An axiomatic approach to valuation in life insurance

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Abstract

The classical Principle of Equivalence ensures that a life insurance company can accomplish that the mean balance per contract converges to zero almost surely for an increasing number of clients. In an axiomatic approach, this idea is adapted to the general case of stochastic financial markets. In accordance with existing results, the implied minimum fair price of general life insurance products is then uniquely determined by the product of the given equivalent martingale measure of the financial market with the probability measure of the biometric state space. A detailed historical example concerning contract pricing and valuation is given.

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1 Introduction

In traditional life insurance mathematics, financial markets are assumed to be deterministic. Under this assumption, the philosophy behind the classical Principle of Equivalence is that a life insurance company should be able to accomplish that the mean balance per contract converges to zero almost surely for an increasing number of clients. Roughly speaking, premiums are chosen

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such that incomes and losses are “balanced in the mean”. This idea leads to a valuation method usually called “Expectation Principle” and relies on two important ingredients: the stochastic independence of individual lifes and the Strong Law of Large Numbers. In modern life insurance mathematics, where financial markets are sensibly assumed to be stochastic and where more general products (e.g. unit-linked ones) are taken into consideration, the valuation principle which is widely accepted is also an expectation principle. However, the respective probability measure is different as the minimum fair price (or present value) of an insurance claim is determined by the no-arbitrage pricing method as it is known from financial mathematics. The respective equivalent martingale measure (EMM) is the product of the given EMM of the financial market with the probability measure of the biometric state space. Although research on the valuation of unit-linked products already started in the late 1960s, one of the first results (for a particular contract) that was in its core identical to the mentioned product measure approach was Brennan and Schwartz (1976). The most recent papers mainly dedicated to valuation by this approach are (for the Black-Scholes model) Aase and Persson (1994) and (for a simple stochastic interest rate model) Persson (1998). A brief history of valuation in (life) insurance can be found in Møller (2002) (the works Møller (2002, 2003a and 2003b) also consider valuation, but focus on hedging, resp. advanced premium principles). Again, one should have a look at the assumptions concerning the considered valuation principle. Aase and Persson (1994), but also other authors, a priori suppose independence of financial and biometric events. An arbitrage-free and complete financial market ensures the uniqueness of the (financial) EMM. The use of the product measure as mentioned above is usually explained by the risk-neutrality of the insurer towards biometric risks (cf. Aase and Persson, 1994; Persson, 1998). In Møller (2001), a further good reason can be found: the product measure coincides trivially with the so-called minimal martingale measure (cf. Schweizer, 1995).

Apart from these reasons for the considered product measure approach, the aim of this paper is the deduction of a valuation principle by an adaption of the classical demand for convergence of mean balances due to the Law of Large Numbers. This idea seems to be new. In a discrete finite time framework, it is carried out by an axiomatic approach which mainly reflects the commonly accepted assumptions in the modern theory of life insurance (as already mentioned: independence of individuals, independence of biometric and financial events, no-arbitrage pricing etc.). The resulting valuation principle is in accordance with the above mentioned results since the implied minimum fair price for general life insurance products is uniquely determined by the equivalent martingale measure that is given by the product of the EMM of the financial market with the probability measure of the biometric state space. In fact, due to the no-arbitrage pricing, the complete price process is determined. Under the mentioned axioms, it is shown how a life insurance company can accomplish that the mean balance per contract at any future time t converges to zero almost surely for an increasing number of customers. The respective (purely financial and self-financing) hedging strategy can be financed (the initial costs, of course) by the minimum fair premiums. The considered hedging method is different from the risk-minimizing and mean-variance hedging strategies considered e.g. in Møller (1998, 2001 and 2002). In fact, the method is a (discrete) generalization of the matching approach in Aase and Persson (1994). Even though that this hedging method is less sophisticated than e.g. risk minimizing strategies (which are unfortunately not self-financing), it surely is of practical use since it is easier to realize (not every single life has to be observed over the whole time axis). Examples for the pricing and hedging of different types of contracts are given. A more detailed example shows for a traditional life insurance and an endowment contract the historical development of the ratio of the minimum fair annual premium per benefit. Assuming that premiums are calculated by a conservatively chosen constant technical rate of interest, the example also considers the development of the present values of these contracts.

Although the model considered in this paper is restricted to a finite set of points of time, the approach is quite general in the sense that it does not propose particular models for the dynamics of the financial securities or the biometric events. The concept of a life insurance contract is introduced in a very general way and the presented methods are not restricted to a particular type of contract. Further, all methods and results of the paper can be applied to non-life insurance as long as the assumptions are also appropriate in the considered cases.

The section content is as follows. In Section 2, the principles which are considered to be reasonable for a modern theory of life insurance are briefly discussed in an enumerated list. Section 3 introduces the considered model and some first axioms concerning the common probability space of financial and biometric risks. Section 4 contains a definition of general life insurance contracts and the statement of a generalized Principle of Equivalence. (The paper makes a difference between the classical Expectation Principle, which is a valuation method, and the Principle of Equivalence, which is an economic "fairness" argument.) In Section 5, the case of classical life insurance mathematics is briefly reviewed. Section 6 contains the axiomatic approach to valuation in the general case and the deduction of the minimum fair price. Section 7 is on the topic of hedging, i.e. on the convergence of the mean balances. Examples are given in this section, too. In Section 8, it is shown how parts of the results can be adapted to the case of incomplete markets, even for markets with arbitrage opportunities something can be rescued. The last section is dedicated to the numerical pricing example mentioned above and confirms the importance of modern valuation principles.

2 Principles of life insurance mathematics

In the author's opinion, the following eight assumptions are crucial for a modern theory of life insurance mathematics. The principles are given in an informal manner, the mathematically precise formulation follows later.

1. Independence of technical and financial events. One of the basic assumptions is that the technical (biometric) events, for instance death or injury of persons, are independent of the events of the financial markets (cf. Aase and Persson, 1994). In contrast to reinsurance companies, where the movements on the financial markets can be highly correlated to the payments of the insurer, it is common sense that such effects can be neglected in the case of life insurance.

2. Complete, arbitrage-free financial markets. Except for Section 8, where incomplete markets are examined, complete, arbitrage-free *financial* markets are considered throughout the paper. Even though this might be an unrealistic assumption from the viewpoint of finance, it is a realistic one from the perspective of life insurance. The reason is that a life insurance company usually does not invent purely financial products as this is the working field of banks. Therefore, it can be assumed that all considered *financial* products are either traded on the market, can be bought from banks or can be replicated by self-financing strategies. Nonetheless, it is self-evident that a claim which also depends on a technical event (e.g. the death of a person) *cannot* be hedged by financial securities. Hence, the joint market of financial and technical risks is not complete. In the literature, completeness of financial markets is often assumed by the use of the Black-Scholes model (cf. Aase and Persson, 1994; Møller, 1998). It figures out that parts of the results of the paper are also valid in the case of incomplete financial markets - which allows for more models. However, in this case financial portfolios will be restricted to replicable ones and also the considered life insurance contracts are restricted in a similar way.

3. Biometric states of individuals are independent.

4. Large classes of similar individuals. Concerning the Law of Large Numbers as applied in classical life insurance mathematics, an implicit assumption is a large number of persons under contract in a particular company. Even stronger, it can be assumed that classes of "similar" persons, e.g. of the same age, are large. An insurance company should be able to cope with such a large class of similar persons even if all members of the class have the same kind of contract (cf. Principle 7 below).

5. Similar individuals cannot be distinguished. For fairness reasons, any two individuals with similar biometric development to be expected should pay the same price for the same kind of contract. Further, any activity (e.g. hedging) of an insurance company due to two individuals having the same kind of contract is assumed to be identical as long as their probable future biometric development is independently identical from the stochastic point of view.

6. No-arbitrage pricing. As we know from the theory of financial markets, an important property of a reasonable pricing system is the absence of arbitrage, i.e. the absence of riskless wins. In particular, it should not be possible to beat the market by selling and buying (life) insurance products in an existing or hypothetical reinsurance market (see e.g. Delbaen and Haezendonck, 1989). Hence, any product and cashflow will be priced by the no-arbitrage principle.

7. Minimum fair prices allow hedging such that mean balances converge to zero a.s. The principle of independence of the biometric state spaces is closely related to the Expectation Principle of classical life insurance mathematics. In the classical case, where financial markets are assumed to be deterministic, this principle states that the present value of a cashflow is the expectation of the sum of its discounted payoffs. The connection between the two principles is the Law of Large Numbers. Present values or prices are determined such that for a large number of contracts due to independent individuals the insurer can accomplish that the mean final balance per contract, but also the mean balance at any time t, converges to zero almost surely. In analogy to the classical case, we generally demand that the minimum fair price of any contract (from the viewpoint of the insurer) should at least cover the price of a purely financial hedging strategy that lets the mean balance per contract converge to zero a.s. for an increasing number of clients.

8. Principle of Equivalence. Under a reasonable valuation principle (cf. Principle 7), the Principle of Equivalence demands that the future payments to the insurer (premiums) should be determined such that their present value equals the present value of the future payments to the insured (benefits). The idea is that the liabilities (benefits) can somehow be hedged by working with the premiums. This concept will be considered in detail in the coming sections.

