2\{2\delta_0 + g_1 + 4(1+g_0)g_1\}$ by Proposition 5.4. By our assumption we have

$$-[2-\mu^*](1+\omega) + 2\omega - \Lambda \geq 0.$$

Therefore, we get

$$2\varepsilon \iint_{B^*} \left(\nabla\psi^- \cdot \nabla\chi - \mu^* q\psi^-\chi\right) + \varepsilon^2 \iint_{B^*} \left(|\nabla\chi|^2 - \mu^* q\chi^2\right) \geq 0$$

for all $\varepsilon \in \mathbb{R}$. This means

$$\iint_{B^*} \left(\nabla\psi^- \cdot \nabla\chi - \mu^* q\psi^-\chi\right) = 0$$

for all $\chi \in C_0^\infty(B,\mathbb{R})$. The Lemma of Weyl implies the real analyticity of $\psi^- = \psi^-(u,v)$ in B^*. But due to our assumption it vanishes on a domain of positive twodimensional Hausdorff measure. Therefore, this cut off function vanishes identically. The proposition is proved. □

Remark 6.4 *It is the interplay between the last three results that makes the constant g_0, g_1, γ_0, and δ_0 small, while certain variations of their particular proofs are in fact conceivable.*

7 Curvature estimates

From [15] we adopt

Lemma 7.1 *Let $X \in \mathcal{C}(B,\mathbb{R}^3)$ be a critical point of the functional $\mathcal{H}[X]$, given in weighted conformal parameters $(u,v) \in B$. Then there exists a constant $c_* = c_*(g_0) \in (0,1)$, such that the plane mapping*

$$f(u,v) := \Big(x^1(u,v), x^2(u,v)\Big), \qquad (u,v) \in \overline{B},$$

satisfies the inequalities

$$c_*(g_0)|\nabla X(u,v)|^2 \le |\nabla f(u,v)|^2 \le |\nabla X(u,v)|^2 \qquad \text{in} \quad B.$$

Theorem 7.2 *Let $X \in \mathcal{C}(B,\mathbb{R}^3)$ be a stable critical point of the functional $\mathcal{H}[X]$, given in weighted conformal parameters $(u,v) \in B$. Let $g_0, g_1 \ge 0$ and $\delta_0 > 0$ be small enough such that the assumptions of Lemma 6.1, Lemma 6.2, and Proposition 6.3 are satisfied. Finally, assume that the immersion represents a geodesic disc $\mathfrak{B}_r(X_0)$ of radius $r > 0$ centered at $X_0 := X(0,0)$. Then, there exists a constant $\Theta = \Theta(g_0, g_1, \delta_0) \in (0, +\infty)$, such that*

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \le \frac{1}{r^2}\,\Theta(g_0,g_1,\delta_0) + 4(2g_0+g_0^2)(1+g_0)^3\gamma_0^2$$

holds true for the principle curvatures $\kappa_1 = \kappa_1(u,v)$ and $\kappa_2 = \kappa_2(u,v)$.

Proof: We essentially follow the lines of [15]. Therefore, we only give a sketch.

1. First, we calculate

$$N_u \circ \mathbf{G}(N)^{-1} \circ N_u^t + N_v \circ \mathbf{G}(N)^{-1} \circ N_v^t = 2\{2H_G(N)^2 - K\}W$$

using the linear dependence from paragraph 4.1 between the weighted fundamental forms. Squaring $\varrho_1\kappa_1 + \varrho_2\kappa_2 = 2H_G(N)$ yields

$$-2K = \frac{\varrho_1}{\varrho_2}\,\kappa_1^2 + \frac{\varrho_2}{\varrho_1}\,\kappa_2 - \frac{4H_G(N)^2}{\varrho_1\varrho_2}\,.$$

Therefore, we have

$$\begin{aligned} N_u \circ \mathbf{G}(N)^{-1} \circ N_u^t + N_v \circ \mathbf{G}(N)^{-1} \circ N_v^t \;&=\; 4H_G(N)^2W + \left(\frac{\varrho_1}{\varrho_2}\,\kappa_1^2 + \frac{\varrho_2}{\varrho_1}\,\kappa_2^2\right)W - \frac{4H_G(N)^2}{\varrho_1\varrho_2}\,W \\ \ge\; \frac{1}{(1+g_0)^2}\,(\kappa_1^2+\kappa_2^2)W - 4(2g_0+g_0^2)H_G(N)^2W. \end{aligned}$$

