Weak Phan systems of type C_n

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Abstract

The present article provides a classification of non-collapsing Phan amalgams of type C_n and of shape $S \supseteq S_2$ over finite fields of odd square order. Together with the results of [4] this completes the proof of a Phan-type theorem for the group $Sp_{2n}(q)$ for sufficiently large n and q.

1 Introduction

The modern approach to Phan-type theorems, as outlined in [1], falls into two parts. On one hand one has to prove the simple connectedness of some suitable geometry, on the other hand one has to classify related amalgams. The so-called flipflop geometry Γ of type C_n over finite fields of square order has been introduced and studied in [4]. The Main Theorem of that paper states that this flipflop geometry is 2-simply connected for $n \geq 3$ and $q \geq 8$. By Tits' lemma (Corollaire 1 of [7]) this implies that the amalgam consisting of the rank-1- and rank-2-parabolics of some flag-transitive group G of automorphisms of Γ admits G as its universal completion. We refer to [4] for details; see also Section 2 for a definition of Γ and a short summary of the setting.

The purpose of the present paper is to classify the non-collapsing Phan amalgams of type C_n and of shape $S \supseteq S_2$, which is achieved in the following theorem. For definitions we again refer to Section 2.

Theorem 1
Let $n \geq 3$, let q be odd, and let $\mathcal{A}$ be a non-collapsing Phan amalgam of type C_n and of shape $S \supseteq S_2$ over $\mathbb{F}_{q^2}$. Then the unique unambiguous covering $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is isomorphic to the standard Phan amalgam $\hat{\mathcal{A}}_S$.

The above theorem constitutes the final step of the proof of the Phan-type Theorem 2. The latter is an immediate consequence of Theorem 1 and the results of [4], especially Part (1) of Theorem 2.

Theorem 2
Let $n \geq 3$, let $q \geq 8$, and let G be a group that admits a weak Phan system of type C_n over $\mathbb{F}_{q^2}$. Then G is a homomorphic image of $Sp_{2n}(q)$ (the universal Chevalley group of type $C_n(q)$).

The bound on q in Theorem 2 comes from the results of [4]. The particular proof of 2-simple connectedness fails for $q \leq 7$. Currently it is only known that Γ is not 2-simply connected for $q = 2$; nothing is known for $3 \leq q \leq 7$.

This article is organized as follows. In Section 2 we provide all necessary definitions and a number of results on Phan systems and Phan amalgams. In Section 3 we give the proof of Theorem 1.

Remark: Analogues of Theorem 1 should hold for any non-collapsing Phan amalgam of type Δ (and level two), where Δ is an arbitrary Dynkin diagram admitting single and double edges.

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2 Phan systems and Phan amalgams

Let $G = Sp_{2n}(q^2)$ and let V be the natural symplectic space for G. Following Section 3 of [4], denote by $(\cdot,\cdot)$ the corresponding alternating form. Let the bar denote the involutory field automorphism of $\mathbb{F}_{q^2}$ and let σ be a flip, i.e., $(\lambda v)^\sigma = \bar{\lambda} v^\sigma$, $(u^\sigma, v^\sigma) = \overline{(u,v)}$, and $\sigma^2 = -\mathrm{id}$. Then, by Proposition 3.1 of [4], we can find a canonical basis $\mathcal{B} = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ satisfying $(e_i, e_j) = (f_i, f_j) = 0$, $(e_i, f_j) = \delta_{ij}$ and $e_i^\sigma = f_i$, $f_i^\sigma = -e_i$ for $1 \leq i, j \leq n$. The centralizer G_σ of σ in G is isomorphic to $Sp_{2n}(q)$ (Proposition 3.8 of [4]) and acts flag-transitively on the geometry Γ which consists of the subspaces U of V that are totally isotropic with respect to $(\cdot,\cdot)$ and that satisfy $V = U^\perp \oplus U^\sigma$ (Proposition 4.2 of [4]).

Let $\mathcal{B} = \{e_1, f_1, \ldots, e_n, f_n\}$ be a canonical basis of V and let F be the flag $\langle e_1, f_1\rangle$, $\langle e_1, f_1, e_2, f_2\rangle, \ldots, \langle e_1, f_1, \ldots, e_n, f_n\rangle$. A decomposition $V = \oplus_{i=0}^k V_i$ is called **compatible with $\mathcal{B}$** if each V_i is spanned by a subset of $\mathcal{B}$ of the form $\{e_j, f_j, e_{j+1}, f_{j+1}, \ldots, e_t, f_t\}$ for some $1 \leq j \leq t \leq n$. The compatible decompositions are indexed by subsets J of the set $I = \{1, \ldots, n\}$. We denote the decomposition corresponding to the set J by Δ_J. For $J = \{i_1 < i_2 < \cdots < i_k\}$ this decomposition Δ_J is $V_0 = \langle e_1, f_1, \ldots, e_{i_1}, f_{i_1}\rangle$, $V_1 = \langle e_{i_1+1}, f_{i_1+1}, \ldots, e_{i_2}, f_{i_2}\rangle, \ldots, V_k = \langle e_{i_k+1}, f_{i_k+1}, \ldots, e_n, f_n\rangle$. Note that $V_k = \{0\}$ if $i_k = n$.