Concerning premium calculation, the classical Expectation Principle (cf. Principle 7) is usually seen as a minimum premium principle since any insurance company must be able to cope with higher expenses than the expected (cf. Embrechts, 2000). We refer to the literature for more information on the topic (e.g. Delbaen and Haezendonck, 1989; Gerber, 1997; Goovaerts, De Vylder and Haezendonck, 1984; but also Møller, 2002-2003b; Schweizer, 2001).

3 The model

Let $(F, \mathcal{F}_T, \mathbb{F})$ be a probability space equipped with the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$, where $\mathbb{T} = \{0, 1, 2, \dots, T\}$ denotes the discrete finite time axis. Assume that $\mathcal{F}_0$ is trivial, i.e. $\mathcal{F}_0 = \{\emptyset, F\}$. Let the price dynamics of d securities of a frictionless financial market be given by an adapted $\mathbb{R}^d$-valued process $S = (S_t)_{t\in\mathbb{T}}$. The d assets with price processes $S^0, S^1, \dots, S^{d-1}$ are traded at times $t \in \mathbb{T} - \{0\}$. The first asset with price process S^0 is called the *money account* and has the properties $S_0^0 = 1$ a.s. and $S_t^0 > 0$ for $t \in \mathbb{T}$. The tuple $M^F = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}, \mathbb{T}, S)$ is called a *securities market model.* A *portfolio* due to M^F is given by a d-dimensional vector θ of real-valued random variables on $(F, \mathcal{F}_T, \mathbb{F})$. A *t-portfolio* is a portfolio θ_t which is $\mathcal{F}_t$-measurable. As usual, $\mathcal{F}_t$ is interpreted as the information available at time t. As an economic agent takes decisions due to the available information, a *trading strategy* is a vector $\theta_{\mathbb{T}} = (\theta_t)_{t\in\mathbb{T}}$ of t-portfolios θ_t. The discounted total gain (or loss) of such a strategy is given by $\sum_{t=0}^{T-1}\langle\theta_t, \overline{S}_{t+1} - \overline{S}_t\rangle$, where $\overline{S} := (S_t/S_t^0)_{t\in\mathbb{T}}$ denotes the price process discounted by the money account and $\langle .\, ,.\rangle$ denotes the inner product on $\mathbb{R}^d$. We can now define the set of all discounted gains

$$G = \left\{ \sum_{t=0}^{T-1}\langle\theta_t, \overline{S}_{t+1} - \overline{S}_t\rangle : \theta_t \text{ is a } t\text{-portfolio for } t \in \mathbb{T} - \{T\} \right\}. \tag{1}$$

G is a subspace of the space of all real-valued random variables $L^0(F, \mathcal{F}_T, \mathbb{F})$ where two elements are identified if they are equal $\mathbb{F}$-a.s. The process S satisfies the so-called *no-arbitrage condition* (NA) if $G \cap L^0_+ = \{0\}$, where L^0_+ are the non-negative elements of $L^0(F, \mathcal{F}_T, \mathbb{F})$ (the notation follows Delbaen, 1999). The Fundamental Theorem of Asset Pricing (Dalang, Morton and Willinger, 1990) states that the price process S satisfies (NA) if and only if there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{F}$ such that under $\mathbb{Q}$ the process $\overline{S}$ is a martingale. Moreover, $\mathbb{Q}$ can be found with bounded Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{F}$. A very readable overview concerning some of the existing proofs of the theorem is Delbaen (1999). The probability measure $\mathbb{Q}$ as considered above is called (risk-neutral) *equivalent martingale measure* (EMM).

DEFINITION 3.1. *A* **valuation principle** π^F *on a set* Θ *of portfolios due to* M^F *is a linear mapping which maps each* $\theta \in \Theta$ *to an adapted stochastic process (price process)* $\pi^F(\theta) = (\pi^F_t(\theta))_{t\in\mathbb{T}}$ *such that*

$$\pi^F_t(\theta) = \langle\theta, S_t\rangle = \sum_{i=0}^{d-1}\theta^i S^i_t \tag{2}$$

for any $t \in \mathbb{T}$ *for which* θ *is* $\mathcal{F}_t$*-measurable.*

For the moment, the set Θ is not specified any further.

Consider an arbitrage-free market with price process S as given above and a portfolio θ with price process $\pi^F(\theta)$. Assume that θ's price is not determined by the market. From the Fundamental Theorem it is known that the enlarged market with price dynamics $S' = (S^0, \ldots, S^{d-1}, \pi^F(\theta))$ is arbitrage-free if and only if there exists an EMM $\mathbb{Q}$ such that $\overline{S'}$ becomes a $\mathbb{Q}$-martingale. Hence, one obtains the valuation principle

$$\pi^F_t(\theta) = S^0_t \cdot \mathbf{E}_{\mathbb{Q}}[\langle\theta, S_T\rangle/S^0_T|\mathcal{F}_t]. \tag{3}$$

As is well-known, the no-arbitrage condition does not imply a unique price process for θ when it cannot be replicated by a self-financing strategy $\theta_{\mathbb{T}}$, i.e. a strategy such that $\langle\theta_{t-1}, S_t\rangle = \langle\theta_t, S_t\rangle$ for each $t > 0$ and $\theta_T = \theta$. However, in a *complete* market M^F, i.e. a market which features a self-financing replicating strategy for any portfolio θ, the no-arbitrage condition implies for $|\mathcal{F}_T| < \infty$ a unique EMM $\mathbb{Q}$ and therefore unique prices (Taqqu and Willinger, 1987).

DEFINITION 3.2. *A* **t-claim** *with payoff C_t at time t is a t-portfolio of the form $\frac{C_t}{S_t^0}e_0$ where C_t is a $\mathcal{F}_t$-measurable random variable and e_0 denotes the first canonical base vector in $\mathbb{R}^d$. A* **cashflow** *over the time period $\mathbb{T}$ is a vector $(\frac{C_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ of t-claims.*

A t-claim is interpreted as the right on the amount C_t of cash on the money account S^0 at time t. That means the owner is actually given C_t in cash at t. The interpretation of a cashflow is obvious.

We will now introduce axioms which concern the properties of market models (not of valuation principles) that include biometric events (Principles 1 to 4 of Section 2). Assume to be given a filtered probability space $(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$ which describes the development of the biological states of all considered human beings. *No particular model for the development of the biometric information is chosen.*

AXIOM 1. *A common filtered probability space*

$$(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P}) = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}) \otimes (B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B}) \tag{4}$$

of financial and biometric events is given, i.e. $M = F \times B$, $\mathcal{M}_t = \mathcal{F}_t \otimes \mathcal{B}_t$ and $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$.

AXIOM 2. *A complete financial market*

$$M^F = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}, \mathbb{T}, {}_FS) \tag{5}$$

together with a unique equivalent martingale measure $\mathbb{Q}$ is given. The common market of financial and biometric risks is denoted by

$$M^C = (M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P}, \mathbb{T}, S), \tag{6}$$

where $S(f,b) = {}_FS(f)$ for all $(f,b) \in M$.

In the following, M^C is understood as a securities market model. The notions portfolio, no-arbitrage etc. are used as above. Usually, a non-deterministic financial market will be considered, i.e. $2 < |\mathcal{F}_T| < \infty$ can be assumed.

REMARK 3.3. S is the canonical embedding of ${}_FS$ into $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$. In the following, we will often use the same symbol for a random variable X in $(F, \mathbb{F})$ and a random variable Y in $(M, \mathbb{P})$ when we have that Y is the embedding of X into $(M, \mathbb{P})$, i.e. $Y(f, b) = X(f)$ for all $(f, b) \in M$. Now, any portfolio θ of the complete financial market M^F can be replicated by some self-financing trading strategy $\theta_\mathbb{T} = (\theta_t)_{t\in\mathbb{T}}$. The unique price process $\pi^F(\theta)$ of the portfolio is given by

$$\pi_t^F(\theta) = {}_FS_t^0 \cdot \mathbf{E}_\mathbb{Q}[\langle\theta, {}_FS_T\rangle / {}_FS_T^0 | \mathcal{F}_t]. \tag{7}$$

As S is the embedding of ${}_FS$ into $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$, the (embedded) portfolio θ in M^C is also replicated by the (embedded) trading strategy $\theta_\mathbb{T} = (\theta_t)_{t\in\mathbb{T}}$ in M^C. Hence, to avoid arbitrage opportunities, the price process $\pi(\theta)$ in M^C must fulfill $\pi_t(\theta) = \pi_t^F(\theta)$ $\mathbb{P}$-a.s. for any $t \in \mathbb{T}$. Since $\mathbf{E}_\mathbb{Q}[X|\mathcal{F}_t] = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_0]$ $\mathbb{P}$-a.s. for *any* random variable X in $(F, \mathbb{F})$, we must have $\mathbb{P}$-a.s.

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0]. \tag{8}$$

AXIOM 3. *There are infinitely many human individuals and we have*

$$(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B}) = \bigotimes_{i=1}^{\infty} (B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i), \tag{9}$$

where $B_H = \{(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i), i \in \mathbb{N}^+\}$ *is the set of probability spaces describing the state of the i-th individual* $(\mathbb{N}^+ := \mathbb{N} \setminus \{0\})$. *Each* $\mathcal{B}_0^i$ *is trivial.*

It follows that $\mathcal{B}_0$ is also trivial, i.e. $\mathcal{B}_0 = \{\emptyset, B\}$.

AXIOM 4. *For any space* $(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i)$ *in* B_H *there are infinitely many isomorphic (identical except for the index) ones in* B_H.