Furthermore, we estimate

$$|\nabla N|^2 \ge \frac{1}{(1+g_0)^3}\,(\kappa_1^2+\kappa_2^2)W - 4\,\frac{2g_0+g_0^2}{1+g_0}\,H_G(N)^2W,$$

and, thus, by $H_G(N) \le \sqrt{1+g_0}\,\gamma_0$ from paragraph 5.3,

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \le (1+g_0)^3\,\frac{|\nabla N(0,0)|^2}{W(0,0)} + 4(2g_0+g_0^2)(1+g_0)^3\gamma_0^2\,.$$

We will estimate the gradient $\nabla N(0,0)$ from above, and the area element $W(0,0)$ from below.

2. For the area $\mathcal{A}[X]$ of the immersion $X = X(u,v)$ we have

$$\begin{aligned} \mathcal{A}[X] \;&=\; \iint\limits_B |X_u \wedge X_v|\,dudv \;=\; \frac{1}{2}\iint\limits_B \{X_u \circ \mathbf{G}(N) \circ X_u^t + X_v \circ \mathbf{G}(N) \circ X_v^t\}\,dudv \\ &\ge\; \frac{1}{2(1+g_0)}\iint\limits_B \{|X_u|^2 + |X_v|^2\}\,dudv \;=:\; \frac{1}{2(1+g_0)}\,\mathcal{D}[X] \end{aligned}$$

with the Dirchlet integral $\mathcal{D}[X]$.

Investing Lemma 6.1 we find

$$\mathcal{D}[X] \le c_1(g_0) r^2 \,, \quad c_1(g_0) := \frac{4(1+g_0)\pi\mu}{2\mu-(1+g_0)} \,.$$

3. Let $\Gamma(\overline{B})$ be the set of all continuous and piecewise differentiable curves $\gamma(t) : [0,1] \longrightarrow \overline{B}$, where $\gamma(0) = (0,0)$, $\gamma(1) \in B$. We have

$$\inf_{\gamma \in \Gamma(\overline{B})} \int_0^1 \left| \frac{d}{dt} X(\gamma(t)) \right| dt \ge r.$$

By Lemma 1 from [15], there is a point $w^* \in B$ with $|w^*| \le 1-\nu_0$, $\nu_0 := e^{-4\pi c_1}$, such that

$$\frac{W(w^*)}{r^2} \ge c_2(g_0) > 0$$

with a real a priori constant $c_2 = c_2(g_0) \in (0,+\infty)$.

4. Let $w_0 \in B_{1-\frac{1}{2}\nu_0}(0,0)$, and let $\varrho > 0$ such that $B_\varrho(w_0) \subset B_{1-\frac{1}{2}\nu_0}(0,0)$. Investing Lemma 6.2, for given $\lambda > 0$ there exist a $\delta = \delta(\nu_0, g_0; \lambda) \in (0, (1-\frac{1}{2}\nu_0)^2)$ and a $\delta^* \in [\delta, \sqrt{\delta}]$, such that

$$\int_{\partial B_{\delta^*\varrho}(w_0)} |dN(w)| \le 2\sqrt{\frac{8\pi^2(1+g_0)}{(-\log\delta)\mu^*\nu_0^2}} \le 2\lambda.$$

By Proposition 6.3 we get the following modulus of continuity for the spherical mapping $N = N(u,v)$: For $\lambda = \lambda(g_0, g_1, \delta_0)$ sufficiently small, there exists a $\delta_1 = \delta_1(g_0, g_1, \delta_0) \in (0, \frac{1}{4}\nu_0^2)$, such that for all $w_0 \in \overline{B}_{1-\frac{1}{2}\nu_0}(0,0)$ the estimate

$$|N(w) - N(w_1)| \le \lambda(g_0, g_1, \delta_0) \quad \text{for all } w \in \overline{B}_{\delta_1}(w_0)$$

holds true, where $w_1 \in B_{\delta_1}(w_0)$ is an arbitrary point. Setting

$$\mathfrak{N}(w) := \frac{1}{\lambda} \{N(w) - N(w_1)\} \, : \, \overline{B}_{\delta_1}(w_0) \longrightarrow \mathbb{R}^3$$

we have $|\mathfrak{N}(w)| \le 1$ in $\overline{B}_{\delta_1}(w_0)$. Due to Corollary 5.5 we have

$$|\triangle \mathfrak{N}| \le \lambda(1+g_0)\{1 + (1+g_0)^2[2\delta_0 + g_1 + 4(1+g_0)g_1]\}|\nabla \mathfrak{N}|^2 \,.$$