For a nondegenerate σ-invariant subspace U of V, let $Sp(U)$ denote the full subgroup of G_σ that preserves the form $\mu\lambda(\cdot,\cdot)|_{U_\lambda \times U_\lambda}$ (cf. Lemma 3.6, Lemma 3.7, Proposition 3.8 of [4]) and acts trivially on $U^\perp$; let $SU(U)$ denote the special unitary group that is embedded naturally into $Sp(U)$. Clearly, $Sp(U) \cong Sp(2m, q)$ and $SU(U) \cong SU(m, q^2)$ where $2m$ is the dimension of U. For $J \subset I$ let $L_J = Sp(V_k)\prod_{i=0}^{k-1} SU(V_i)$ where $V = \oplus_{i=0}^{k} V_i$ is the decomposition Δ_J (and with the understanding that $Sp(V_k)$ is the trivial group in case $V_k = \{0\}$). If F_0 is the subflag of F of type J, then the parabolic of G_σ corresponding to F_0 is equal to $L_J D$ where D is the Borel subgroup corresponding to F. The **level** of a subgroup L_J is $n - |J|$, the corank of J in I. The level of L_J coincides with the rank of the parabolic $L_J D$. In case $n = 2$, the groups $L_{\{2\}} \cong SU_2(q^2)$ and $L_{\{1\}} \cong Sp_2(q)$ are called a **standard pair** in $G_\sigma \cong Sp_4(q)$. Following [2], we say that subgroups L_1 and L_2 of $SU_3(q^2)$ form a **standard pair** whenever each L_i is the stabilizer in $SU_3(q^2)$ of a nonsingular vector v_i of the natural module of $SU_3(q^2)$ and, moreover, v_1 and v_2 are perpendicular.

Let S be a subset of the power set of $I = \{1, ..., n\}$ that is closed under taking supersets. The **standard Phan amalgam of type C_n and of shape S** is the amalgam $\hat{\mathcal{A}}_S = \cup_{J\in S} L_J$. In the particular case where $S = S_k$ consists of all subsets $J \subseteq I$ with $|I \backslash J| \leq k$, we will call $\hat{\mathcal{A}}_S$ the **standard Phan amalgam of type C_n and of level k and rank n**; it is denoted by $\hat{\mathcal{A}}_k = \hat{\mathcal{A}}(n, k, q)$. The shape S_k will be called the **straight level k shape**. If $S \supseteq S_k$ then we say that S is of **level k**.

By an arbitrary **Phan amalgam of type C_n and of shape S** we will understand an amalgam $\mathcal{A} = \cup_{J\in S} K_J$ where K_J is a group isomorphic to a quotient of L_J over a subgroup of the center of L_J. Furthermore, if $J \subset J'$, then we require that $K_{J'}$ be contained in K_J, namely that $K_{J'}$ be the image of $L_{J'}$ under the natural homomorphism from L_J onto K_J.

The definition of a Phan amalgam leaves some ambiguity as to what is the exact structure of each K_J. For example, in the straight level two case, when $j - i > 1$, either $K_{I\backslash\{i\}}$ and $K_{I\backslash\{j\}}$ have trivial intersection or they have a common central involution. Similarly, when $j - i = 1$, the group $K_{I\backslash\{i,j\}}$ may be any quotient of $L_{I\backslash\{i,j\}}$ over a subgroup of $Z(L_{I\backslash\{i,j\}})$. Finally, the intersections of the members of the amalgam might be larger than expected. We call a Phan amalgam **unambiguous** if every K_J is isomorphic to the corresponding L_J and if $K_J \cap K_{J'} = K_{J\cup J'}$ for all J and J'. By a **covering** of a Phan amalgam $\mathcal{A} = \cup_{J\in S} K_J$ of shape S we mean a second Phan amalgam $\tilde{\mathcal{A}} = \cup_{J\in S} \tilde{K}_J$ of the same shape S, together with an amalgam homomorphism $\pi : \tilde{\mathcal{A}} \to \mathcal{A}$, such that π induces a surjective homomorphism of $\tilde{K}_J$ onto K_J for every $J \in S$. We call two coverings $(\tilde{\mathcal{A}}_1, \pi_1)$ and $(\tilde{\mathcal{A}}_2, \pi_2)$ of $\mathcal{A}$ **equivalent** if there is an isomorphism ϕ of $\mathcal{A}_1$ onto $\mathcal{A}_2$ such that $\pi_1 = \pi_2\phi$.