4 Life insurance contracts

By definition, the biometric development has no influence on the price process S of the financial market. A portfolio θ that contains technical risk - that is a portfolio which is not of the form $\theta(f, b) = {}_F\theta(f)$ a.s. with ${}_F\theta$ an M^F-portfolio

- cannot be replicated by purely financial products. Hence, relative pricing of life insurance products due to M^F is not possible. In general, life insurance policies are not traded and the possibility of a valuation of such contracts by the market is not given. The market M^C of financial and biometric risks is incomplete. Nonetheless, the products have to be priced as the insured usually have the right to dissolve any contract at any time of its duration. We are therefore in the need of a reasonable valuation principle π for the considered portfolios Θ of the market M^C and in particular for general life insurance products.

DEFINITION 4.1. *A general* **life insurance contract** *is a vector* $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ *of pairs* (γ_t, δ_t) *of t-portfolios out of* Θ*. For any* $t \in \mathbb{T}$*, the portfolio* γ_t *is interpreted as a payment of the insurer to the insurant* (**benefit**) *and* δ_t *as a payment of the insurant to the insurer* (**premium**)*, respectively taking place at t. The notation* $({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}}$ *means that the contract depends on the i-th individual's life, i.e. for all* $(f, x), (f, y) \in M$

$$({}^i\gamma_t(f, x), {}^i\delta_t(f, x))_{t\in\mathbb{T}} = ({}^i\gamma_t(f, y), {}^i\delta_t(f, y))_{t\in\mathbb{T}} \tag{10}$$

whenever $p^i(x) = p^i(y)$*,* p^i *being the projection from* B *onto* B^i*.*

For any contract $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ between a life insurance company and an individual, this stream of payments is from the viewpoint of the insurer equivalent to holding the portfolios $(\delta_t - \gamma_t)_{t\in\mathbb{T}}$. Even though that there has not been considered any particular valuation principle until now, it is assumed that a suitable principle π is a minimum fair price in the heuristic sense given in Section 2, Principle 7. The properties of a minimum fair price will be further developed in Section 6.

AXIOM 5. *Consider a suitable valuation principle* π *on* Θ*. For any life insurance contract* $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ *the* **Principle of Equivalence** *demands that*

$$\pi_0\left(\sum_{t=0}^{T} \gamma_t\right) = \pi_0\left(\sum_{t=0}^{T} \delta_t\right). \tag{11}$$

As already mentioned in Section 2 (Principle 8), the idea of equation (11) is that the liabilities $(\gamma_t)_{t\in\mathbb{T}}$ can somehow be hedged by working with the premiums $(\delta_t)_{t\in\mathbb{T}}$ since their present values are identical. For the classical case, this idea is explained in the section below.

5 Valuation I - The classical case

In classical life insurance mathematics, the financial market is assumed to be deterministic. We realize the assumption by $|\mathcal{F}_T| = 2$, i.e. $\mathcal{F}_T = \{\emptyset, F\}$, and identify $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$ with $(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$. As the market is assumed to be free of arbitrage, all assets must show the same dynamics. We therefore have $S = (S^0)$, i.e. $d = 1$ and the only asset is the money account as a deterministic function of time. In the classical framework, it is common sense that the fair present value of a $\mathbb{B}$-integrable payoff C_t at t is the (conditional) expectation of the discounted payoff, i.e. for a t-claim C_t/S_t^0 (cf. Definition 3.2) we have

$$\pi_s(C_t/S_t^0) := S_s^0 \cdot \mathbf{E}_{\mathbb{B}}[C_t/S_t^0|\mathcal{B}_s], \quad s \in \mathbb{T}. \tag{12}$$

Under the *Expectation Principle* (12), the well-known classical Principle of Equivalence is given by (11). As the discounted price processes are $\mathbb{B}$-martingales, the classical financial market together with a finite number of (classical) price processes of life insurance policies is free of arbitrage opportunities.

Let us have a closer look at the logic behind valuation principle (12). Assume that Θ is given by the $\mathbb{B}$-integrable portfolios. Suppose Axiom 1 to 3 and consider a set of portfolios $\{({}^i\gamma_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ where ${}^i\gamma_t$ depends on the i-th individual's life, only (cf. Definition 4.1). Suppose that for all $t \in \mathbb{T}$ there is a $c_t \in \mathbb{R}^+$ such that

$$||{}^i\gamma_t||_2 \leq c_t \tag{13}$$

for all $i \in \mathbb{N}^+$, where $||.||_2$ denotes the norm on the Hilbert space $L^2(M, \mathcal{M}_T, \mathbb{P})$. Now, buy for all $i \in \mathbb{N}^+$ and all $t \in \mathbb{T}$ the portfolios $-\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$, where $-\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$ is interpreted as a financial product (a t-portfolio) which matures at time t, i.e. the right on $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$ in cash at t is sold at 0. Consider the balance of wins and losses at time t. The mean total payoff at t for the first m contracts is given by

$$\frac{1}{m}\sum_{i=1}^{m}({}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) \cdot S_t^0. \tag{14}$$

Clearly, (14) converges $\mathbb{B}$-a.s. to 0 as we can apply the Strong Law of Large Numbers by Kolmogorov's Criterion. Furthermore, it follows directly from (12) that we have $\pi_0(-\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) = -\pi_0({}^i\gamma_t)$ for all $i \in \mathbb{N}^+$. Hence, in the classical case, the fair present value of any claim equals the price of a hedge at time 0 such that for an increasing number of independent claims the mean balance of claims and hedges converges to zero almost surely.

Now, consider the set of life insurance contracts $\{({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ with the deltas being defined in analogy to the gammas above. Since for the company a contract can be considered as a vector $({}^i\delta_t - {}^i\gamma_t)_{t\in\mathbb{T}}$ of portfolios, the analogous hedge is given by $(\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t] - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t])_{t\in\mathbb{T}}$. Under Axiom 5, the Equivalence Principle (11) states that the contract itself has value zero. From the Expectation Principle (12) we therefore obtain for all $i \in \mathbb{N}^+$

$$\sum_{t=0}^{T} \pi_0(\mathbf{E}_{\mathbb{B}}[{}^i\delta_t] - \mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) = \sum_{t=0}^{T} \pi_0({}^i\delta_t - {}^i\gamma_t) = 0. \tag{15}$$

Hence, under (12) and Axiom 1, 2, 3 and 5, a life insurance company can (without any costs at time 0) buy a hedge such that the mean balance per contract at any time t converges to zero almost surely for an increasing number of individual contracts:

$$\frac{1}{m}\sum_{i=1}^{m}({}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t] + \mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) \cdot S_t^0 \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{16}$$

As a direct consequence, the mean of the *final* balance converges, too:

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{t=0}^{T}({}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t] + \mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) \cdot S_T^0 \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{17}$$

REMARK 5.1. Roughly speaking, the Expectation Principle (12) implies that the price of any claim at least covers the costs of a purely financial hedge such that for an increasing number of independent claims the mean balance of claims and hedges converges to zero almost surely. The Equivalence Principle (11) induces that the hedge of any insurance contract costs nothing at time 0, which is important as the contract itself is for free, too (cf. equation (15)).

6 Valuation II - The general case

Before it comes to the topic of valuation in the general case, two technical lemmas have to be proven and some further notion has to be introduced.

Let the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be equipped with the usual Borel-σ-algebra and recall that a function g into $\overline{\mathbb{R}}$ is called *numeric*.

LEMMA 6.1. *Consider $n > 1$ measurable numeric functions g_1 to g_n on the product $(M, \mathcal{M}, \mathbb{P}) = (F, \mathcal{F}, \mathbb{F}) \otimes (B, \mathcal{B}, \mathbb{B})$ of two probability spaces. Then $g_1 = \ldots = g_n$ $\mathbb{P}$-a.s. if and only if $\mathbb{F}$-a.s. $g_1(f, .) = \ldots = g_n(f, .)$ $\mathbb{B}$-a.s.*

Proof. For any $Q \in \mathcal{M}$ it is well-known that $\mathbb{P}(Q) = \int \mathbb{B}(Q_f)d\mathbb{F}$, where $Q_f = \{b \in B : (f,b) \in Q\}$ and the function $\mathbb{B}(Q_f)$ on F is $\mathcal{F}$-measurable. As for $i \neq j$ the difference $g_{i,j} := g_i - g_j$ is measurable, the set $Q := \bigcap_{i \neq j} g_{i,j}^{-1}(0)$ is $\mathcal{M}$-measurable. Now, $g_1 = \ldots = g_n$ a.s. is equivalent to $\mathbb{P}(Q) = 1$ and this again is equivalent to $\mathbb{B}(Q_f) = 1$ $\mathbb{F}$-a.s. However, $\mathbb{B}(Q_f) = 1$ is equivalent to $g_1(f,.) = \ldots = g_n(f,.)$ $\mathbb{B}$-a.s. □

LEMMA 6.2. *Let $(g_n)_{n \in \mathbb{N}}$ and g be a sequence, respectively a function out of $L^0(M, \mathcal{M}, \mathbb{P})$, i.e. the real valued measurable functions on M, where $(M, \mathcal{M}, \mathbb{P})$ is as above. Then $g_n \to g$ $\mathbb{P}$-a.s. if and only if $\mathbb{F}$-a.s. $g_n(f,.) \to g(f,.)$ $\mathbb{B}$-a.s.*

Proof. The elements of $L^0(M, \mathcal{M}, \mathbb{P})$ are also numeric measurable functions. Now, recall that for any sequence of real numbers $(h_n)_{n \in \mathbb{N}}$ and any $h \in \mathbb{R}$ the property $h_n \to h$ is equivalent to $\limsup h_n = \liminf h_n = h$. As the limes superior and the limes inferior of a measurable numeric function always exist and are measurable, one obtains from Lemma 6.1 that

$$\limsup_{n \to \infty} g_n = \liminf_{n \to \infty} g_n = g \quad \mathbb{P}\text{-a.s.} \tag{18}$$

if and only if $\mathbb{F}$-a.s.

$$\limsup_{n \to \infty} g_n(f,.) = \liminf_{n \to \infty} g_n(f,.) = g(f,.) \quad \mathbb{B}\text{-a.s.} \tag{19}$$

□

As we have seen in Section 4, there is the need for a suitable set Θ of portfolios on which a particular valuation principle should work. Further, a mathematical precise description of what was called "similar" in Section 2, Principle 5 has to be introduced.

