L^q–Theory of a Singular "Winding" Integral Operator Arising from Fluid Dynamics

Reinhard Farwig
Department of Mathematics, Darmstadt University of Technology,
Schlossgartenstr. 7, D-64289 Darmstadt
farwig@mathematik.tu-darmstadt.de

Toshiaki Hishida*
Faculty of Engineering, Niigata University, Niigata 950–2181, Japan
hishida@eng.niigata-u.ac.jp

Detlef Müller
Mathematisches Seminar, Universität Kiel, D-24118 Kiel
mueller@math.uni-kiel.de

We analyze in classical $L^q(\mathbb{R}^n)$–spaces, $n = 2$ or $n = 3$, $1 < q < \infty$, a singular integral operator arising from the linearization of a hydrodynamical problem with a rotating obstacle. The corresponding system of partial differential equations of second order involves an angular derivative which is not subordinate to the Laplacian. The main tools are Littlewood-Paley theory and a decomposition of the singular kernel in Fourier space.
Key words: Fluid dynamics, Littlewood–Paley theory, rotating obstacle, singular integral operator

1 Introduction

Consider a three-dimensional rotating rigid body with angular velocity $\omega = (0, 0, 1)^T$ and assume that the complement, a time-dependent exterior domain $\Omega(t) \subset \mathbb{R}^3$, is filled with a viscous incompressible fluid modelled by the Navier-Stokes equations. By a simple coordinate transform we are led to the

*Supported in part by an Alexander von Humboldt research fellowship, Germany

nonlinear system [6]

$$\begin{array}{rcll} u_t - \nu\Delta u + u\cdot\nabla u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &=& f & \text{in } \Omega \\ \operatorname{div} u &=& 0 & \text{in } \Omega \\ u &=& \omega\wedge x & \text{on } \partial\Omega \\ u &\to& 0 & \text{at } \infty \end{array} \tag{1.1}$$

for the unknown velocity u and pressure function p in a time-independent exterior domain $\Omega \subset \mathbb{R}^3$ where $\nu > 0$ is the coefficient of viscosity. Looking for stationary solutions of (1.1), i.e. for time-periodic solutions of the original problem, and ignoring the nonlinear term $u\cdot\nabla u$ we arrive at a linear stationary partial differential equation in Ω.

The first step to analyze this problem is the L^q–theory of the system

$$\begin{array}{rcll} -\nu\Delta u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &=& f & \text{in } \mathbb{R}^3 \\ \operatorname{div} u &=& g & \text{in } \mathbb{R}^3 \end{array} \tag{1.2}$$

in the whole space. Here for later applications we allow $\operatorname{div} u$ to equal an arbitrarily given function g. The Coriolis force $\omega\wedge u = (-u_2, u_1, 0)^T$ can be considered as a perturbation of the Laplacian. But the first order partial differential operator $(\omega\wedge x)\cdot\nabla u$ is *not* subordinate to the Laplacian due to the increasing term $\omega\wedge x = (-x_2, x_1, 0)^T$. Using cylindrical coordinates $(r, \theta, x_3) \in (0,\infty)\times[0,2\pi)\times\mathbb{R}$ we get

$$(\omega\wedge x)\cdot\nabla u = -x_2\partial_1 u + x_1\,\partial_2 u = \partial_\theta u$$

showing that the crucial term $(\omega\wedge x)\cdot\nabla u$ is "just" an angular derivative of u w.r.t. θ. Since

$$\operatorname{div}\big((\omega\wedge x)\cdot\nabla u - \omega\wedge u\big) = (\omega\wedge x)\cdot\nabla\operatorname{div} u = \partial_\theta\, g\,,$$

the pressure p will satisfy the equation

$$\Delta p = \operatorname{div} f + \nu\Delta g + \partial_\theta g \quad \text{in } \mathbb{R}^3$$

which can easily be solved in L^q–spaces. Given p and ignoring $(1.2)_2$ we arrive at the system

$$-\nu\Delta u - \partial_\theta u + \omega\wedge u = f \quad \text{in } \mathbb{R}^3 \tag{1.3}$$

with another right-hand side f. Note that (1.3) also makes sense for a two-dimensional vector field u on $\mathbb{R}^2$; then $\omega\wedge u = (-u_2, u_1)^T$ and $(r,\theta) \in (0,\infty)\times[0,2\pi)$ denote polar coordinates in $\mathbb{R}^2$.

Theorem 1.1 (1) *Let $f \in L^q(\mathbb{R}^n)^n$, $n = 2$ or $n = 3$, $1 < q < \infty$. Then (1.3) has a solution $u \in L^1_{loc}(\mathbb{R}^n)^n$ satisfying the estimate*

$$\|\nu\nabla^2 u\|_q + \|\partial_\theta u - \omega \wedge u\|_q \le c\,\|f\|_q \,. \tag{1.4}$$

Its equivalence class in the homogeneous Sobolev space $\hat{H}^{2,q}(\mathbb{R}^n)^n$ is unique.

(2) *Let $f \in L^{q_1}(\mathbb{R}^3)^3 \cap L^{q_2}(\mathbb{R}^3)^3$, $1 < q_1, q_2 < \infty$, and let u_1 and u_2 be solutions as given by (1) corresponding to $q = q_1$ and $q = q_2$, respectively. Then there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that u_1 coincides with u_2 up to an affine linear vector field $\alpha\omega + \beta\omega\wedge x + (\gamma x_1, \gamma x_2, \delta x_3)^T$, and any solution remains a solution if one adds such a term. For $n = 2$ the terms $\alpha\omega$ and $(0, 0, \delta x_3)^T$ have to be omitted.*

(3) *Let $f \in L^q(\mathbb{R}^n)^n$, $n = 2$ or $n = 3$, and let $g \in H^{1,q}_{\rm loc}(\mathbb{R}^n)$ such that $(\omega \wedge x)g, \nabla g \in L^q(\mathbb{R}^n)^n$, $1 < q < \infty$. Then (1.2) has a locally integrable solution (u, p) satisfying the estimate*

$$\|\nu\nabla^2 u\|_q + \|\partial_\theta u - \omega \wedge u\|_q + \|\nabla p\|_q \le c\left(\|f\|_q + \|\nu\nabla g + (\omega \wedge x)g\|_q\right)$$

where $(1.2)_2$ has to be understood in the sense $\nabla \mathrm{div}\, u = \nabla g$. Its equivalence class in $\hat{H}^{2,q}(\mathbb{R}^n)^n \times \hat{H}^{1,q}(\mathbb{R}^n)$ is unique. Moreover, if (u_1, p_1) and (u_2, p_2) are two such solutions, then p_1 equals p_2 up to a constant and u_1 equals u_2 up to an affine linear vector field of the form $\alpha\omega + \beta\omega\wedge x + (\gamma x_1, \gamma x_2, -2\gamma x_3)^T$, $\alpha, \beta, \gamma \in \mathbb{R}$, and any solution remains a solution if one adds such terms. For $n = 2$, u_1 equals u_2 up to the linear term $\beta(-x_2, x_1)^T$, $\beta \in \mathbb{R}$.