Definition 2.1 Let $n \geq 2$, let Δ be the Dynkin diagram C_n, and let $I = \{1, \ldots, n\}$. A group G admits a **weak Phan system of type C_n** if G contains subgroups $U_i \cong SL_2(q) \cong Sp_2(q) \cong SU_2(q^2)$, $i \in I$, and $U_{i,j}$, $i \neq j \in I$, so that the following hold:

(wP1) If (i, j) is not an edge of Δ, then $U_{i,j}$ is a central product of U_i and U_j;

(wP2) if (i, j) is an edge of Δ, then U_i and U_j are contained in $U_{i,j}$ which is isomorphic to a quotient of $SU_3(q^2)$ over a subgroup of its center if (i, j) is a single edge and isomorphic to a quotient of $Sp_4(q)$ over a subgroup of its center if (i,j) is a double edge; moreover, U_i and U_j form a standard pair in $U_{i,j}$; and

(wP3) the subgroups $U_{i,j}$, $i \neq j \in I$, generate G.

If a group G contains a weak Phan system $U_1, \ldots, U_n$, then $\mathcal{A} = \cup_{i\neq j \in I} U_{i,j}$ is a Phan amalgam of level two. This amalgam does not collapse, because G is a quotient of its universal completion. The converse is proved in the following lemma:

Lemma 2.2
Suppose $\mathcal{A}$ is a Phan amalgam of type C_n and of shape $S \supseteq S_2$. Suppose further that G is a nontrivial completion of $\mathcal{A}$ via some map π. Then $\pi|_{K_{I\setminus\{i\}}}$ is injective. In particular, $\pi(K_{I\setminus\{1\}}), \ldots, \pi(K_{I\setminus\{n\}})$ form a weak Phan system in G.

Proof. By way of contradiction suppose $1 \neq u \in K_{I\setminus\{i\}}$ and $\pi(u) = 1$. Let j be equal to $i-1$ or $i+1$. By Lemma 2.3 of [2] and Lemma 4.6 of [4] (i.e., connectedness of the respective geometries) we have $\langle K_{I\setminus\{i\}}, K_{I\setminus\{j\}} \rangle$ equal to a quotient of $SU_3(q^2)$ or $Sp_4(q)$ over a subgroup of the center. As $1 \neq u \in K_{I\setminus\{i\}}$, we have $u \notin Z(\langle K_{I\setminus\{i\}}, K_{I\setminus\{j\}} \rangle)$, whence $\pi(\langle K_{I\setminus\{i\}}, K_{I\setminus\{j\}} \rangle) = 1$ and, thus, $\pi(K_{I\setminus\{i\}}) = 1 = \pi(K_{I\setminus\{j\}})$. It follows that $\pi(G) = 1$, a contradiction. $\square$

The following results deal with characteristic completions of amalgams. Suppose $\mathcal{A}$ is an amalgam. A completion G of $\mathcal{A}$ is called **characteristic** if and only if every automorphism of $\mathcal{A}$ extends to an automorphism of G. In [2] the following has been proved. (Roughly speaking, a Phan amalgam of type A_n is what one gets if one throws away all members of a Phan amalgam of type C_{n+1} that are of the form $K_{I\setminus J}$ with $n+1 \in J$. For a precise definition see [2].)

Proposition 2.3 (see [2])
Let $n \geq 2$. The group $G \cong SU(n+1, q^2)$ is a characteristic completion of the standard Phan amalgam of type A_n for any shape $S \supseteq S_2$.

Lemma 2.4 (see [2])
Let $G \cong SU_3(q^2)$, let $U_1 \cong SU_2(q^2)$, $U_2 \cong SU_2(q^2)$ be a standard pair in G, and let T be the joint stabilizer in $\mathrm{Aut}(G)$ of U_1 and U_2. Then T is an extension of a group of order $(q+1)^2$ by the field automorphisms. Moreover, the centralizer in T of U_1 is of order $q+1$.

The next lemma is the C_2-analogue of Lemma 2.4.

Lemma 2.5
Let $G \cong Sp_4(q)$ and let $U_1 \cong SU_2(q^2)$, $U_2 \cong Sp_2(q)$ be a standard pair in G. Then the joint stabilizer T in $\mathrm{Aut}(G)$ of U_1 and U_2 is an extension of a group of order $(q+1)^2$ by the field automorphisms.