DEFINITION 6.3.

(*i*) *Define*

$$\Theta = (L^1(M, \mathcal{M}_T, \mathbb{P}))^d \tag{20}$$

and

$$\Theta^F = (L^1(F, \mathcal{F}_T, \mathbb{F}))^d, \tag{21}$$

where Θ^F can be interpreted as a subset of Θ by the usual embedding.

(*ii*) *A set* $\Theta' \subset \Theta$ *of portfolios in* M^C *is called independently identically distributed due to* $(B, \mathcal{B}_T, \mathbb{B})$*, abbreviated* **B-i.i.d.***, when for almost all* $f \in F$ *the random variables* $\{\theta(f,.) : \theta \in \Theta'\}$ *are i.i.d. on* $(B, \mathcal{B}_T, \mathbb{B})$*. Under Axiom 4, such sets exist and can be countably infinite.*

(*iii*) *Under the Axioms 1 to 3, a set* $\Theta' \subset \Theta$ *satisfies property* **(K)** *if for almost all* $f \in F$ *the elements of* $\{\theta(f,.) : \theta \in \Theta'\}$ *are stochastically independent on* $(B, \mathcal{B}_T, \mathbb{B})$ *and* $||\theta^j(f,.)||_2 < c(f) \in \mathbb{R}^+$ *for all* $\theta \in \Theta'$ *and all* $j \in \{0, \dots, d-1\}$*.*

Sets fulfilling property (B-i.i.d.) or (K) are indexed with the respective symbol. A discussion of the Kolmogorov-Criterion-like (K)-condition can be found below (Remark 7.5). The condition figures out to be quite weak.

Now, the remaining axioms which concern valuation can be stated. The next axiom is motivated by the demand that whenever the market with the original d securities with prices S is enlarged by a finite number of price processes $\pi(\theta)$ due to general portfolios $\theta \in \Theta$, the no-arbitrage condition (NA) should hold on the new market. This axiom corresponds to the sixth principle of section 2.

AXIOM 6. *For any* $t \in \mathbb{T}$ *and* $\theta \in \Theta$

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{M}}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t] \tag{22}$$

for a probability measure $\mathbb{M} \sim \mathbb{P}$*.*

The following axiom is due to the fifth and the seventh principle.

AXIOM 7. *Under the Axioms 1 - 4 and 6, the* **minimum fair price** π *on* Θ *is for any* $\theta \in \Theta$ *given by*

$$\pi_0(\theta) = \pi_0^F(H(\theta)) \tag{23}$$

where

$$H : \Theta \longrightarrow \Theta^F \tag{24}$$

is such that

(*i*) $H({}^1\theta) = H({}^2\theta)$ *for B-i.i.d. portfolios* ${}^1\theta$ *and* ${}^2\theta$*.*

(*ii*) *for t-portfolios* $\{{}^i\theta, i \in \mathbb{N}^+\}_{B-i.i.d.}$ *or* $\{{}^i\theta, i \in \mathbb{N}^+\}_K$ *one has*

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\theta - H({}^i\theta), S_t\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-}a.s. \tag{25}$$

Relation (24) means that the *hedge* $H(\theta)$ is a portfolio of the *financial* market. Recall, that the financial market M^F is complete and any portfolio features a self-financing replicating strategy. However, (24) also implies that the hedging strategy does not react on biometric events happening after time 0. Due to (i), as in the classical case, the *hedging method* H cannot distinguish between similar (B-i.i.d.) individuals (cf. Principle 5, Section 2). Property (ii) is also adopted from the classical case, where pointwise convergence is ensured by the Expectation Principle for appropriate insurance products combined with respective hedges (cf. Principle 7, Section 2 and Section 5). Property (ii) is also related to Principle 4 in Section 2 as insurance companies should be able to cope with large classes of similar contracts.

Now, the main result of this paper can be stated.

PROPOSITION 6.4. *Under the Axioms 1 - 4, 6 and 7, the minimum fair price* π *on* Θ *is uniquely determined by* $\mathbb{M} = \mathbb{Q}\otimes\mathbb{B}$, *i.e. for* $\theta\in\Theta$ *and* $t\in\mathbb{T}$

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle/S_T^0|\mathcal{F}_t\otimes\mathcal{B}_t]. \tag{26}$$

As already mentioned, this product measure approach is quite well established in the existing literature. However, the deduction by an axiomatic approach as well as the generality of the above result seem to be new.

Clearly, (12) is the special case of (26) in the presence of a deterministic financial market ($|\mathcal{F}_T| = 2$). As π is unique, it is at the same time the minimal valuation principle with the demanded properties. Actually, property (ii) of Axiom 7 ensures that insurance companies do not charge more than the costs of an acceptable purely financial hedge for each product which is sold. So to speak, the minimum fair price is fair from the viewpoint of the insured, as well as from the viewpoint of the companies.

The following lemmas are needed in order to prove the proposition.

LEMMA 6.5. *Under Axiom 1 and 2, one has for any* $\theta\in\Theta$

$$H^*(\theta) := \mathbf{E}_{\mathbb{B}}[\theta] \in \Theta^F. \tag{27}$$

There is a self-financing strategy replicating $H^*(\theta)$ and under Axiom 6

$$\pi_t(H^*(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0] \tag{28}$$

for $t \in \mathbb{T}$. If θ is a t-portfolio, then so is $H^*(\theta)$. Moreover, H^* fulfills properties (i) and (ii) of Axiom 7.

Proof. By Fubini's Theorem, $\mathbf{E}_{\mathbb{B}}[\theta(f,.)]$ exists $\mathbb{F}$-a.s. and is $\mathbb{F}$-integrable. Hence, by the completeness of M^F and uniqueness of $\mathbb{Q}$, the portfolio (27) can be hedged by the financial securities in M^F and has (due to Remark 3.3) the price process

$$\pi_t(\mathbf{E}_{\mathbb{B}}[\theta]) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}}[\langle \mathbf{E}_{\mathbb{B}}[\theta], S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0]. \tag{29}$$

Since $\mathbf{E}_{\mathbb{Q}}[\mathbf{E}_{\mathbb{B}}[X]|\mathcal{F}_t] = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_0]$ $\mathbb{P}$-a.s. for *any* $X \in \Theta$, (29) is identical to (28) $\mathbb{P}$-a.s. As we also have $\mathbf{E}_{\mathbb{B}}[\theta] = \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[\theta|\mathcal{F}_T \otimes \mathcal{B}_0]$ $\mathbb{P}$-a.s., $H^*(\theta)$ is a t-portfolio. Let us prove the last statement. Property (i) of Axiom 7 is obviously fulfilled. For any t-portfolios $\{^i\theta, i \in \mathbb{N}^+\}_K$ or $\{^i\theta, i \in \mathbb{N}^+\}_{B-i.i.d.}$, the Strong Law of Large Numbers (in the first case by Kolmogorov's Criterion) implies for almost all $f \in F$ that

$$\frac{1}{m}\sum_{i=1}^{m}\langle ^i\theta(f,.) - H^*(^i\theta)(f), S_t(f)\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{30}$$

Lemma 6.2 completes the proof. ☐

LEMMA 6.6. *Under Axiom 1 and 2, one obtains that for any $\theta \in \Theta$, any $t \in \mathbb{T}$ and for $\mathbb{M} \in \{\mathbb{F}\otimes\mathbb{B}, \mathbb{Q}\otimes\mathbb{B}\}$*

$$\mathbf{E}_{\mathbb{M}}[\langle\theta - H^*(\theta), S_t\rangle] = 0. \tag{31}$$

Proof. By Fubini's Theorem. ☐

LEMMA 6.7. *Under the Axioms 1 - 4 and 6, any $H : \Theta \to \Theta^F$ fulfilling (i) and (ii) of Axiom 7 fulfills for any θ out of some $\Theta_{B-i.i.d.}$*

$$\pi_t(H(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0], \quad t \in \mathbb{T}. \tag{32}$$

Roughly speaking, Lemma 6.7 states that there is no reasonable purely financial hedging method (for the relevant portfolios) with better convergence properties than (27) has. Even a hedging method with stronger than pointwise convergence, e.g. an additional L^p-convergence ($p \geq 1$), must follow (32) and has the same price process as (27).

Proof of Lemma 6.7. Consider to be given such an H as in Lemma 6.7 and a set $\{{}^i\theta, i \in \mathbb{N}^+\}_{B-i.i.d.}$ of t-portfolios that contains a given t-portfolio θ. From Lemma 6.2, one has $\mathbb{F}$-a.s.