The so-called *homogeneous* Sobolev spaces $\hat{H}^{k,q}(\mathbb{R}^n)$ in Theorem 1.1 are defined as follows: Let Π_{k-1} denote the space of polynomials of degree $\le k-1$. Then, using multi-index notation,

$$\hat{H}^{k,q}(\mathbb{R}^n) = \left\{u \in L^1_{\rm loc}(\mathbb{R}^n)/\Pi_{k-1} : \partial^\alpha u \in L^q(\mathbb{R}^n) \quad \text{for all } \alpha \in \mathbb{N}_0^n,\ |\alpha| = k\right\}$$

is equipped with the norm $\sum_{|\alpha|=k} \|\partial^\alpha u\|_q$. Note that elements in $\hat{H}^{k,q}(\mathbb{R}^n)$ are equivalence classes of L^1_{loc}–functions being unique only up to polynomials from Π_{k-1}. Since $\hat{H}^{k,q}(\mathbb{R}^n)$ can be considered as a closed subspace of $L^q(\mathbb{R}^n)^N$ for some $N = N(k, n) \in \mathbb{N}$, it is reflexive for every $q \in (1, \infty)$. For more details on these spaces see Chapter II in [3]. Notice, however, that the space Π_1^n is not completely contained in the kernel of the operator

$$L = -\nu\Delta - \partial_\theta + \omega\wedge$$

arising in (1.3).

We note that separate L^q–estimates of the terms $\omega\wedge u$ and $\partial_\theta u$ in Theorem 1.1 are not possible unless f satisfies an additional set of compatibility conditions, see Remark 2.3 and Proposition 2.4 below; in particular u or $\omega\wedge u$ are not necessarily L^q–integrable. Furthermore Proposition 2.1 indicates that the main solution operator does not define a classical Calderón–Zygmund integral operator.

The underlying problem of the flow around a rotating obstacle has attracted much attention during the last years. Weak solutions have been considered in [1, 2], whereas one of the present authors proved the existence of a unique instationary solution in an L^2–setting using semigroup theory [6, 7]. It is a remarkable fact that the operator $-\nu\Delta u - \partial_\theta u + \omega\wedge u$ does *not* generate an analytic semigroup, but a contractive C^0–semigroup. Several auxiliary linearized equations without the crucial term $\partial_\theta u$ have been considered in [8]. An L^2– and an $L^{3/2}$–theory of problem (1.2) have been established in [4], where the nonlinear problem is also solved for non-Newtonian, second-order fluids and rigid bodies moving due to gravity. Pointwise decay estimates for the linear and nonlinear case are obtained in [5]. For further references on moving bodies in fluids see [4, 5].

2 Preliminaries

To find the fundamental solutions of (1.2) and of (1.3), see also [6, 7], we use the Fourier transform $\mathcal{F} =^\wedge$, i.e.,

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, u(x)dx\,.$$

Note that in $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions, $\widehat{\partial_j u} = i\xi_j\hat{u}$ and $\widehat{x_j u} = i\partial\hat{u}/\partial\xi_j$, $1\le j\le n$. Hence (1.3) is related to the problem

$$\nu s^2\hat{u} - \partial_\varphi\hat{u} + \omega\wedge\hat{u} = \hat{f} \tag{2.1}$$

where $s=|\xi|$ and $\partial_\varphi = -\xi_2\,\partial/\partial\xi_1 + \xi_1\partial/\partial\xi_2 = (\omega\wedge\xi)\cdot\nabla_\xi$ is the angular derivative in Fourier space when using polar or cylindrical coordinates for $\xi\in\mathbb{R}^2$ or $\xi\in\mathbb{R}^3$, resp. Ignoring for a moment the term $\omega\wedge\hat{u}$ the ordinary differential equation $-\partial_\varphi\hat{u} + \nu s^2\hat{u} = \hat{f}$ yields the solution

$$\hat{u}(\varphi) = e^{\nu s^2\varphi}\hat{u}_0 - e^{\nu s^2\varphi}\int_0^\varphi e^{-\nu s^2 t}\hat{f}(t)dt, \quad \hat{u}_0\in\mathbb{R}^n, \tag{2.2}$$

when omitting in $\hat{u}$, $\hat{f}$ the variables $s = |\xi|$ or $s' = (\xi_1^2 + \xi_2^2)^{1/2}, \xi_3$, resp. Due to the 2π–periodicity of $\hat{u}$ w.r.t. φ the unknown $\hat{u}_0$ is given by

$$\hat{u}_0 = \left(1 - e^{-2\pi\nu s^2}\right)^{-1} \int_0^{2\pi} e^{-\nu s^2 t} \hat{f}(t) dt\,.$$

Using for $s \neq 0$ the geometric series expansion of $\left(1 - e^{-2\pi\nu s^2}\right)^{-1}$ and the 2π–periodicity of $\hat{f}$ w.r.t. t we get $\hat{u}_0 = \int_0^\infty e^{-\nu s^2 t} \hat{f}(t) dt$. Then (2.2) yields

$$\hat{u}(\varphi) = \int_0^\infty e^{-\nu s^2 t} \hat{f}(\varphi + t) dt. \tag{2.3}$$

Let $O(t)$ denote the orthogonal matrix

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad O(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

describing the rotation around the ξ_3–axis or in the plane by the angle t, resp. Thus, in the variable ξ,

$$\hat{u}(\xi) = \int_0^\infty e^{-\nu s^2 t} \hat{f}\big(O(t)\xi\big) dt$$

is the solution of (2.1) when $\omega \wedge u$ has been ignored. To deal with the term $\omega \wedge u$ note that $\partial_\varphi O(\varphi) = \omega \wedge O(\varphi)$ in the sense of linear maps. Applying $O(\varphi)^T$ to (2.1) the unknown $\hat{v}(\varphi) = O(\varphi)^T \hat{u}(\varphi)$ will satisfy the ordinary differential equation $\nu s^2 \hat{v}(\varphi) - \partial_\varphi \hat{v}(\varphi) = O(\varphi)^T \hat{f}(\varphi)$. Hence by (2.3) $\hat{v}(\varphi) = \int_0^\infty e^{-\nu s^2 t} O(\varphi + t)^T \hat{f}(\varphi + t) dt$ and consequently

$$\hat{u}(\xi) = \int_0^\infty e^{-\nu s^2 t} O(t)^T \hat{f}(O(t)\xi) dt\,. \tag{2.4}$$

Since $e^{-\nu|\xi|^2 t}$ multiplied by $(2\pi)^{-n/2}$ is the Fourier transform of the heat kernel

$$E_t(x) = \frac{1}{(4\pi\nu t)^{n/2}} e^{-\frac{|x|^2}{4\nu t}}$$

and since $\widehat{f(O(t)x)} = \hat{f}(O(t)\xi)$, (2.4) yields the formal solution

$$u(x) = \int_0^\infty O(t)^T E_t * f(O(t)\cdot)(x) dt \tag{2.5}$$

of (1.3).

Note that for $n = 3$ and $f \in \mathcal{S}(\mathbb{R}^3)^3$, the integrals (2.4) and (2.5) do in fact converge absolutely and define a distributional solution $u \in \mathcal{S}'(\mathbb{R}^3)^3$ of (1.3).