Proof. This follows immediately from the fact that the Borel subgroup of G with respect to the geometry Γ (cf. [4]) has order $(q+1)^2$. The only additional automorphisms are induced by the field automorphisms. $\square$

Proposition 2.6
Let $n \geq 2$. The group $G \cong Sp_{2n}(q)$ is a characteristic completion of the standard Phan amalgam $\hat{\mathcal{A}}_S$ of type C_n for any shape $S \supseteq S_2$.

Proof. We will only prove the proposition for $S = S_2$, the general case being a straightforward induction on the size of S (cf. the proof of Proposition 6.1 of [2]). We want to show that the group A of automorphisms of $\hat{\mathcal{A}}_2$ is of order $(q+1)^n f$ where $q = p^f$, p a prime. Proceed by induction on n. The case $n = 2$ is implied by Lemma 2.5. Now, for arbitrary n, consider the amalgam $\mathcal{B}$ of all members of $\hat{\mathcal{A}}_2$ that are contained in the lower right $(2n-2) \times (2n-2)$-block of G. The claim will follow if we prove that $C_A(\mathcal{B})$ has order at most $q+1$. Let $L = L_{I\backslash\{1,2\}} \cong SU_3(q^2)$ be the member of $\hat{\mathcal{A}}_2$ containing $L_{I\backslash\{1\}}$ and $L_{I\backslash\{2\}}$. Since $C_A(\mathcal{B})$ acts trivially on $L_{I\backslash\{2\}}$ and stabilizes $L_{I\backslash\{1\}}$ we can apply Lemma 2.4, so $C_A(\mathcal{B})$ induces on L a group of order at most $q+1$. The other members of $\hat{\mathcal{A}}_2$ that are not in $\mathcal{B}$ are direct products of $L_{I\backslash\{1\}}$ and a member of $\mathcal{B}$. But clearly every element of $C_A(\mathcal{B})$ that acts trivially on $L_{I\backslash\{1\}}$ acts trivially on every such direct product. We have shown that the group of automorphisms of $\hat{\mathcal{A}}_2$ is of order at most $(q+1)^n f$. This finishes the proof because G induces $(q+1)^n f$ automorphisms of $\hat{\mathcal{A}}_2$. $\square$

Corollary 2.7
Let $J \subset I$ with $|I\backslash J| \geq 3$. Then the group L_J is a characteristic completion of the amalgam $\cup_{J' \supset J} L_{J'}$. $\square$

Lemma 2.8
Let $\mathcal{A}_i$ be an amalgam and let G_i be a completion of $\mathcal{A}_i$ via the map π_i, $i = 1, 2$. Suppose there exist isomorphisms $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ and $\phi : G_1 \to G_2$ such that $\phi\pi_1 = \pi_2\psi$. If G_1 is a characteristic completion of $\mathcal{A}_1$, then for any isomorphism $\psi' : \mathcal{A}_1 \to \mathcal{A}_2$ there exists a unique isomorphism $\phi' : G_1 \to G_2$ such that $\phi'\pi_1 = \pi_2\psi'$.

Proof. Consider $\alpha = (\psi')^{-1}\psi$. This is an automorphism of $\mathcal{A}_1$. Since G_1 is a characteristic completion, α extends to an automorphism of G_1; that is, there is an automorphism β of G_1 such that $\pi_1\alpha = \beta\pi_1$. (Notice that β is the unique automorphism of G_1 with that property, because $\mathcal{A}_1$ generates G_1.) The map $\phi' = \phi\beta^{-1}$ has the required properties. $\square$

3 Proof of Theorem 1

We begin the proof of Theorem 1 with the following proposition.

Proposition 3.1
Every non-collapsing Phan amalgam $\mathcal{A}$ has a unique (up to equivalence) unambiguous (non-collapsing) covering $\tilde{\mathcal{A}}$.