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\theta(f,.) - H(\theta)(f), S_t(f)\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{33}$$

and by the Law of Large Numbers

$$\langle \mathbf{E}_{\mathbb{B}}[{}^i\theta(f,.)], S_t(f)\rangle = \langle H(\theta)(f), S_t(f)\rangle. \tag{34}$$

Condition (NA) and Remark 3.3 imply $\pi_t(H(\theta)) = \pi_t(\mathbf{E}_{\mathbb{B}}[{}^i\theta])$ for $i \in \mathbb{N}^+$ and $t \in \mathbb{T}$. Lemma 6.5 completes the proof. □

Proof of Proposition 6.4. From the Fundamental Theorem we know that $\mathbb{Q} \sim \mathbb{F}$. A direct consequence of Lemma 6.1 and the Radon-Nikodym Theorem is $\mathbb{Q}\otimes\mathbb{B} \sim \mathbb{F}\otimes\mathbb{B}$. From Lemma 6.5 we obtain that (26) exists. Hence, (26) fulfills Axiom 6. The same lemma implies that (26) is a minimum fair price in the sense of Axiom 7. Now, uniqueness will be shown. Suppose that π is a minimum fair price in the sense of Axiom 7 and consider some $\{{}^i\theta,\ i \in \mathbb{N}^+\}_{B-i.i.d.}$. From Lemma 6.7 it is known that $\pi_0({}^i\theta) = \pi_0(H^*({}^i\theta)) = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle {}^i\theta, S_T\rangle / S_T^0]$ for all $i \in \mathbb{N}^+$. One can surely choose a set $\{{}^i\theta,\ i \in \mathbb{N}^+\}_{B-i.i.d.}$ such that ${}^1\theta = (\mathbf{1}_Z, 0, \dots, 0)$, where $\mathbf{1}_Z$ is the indicator function of a cylinder set $Z = (F', B_1, B_2, \dots)$ with $F' \in \mathcal{F}_T$ and $B_j \in \mathcal{B}_T^j$ for $j \in \mathbb{N}^+$ where $B_j \neq B^j$ for only finitely many j (Axiom 4 is crucial for the possibility of this choice!). Clearly, these cylinders form a $\cap$-stable generator for $\mathcal{M}_T$, the σ-algebra of the product space, and M itself is an element of this generator. One obtains $\pi_0({}^1\theta) = \mathbb{Q}\otimes\mathbb{B}(Z) = \mathbb{M}(Z)$ from (28) and (22). $\mathbb{M} = \mathbb{Q}\otimes\mathbb{B}$ follows from the coincidence of the measures on the generator. □

Axiom 7 (together with 6) could be interpreted as a *strong no-arbitrage principle* that fulfills (NA) and also precludes arbitrage-like strategies that have their origin in the Law of Large Numbers.

EXAMPLE 6.8 (Arbitrage-like trading opportunities). Consider a set $\{{}^i\theta, i \in \mathbb{N}^+\}_{B-i.i.d.}$ of portfolios. The minimum fair price for each portfolio is given by (26) ($t = 0$). If an insurance company sells the products $\{{}^1\theta, \dots, {}^m\theta\}$ at that prices, it can buy hedging portfolios such that the mean balance converges to zero almost surely with m (cf. Axiom 7, (*ii*)). However, if the company charges $\pi_0({}^i\theta) + \epsilon$, where $\epsilon > 0$ is an additional fee and π is as in (26), there still is the hedge as explained above, but the gain ϵ per contract was made at $t = 0$. Hence, the safety load ϵ makes in the limit a deterministic money making machine out of the insurance company.

Example 6.8 directly points at the main difference between pricing in life insurance mathematics and financial mathematics. On financial markets such arbitrage-like strategies are not possible as there usually are not enough independent stocks. Furthermore, the stochastic behaviour of securities is by far not as good known as the stochastics of biometric events. Indeed, practitioners say that the probabilities from the biometric probability space are almost known for sure. Hence, biometric expectations can be computed with high accuracy whereas expectations on financial markets have the character of speculation. From this point of view, any possible EMM $\mathbb{M}'$ on the market M^C obtained by the free trading of portfolios in M^C should be expected to be close to $\mathbb{Q} \otimes \mathbb{B}$. Any systematic deviation could give rise to arbitrage-like trading opportunities, as we have seen above.

REMARK 6.9 (Quadratic hedging). Consider an L^2-framework, i.e. the payoff $\langle \theta_t, S_t \rangle$ of any considered t-portfolio θ_t lies in $L^2(M, \mathcal{M}_t, \mathbb{P})$. As $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$, it can easily be shown that $\mathbf{E}_{\mathbb{B}}[.]$ is the orthogonal projection of $L^2(M, \mathcal{M}_t, \mathbb{P})$ onto its (purely financial) subspace $L^2(F, \mathcal{F}_t, \mathbb{F})$. Standard Hilbert space theory implies that the payoff $\langle \mathbf{E}_{\mathbb{B}}[\theta_t], S_t \rangle = \mathbf{E}_{\mathbb{B}}[\langle \theta_t, S_t \rangle]$ of the hedge $H^*(\theta_t)$ is the best L^2-approximation of the payoff $\langle \theta_t, S_t \rangle$ of the portfolio θ_t by a purely financial portfolio out of M^F. Further, it can easily be shown that $\mathbb{M} = \mathbb{Q} \otimes \mathbb{B}$ minimizes $||d\mathbb{M}/d\mathbb{P} - 1||_2$ under the constraint $\mathbf{E}_{\mathbb{B}}[d\mathbb{M}/d\mathbb{P}] = d\mathbb{Q}/d\mathbb{F}$. Under some additional technical assumptions, this property is a characterization of the so-called *minimal martingale measure* in the time continuous case (cf. Schweizer, 1995; Møller, 2001). Hence, $\mathbb{Q} \otimes \mathbb{B}$ can be interpreted as the EMM which lies "next" to $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$ due to the L^2-metric. Beside the convergence properties discussed in this paper, these are the most important (and "natural") reasons for the use of (26). The hedging method H^* considered in this paper is not really the so-called *mean-variance hedge* as it is known from the literature (cf. Bouleau and Lamberton, 1989; Duffie and Richardson, 1991). The difference is that the mean-variance approach generally allows for *all* self-financing trading strategies in M^C, i.e. also biometric events can have influence on the strategy in this case. However, the ideas are of course quite similar. An overview concerning hedging approaches in insurance can be found in Møller (2002).

7 Hedging

In this section, it is shown under quite weak assumptions that a big insurance company is able to hedge in the mean almost all of its risk by products of the financial market.

Suppose Axiom 1 to 4 and a not further specified valuation principle π on Θ (Θ is defined as in (20)). Consider a set of life insurance contracts $\{({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ with $\{{}^i\gamma_t : i \in \mathbb{N}^+\}_K$ and $\{{}^i\delta_t : i \in \mathbb{N}^+\}_K$ for all $t \in \mathbb{T}$. Following hedging method H^* of Lemma (6.5), the portfolios (or strategies replicating) $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$ and $-\mathbf{E}_{\mathbb{B}}[{}^i\delta_t]$ are bought at time 0 for all $i \in \mathbb{N}^+$ and all $t \in \mathbb{T}$. Consider the balance of wins and losses at any time $t \in \mathbb{T}$. For the *mean total payoff per contract at time* t we have

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t - {}^i\gamma_t], S_t\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{35}$$

due to Lemma 6.5. In analogy to Section 5, also the mean *final* balance converges to zero a.s., i.e.

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{t=0}^{T}\langle {}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t - {}^i\gamma_t], S_T\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{36}$$

This kind of risk management is static in the sense that no trading strategy reacts on biometric events happening after time 0. This corresponds to the considerations in the classical case which have taken place in Section 5. It was already mentioned in Remark 6.9 that the considered hedging method is not exactly the so-called mean-variance hedging. Another (more comprehensive, but not self-financing) hedging approach is given by so-called *risk-minimizing strategies* (e.g. Møller, 1998 and 2001).

REMARK 7.1. Due to Lemma 6.6, any of the balances in (35) and (36) has expectation 0 under the physical probability measure $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$.

Until now, premium calculation has not played any role in this section. However, if the Principle of Equivalence (11) is applied under the minimum fair price (26), one obtains for all $i \in \mathbb{N}^+$

$$\sum_{t=0}^{T}\pi_0(\mathbf{E}_{\mathbb{B}}[-{}^i\delta_t + {}^i\gamma_t]) = \sum_{t=0}^{T}\pi_0({}^i\delta_t - {}^i\gamma_t) = 0. \tag{37}$$

REMARK 7.2. Under (11) and (26), a life insurance company can without any costs at time 0 (!) pursue a trading strategy such that the mean balance per contract at any time t converges to zero almost surely for an increasing number of individual contracts.

Remark 7.2 is perhaps the result with the strongest practical impact. In contrast to other, more comprehensive hedging methods, the presented method has the advantage that there is no need for the hedger to take into account the biometric development of each individual. The information available at the time of contract underwriting ($t = 0$) is sufficient and all strategies are self-financing.

