Line-hyperline pairs of projective spaces and fundamental subgroups of linear groups

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Abstract

This article provides a self-contained, purely combinatorial local recognition of the graph on the non-intersecting line-hyperline pairs of the projective space $\mathbb{P}_n(\mathbb{F})$ for $n \geq 8$ and $\mathbb{F}$ a division ring with the exception of the case $n = 8$ and $\mathbb{F} = \mathbb{F}_2$. Consequences of that result are a characterization of the hyperbolic root group geometry of $SL_{n+1}(\mathbb{F})$, $\mathbb{F}$ a division ring, and a local recognition of certain groups containing a central extension of $PSL_{n+1}(\mathbb{F})$, $\mathbb{F}$ a field, using centralizers of p-elements.

1 Introduction and Preliminaries

The characterization of graphs and geometries using certain configurations that do or do not occur in some graph or geometry is a central problem in synthetic geometry. One class of such characterizations are the so-called local recognition theorems of locally homogeneous graphs. A graph Γ is called **locally homogeneous** if $\Gamma(x) \cong \Gamma(y)$ for all vertices $x, y \in \Gamma$, where $\Gamma(x)$ denotes the induced subgraph on the neighbors of x in Γ. A locally homogeneous graph Γ with $\Gamma(x) \cong \Delta$ is also called **locally** Δ. For some fixed graph Δ it is a natural question to ask for a classification of all connected graphs Γ that are locally Δ. A connected locally Δ graph Γ is **locally recognizable** if, up to isomorphism, Γ is the unique graph with that property. Several local recognition results of a lot of classes of graphs can be found in the literature. As an example we refer to the local recognition of the Kneser graphs by Jonathan I. Hall [6]; the Kneser graphs can be considered as 'thin' analogues of the graphs that are studied in this paper.

The present article focuses on graphs on line-hyperline pairs of projective spaces; more precisely, let $\mathbf{L}_n(\mathbb{F})$ denote the graph on the non-intersecting line-hyperline pairs of the projective space $\mathbb{P}_n(\mathbb{F})$ (where n is a natural number and $\mathbb{F}$ a division ring) in which two vertices are adjacent if the line of one vertex is contained in the hyperline of the other vertex and vice versa. Then the following holds.

Theorem 1
Let $n \geq 7$ and let $\mathbb{F}$ be a division ring, or let $n \geq 6$ and let $\mathbb{F}$ be a division ring of order at least three. If Γ is a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph, then Γ is isomorphic to $\mathbf{L}_{n+2}(\mathbb{F})$.

I believe that the exception $n = 6$ and $\mathbb{F} = \mathbb{F}_2$ in Theorem 1 merely arises as an exception of my particular proof, and is not a real exception. Note that, besides this exception, Theorem 1 is in a sense optimal. Indeed, if $\mathbb{F}$ is a field, besides the graph $\mathbf{L}_7(\mathbb{F})$ also the graph on the fundamental SL_2's of the group $E_6(\mathbb{F})$ with commuting as adjacency is connected and locally $\mathbf{L}_5(\mathbb{F})$, as can be read off the extended Dynkin diagram of type E_6. In fact, the graph $\mathbf{L}_n(\mathbb{F})$ can be described as a 'commuting fundamental SL_2' graph as well. There is a one-to-one correspondence between the non-intersecting line-hyperline pairs of $\mathbb{P}_n(\mathbb{F})$ and the fundamental SL_2's of the group $SL_{n+1}(\mathbb{F})$ by assigning a fundamental SL_2 to the pair consisting of its commutator and its centralizer in the natural module of $SL_{n+1}(\mathbb{F})$; two fundamental SL_2's commute if and only if the commutator of one is contained in the centralizer of the other and vice versa. The root subgroups of $SL_{n+1}(\mathbb{F})$ together with the fundamental SL_2's (and symmetrized containment as incidence) form a point-line geometry, called the **hyperbolic root group geometry**. This geometry is characterized in the following theorem as a consequence of Theorem 1.