Proof. The proof is by induction on $|S|$ where S is the shape of $\mathcal{A} = \cup_{J \in S} K_J$. The claim holds in case $S = \emptyset$, which corresponds to the empty amalgam. Suppose that S is a non-empty shape, and that the claim holds for every shape $S' \subset S$. Let J be a minimal (under inclusion) element of S and set $S' = S \backslash \{J\}$ and $\mathcal{A}' = \cup_{J' \in S'} K_{J'}$. Then S' is a shape, and $\mathcal{A}'$ is a Phan sub-amalgam in $\mathcal{A}$ of shape S'. By the inductive assumption there is a (unique) unambiguous covering Phan amalgam $(\tilde{\mathcal{A}}' = \cup_{J' \in S'} K'_{J'}, \pi')$ of $\mathcal{A}'$. We will construct an unambiguous covering $(\tilde{\mathcal{A}}, \pi)$ of $\mathcal{A}$ by gluing a copy of L_J to $\tilde{\mathcal{A}}'$ and by extending π' to the new member of the amalgam. To glue L_J to the amalgam $\tilde{\mathcal{A}}'$ we need to construct an isomorphism from the sub-amalgam $\mathcal{B} = \cup_{J' \supset J} K'_{J'}$ of $\tilde{\mathcal{A}}'$ onto the corresponding amalgam $\mathcal{C} = \cup_{J' \supset J} L_{J'}$ of proper subgroups of L_J. By the definition of a Phan amalgam there is a homomorphism ψ from L_J onto K_J mapping $\mathcal{C}$ onto $\mathcal{D} = \cup_{J' \supset J} K_{J'}$. Note that $\mathcal{D}$ is a Phan amalgam of shape $\{J' \mid J' \supset J\}$. Note further that $(\mathcal{B}, \pi'|_{\mathcal{B}})$ and $(\mathcal{C}, \psi)$ are two unambiguous coverings of $\mathcal{D}$. By induction, the uniqueness of the unambiguous covering holds so that there is an amalgam isomorphism ϕ from $\mathcal{B}$ onto $\mathcal{C}$ such that $\psi\phi = \pi'|_{\mathcal{B}}$. The map ϕ tells us how to glue L_J to $\tilde{\mathcal{A}}'$ to produce $\tilde{\mathcal{A}}$, and furthermore, as π we can take the union of ψ and π'. The condition $\psi\phi = \pi'|_{\mathcal{B}}$ guarantees that ψ and π' agree on the intersection $\mathcal{B} = \mathcal{C}$ (identified via ϕ). Finally, notice that $\tilde{\mathcal{A}}$ is an unambiguous Phan amalgam of type S, so $(\tilde{\mathcal{A}}, \pi)$ is an unambiguous covering of $\mathcal{A}$.

This completes the proof of the existence of an unambiguous covering $\tilde{\mathcal{A}}$. Now we will prove the uniqueness. Suppose we have two such coverings $\tilde{\mathcal{B}} = \cup_{J \in S} B_J$ and $\tilde{\mathcal{C}} = \cup_{J \in S} C_J$ with corresponding amalgam homomorphism π_1 and π_2 onto $\mathcal{A}$. Select J as in the previous paragraph, and define $S' = S \setminus \{J\}$. Let $\mathcal{A}'$, $\tilde{\mathcal{B}}'$ and $\tilde{\mathcal{C}}'$ be the sub-amalgams of shape S' in $\mathcal{A}$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$, respectively. By induction, there exists an isomorphism ϕ from $\tilde{\mathcal{B}}'$ onto $\tilde{\mathcal{C}}'$ such that $\pi_1|_{\tilde{\mathcal{B}}'} = \pi_2|_{\tilde{\mathcal{C}}'}\phi$. It suffices to extend ϕ to B_J.

We have two cases: First, let us assume that the decomposition Δ_J has more than one summand of dimension greater than two. In this case, $B_J \cong C_J \cong L_J$ is isomorphic to a direct product of $L_{J'}$ and $L_{J''}$ for suitable supersets J' and J'' of J. Clearly ϕ is already known on $B_{J'}$ and $B_{J''}$, and so ϕ extends uniquely to B_J. Since every member B_K with $K \supset J$ is a direct product of its intersections with $B_{J'}$ and $B_{J''}$, this extension, which we will also denote by ϕ, will be a well-defined amalgam isomorphism from $\mathcal{B}$ to $\mathcal{C}$, and furthermore, $\pi_2 = \pi_1\phi$.

In the second case, Δ_J has a unique summand of dimension m greater than two. In this case $B_J \cong C_J \cong L_J$ is isomorphic to $SU(m, q^2)$ or $Sp(2m, q)$. Choose an arbitrary isomorphism $\psi : B_J \longrightarrow C_J$, and consider the following map α from K_J to K_J: For $u \in K_J$, $\alpha(u)$ is defined to be $\pi_2\psi\pi_1^{-1}(u)$. Notice that α is a well-defined automorphism of K_J, because the fibers of π_1 are cosets of the kernel of π_1, and ψ takes them to cosets of the kernel of π_2 (since ψ takes the kernel of π_1 to the kernel of π_2—those being subgroups of equal order in the cyclic centers of B_J and C_J, respectively). Notice that every automorphism of K_J lifts to a unique automorphism of L_J which is isomorphic to $SU(m,q^2)$ or $Sp(2m,q)$. Thus, there is an automorphism β of C_J such that $\pi_2|_{C_J}\beta = \alpha\pi_2|_{C_J}$. Define $\theta : B_J \to C_J$ as $\beta^{-1}\psi$. First of all we have $\pi_1|_{B_J} = \pi_2|_{C_J}\theta$. Indeed, $\pi_2|_{C_J}\theta = \pi_2|_{C_J}\beta^{-1}\psi = \alpha^{-1}\pi_2|_{C_J}\psi = \pi_1|_{B_J}\psi^{-1}\pi_2^{-1}|_{K_J}\pi_2|_{C_J}\psi = \pi_1|_{B_J}$. Second, for every $J' \supset J$ we have that $\theta^{-1}\phi|_{B_{J'}}$ is a lifting to $B_{J'}$ of the identity automorphism of $K_{J'}$, and hence it is the identity. For, $\theta^{-1}\phi = \psi^{-1}\beta\phi = \psi^{-1}\pi_2^{-1}\alpha\pi_2\phi = \psi^{-1}\pi_2^{-1}\pi_2\psi\pi_1^{-1}\pi_2\phi = \pi_1^{-1}\pi_2\phi = \mathrm{id}$ (to enhance legibility we omitted the restrictions). This shows that ϕ and θ agree on every subgroup $B_{J'}$, which allows us to extend ϕ to the entire $\tilde{\mathcal{B}}$ by defining it on B_J as θ. $\square$