EXAMPLE 7.3 (Traditional contracts with stochastic interest rates). Consider a life insurance contract which is for the i-th individual given by two cashflows $({}^i\gamma_t)_{t\in\mathbb{T}} = (\frac{{}^iC_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ and $({}^i\delta_t)_{t\in\mathbb{T}} = (\frac{{}^iD_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ with $\mathbb{T} = \{0, 1, \ldots, T\}$ in years. Assume that each ${}^iC_t$ is given by ${}^iC_t(f, b) = {}^ic\,{}^i\beta_t^\gamma(b^i)$ for all $(f, b) \in M$ where ${}^ic$ is a positive constant. Let $({}^i\delta_t)_{t\in\mathbb{T}}$ be defined analogously with the variables ${}^iD_t, {}^id$ and ${}^i\beta_t^\delta$. Suppose that ${}^i\beta_t^{\gamma(\delta)}$ is $\mathcal{B}_t^i$-measurable with ${}^i\beta_t^{\gamma(\delta)} \in \{0, 1\}$ for all $b^i \in B^i$. The portfolio e_0/S_t^0 can be interpreted as the guaranteed payoff of one currency unit at time t. This kind of contract is called a *zero-coupon bond with maturity* t and its price at time $s < t$ is denoted by $p(s, t - s) = \pi_s(e_0/S_t^0)$ where $t - s$ is the time to maturity and $p(s, 0) := 1$ for all $s \in \mathbb{T}$.

1. Traditional life insurance. Suppose that ${}^i\beta_t^\gamma = 1$ if and only if the i-th individual has died in $(t - 1, t]$ and for $t < T$ that ${}^i\beta_t^\delta = 1$ if and only if the i-th individual is still alive at t, but ${}^i\beta_T^\delta \equiv 0$. Assume that i is alive at $t = 0$. Clearly, this contract is a life insurance with fixed annual premiums ${}^id$ and the benefit ${}^ic$ in the case of death. $\mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\gamma]$ and $\mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\delta]$ are mortality, respectively survival probabilities. This data can be obtained from so-called mortality tables. Usually, the notation is ${}_{t-1|1}q_x = \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\gamma]$ $(t > 0)$ and ${}_tp_x = \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\delta]$ $(0 < t < T)$ for an individual of age x (cf. Gerber, 1997; for convenience reasons, the notation ${}_{-1|1}q_x = 0$ and ${}_0p_x = 1$ is used in the following). By Fubini's Theorem, the hedge H^* for ${}^i\delta_t - {}^i\gamma_t$ is for $t < T$ given by the number of $({}^ic\,{}_{t-1|1}q_x - {}^id\,{}_tp_x)$ zero-coupon bonds with maturity t, and for $t = T$ by ${}^ic\,{}_{T-1|1}q_x$ zero-coupon bonds with maturity T.

2. Endowment. Assume for $t < T$ that ${}^i\beta_t^\gamma = 1$ if and only if the i-th individual has died in $(t - 1, t]$, but ${}^i\beta_T^\gamma = 1$ if and only if i has died in $(T - 1, T]$ or is still alive at T. Further, ${}^i\beta_t^\delta = 1$ if and only if the i-th individual is still alive at $t < T$, but ${}^i\beta^\delta_T \equiv 0$. Assume that i is alive at $t = 0$. This contract is a so-called endowment that features fixed annual premiums ${}^id$ and the benefit ${}^ic$ in the case of death, but also the payoff ${}^ic$ when i is alive at T. The hedge H^* due to ${}^i\delta_t - {}^i\gamma_t$ is for $t < T$ given by the number of $({}^ic\,{}_{t-1|1}q_x - {}^id\,{}_tp_x)$ zero-coupon bonds with maturity t, and for $t = T$ by ${}^ic\,({}_{T-1|1}q_x + {}_Tp_x)$ zero-coupon bonds with maturity T.

Actually, in the case of traditional contracts, all hedging can be done by zero-coupon bonds (which is also called *matching*).

EXAMPLE 7.4 (Unit-linked products). The case of a unit-linked product is interesting if and only if the product is not the sum of a traditional life insurance contract and a simple funds policy (which is often the case in practice). So, let us assume that the contract is given by a cashflow of constant premiums $({}^i\delta_t)_{t\in\mathbb{T}}$ as in Example 7.3 and a flow of benefits $({}^i\gamma_t)_{t\in\mathbb{T}}$ such that ${}^i\gamma_t(f,b) = {}^i\theta_t \cdot {}^ic\,{}^i\beta^\gamma_t(b^i)$ for all $(f,b) \in M$ where ${}^i\theta_t \in \Theta^F$ is an arbitrary financial t-portfolio and all other notations are the same as in the introduction of Example 7.3. For instance, one could consider a number of shares of an index, or a number of assets together with the respective European Puts which ensure a certain level of benefit (i.e. a "unit-linked product with guarantee"). The strategy due to ${}^i\delta_t - {}^i\gamma_t$ is given by ${}^ic \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta^\gamma_t]$ times the replicating strategy of ${}^i\theta_t$ minus $({}^id \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta^\delta_t])$ zero-coupon bonds maturing at time t. In particular, for ${}^i\theta_t$ being a constant portfolio, the strategy is obviously very simple as the portfolio must not be replicated, but can be bought directly.

REMARK 7.5. The technical assumption (K) which is sufficient for the convergence of (35) (cf. Definition 6.3 (iii)) and which is demanded at the very beginning of the section will be discussed now. In the case of traditional life insurances as in Example 7.3, the realistic condition ${}^ic, {}^id \le c \in R^+$ for all $i \in \mathbb{N}^+$ implies (K). In the case of unit-linked products, suppose that there are only finitely many possible portfolios ${}^i\theta_t$ for each $t \in \mathbb{T}$ (which is also quite realistic as often shares of one single funds are considered). Under this assumption, again ${}^ic, {}^id \le c \in R^+$ for all $i \in \mathbb{N}^+$ implies (K). Hence, (K) is no drawback for practical purposes.

8 Incomplete financial markets

Until now, the theory presented in this paper assumed complete and arbitrage-free markets (cf. Axiom 2), which reduces the number of explicit market models that can be considered. However, some of the concepts work (under some restrictions) with incomplete market models.

In particular, it is now assumed that in Axiom 2 completeness of the market model M^F and uniqueness of the EMM $\mathbb{Q}$ is *not* demanded. Let us enumerate the altered axiom by 2' and define

$$\Theta^F = \{\theta : \theta \text{ replicable by a self-financing strategy in } M^F\} \tag{38}$$

$$\Theta = \{\theta : \theta \text{ is an } M^C\text{-portfolio and } \mathbf{E}_{\mathbb{B}}[\theta] \in \Theta^F\}. \tag{39}$$

It is well-known from the theory of financial markets that *any* EMM $\mathbb{Q}$ fulfills pricing formula (3) for any replicable portfolio $\theta \in \Theta^F$. Now, with Θ^F and Θ as defined above and Axiom 2 replaced by 2', it can easily be checked that the Lemmas 6.5, 6.6 and 6.7 still hold. Concerning Proposition 6.4, π as defined in (26) is for *any* financial EMM $\mathbb{Q}$ a minimum fair price. Hence, uniqueness gets lost. However, for any minimum fair price one still has that π_0 is unique (on the new set Θ). The reason is that for any $\theta \in \Theta$ and any two EMM $\mathbb{Q}$ and $\underline{\mathbb{Q}}$ of M^F

$$\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0] = \mathbf{E}_{\underline{\mathbb{Q}}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0] \tag{40}$$

due to Fubini's Theorem and the (NA)-condition. Hence, pricing at time $t = 0$ (i.e. present values) and hedging (cf. Section 7) still work as in the case of complete financial markets.

In the presence of arbitrage opportunities, the existence of an equivalent martingale measure gets lost. Nonetheless, assume a financial market model M^F which is neither necessarily arbitrage-free, nor complete and suppose that there is a valuation principle π^F used in M^F on a set Θ^F of purely financial portfolios which are taken into consideration (this does not mean absence of arbitrage). Under the considered Θ^F, define Θ by (39) and for any $\theta \in \Theta$

$$\pi_0(\theta) = \pi_0^F(\mathbf{E}_{\mathbb{B}}[\theta]), \tag{41}$$

which is the price of the hedge H^* at time 0 (compare with (23) and (28) for t = 0). In a L^2-framework as in Remark 6.9, i.e. if we have for any t that $H_t(\Theta^F) = \langle\Theta^F, S_t\rangle$ is a closed subspace of $L^2(F, \mathcal{F}_T, \mathbb{F})$, the operator $\mathbf{E}_{\mathbb{B}}[.]$ is again the orthogonal projection of the subspace $H_t(\Theta) = \langle\Theta, S_t\rangle$ of $L^2(M, \mathcal{M}_T, \mathbb{P})$ onto its (purely financial) subspace $H_t(\Theta^F)$. Thus, $\mathbf{E}_{\mathbb{B}}[\theta]$ is the best approximation to any $\theta \in \Theta$ in the L^2-sense (cf. Remark 6.9). Even if we do not assume the L^2-framework, the properties (i) and (ii) of Axiom 7 are still fulfilled for the above defined Θ and for H^* as in (27). Hence, π_0 satisfies the demand for converging balances as stated in Principle 7 of Section 2 and the expressions (35) and (36) are still valid. For these two reasons, (41) is a quite sensible valuation principle.