Theorem 2
Let $n \geq 8$ and let $\mathbb{F}$ be a division ring, or let $n \geq 7$ and let $\mathbb{F}$ be a division ring of order at least three. Moreover, let $(\mathcal{P}, \mathcal{L}, \perp)$ be a partial linear space endowed with a symmetric relation $\perp$ on the point set such that $x \perp p$ and $x \perp q$, for distinct points p, q on some line l and an arbitrary point x, implies $x \perp y$ for all points y of l. Furthermore, suppose that, for any line $k \in \mathcal{L}$, the space $k^\perp$ is isomorphic to the hyperbolic root group geometry of $SL_n(\mathbb{F})$ with $l \perp m$ if and only if $[l, m] = 1$ for lines l, m inside $k^\perp$. If the graph $(\mathcal{L}, \perp)$ is connected, then $(\mathcal{P}, \mathcal{L})$ is isomorphic to the hyperbolic root group geometry of $SL_{n+2}(\mathbb{F})$.

The proofs of Theorem 1 and Theorem 2 are based on the reconstruction of the projective space $\mathbb{P}_n(\mathbb{F})$ from an arbitrary graph Γ isomorphic to $\mathbf{L}_n(\mathbb{F})$, called the interior projective space on Γ, cf. Section 3. The following theorem relates the automorphism group of $\mathbf{L}_n(\mathbb{F})$ to the automorphism group of $\mathbb{P}_n(\mathbb{F})$ and is an immediate consequence of that reconstruction.

Theorem 3
Let $n \geq 5$, and let Γ be a graph isomorphic to $\mathbf{L}_n(\mathbb{F})$. Then the interior projective space on Γ is isomorphic to $\mathbb{P}_n(\mathbb{F})$ or $\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}}$. In particular, the automorphism group of Γ is of the form $P\Gamma L_{n+1}(\mathbb{F}).2$ or $P\Gamma L_{n+1}(\mathbb{F})$ depending on whether or not the space $\mathbb{P}_n(\mathbb{F})$ admits a duality.

In case $\mathbb{P}_n(\mathbb{F})$ admits a duality, it is not clear whether $P\Gamma L_{n+1}(\mathbb{F}).2$ actually is a semidirect product or not. Of course, this problem is equivalent to the famous open problem whether a projective space that admits a duality also admits a polarity.

The final two results of this article are applications of Theorem 1 to group theory. If a locally recognizable graph Γ admits a group G of automorphisms that acts transitively on the set of all ordered triangles of Γ, then the local recognition of Γ implies a local recognition of G. The local recognition of the Kneser graphs in [6], for example, implies a local recognition of the symmetric groups along the lines of Theorem 27.1 of [3] or Theorem 2.5.5 of [5]. Line-hyperline graphs also admit a highly transitive group of automorphisms by Theorem 3.

If the division ring $\mathbb{F}$ has characteristic distinct from two, then $SL_2(\mathbb{F})$ admits a central involution, and one can prove the following local recognition theorem.

Theorem 4
Let $n \geq 7$, and let $\mathbb{F}$ be a division ring of characteristic distinct from 2. Let G be a group with subgroups A and B isomorphic to $SL_2(\mathbb{F})$, and denote the central involution of A by x and the central involution of B by y. Furthermore, assume the following holds:

- *$C_G(x) = X \times K$ with $K \cong GL_n(\mathbb{F})$ and $A \leq X$;*
- *$C_G(y) = Y \times J$ with $J \cong GL_n(\mathbb{F})$ and $B \leq Y$;*
- *A is a fundamental SL_2 of J;*
- *B is a fundamental SL_2 of K; and*
- *there exists an involution in $J \cap K$ that is the central involution of a fundamental SL_2 of both J and K.*

If $G = \langle J, K \rangle$, then (up to isomorphism) $PSL_{n+2}(\mathbb{F}) \leq G/Z(G) \leq PGL_{n+2}(\mathbb{F})$.

If the division ring $\mathbb{F}$ is finite, then $\mathbf{L}_n(\mathbb{F})$ is finite and so is its automorphism group. In order to state the final theorem recall some terminology from finite group theory, see, e.g., Section B of [4]. A finite group G is called **quasisimple** if and only if $G/Z(G)$ is simple and $G = [G, G]$; it is **semisimple** if and only if $G = G_1 \cdots G_r$ with G_i quasisimple and $[G_i, G_j] = 1$ for distinct $1 \leq i, j \leq r$. A **component** of a group G is a quasisimple subnormal subgroup of G. The **layer** of G is the subgroup $E(G)$ generated by all the components of G, with the understanding that $E(G) = 1$ if G does not have a component. By Theorem 3.5 of [4], the components of a group G commute pairwise, the layer $E(G)$ is the unique maximal normal semisimple subgroup of G, and any automorphism of G permutes the set of components of G.