In view of the preceding proposition let $\mathcal{A} = \cup_{\{i,j\}\in I} K_{I\setminus\{i,j\}}$ be a non-collapsing, unambiguous Phan amalgam of shape S_2. We will prove the uniqueness of $\mathcal{A}$ (up to isomorphism) in a series of lemmas. Clearly, when $n = 2$ the amalgam is unique by definition.

The case $n = 3$

Since $\mathcal{A}$ is unambiguous, each subgroup $K_{I\setminus\{i\}}$ coincides with $K_{I\setminus\{i,j\}} \cap K_{I\setminus\{i,k\}}$ for $\{i,j,k\} = \{1,2,3\}$. Define $D_1 = N_{K_{\{2,3\}}}(K_{\{1,3\}})$ (both groups $K_{\{1,3\}}$, $K_{\{2,3\}}$ considered as subgroups of $K_{\{3\}}$) and $D_3 = N_{K_{\{1,2\}}}(K_{\{1,3\}})$. Since $K_{\{1,3\}}$ and $K_{\{2,3\}}$ form a standard pair in $K_{\{3\}}$, it follows that D_1 has order $q+1$, and it is a maximal torus in $K_{\{2,3\}} \cong SU_2(q^2)$. Similarly, D_3 is a maximal torus of order $q+1$ in $K_{\{1,2\}} \cong Sp_2(q)$. We also define $D_2^1 = N_{K_{\{1,3\}}}(K_{\{2,3\}})$ and $D_2^3 = N_{K_{\{1,3\}}}(K_{\{1,2\}})$. Again, these are two maximal tori of size $q+1$ in $K_{\{1,3\}} \cong SU_2(q^2)$. The following lemma gives us an extra condition on $\mathcal{A}$ that holds because $\mathcal{A}$ does not collapse.

Lemma 3.2
We have $D_2^1 = D_2^3$.

Proof. Let G be a non-trivial completion of $\mathcal{A}$ and let π be the corresponding map from $\mathcal{A}$ to G. Since $\mathcal{A}$ is non-collapsing, we may assume that π is injective on every $K_{I\setminus\{i\}}$, by Lemma 2.2. Observe that $D_2^i = C_{K_{\{1,3\}}}(D_i)$ for $i = 1,3$. Thus, $\pi(D_2^i) = C_{\pi(K_{\{1,3\}})}(\pi(D_i))$. Since D_1 and D_3 commute elementwise in $K_{\{2\}}$, we have that $\pi(D_1)$ and $\pi(D_3)$ commute elementwise as well. Since $K_{\{1,3\}}$ is invariant under $D_3 = N_{K_{\{1,2\}}}(K_{\{1,3\}})$ (in $K_{\{1\}}$) and since π is injective on $K_{\{1,3\}}$ (by Lemma 2.2), it follows that $D_2^1 = C_{K_{\{1,3\}}}(D_1)$ is invariant under D_3 (again as subgroups of $K_{\{1\}}$). Hence $\pi(D_2^1) = C_{\pi(K_{\{1,3\}})}(\pi(D_1))$ is invariant under $\pi(D_3)$. Similarly, $\pi(D_2^3) = C_{\pi(K_{\{1,3\}})}(\pi(D_3))$ is invariant under $\pi(D_1)$. Notice that D_2^3 and D_1 are both cyclic of order $q+1$. Since the order of $\mathrm{Aut}(C_{q+1})$ equals Euler's $\varphi(q+1)$ which is smaller than $q+1$, the group D_1 contains a non-trivial element d acting trivially on D_2^3. Analysis of $K_{\{3\}} \cong SU_3(q^2)$ shows that the only elements commuting with d in $K_{\{1,3\}}$ are those contained in D_2^1. Hence $D_2^1 = D_2^3$ as both groups are of order $q+1$. □

In view of the lemma we write D_2 for $D_2^1 = D_2^3$.