9 Historical pricing example

Let us consider the traditional contracts as described in Example 7.3. Due to the Equivalence Principle (11), we demand

$$\pi_0\left(\sum_{t=0}^{T} {}^i c\, {}^i\beta_t^\gamma e_0/S_t^0\right) = \pi_0\left(\sum_{t=0}^{T} {}^i d\, {}^i\beta_t^\delta e_0/S_t^0\right). \tag{42}$$

Now, suppose that the minimum fair price π from (26), respectively the valuation principle (41), is applied for premium calculation. Clearly,

$$\frac{{}^i d}{{}^i c} = \sum_{t=0}^{T} p(0,t)\cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\gamma] \Big/ \sum_{t=0}^{T} p(0,t)\cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\delta] \tag{43}$$

where $p(0,t)$ is the price of a zero-coupon bond as defined in Section 7. An important consequence of (43) is that the quotient ${}^i d/{}^i c$ (minimum fair premium/benefit) depends on the zero-coupon bond prices (or yield curve) at time 0. As the term structure of interest rates indeed varies from day to day, this in particular means that ${}^i d/{}^i c$ varies from day to day and therefore depends on the day of underwriting (actually, it depends on the exact time). Insurance companies do not determine the prices for products daily. Hence, they give rise to financial risks as the contracts may be over-valued.

Now, assume that any time value is given in fractions of years. The so-called *spot (interest) rate* $R(t,\tau)$ for the time interval $[t, t+\tau]$ is defined by

$$R(t,\tau) = -\frac{\log p(t,\tau)}{\tau}. \tag{44}$$

The *short rate* $r(t)$ at t is defined by $r(t) = \lim_{\tau\to 0} R(t,\tau)$, where the limit is assumed to exist. The *yield curve* at time t is the mapping with $\tau \mapsto R(t, t+\tau)$ for $\tau > 0$ and $0 \mapsto r(t)$. Figure 5 on page 32 shows the historical yield structure (i.e. the set of yield curves) of the German debt securities market from September 1972 to February 2003 (the 366 values are taken from the end of each month). The maturities' range is 0 to 28 years. The values for $\tau > 0$ were computed via a parametric presentation of yield curves (the so-called Svensson-method; cf. Schich, 1997) for which the parameters can be taken from the internet page of the German Federal Reserve (*Deutsche Bundesbank*; `http://www.bundesbank.de`). The implied Bundesbank values R' are estimates of *discrete* interest rates on notional zero-coupon bonds based on German Federal bonds and treasuries (cf. Schich, 1997) and have to be converted to continuous interest rates (as implicitly used in (44)) by $R = \ln(1+R')$. As an approximation for the short rate, the day-to-day money rates from the Frankfurt market (*Monatsdurchschnitt des Geldmarktsatzes für Tagesgeld am Frankfurter Bankplatz*; also available at the Bundesbank homepage) are taken and converted into continuous rates.

Equation (44) shows that interest rates (yields) and zero-coupon bond prices contain the same information, namely the present value of a non-defaultable future payoff. As there is a yield curve given for any time t of the historical time axis, it is possible to compute the historical value of ${}^i d/{}^i c$ for any t via (44). Doing so, one obtains

$$\frac{{}^i d}{{}^i c}(t) = \sum_{\tau=0}^{T} p(t,\tau)\, {}_{\tau-1|1}q_x(t) \Big/ \sum_{\tau=0}^{T-1} p(t,\tau)\, {}_{\tau}p_x(t) \tag{45}$$

for the traditional life insurance and

$$\frac{{}^i d}{{}^i c}(t) = \left(p(t,T)\, {}_{T}p_x(t) + \sum_{\tau=0}^{T} p(t,\tau)\, {}_{\tau-1|1}q_x(t) \right) \Big/ \sum_{\tau=0}^{T-1} p(t,\tau)\, {}_{\tau}p_x(t) \tag{46}$$

for the endowment. In this example, the values ${}_{\tau-1|1}q_x$ ($\tau > 0$) and ${}_{\tau}p_x$ ($0 < \tau < T$) are taken from (or computed by) the DAV (*Deutsche Aktuarvereinigung*) mortality table "1994 T" (Loebus, 1994), the value ${}_{T}p_x$ is computed by the table "1994 R" (Schmithals and Schütz, 1995). The reason for the different tables is that in actuarial practice mortality tables contain safety loads which depend on whether the death of a person is in (financial) favour of the insurance company, or not. All probabilities mentioned above are considered to be constant in time. Especially, to make things easier, there is no "aging shift" applied to table "1994 R".

Now, consider a man of age $x = 30$ years and the time axis $\mathbb{T} = \{0, 1, \ldots, 10\}$ (in years). In Figure 1, the rescaled quotients (45) and (46) are plotted for the above setup. For comparison reasons: the absolute values at the starting point (September 1972) are ${}^i d/{}^i c = 0.063792$ for the endowment, respectively ${}^i d/{}^i c = 0.001587$ for the life insurance. The plot nicely shows the dynamics of the quotients and hence of the minimum fair premiums ${}^i d$ if the benefit ${}^i c$ is assumed to be constant. The premiums of the endowment seem to be much more subject to the fluctuations of the interest rates than the premiums of the traditional life insurance. For instance, the minimum fair annual premium ${}^i d$ for the 10-years endowment with a benefit of ${}^i c = 100,000$ Euros was 5,285.55 Euros at the 31st July 1974 and 8,072.26 at the 31st January 1999. For the traditional life insurance (with the same benefit), one obtains ${}^i d = 152.46$ Euros at the 31st July 1974 and 168.11 at the 31st January 1999.

If one assumes a discrete technical rate of interest R'_{tech}, e.g. 0.035, which is the legally guaranteed rate of interest by German life insurers, one can compute "technical" quotients ${}^i d_{\text{tech}}/{}^i c$ by computing the "technical" values of zero-coupon bonds, i.e. $p_{\text{tech}}(t,\tau) = (1+R'_{\text{tech}})^{-\tau}$, and plugging them into (45), resp. (46). If a life insurance company charges the "technical" premiums ${}^i d_{\text{tech}}$ instead of the minimum fair premiums ${}^i d$ and if one considers the valuation principle (26), respectively (41), to be a reasonable choice, the *present value* of the considered insurance contract is

$$ {}^i V = ({}^i d_{\text{tech}} - {}^i d) \cdot \sum_{\tau=0}^{T-1} p(t,\tau)\, {}_\tau p_x(t) \tag{47} $$

due to the Principle of Equivalence, respectively (42). In particular, this means that the insurance company can book the gain or loss (47) in the limit, mean or expectation (cf. Example 6.8 and Remark 7.1) at time 0 as long as proper risk management (as described in Section 7) takes place afterwards. Thus, the present value (47) is a measure for the profit, or simply *the expected discounted profit* of the considered contract. Figure 2 shows the historical development of ${}^i V/{}^i c$ (present value/benefit) for the 10-years endowment as described above (solid line). For instance, the present value ${}^i V$ of a 10-years endowment with a benefit of ${}^i c = 100,000$ Euros was 20,398.70 Euros at July 31, 1974. At the 31st January 1999, it was worth 2,578.55 Euros, only. The situation gets even worse in the case of a technical (or promised) rate of interest $R'_{\text{tech}} = 0.050$ (dashed line) - which is quite little in contrast to formerly promised returns of e.g. German life insurers. At the 31st January 1999, such a contract was worth -3,141.95 Euros, i.e. the contract actually produced a loss in the mean. More recent values from February 28, 2003 are 4,592.69 Euros for a technical interest of 0.035 and -1,127.39 Euros in the other case. Some present values of the 10-years traditional life insurance can be found in Table 1 on page 29.

All computations from above have also been carried out for a 25-years endowment, respectively life insurance (cf. Table 1). The corresponding figures are 3 and 4. Concerning Figure 3, the absolute values at the starting point (September 1972) are ${}^i d/{}^i c = 0.013893$ for the endowment, respectively ${}^i d/{}^i c = 0.002553$ for the life insurance. The minimum fair premium ${}^i d$ for the 25-years endowment with benefit ${}^i c = 100,000$ Euros was 808.39 Euros at the 31st July 1974 and 2,177.32 Euros at the 31st January 1999. For the traditional life insurance (with the same benefit), one obtains ${}^i d = 216.37$ Euros at the 31st July 1974 and 303.90 at the 31st January 1999. Hence, the premium-to-benefit ratio for both types of contracts seems to be more dependent on the yield structure than in the 10-years case. However, compared to the 10-years contracts, the longer running time seems to stabilize the present values of the contracts (cf. Table 1 and Figure 4). Nonetheless, they are still strongly depending on the yield structure.

The examples have shown the importance of realistic valuation principles in life insurance. Any premium calculation method and all related parameters (like e.g. technical rates of interest, which have to be determined in some way) should be carefully examined in order to be properly prepared for the fluctuations of financial markets. There is no doubt that many of the financial problems of life insurance companies that have arisen in the past few years could have been avoided by a proper use of modern valuation principles and - perhaps even more important - modern financial hedging strategies.