However, if $n = 2$, then both integrals fail to converge in $\mathcal{S}'(\mathbb{R}^2)^2$, even when $f \in \mathcal{S}(\mathbb{R}^2)^2$. This is not surprising, in view of a similar phenomenon for the Poisson equation in dimension 2. In this case, we need to modify (2.4), by defining a solution $u \in \mathcal{S}'(\mathbb{R}^2)^2$ e.g. by means of the convergent integral

$$\begin{aligned} \langle u, \varphi \rangle &= \langle \hat{u}, \check{\varphi} \rangle = \int_{|\xi| \geq 1} \int_0^\infty e^{-\nu s^2 t}\, O(t)^T \hat{f}(O(t)\xi) \cdot \check{\varphi}(\xi)\, dt\, d\xi \\ &\quad + \int_{|\xi|<1} \int_0^\infty e^{-\nu s^2 t}\, O(t)^T \hat{f}(O(t)\xi) \cdot (\check{\varphi}(\xi) - \check{\varphi}(0))\, dt\, d\xi \end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^2)^2$; here $\check{}$ denotes the inverse Fourier transform.

Then, in both dimensions $n = 2, 3$, for $f \in \mathcal{S}(\mathbb{R}^n)^n$, we have constructed a solution $u \in \mathcal{S}'(\mathbb{R}^n)^n$ of (1.3). Moreover, in the next section we shall prove that u satisfies inequality (1.4) in Theorem 1.1(1). In particular, $||\nabla^2 u||_q \leq c||f||_q < \infty$ for $1 < q < \infty$, yielding $u \in L^1_{loc}(\mathbb{R}^n)^n$. We will conclude that, for any $f \in L^q(\mathbb{R}^n)^n$, there is a solution $u \in L^1_{loc}(\mathbb{R}^n)^n$ of (1.3) satisfying (1.4).

To this end, consider the sequence of balls $B_m(0) \subset \mathbb{R}^n$ and choose a sequence $\{f_j\} \subset \mathcal{S}(\mathbb{R}^n)^n$ converging to f in $L^q(\mathbb{R}^n)^n$. Let u_j be the solution of (1.3) corresponding to f_j. The proof of completeness of $\hat{H}^{2,q}(\mathbb{R}^n)$ in [3] reveals that we can find a sequence of polynomials $\{r_j\} \subset \Pi_1^n$ and $\tilde{u} \in L^1_{loc}(\mathbb{R}^n)^n$ such that for $j \to \infty$

$$||\nabla^2 \left((u_j + r_j) - \tilde{u} \right) ||_q \to 0$$

and

$$(u_j + r_j)|_{B_m} \to \tilde{u}|_{B_m} \text{ in } L^q(B_m)^n \quad \text{for all} \quad m \in \mathbb{N}. \tag{2.6}$$

Then (2.6) implies that $Lu_j + Lr_j \to L\tilde{u}$ in the sense of distributions, which shows that $Lr_j \to L\tilde{u} - f$ in $\mathcal{D}'(\mathbb{R}^n)^n$. And, since $L\,\Pi_1^n$ is closed, as a linear subspace of the finite-dimensional space Π_1^n, we see that $L\tilde{u} - f = Lr$, for some $r \in \Pi_1^n$. Thus, if we put $u = \tilde{u} - r$, then $u \in L^1_{loc}(\mathbb{R}^n)^n$ and $||\nabla^2 u||_q \leq c||f||_q$, so that u satisfies (1.4).

Observe next that formula (2.5) may be rewritten by using

$$E_t * f(O(t)\cdot)(x) = (E_t * f)(O(t)x)\,,$$

the proof of which is based on the radial symmetry of $E_t(\cdot)$.

For $n = 3$ we arrive at the identity

$$u(x) = \int_{\mathbb{R}^3} \Gamma(x,y) f(y) dy \tag{2.7}$$

with the fundamental solution

$$\Gamma(x,y) = \int_0^\infty O(t)^T E_t(O(t)x - y) dt \,. \tag{2.8}$$

Furthermore $\Delta u(x)$ can be represented – as $u(x)$ in (2.7) – with the help of the kernel

$$\begin{aligned} K(x,y) = \Delta_x \Gamma(x,y) &= \int_0^\infty \Delta_x O(t)^T E_t(O(t)x - y) dt \\ &= \int_0^\infty O(t)^T \frac{1}{(4\pi\nu t)^{n/2}} \Big(-\frac{n}{2\nu t} + \frac{|O(t)x-y|^2}{(2\nu t)^2} \Big) \exp \Big(\frac{-|O(t)x-y|^2}{4\nu t} \Big) dt \,, \end{aligned} \tag{2.9}$$

for $n = 2$ or $n = 3$, cf. (3.4) below.

The following proposition indicates that $K(x,y) = \Delta_x \Gamma(x,y)$ does *not* define a classical Calderón–Zygmund integral operator.

Proposition 2.1 (1) *Let $n = 3$. Then, for $|x|$, $|y| \to \infty$, the fundamental solution $\Gamma(x,y)$ is not bounded by $C|x-y|^{-1}$. Actually there exists an $\alpha > 0$ such that for suitable x, $y \in \mathbb{R}^3$ with $|x|$, $|y| \to \infty$*

$$|\Gamma(x,y)| \geq \alpha \, \frac{\log|x-y|}{|x-y|} \,.$$

(2) *Let $n = 2$ or $n = 3$. Then there exists an $\alpha > 0$ and suitable x, $y \in \mathbb{R}^n$ with $|x|$, $|y| \to \infty$ such that the kernel $K_1(x,y) = \int_0^\infty t^{-n/2} \frac{1}{t} e^{-|O(t)x-y|^2/t} dt$ satisfies the estimate*

$$K_1(x,y) \geq \frac{\alpha}{|x-y|} \,.$$

The same result holds for the kernel $K_2(x,y)$ where the term $\frac{1}{t}$ in the definition of K_1 is replaced by $|O(t)x-y|^2/t^2$, cf. (2.9).

Proof (1) Considering only the component $\Gamma_{3,3}(x,y)$ and points $x, y \in \mathbb{R}^3$ with equal third component $x_3 = y_3$ and of equal norm $r = |x| = |y|$ we use complex notation. Thus we may omit the third component of x, y and we restrict ourselves to complex numbers $x = r$ and $y = re^{i\theta}$, $0 < \theta < \pi$, yielding

$$|O(t)x - y| = r|e^{it} - e^{i\theta}| = 2r\Big|\sin\frac{\theta - t}{2}\Big|$$

and $|x-y| = 2r|\sin\frac{\theta}{2}|$. Now $\Gamma_{3,3}(x,y)$ is bounded from below by $\sum_{k=0}^{N} I_k(r,\theta)$, where $N = [2r^2\sin^2\frac{\theta}{2}]$ and

$$I_k(r,\theta) = \int_{\theta/2+2k\pi}^{3\theta/2+2k\pi} \frac{1}{(4\pi\nu t)^{3/2}} \exp\Big(-r^2\sin^2\Big|\frac{\theta-t}{2}\Big|/(\nu t)\Big)\,dt.$$