We want to prove the uniqueness of the amalgam $\mathcal{A}$. Assume there exists another amalgam $\mathcal{A}' = K'_{\{1\}} \cup K'_{\{2\}} \cup K'_{\{3\}}$. According to Goldschmidt's lemma 2.7 of [3] the amalgams $\mathcal{B} = K_{\{1\}} \cup K_{\{3\}}$ and $\mathcal{B}' = K'_{\{1\}} \cup K'_{\{3\}}$ are isomorphic via some amalgam isomorphism ψ. Clearly, $\psi(K_{\{1,3\}}) = \psi(K_{\{1\}} \cap K_{\{3\}}) = K'_{\{1\}} \cap K'_{\{3\}} = K'_{\{1,3\}}$. Also, by [2], we can assume that $\psi(K_{\{2,3\}}) = K'_{\{2,3\}}$. Let W be the natural module of $K_{\{3\}} \cong SU_3(q^2)$ considered as an $\mathbb{F}_q$-vector space. Denote the basis of W by $\{e_1, f_1, e_2, f_2, e_3, f_3\}$. Note that $K_{\{1\}}$ acts on the subspace of W spanned by $\{e_2, f_2, e_3, f_3\}$ (although it does not preserve the unitary form on that space). As $N_{K_{\{1,3\}}}(K_{\{1,2\}}) = D_2 = N_{K_{\{1,3\}}}(K_{\{2,3\}})$ we have $N_{K'_{\{1,3\}}}(\psi(K_{\{1,2\}})) = \psi(D_2) = N_{K'_{\{1,3\}}}(K'_{\{2,3\}})$. Moreover, $\psi(K_{\{1,2\}}) \subseteq \psi(K_{\{1\}}) = K'_{\{1\}}$. Via the isomorphism ψ, the groups $K'_{\{3\}}$ and $K'_{\{1\}}$ act on W. In particular, the latter also acts on $\langle e_2, f_2, e_3, f_3\rangle$. Furthermore note that D_2 and $\psi(D_2)$ act on $\langle e_2, f_2, e_3, f_3\rangle$. The only two-dimensional subspaces of $\langle e_2, f_2, e_3, f_3\rangle$ that are stabilized by $K_{\{1,2\}}$ and by $D_2 = N_{K_{\{1,3\}}}(K_{\{1,2\}})$ are $\langle e_2, f_2\rangle$ and $\langle e_3, f_3\rangle$. Therefore the same holds true for $\psi(D_2)$ and $\psi(K_{\{1,2\}})$. But the only two-dimensional subspace of W that is centralized by $K_{\{2,3\}}$ is the space $\langle e_3, f_3\rangle$. This coincides with the unique two-dimensional subspace of $\langle e_2, f_2, e_3, f_3\rangle$ that is normalized (and not centralized) by $K_{\{1,2\}}$. Therefore also $\psi(K_{\{1,2\}})$ normalizes $\langle e_3, f_3\rangle$ and centralizes $\langle e_2, f_2\rangle$. We have proved the following:

Proposition 3.3
If $n = 3$, then the unambiguous, non-collapsing Phan amalgam $\mathcal{A}$ of shape S_2 is unique up to isomorphism. □

The case $n > 3$

We will proceed by induction, using the case $n = 3$ as basis. Let $n > 3$ and let $\mathcal{A}$ be an unambiguous, non-collapsing Phan amalgam of type C_n and of shape S_2.

Lemma 3.4
There exists a unique amalgam $\mathcal{B} = \mathcal{A} \cup H_1 \cup H_2$ where $H_1 \cong SU_n(q^2)$ is generated by the subgroups $K_{I\setminus\{i,j\}}$, $1 \le i < j \le n-1$, and $H_2 \cong Sp_{2n-2}(q)$ is generated by the subgroups $K_{I\setminus\{i,j\}}$, $2 \le i < j \le n$.

Proof. Let $\mathcal{B}_1 = \cup_{1\le i<j\le n-1} K_{I\setminus\{i,j\}}$, $\mathcal{B}_2 = \cup_{2\le i<j\le n} K_{I\setminus\{i,j\}}$, and $\mathcal{C} = \mathcal{B}_1 \cap \mathcal{B}_2$. By the inductive assumption, $\mathcal{B}_1$ is isomorphic to the amalgam found in $SU_n(q^2)$ and $\mathcal{B}_2$ is isomorphic to the amalgam found in $Sp_{2n-2}(q)$. Furthermore, by Propositions 2.3 and 2.6 the groups $SU_n(q^2)$ and $Sp_{2n-2}(q)$ are characteristic completions of $\mathcal{B}_1$, resp. $\mathcal{B}_2$, whence there exist injective amalgam homomorphisms $\pi_1 : \mathcal{B}_1 \to H_1$ and $\pi_2 : \mathcal{B}_2 \to H_2$. We want to glue H_1 and H_2 to $\mathcal{A}$ via the maps π_1 and π_2. Notice that π_1 and π_2 send