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A Figures and tables

Date	1974/07/31	1999/01/31
Traditional life insurance: 10 years		
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	168.94	
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	165.45	
Minimum fair annual premium ${}^i d$	152.46	168.11
Present value ${}^i V$ ($R'_{\text{tech}} = 0.035$)	108.90	7.17
Present value ${}^i V$ ($R'_{\text{tech}} = 0.050$)	85.84	-22.80
Traditional life insurance: 25 years		
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	328.02	
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	303.27	
Minimum fair annual premium ${}^i d$	216.37	303.90
Present value ${}^i V$ ($R'_{\text{tech}} = 0.035$)	1,009.56	376.84
Present value ${}^i V$ ($R'_{\text{tech}} = 0.050$)	785.80	-9.83
Endowment: 10 years		
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	8,372.65	
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	7,706.24	
Minimum fair annual premium ${}^i d$	5,285.55	8,072.26
Present value ${}^i V$ ($R'_{\text{tech}} = 0.035$)	20,398.70	2,578.55
Present value ${}^i V$ ($R'_{\text{tech}} = 0.050$)	15,995.27	-3,141.95
Endowment: 25 years		
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	2,760.85	
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	2,255.93	
Minimum fair annual premium ${}^i d$	808.39	2,177.32
Present value ${}^i V$ ($R'_{\text{tech}} = 0.035$)	17,655.42	9,118.39
Present value ${}^i V$ ($R'_{\text{tech}} = 0.050$)	13,089.53	1,228.34

Table 1: Selected (extreme) values due to different contracts for a 30 year old man (fixed benefit: ${}^i c = 100,000$ Euros)

Figure 1: Rescaled plot of the quotient ${}^{i}d/{}^{i}c$ (minimum fair annual premium/benefit) for the 10-years endowment (solid), resp. life insurance (dashed), for a 30 year old man

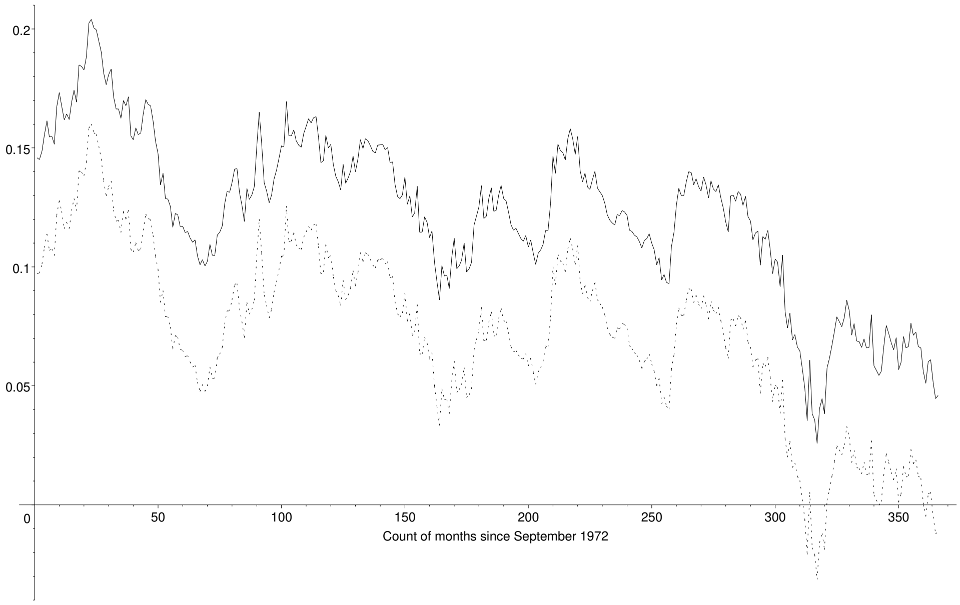

Figure 2: ${}^{i}V/{}^{i}c$ (present value/benefit) for the 10-years endowment under a technical interest rate of 0.035 (solid) and 0.050 (dashed)

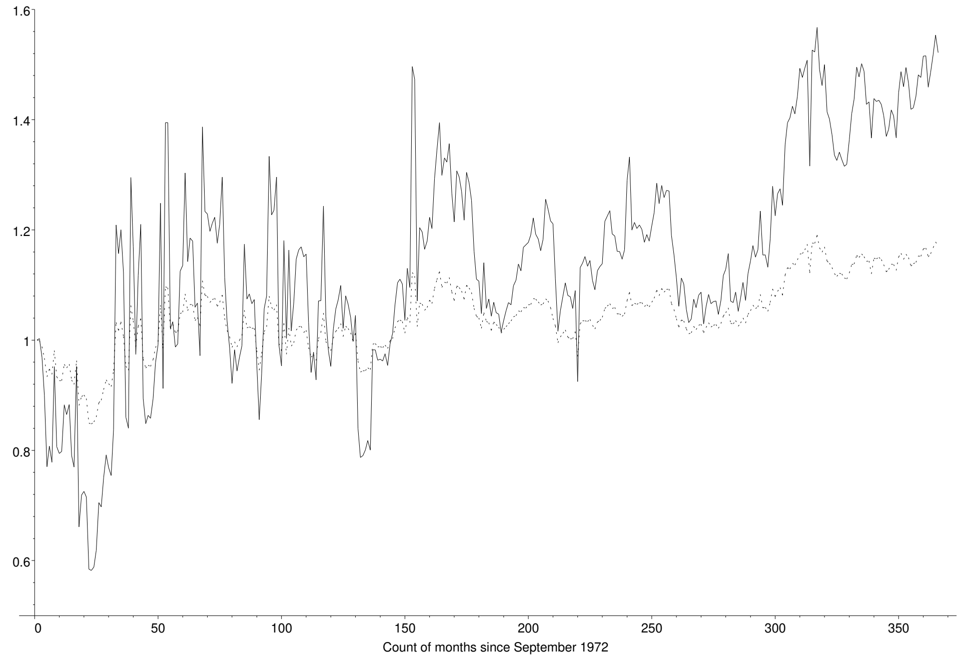

Figure 3: Rescaled plot of the quotient ${}^i d / {}^i c$ (minimum fair annual premium/benefit) for the 25-years endowment (solid), resp. life insurance (dashed), for a 30 year old man

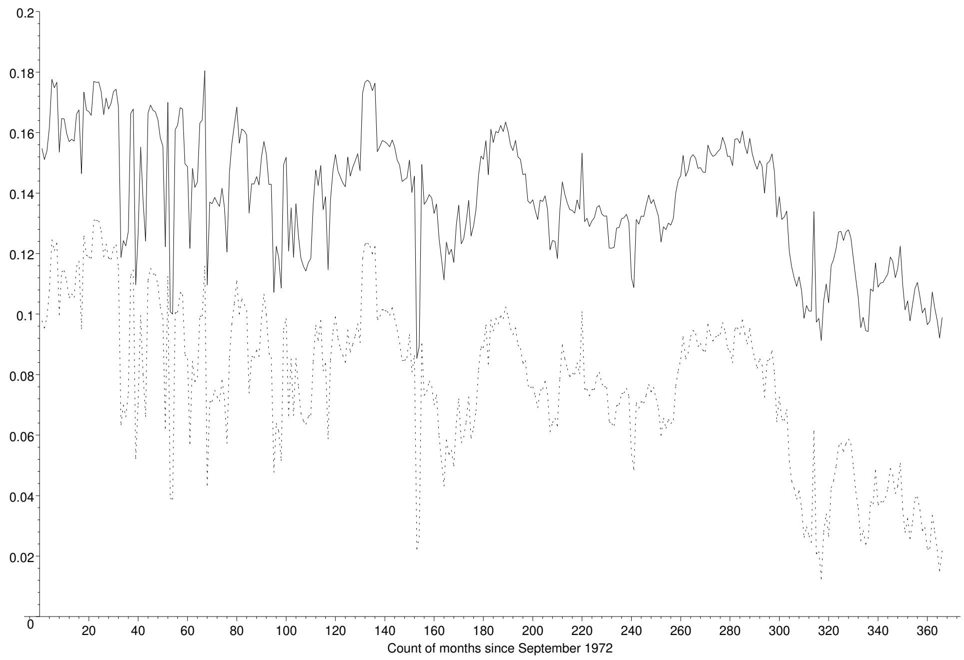

Figure 4: ${}^i V / {}^i c$ (present value/benefit) for the 25-years endowment under a technical interest rate of 0.035 (solid) and 0.050 (dashed)

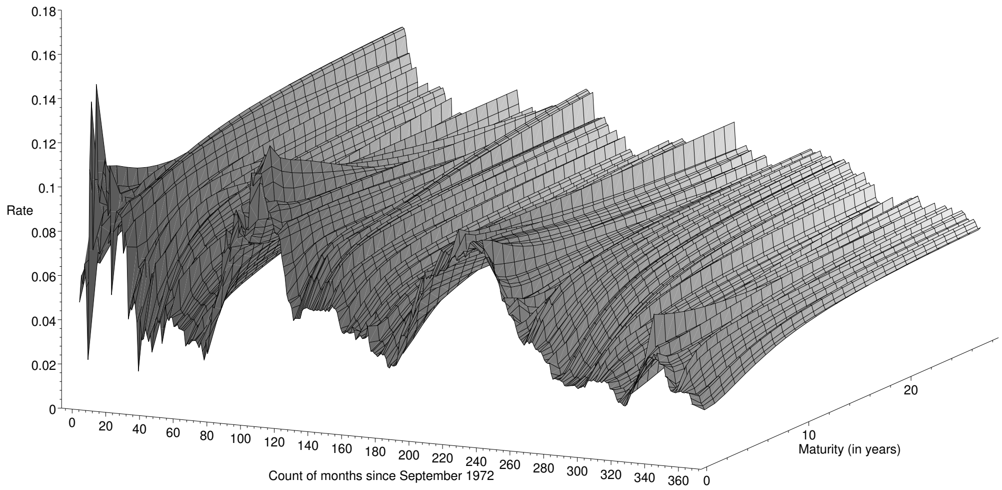

Figure 5: Historical yields of the German debt securities market