We find constants $\alpha_j > 0$ independent of r, θ and of k such that for $k \geq 1$

$$\begin{aligned} I_k(r,\theta) &\geq \frac{\alpha_1}{k^{3/2}} \int_{-\theta/2}^{\theta/2} \exp\big(-\alpha_2 r^2 t^2/k\big)\,dt \\ &= \frac{2\alpha_1}{rk} \int_0^{r\theta/(2\sqrt{k})} \exp\big(-\alpha_2 s^2\big)\,ds\,. \end{aligned}$$

For $1 \leq k \leq N \sim r^2\theta^2$ and $r\theta \gg 1$, we find $\alpha_3 > 0$ such that $I_k(r,\theta) \geq \frac{\alpha_3}{rk}$. Summing up we are led to the inequality

$$\Gamma_{3,3}(x,y) \geq \sum_{k=1}^{N} I_k(r,\theta) \geq \alpha_3 \sum_{k=1}^{N} \frac{1}{rk} \geq \alpha_4 \frac{\log(r\theta)}{r}$$

with a constant $\alpha_4 > 0$ independent of r and of θ when $r\theta \gg 1$.

(2) Again we use complex notation and consider points $x = r$, $y = re^{i\theta}$, $0 < \theta < \pi$, where now $r^2\theta \gg 1$. Then $K_1(x,y)$ is bounded from below by

$$\begin{aligned} &\int_{\theta-\sqrt{\theta}/r}^{\theta+\sqrt{\theta}/r} t^{-n/2} \exp\Big(-4r^2\sin^2\Big|\frac{\theta-t}{2}\Big|/t\Big)\frac{dt}{t} \\ &\quad\geq \frac{\alpha_1}{\theta^{1+n/2}} \int_0^{\sqrt{\theta}/r} \exp\big(-\alpha_2 r^2 t^2/\theta\big)\,dt \\ &\quad\geq \frac{\alpha_1}{r\theta^{1/2+n/2}} \int_0^1 e^{-\alpha_2 s^2}\,ds\,. \end{aligned}$$

Hence $K_1(x,y) \geq \frac{\alpha_3}{\theta^{n/2-1/2}|x-y|}$. The kernel $K_2(x,y)$ can be estimated analogously. ∎

Before proving Theorem 1.1 in Section 3 below we consider the much simpler case $q = 2$, the question of separate estimates for u_θ and $\omega \wedge u$ and a variation of (2.10) when the integrals w.r.t. t extend from 2π to ∞.

Proposition 2.2 *Given $f \in L^2(\mathbb{R}^n)^n$, $n = 2$ or $n = 3$, the solution u of* (1.3) *given by* (2.5) *satisfies the estimate*

$$\|\nabla^2 u\|_2 + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_2 \leq c\|f\|_2 \,. \tag{2.10}$$

Proof By Plancherel's theorem, Fubini's theorem and the inequality of Cauchy– Schwarz (with $s = |\xi|$)

$$\begin{aligned}
\|\Delta u\|_2^2 &= \int_{\mathbb{R}^n} s^4 \Big| \int_0^\infty e^{-\nu s^2 t}\, O(t)^T \hat{f}(O(t)\xi)\, dt \Big|^2 d\xi \\
&\leq \int_{\mathbb{R}^n} \Big(\int_0^\infty s^2 e^{-\nu s^2 t}\, dt \Big) \cdot \Big(\int_0^\infty s^2 e^{-\nu s^2 t} |\hat{f}(O(t)\xi)|^2 dt \Big) d\xi \\
&= \frac{1}{\nu} \int_0^\infty \Big(\int_{\mathbb{R}^n} s^2 e^{-\nu s^2 t} |\hat{f}(O(t)\xi)|^2\, d\xi \Big)\, dt \\
&= \frac{1}{\nu} \int_0^\infty \Big(\int_{\mathbb{R}^n} s^2 e^{-\nu s^2 t} |\hat{f}(\xi)|^2\, d\xi \Big)\, dt \\
&= \frac{1}{\nu^2} \|f\|_2^2 \,.
\end{aligned}$$

Furthermore, for any second order partial derivative

$$\|\partial_j \partial_k u\|_2 = \|\xi_j \xi_k \hat{u}\|_2 \leq \| |\xi|^2 \hat{u}\|_2 = \|\Delta u\|_2 \leq \frac{1}{\nu} \|f\|_2 \,.$$

∎

Remark 2.3 Inequality (2.10) cannot be improved in the sense that both $\|\omega \wedge u\|_2$ and $\|(\omega \wedge x) \cdot \nabla u\|_2$ are finite or can even be estimated by $\|f\|_2$. In the two-dimensional case let

$$u(x) = u(r,\theta) = a(r) \frac{1}{r} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = a(r) \frac{1}{r^2} x^\perp$$

where $x^\perp$ is obtained from x by rotation with the angle $\frac{\pi}{2}$ and $a \in C^\infty(\overline{\mathbb{R}_+})$ satisfies $a = 1$ for large r and $a = 0$ for $r \in [0,1)$. Obviously $u \in C^\infty(\mathbb{R}^2)^2$ is solenoidal, $|\nabla^2 u(x)| \sim \frac{1}{r^3}$ for large r yielding $\nabla^2 u \in L^2(\mathbb{R}^2)^4$, $\operatorname{supp} \Delta u \subset \operatorname{supp} a$ and $\omega \wedge u = \frac{a(r)}{r} \binom{-\cos\theta}{-\sin\theta} = u_\theta$. Consequently $\omega \wedge u - u_\theta \equiv 0$ and the right-hand side $f = -\nu \Delta u \in L^2(\mathbb{R}^2)^2$, but $|\omega \wedge u| \sim \frac{1}{r} \notin L^2(\mathbb{R}^2)$. An analogous result holds in L^q–spaces, $q \neq 2$, when choosing $u(x) = a(r) r^{-\lambda} x^\perp$ for suitable $\lambda > 0$.

Proposition 2.4 *Let $f \in L^q(\mathbb{R}^2)^2$ satisfy the compatibility conditions*

$$f_m(r) := \frac{1}{2\pi} \int_0^{2\pi} O(\theta)^T f(r,\theta)\, d\theta = 0 \quad \textit{for a.a.} \quad r > 0. \tag{2.11}$$

Then one can find a suitable representative u of the unique solution in $\hat{H}^{2,q}(\mathbb{R}^2)^2$ of (1.3) given by Theorem 1.1, satisfying the estimate

$$\|\nabla^2 u\|_q + \|\partial_\theta u\|_q + \|u\|_q \le c\|f\|_q.$$

An analogous result holds for $n = 3$ where (2.11) is replaced by the assumption $\frac{1}{2\pi}\int_0^{2\pi} O(\theta)^T f(r, \theta, x_3)\, d\theta = 0$ for a.a. $r = \sqrt{x_1^2 + x_2^2} > 0$, $x_3 \in \mathbb{R}$.