Choose $\lambda = \lambda(g_0, g_1, \delta_0) > 0$, such that $\lambda(1+g_0)\{1+(1+g_0)^2[2\delta_0 + g_1 + 4(1+g_0)g_1]\} \le \frac{1}{2}$. Then,

$$|\triangle \mathfrak{N}(w)| \le \frac{1}{2} |\nabla \mathfrak{N}(w)|^2 \,, \quad |\mathfrak{N}(w)| \le 1 \quad \text{in } \overline{B}_{\delta_1}(w_0).$$

The gradient estimate by E. Heinz from [8], Theorem 2, yields a constant $c_3 = c_3(g_0, g_1, \delta_0) \in (0,+\infty)$, such that

$$|\nabla N(w_0)| \le c_3(g_0, g_1, \delta_0) \quad \text{for all } w_0 \in \overline{B}_{1-\frac{1}{2}\nu_0}(0,0).$$

5. From Proposition 5.4 we infer the linear differential inequality

$$|\triangle Y| \le c_4(g_0, g_1, \delta_0) \{|Y_u| + |Y_v|\} \quad \text{in } \overline{B}_{1-\frac{1}{2}\nu_0}(0,0)$$

with a real constant $c_4 = c_4(g_0, g_1, \delta_0) \in (0,+\infty)$. Here, $Y := r^{-1}X$. Moreover, for $\nu \in (0, \frac{1}{4}\nu_0)$ with $2\nu\sqrt{c_3} \le 1$ and all $w_0 \in \overline{B}_{1-\nu_0}(0,0)$ the modulus of projection

$$|N(w) - N(w_0)| \le 2\nu\sqrt{c_3} \le 1, \quad w \in \overline{B}_{2\nu}(w_0),$$

is controlled by the gradient estimate from point 4. Now, the iterative scheme from the proof in [15] yields an a priori constant $c_5 = c_5(g_0, g_1, \delta_0) \in (0, +\infty)$ such that

$$\frac{W(0,0)}{r^2} \geq c_5(g_0, g_1, \delta_0) > 0.$$

From the curvature inequality given in the first point and the gradient estimate for $N = N(u,v)$ from the fourth point, we find

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \leq \frac{1}{r^2}\,(1+g_0)^3 \frac{c_3(g_0,g_1,\delta_0)}{c_5(g_0,g_1,\delta_0)} + 4(2g_0+g_0^2)(1+g_0)^3\gamma_0^2$$

for $w_0 = (0,0)$. Setting

$$\Theta(g_0,g_1,\delta_0) := (1+g_0)^3 \frac{c_3(g_0,g_1,\delta_0)}{c_5(g_0,g_1,\delta_0)}\,,$$

the proof is complete. □

Remark 7.3 *1. In the case of vanishing weighted mean curvature we infer $K \leq 0$ for the Gauß curvature. By a theorem of Hadamard (cp. [11], Theorem 6.4.4.), the exponential map into the manifold is injectiv for all values of the geodesic radius $r > 0$. Although we have excluded $\gamma_0 = 0$, a few modifications yield the following Bernstein type result: $K \equiv 0$ for $r \to \infty$, that is a complete stable immersion of minimal surface type is a plane. For details we refer to [5].*

2. If $K \leq K_0 \in (0, +\infty)$ for the Gauß curvature, the exponential map is a local diffeomorphism if $r\sqrt{K_0} < \pi$ holds true (cp. [11]). Thus, we are not allowed to apply Theorem 7.2 for all values of the geodesic radius. Our method of proof yields no result of Bernstein type even for special functionals.

3. The weighted mean curvature depends on the definition of the weight matrix. Therefore, variants of the methods presented here are needed to improve the numerical quality of our results. In this context compare [3].

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