$\mathcal{C}$ into subgroups $K_1 \leq H_1$, resp. $K_2 \leq H_2$ that are isomorphic to $SU_{n-1}(q^2)$. Since the copies of $\mathcal{C}$ in K_1 and K_2 are standard Phan amalgams of type A_{n-2}, there is an isomorphism $\phi : K_1 \to K_2$ that takes $\pi_1(\mathcal{C})$ to $\pi_2(\mathcal{C})$. Let ψ be the restriction of ϕ to $\mathcal{C}$. Consider $\mathcal{A}_1 = \pi_1(\mathcal{C})$ and $\mathcal{A}_2 = \pi_2(\mathcal{C})$ together with their embeddings into K_1, resp. K_2. Applying Lemma 2.8 with ϕ and ψ as above and $\psi' = \pi_2|_{\mathcal{C}}(\pi_1|_{\mathcal{C}})^{-1}$, there exists a unique isomorphism $\phi' : K_1 \to K_2$ such that $\phi'|_{\mathcal{A}_1} = \psi'$. Thus, $\phi'|_{\mathcal{A}_1}\pi_1|_{\mathcal{C}} = \pi_2|_{\mathcal{C}}$. Identifying K_1 with K_2 via ϕ' we obtain our unique amalgam $\mathcal{B}$. □

Let us now turn to the uniqueness of the amalgam $\mathcal{A}$. Suppose we have two non-collapsing, unambiguous Phan amalgams $\mathcal{A}$ and $\mathcal{A}'$ of type C_n and of shape S_2. Extend $\mathcal{A}$ and $\mathcal{A}'$ to amalgams $\mathcal{B} = \mathcal{A} \cup H_1 \cup H_2$ and $\mathcal{B}' = \mathcal{A}' \cup H_1' \cup H_2'$ as in Lemma 3.4. Observe that by Goldschmidt's lemma 2.7 of [3] there exists an isomorphism ϕ from $H_1 \cup H_2$ onto $H_1' \cup H_2'$. By the inductive assumption, the $K_{I\backslash\{i,j\}}$, $2 \leq i < j \leq n-1$, form a standard Phan amalgam of type A_{n-2} in $H_1 \cap H_2$; similarly the $K'_{I\backslash\{i,j\}}$, $2 \leq i < j \leq n-1$, form a standard Phan amalgam of type A_{n-2} in $H_1' \cap H_2'$. This implies that $\cup_{2 \leq i<j \leq n-1} K'_{I\backslash\{i,j\}}$ and $\cup_{2 \leq i<j \leq n-1} \phi(K_{I\backslash\{i,j\}})$ are standard Phan amalgams of type A_{n-2} in $H_1' \cap H_2'$. The two amalgams correspond to two choices of an orthonormal basis in the natural unitary space for $H_1' \cap H_2'$. Correcting ϕ, if necessary, by an inner automorphism of $H_1' \cap H_2'$, we may assume that $\phi(K_{I\backslash\{i,j\}}) = K'_{I\backslash\{i,j\}}$ for $2 \leq i < j \leq n-1$. Studying centralizers in H_1 and H_2 we see that $\phi(K_{I\backslash\{1\}}) = K'_{I\backslash\{1\}}$ and $\phi(K_{I\backslash\{n\}}) = K'_{I\backslash\{n\}}$. Therefore ϕ extends to an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$. Indeed, ϕ is already defined on all $K_{I\backslash\{i,j\}}$ with $2 \leq i < j \leq n-1$. Also, inside H_1' we see that $\phi(K_{I\backslash\{1,i\}})$, $i < n$, is $K'_{I\backslash\{1,i\}}$, since $K_{I\backslash\{1,i\}} = \langle K_{I\backslash\{1\}}, K_{I\backslash\{i\}} \rangle$. Similarly, in H_2' we see that $\phi(K_{I\backslash\{i,n\}})$, $1 < i$, is $K'_{I\backslash\{i,n\}}$. It remains to notice that $K_{I\backslash\{1,n\}}$ is the direct product of $K_{I\backslash\{1\}}$ and $K_{I\backslash\{n\}}$ so that ϕ extends to an isomorphism of $\mathcal{A}$ to $\mathcal{A}'$. Thus we have shown:

Proposition 3.5
If $n > 3$, then the amalgam $\mathcal{A}$ is unique up to isomorphism. □

We leave the proof of uniqueness for arbitrary shape to the reader. Theorem 1 follows.

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