Proof The main idea is to show that the integral mean

$$u_m(r) = \frac{1}{2\pi}\int_0^{2\pi} O(\theta)^T u(r, \theta)\, d\theta$$

vanishes for a.a. $r > 0$, for a suitable representative u; for $n = 3$ the integral mean $u_m(r, x_3)$ is defined analogously. Then the identity $O(\theta)\partial_\theta(O(\theta)^T u) = \partial_\theta u - \omega \wedge u$ and Wirtinger's inequality will imply that

$$\|u\|_q^q = \int_0^\infty r \int_0^{2\pi} |O(\theta)^T u(r, \theta)|^q\, d\theta dr \le c\|\partial_\theta(O(\theta)^T u)\|_q^q \le c\|\partial_\theta u - \omega \wedge u\|_q^q,$$

and Theorem 1.1(1) will complete the proof for $n = 2$ and also for $n = 3$.

In order to prove that $u_m(r) \equiv 0$ notice that, for $n = 2$, $\tilde{u}(x) = O(\theta)u_m(r)$ satisfies (1.3) with f replaced by $f = 0$ since

$$L(\tilde{u}) = L(O(\theta)u_m(r)) = O(\theta)(Lu)_m(r) = O(\theta)f_m(r) = 0.$$

Furthermore, since $\tilde{u} \in \mathcal{S}'(\mathbb{R}^2)^2$, the proof of Theorem 1.1(2), see Section 3 below, implies that $\tilde{u} \in \Pi_1^2$. Replacing u by $u - \tilde{u}$, we may then assume that $u_m = 0$. This argument easily extends to the case $n = 3$. ∎

Remark 2.5 The difficulties in the proof of Theorem 1.1 when estimating Δu with u given by (2.5) arise from the corresponding integrals on $(0, \varepsilon)$, $\varepsilon > 0$. Actually, consider the operator S on $L^q(\mathbb{R}^n)$ given by

$$Sf(x) = \int_{2\pi}^\infty (-\Delta)O(t)^T E_t * f(O(t)\cdot)(x)dt,$$

i.e., in Fourier space

$$\widehat{Sf}(\xi) = \int_{2\pi}^\infty s^2 e^{-\nu s^2 t} O(t)^T \hat{f}(O(t)\xi)dt, \quad s = |\xi|.$$

Since $O(t)$ is 2π–periodic and $s^2\sum_{k=1}^\infty e^{-2k\pi\nu s^2} = s^2 e^{-2\pi\nu s^2}(1 - e^{-2\pi\nu s^2})^{-1} =: m(\xi)$, we get that

$$\begin{aligned}\widehat{Sf}(\xi) &= m(\xi)\int_0^{2\pi} e^{-\nu s^2 t} O(t)^T \hat{f}(O(t)\xi)\, dt \\ &= m(\xi)\mathcal{F}\Big(\int_0^{2\pi} O(t)^T E_t * f(O(t)\cdot)(x)dt\Big).\end{aligned}$$

Obviously $m(\xi)$ satisfies the classical Michlin-Hörmander multiplier condition, cf. [9], and due to properties of the heat kernel

$$\| \int_0^{2\pi} O(t)^T E_t * f(O(t)\cdot)(x)dt\|_q \le \int_0^{2\pi} \|f(O(t)\cdot)\|_q \, dt = 2\pi\|f\|_q \, .$$

Then multiplier theory yields the estimate $\|Sf\|_q \le c\|f\|_q$ for every $q \in (1,\infty)$ with a constant $c = c(m,q)$.

3 Proof of Theorem 1.1

Due to the well-known estimate $\|\partial_j\partial_k u\|_q \le c\|\Delta u\|_q$, $1 < q < \infty$, $1 \le j,k \le n$, cf. [9], it suffices to consider only Δu. The main ideas are Littlewood-Paley theory and a decomposition of the integral operator

$$Tf(x) = \int_0^\infty (-\Delta)O(t)^T(E_t * f)(O(t)x)dt = \int_{\mathbb{R}^n} K(x,y)f(y)dy \qquad (3.1)$$

in Fourier space where each integral kernel has compact support. Since

$$\mathcal{F}\big(-\Delta O(t)^T(E_t * f)(O(t)\cdot)\big)(\xi) = O(t)^T|\xi|^2e^{-\nu|\xi|^2t}\hat{f}(O(t)\xi)$$

define $\psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{\psi}(\xi) = (2\pi)^{-n/2}|\xi|^2e^{-\nu|\xi|^2} = \widehat{(-\Delta)E_1} \qquad (3.2)$$

and

$$\psi_t(x) = t^{-n/2}\psi\big(\frac{x}{\sqrt{t}}\big)\,, \quad \hat{\psi}_t(\xi) = \hat{\psi}(\sqrt{t}\xi) = (2\pi)^{-n/2}t|\xi|^2e^{-\nu t|\xi|^2}\,. \qquad (3.3)$$

Thus the kernel $K(x,y)$ may be written in the form

$$K(x,y) = \int_0^\infty O(t)^T\,\psi_t(O(t)x-y)\,\frac{dt}{t}\,. \qquad (3.4)$$

To decompose $\hat{\psi}_t$ choose $\tilde{\varphi}$, $\tilde{\chi} \in C_0^\infty(\frac{1}{2},2)$ such that $0 \le \tilde{\varphi}$, $\tilde{\chi} \le 1$ and

$$\sum_{j=-\infty}^{\infty} \tilde{\chi}(2^{-j}r) = 1\,, \quad \int_0^\infty \tilde{\varphi}(sr)^2\frac{ds}{s} = \frac{1}{2} \quad \text{forall} r > 0\,.$$

Then define for $\xi \in \mathbb{R}^n$ and for $j \in \mathbb{Z}$, $s > 0$

$$\hat{\chi}_j(\xi) = \tilde{\chi}(2^{-j}|\xi|)\,, \quad \hat{\varphi}_s(\xi) = \tilde{\varphi}(\sqrt{s}|\xi|)$$

yielding

$$
\begin{aligned}
&\operatorname{supp} \hat{\chi}_j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}\,, \\
&\operatorname{supp} \hat{\varphi}_s \subset A\big(\tfrac{1}{2\sqrt{s}}\,,\,\tfrac{2}{\sqrt{s}}\big)\,;
\end{aligned}
\tag{3.5}
$$

moreover $\int_{\mathbb{R}^n} \varphi_s(x)dx = 0$ and

$$
\sum_{j=-\infty}^{\infty} \hat{\chi}_j(\xi) = 1\,, \qquad \int_0^\infty \hat{\varphi}_s(\xi)^2\,\frac{ds}{s} = 1 \quad (\xi \neq 0)\,. \tag{3.6}
$$

The family of functions $\{\varphi_s : s > 0\}$ will be used in Littlewood-Paley theory, see I§8.23 in [10], yielding the inequalities

$$
c_1\|f\|_q \le \Big\| \Big(\int_0^\infty |\varphi_s * f(\cdot)|^2\,\frac{ds}{s} \Big)^{1/2} \Big\|_q \le c_2\|f\|_q \tag{3.7}
$$

with constants $c_1, c_2 > 0$ depending on $q \in (1,\infty)$, but independent of $f \in L^q(\mathbb{R}^n)^n$. Furthermore we decompose K by defining $\psi^j \in \mathcal{S}(\mathbb{R}^n)$ by

$$
\psi^j = (2\pi)^{-n/2}\chi_j * \psi \quad \text{or equivalently} \quad \hat{\psi}^j = \hat{\chi}_j \cdot \hat{\psi}\,, \quad j \in \mathbb{Z}\,, \tag{3.8}
$$

yielding $\psi = \sum_{j=-\infty}^{\infty} \psi_j$ and, cf. (3.4),

$$
K_j(x,y) = \int_0^\infty O(t)^T\,\psi_t^j(O(t)x - y)\,\frac{dt}{t}\,, \quad j \in \mathbb{Z}\,. \tag{3.9}
$$

Given K_j we define the operator

$$
T_j f(x) = \int_{\mathbb{R}^n} K_j(x,y)\,f(y)dy = \int_0^\infty O(t)^T\,(\psi_t^j * f)(O(t)x)\frac{dt}{t} \tag{3.10}
$$

such that formally and even w.r.t to the operator norm topology $T = \sum_{j=-\infty}^{\infty} T_j$, see the proof below.

Lemma 3.1 *The functions ψ_t^j have the following properties:*

(1) *For $j \in \mathbb{Z}$ and $t > 0$*

$$
\operatorname{supp} \hat{\psi}_t^j \subset A\Big(\frac{2^{j-1}}{\sqrt{t}}\,,\,\frac{2^{j+1}}{\sqrt{t}}\Big)\,.
$$

(2) *For* $m > \frac{n}{2}$ *let* $h(x) = (1+|x|^2)^{-m}$ *and, cf.* (3.3), $h_t(x) = t^{-n/2}\, h\big(\frac{x}{\sqrt{t}}\big)$. *Then there exists a constant* $c > 0$ *independent of* $j \in \mathbb{Z}$ *such that*

$$|\psi^j(x)| \le c\, 2^{-2|j|} h_{2^{-2j}}(x) \quad \textit{for all } x \in \mathbb{R}^n\,.$$

In particular

$$\|\psi^j\|_1 \le c\, 2^{-2|j|}\,.$$

Proof (1) is obvious due to (3.3), (3.5) and (3.8). To prove (2) we show first of all the pointwise estimate

$$|2^{j|\alpha|}\partial^\alpha \hat{\psi}^j(\xi)| \le c_\alpha\, 2^{-2|j|}\eta(2^{-j}|\xi|) \tag{3.11}$$

for all $\xi \in \mathbb{R}^n$, $j \in \mathbb{Z}$, for all multi-indices $\alpha \in \mathbb{N}_0^n$ and with a function $\eta \in C_0^\infty\big(\frac{1}{4}, 4\big)$, $0 \le \eta \le 1$. By the definition of $\hat{\chi}_j$, (3.5) and the pointwise estimates

$$|\partial^\beta \hat{\psi}(\xi)| \le c_{\beta,N} \begin{cases} |\xi|^{\max(0,2-|\beta|)} & , \ |\xi| < 1 \\ |\xi|^{-N} & , \ |\xi| \ge 1 \end{cases} \quad , \quad \beta \in \mathbb{N}_0^n\,,$$

for every $N \in \mathbb{N}$, cf. (3.2), Leibniz's formula yields the estimate

$$\begin{aligned} |2^{j|\alpha|}\partial^\alpha \hat{\psi}^j(\xi)| &\le c \sum_{0\le\beta\le\alpha} 2^{j|\alpha|}|\partial^{\alpha-\beta}\tilde{\chi}(2^{-j}|\xi|)|\, |\partial^\beta \hat{\psi}(\xi)| \\ &\le c \sum_{0\le\beta\le\alpha} 2^{j|\beta|}\eta(2^{-j}|\xi|)\, |\partial^\beta \hat{\psi}(\xi)|\,. \end{aligned}$$

For $j \ge 0$ where only $|\xi| \sim 2^j$ has to be considered, we get (3.11) immediately, even with $2^{-N|j|}$ replacing $2^{-2|j|}$. For $j < 0$ and $|\xi| \sim 2^j < 1$ the right-hand side of the last inequality is bounded by

$$c \sum_{0\le\beta\le\alpha} \eta(2^{-j}|\xi|)\, 2^{j\,\max(|\beta|,2)} \le c\, 2^{-2|j|}\eta(2^{-j|\xi|})\,.$$

Now (3.11) is proved.

To estimate $\psi^j(x)$ we use for $m > \frac{n}{2}$ the identity

$$(1+|2^j x|^2)^m\, \psi^j(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 - 2^{2j}\Delta)^m\, \hat{\psi}_j(\xi)\, e^{ix\cdot\xi}\, d\xi\,.$$

By (3.11)

$$|(1-2^{2j}\Delta)^m\, \hat{\psi}_j^j(\xi)| \le C_{m,N}\, 2^{-2|j|}\eta(2^{-j}|\xi|)$$

for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. Hence

$$\|(1-2^{2j}\Delta)^m\, \hat{\psi}^j\|_1 \le C_m\, 2^{nj-2|j|}$$

and consequently $|(1+|2^j x|^2)^m\, \psi^j(x)| \le c\, 2^{nj-2|j|}$ proving part (2). ■

Lemma 3.2 *For $j \in \mathbb{Z}$ let $\mathcal{M}^j$ denote the maximal operator*

$$\mathcal{M}^j g(x) = \sup_{r>0} \int_{A_r} (|\psi_t^j| * |g|)(O(t)^T x) \, \frac{dt}{t}$$

where $A_r = \left[\frac{r}{16}, 16r\right]$. Then for $q \in (2,\infty)$ the operator T_j satisfies the estimate

$$\|T_j f\|_q \le c \, \|\psi^j\|_1^{1/2} \, |||\mathcal{M}^j|||_{(q/2)'}^{1/2} \, \|f\|_q$$

with a constant $c > 0$ independent of $j \in \mathbb{N}$. The term $|||\mathcal{M}^j|||_{(q/2)'}$ denotes the operator norm of the sublinear operator $\mathcal{M}^j$ on $L^{(q/2)'}(\mathbb{R}^n)$, where $\frac{1}{(q/2)'} + \frac{1}{q/2} = 1$.

Proof To estimate $\|T_j f\|_q$ we use the Littlewood-Paley decomposition (3.7) of $T_j f$ and find a function $0 \le g \in L^{(q/2)'}(\mathbb{R}^n)$ with $\|g\|_{(q/2)'} = 1$ (note that $q > 2$) such that

$$\begin{aligned} \|T_j f\|_q^2 &\le \frac{1}{c_1^2} \Big\| \int_0^\infty |\varphi_s * T_j f(\cdot)|^2 \, \frac{ds}{s} \Big\|_{q/2} \\ &= \frac{1}{c_1^2} \int_0^\infty \int_{\mathbb{R}^n} |\varphi_s * T_j f|^2 g \, dx \, \frac{ds}{s} \, . \end{aligned}$$

By (3.9), (3.10)

$$\varphi_s * T_j f(x) = \int_0^\infty O(t)^T (\varphi_s * \psi_t^j * f)(O(t)x) \, \frac{dt}{t} \, ,$$

where due to (3.5) $\varphi_s * \psi_t^j = 0$ unless $t \in A(s,j) := [2^{2j-4}s, \, 2^{2j+4}s]$. Since $\int_{t \in A(s,j)} \frac{dt}{t} = \log 2^8$ for every $j \in \mathbb{Z}$, $s > 0$, the inequality of Cauchy-Schwarz and the associativity of convolutions yield

$$\begin{aligned} |\varphi_s * T_j f(x)|^2 &\le c \int_{A(s,j)} |\big(\psi_t^j * (\varphi_s * f)\big)(O(t)x)|^2 \, \frac{dt}{t} \\ &\le c \, \|\psi^j\|_1 \int_{A(s,j)} (|\psi_t^j| * |\varphi_s * f|^2)(O(t)x) \, \frac{dt}{t} \, . \end{aligned}$$

Here we used the inequality

$$|\big(\psi_t^j * (\varphi_s * f)\big)(y)|^2 \le \|\psi_t^j\|_1 (|\psi_t^j| * |\varphi_s * f|^2)(y)$$

and that $\|\psi_t^j\|_1 = \|\psi^j\|$ for all $t > 0$. Thus

$$\|T_j f\|_q^2 \le c \|\psi^j\|_1 \int_0^\infty \int_{A(s,j)} \int_{\mathbb{R}^n} (|\psi_t^j| * |\varphi_s * f|^2)(x) \, g(O(-t)x) dx \, \frac{dt}{t} \, \frac{ds}{s} \, .$$

In the inner integral on $\mathbb{R}^n$ note that $\phi = |\psi_t^j|$ is radially symmetric; thus for arbitrary functions f and h we get $\int (\phi * f) h\, dx = \int f\, \phi * h\, dx$. Then the elementary identity $\phi * [g(O(-t)\cdot)] = (\phi * g)(O(-t)\cdot)$ implies that

$$\|T_j f\|_q^2 \le c\, \|\psi^j\|_1 \int_{\mathbb{R}^n} \int_0^\infty |\varphi_s * f|^2(x) \int_{A(s,j)} (|\psi_t^j| * g)(O(-t)x)\, \frac{dt}{t}\, \frac{ds}{s}\, dx\,.$$

Here the inner integral on $A(s,j)$ is bounded by $\mathcal{M}^j g(x)$ uniformly in $s > 0$. Now Hölder's inequality and (3.7) show that

$$\begin{aligned} \|T_j f\|_q^2 &\le c\, \|\psi^j\|_1 \Big(\int_{\mathbb{R}^n} \Big(\int_0 |\varphi_s * f|^2 \, \frac{ds}{s} \Big)^{q/2} dx \Big)^{2/q} \|\mathcal{M}^j g\|_{(q/2)'} \\ &\le c c_2 \|\psi^j\|_1\, \|f\|_q^2\, |||\mathcal{M}^j|||_{(q/2)'}\, \|g\|_{(q/2)'}\,. \end{aligned}$$

Since $\|g\|_{(q/2)'} = 1$, the proof is complete. ■

Lemma 3.3 *Let $\mathcal{M}$ denote the classical Hardy-Littlewood maximal operator on $\mathbb{R}^n$, i.e.,*

$$\mathcal{M}g(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)|\, dy\,,$$

and let $\widetilde{\mathcal{M}}_\theta g$ denote the "angular" maximal operator

$$\widetilde{\mathcal{M}}_\theta g(x) = \sup_{r>0} \int_{A_r} |g(O(t)^T x)| \frac{dt}{t}\,,$$

where $A_r = [\frac{r}{16}, 16r]$. Then $\mathcal{M}^j$ in Lemma 3.2 satisfies the estimates

$$\begin{aligned} \mathcal{M}^j g(x) &\le c\, 2^{-2|j|}\, \mathcal{M}(\widetilde{\mathcal{M}}_\theta g)(x) && \textit{for a.a. } x \in \mathbb{R}^n\,, \\ \|\mathcal{M}^j g\|_q &\le c\, 2^{-2|j|}\, \|g\|_q && \textit{for } 1 < q < \infty\,. \end{aligned}$$

Proof By Lemma 3.1 (2) $|\psi_t^j(x)| \le c\, 2^{-2|j|} h_{t2^{-2j}}(x)$ and consequently

$$\mathcal{M}^j g(x) \le c\, 2^{-2|j|} \sup_{r>0} \int_{A_r} (h_{t2^{-2j}} * |g|)(O(t)^T x)\, \frac{dt}{t}\,.$$

There exists a constant $c > 0$ independent of r, j such that $h_{t2^{-2j}} \le c h_{r2^{-2j}}$ for all $t \in A_r$. Hence

$$\begin{aligned} \mathcal{M}^j g(x) &\le c\, 2^{-2|j|} \sup_{r>0}\, h_{r2^{-2j}} * \int_{A_r} |g|(O(t)^T x)\, \frac{dt}{t} \\ &\le c\, 2^{-2|j|} \sup_{t>0}\, h_t * \widetilde{\mathcal{M}}_\theta g(x)\,. \end{aligned}$$

Note that h is a nonnegative, radially decreasing function and that $\int h_t\,dx \equiv c_0 > 0$ for all $t > 0$. Therefore we conclude by II§2.1 in [10] that

$$\sup_{t>0} h_t * \widetilde{\mathcal{M}}_\theta g(x) \le c_0 \mathcal{M}(\widetilde{\mathcal{M}}_\theta g)(x)$$

proving the first assertion.

For $q \in (1,\infty)$ the maximal operator $\mathcal{M}$ is bounded on $L^q(\mathbb{R}^n)$. Concerning $\widetilde{\mathcal{M}}_\theta$ we consider for given $g \in L^q(\mathbb{R}^n)$ its restriction

$$g_r(\theta) = g(r,\theta) \quad \text{or} \quad g_{r,x_3}(\theta) = g(r,\theta,x_3)$$

for $n = 2$ or $n = 3$, resp., when using polar or cylindrical coordinates. For $n = 2$ $g_r(\theta) \in L^q(0,2\pi)$ for a.a. $r > 0$ by Fubini's theorem, and with the classical one-dimensional Hardy-Littlewood maximal operator $\mathcal{M}_1$ on $L^q(0,2\pi)$

$$|\widetilde{\mathcal{M}}_\theta g(r,\theta)| \le c(\mathcal{M}_1 g_r)(\theta) \quad \text{for a.a. } r > 0\,. \tag{3.12}$$

Thus

$$\|\widetilde{\mathcal{M}}_\theta g\|_q^q \le c\int_0^\infty r\|\mathcal{M}_1 g_r\|^q_{L^q(0,2\pi)}dr \le c\int_0^\infty r\|g_r\|^q_{L^q(0,2\pi)}dr = c\,\|g\|_q^q$$

due to the L^q–boundedness of $\mathcal{M}_1$. For $n = 3$ the proof is analogous. ∎

End of the proof of Theorem 1.1 (1) Let $q \in (2,\infty)$. Then by Lemmata 3.1 – 3.3

$$\|T_j f\|_q \le c\,2^{-|j|}\cdot 2^{-|j|}\,\|f\|_q\,.$$

Thus $\sum_{j\in\mathbb{Z}} T_j$ converges in the L^q–operator norm and $T = \sum_{j\in\mathbb{Z}} T_j$ is bounded on $L^q(\mathbb{R}^n)^n$ for $q > 2$.

Closely related to T is the operator $T^*f(x) = \int K^*(x,y)f(y)dy$ with kernel

$$K^*(x,y) = \int_0^\infty \psi_t(O(t)y - x)O(t)\,\frac{dt}{t}\,.$$

Analogous arguments as before show that T^* is bounded on $L^q(\mathbb{R}^n)^n$ for every $q > 2$. Now let $q \in (1,2)$. Then for $f \in L^q(\mathbb{R}^n)^n$, $g \in L^{q'}(\mathbb{R}^n)^n$

$$|\langle Tf, g\rangle| = |\langle f, T^*g\rangle| \le \|f\|_q\, c\|g\|_{q'}$$

implying the L^q–boundedness of T. The case $q = 2$ had been considered in Proposition 2.2. ∎

Proof of Theorem 1.1 (2) It suffices to prove that every solution $u \in \mathcal{S}'(\mathbb{R}^3)^3$ of (1.3) when $f = 0$ and $\nabla^2 u \in L^q(\mathbb{R}^3)$ equals a polynomial of the form

$\alpha\omega+\beta\omega\wedge x+(\gamma x_1,\gamma x_2,\delta x_3)^T$. Given u define $\hat{v}(s',\varphi,\xi_3)=O(\varphi)^T\,\hat{u}(s',\varphi,\xi_3)\in\mathcal{S}'(\mathbb{R}^3)^3$ using cylindrical coordinates for $\xi\in\mathbb{R}^3$ and $s'=\sqrt{(\xi_1^2+\xi_2^2)}$. Then, cf. Section 2,

$$\nu|\xi|^2\hat{v}-\partial_\varphi\hat{v}=0\quad\text{in } \mathcal{S}'(\mathbb{R}^3)^3\,.$$

Let us show that $\langle\hat{v},\psi\rangle=0$ for all $\psi\in C_0^\infty(\mathbb{R}^3\backslash\{0\})^3$. Given ψ define

$$\psi_0(s',\varphi,\xi_3)=e^{-\nu|\xi|^2\varphi}\int_{-\infty}^{\varphi}e^{\nu|\xi|^2\varphi'}\psi(s',\varphi',\xi_3)\,d\varphi'\,.$$

Obviously $\psi_0\in C_0^\infty(\mathbb{R}^3\backslash\{0\})^3$ and $(\nu|\xi|^2+\partial_\varphi)\psi_0=\psi$. Consequently

$$\langle\hat{v},\psi\rangle=\langle\hat{v},(\nu|\xi|^2+\partial_\varphi)\psi_0\rangle=\langle(\nu|\xi|^2-\partial_\varphi)\hat{v},\psi_0\rangle=0$$

proving that $\operatorname{supp}\hat{v}\subset\{0\}$ and also $\operatorname{supp}\hat{u}\subset\{0\}$. Hence u is a polynomial. Since $\nabla^2u\in L^q(\mathbb{R}^3)$, u is even affine linear, $u(x)=a+Bx$ for $a\in\mathbb{R}^3$, $B\in\mathbb{R}^{3,3}$. Then (1.3) with $f=0$, i.e., $(\omega\wedge x)\cdot\nabla u=\omega\wedge u$, shows that $\omega\wedge a=0$ or equivalently $a=\alpha\omega$, $\alpha\in\mathbb{R}$. Furthermore Bx must be of the form $Bx=\beta\omega\wedge x+(\gamma x_1,\gamma x_2,\delta x_3)^T$ with constants $\beta,\gamma,\delta\in\mathbb{R}$. For $n=2$ one easily obtains that $a=0$ and $Bx=\beta\omega\wedge x+\gamma x$. ∎

Proof of Theorem 1.1 (3) As explained in Section 1 problem (1.2) may be reduced to (1.3) by solving the equation

$$\Delta p=\operatorname{div}f+\nu\Delta g+\partial_\theta g=\operatorname{div}F\quad\text{in } \mathbb{R}^n \tag{3.13}$$

where $F=f+\nu\nabla g+(\omega\wedge x)g$ satisfies the estimate $\|F\|_q\le c(\|f\|_q+\|\nu\nabla g+(\omega\wedge x)g\|_q)$. Thus $\operatorname{div}F$ may be considered as a continuous linear functional on $\hat{H}^{1,q'}(\mathbb{R}^n)$. Since the operator Δ is easily seen to be an isomorphism from $\hat{H}^{1,q}(\mathbb{R}^n)$ to its dual $\hat{H}^{1,q'}(\mathbb{R}^n)^*$ there exists a unique $p\in\hat{H}^{1,q}(\mathbb{R}^n)$ solving $\Delta p=\operatorname{div}F$ and satisfying $\|\nabla p\|_q\le c\|F\|_q$. Then part (1) yields a $u\in\hat{H}^{2,q}(\mathbb{R}^n)^n$ satisfying $-\nu\Delta u-\partial_\theta u+\omega\wedge u=f-\nabla p$ and the estimate $\|\nabla^2u\|_q+\|\partial_\theta u-\omega\wedge u\|_q\le c(\|f\|_q+\|\nabla p\|_q)$. In particular $(-\nu\Delta-\partial_\theta)\operatorname{div}u=\operatorname{div}f-\Delta p$ and consequently $(-\nu\Delta-\partial_\theta)(\operatorname{div}u-g)=0$. By the reasoning of part (2) we may conclude that $\operatorname{div}u-g$ is a polynomial and due to the integrability assumptions even a constant. Replacing u by $u-\gamma(x_1,x_2,0)^T$, if necessary, we get a solution (u,p) of (1.2) satisfying also $\operatorname{div}u=g$. The uniqueness assertion is proved as in part (2). ∎

References

[1] W. Borchers: Zur Stabilität und Faktorisierungsmethode für die Navier-Stokes-Gleichungen inkompressibler viskoser Flüssigkeiten. Habilitation Thesis, Univ. of Paderborn 1992

[2] Z.M. Chen, T. Miyakawa: Decay properties of weak solutions to a perturbed Navier-Stokes system in $\mathbb{R}^n$. Adv. Math. Sci. Appl. 7, 741–770 (1997)

[3] G.P. Galdi: An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I. Linearized Steady Problems. Springer Tracts in Natural Philosophy 38, 2nd edition 1998

[4] G.P. Galdi: On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications. In: S. Friedlander, D. Serre (eds.), Handbook of Mathematical Fluid Mechanics, Elsevier Science, 653–791 (2002)

[5] G.P. Galdi: Steady flow of a Navier–Stokes fluid around a rotating obstacle. Preprint 2002

[6] T. Hishida: An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle. Arch. Rational Mech. Anal. 150, 307–348 (1999)

[7] T. Hishida: The Stokes operator with rotation effect in exterior domains. Analysis 19, 51–67 (1999)

[8] Š. Nečasova: Some remarks on the steady fall of body in Stokes and Oseen flow. Acad. Sciences Czech Republic, Math. Institute, Preprint 143 (2001)

[9] E.M. Stein: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, N.J., 1970

[10] E.M. Stein: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton, N.J., 1993