Theorem 5
Let $n \geq 8$ and let $\mathbb{F}$ be a finite field, or let $n \geq 7$ and let $\mathbb{F}$ be a finite field of order at least three. If q denotes the order of $\mathbb{F}$, let p be a prime divisor of $q^2 - 1$. Furthermore let G be a group containing p-elements x and y, and assume the following holds:

- *$C_G(x)$ has a characteristic component K with $K/Z(K) \cong PSL_n(\mathbb{F})$;*

- *$C_G(y)$ has a characteristic component J with $J/Z(J) \cong PSL_n(\mathbb{F})$;*
- *x is contained in a subgroup A of J which is a fundamental SL_2 of J;*
- *y is contained in a subgroup B of K which is a fundamental SL_2 of K;*
- *the groups A and B commute with $E(J \cap K)$;*
- *there exists a group $C \cong SL_2(\mathbb{F})$ in $E(J \cap K)$ which is a fundamental SL_2 of both J and K; and*
- *there exist $k \in K$, $j \in J$, and a p-element $z \in C$ such that conjugation with j interchanges $\langle z \rangle$ and $\langle x \rangle$ and conjugation with k interchanges $\langle z \rangle$ and $\langle y \rangle$.*

If $G = \langle J, K \rangle$, then $G/Z(G) \cong PSL_{n+2}(\mathbb{F})$.

This article is organized as follows. Section 2 is a preliminary section, introducing some basic notation and providing some elementary facts. The purpose of Section 3 is to reconstruct the projective space $\mathbb{P}_n(\mathbb{F})$ from an arbitrary graph isomorphic to $\mathbf{L}_n(\mathbb{F})$ and to give a proof of Theorem 3, whereas Section 4 deals with the reconstruction of the (hyperbolic) root group geometry of $SL_{n+1}(\mathbb{F})$. Theorem 1 is proved in Section 5, and Theorem 2 is proved in Section 6. Finally, Section 7 provides proofs of Theorem 4 and Theorem 5.

A weaker version of Theorem 1 (for $n \geq 7$) can be found as Theorem 2.5.1 in the author's PhD thesis [5]. Analogues on graphs on commuting fundamental SL_2's of symplectic and unitary groups are contained in [5] as well, see Theorem 4.4.22 (locally $Sp_{2n}(\mathbb{F})$ for $n \geq 4$ and $\mathbb{F}$ a field) and Theorem 4.5.3 (locally $SU_n(\mathbb{K})$ for $n \geq 8$ and $\mathbb{K}$ a quadratic extension of some finite field $\mathbb{F}$); while Theorem 4.4.22 is optimal (the centralizer of a fundamental SL_2 of $F_4(\mathbb{F})$ is isomorphic to $Sp_6(\mathbb{F})$), there is still room for improvement of Theorem 4.5.3 (the centralizer of a fundamental SL_2 of ${}^2E_6(\mathbb{K})$ is isomorphic to $SU_6(\mathbb{K})$). Similar questions for graphs on reflection tori of linear groups are answered in [1] (Theorem 1.1) or [5] (Theorem 1.3.21), whereas graphs on reflection tori of unitary and orthogonal groups are treated in [2]. Notice that the methods used to prove Theorem 1.1 of [1] (or Theorem 1.3.21 of [5]) can be literally transscribed into a proof of Theorem 1 of the present article for $n \geq 7$. The case $n = 6$ needs a slightly different approach, which also covers the case $n \geq 7$. This approach is presented in this article.

2 Line-hyperline graphs of projective spaces

This section introduces precise notation and provides some general properties of line-hyperline graphs that are used throughout the whole article.

Definition 2.1 Let $n \in \mathbb{N}$, and let $\mathbb{F}$ be a division ring. Consider the projective space $\mathbb{P}_n(\mathbb{F})$ of (projective) dimension n over $\mathbb{F}$. The **line-hyperline graph** $\mathbf{L}(\mathbb{P}_n(\mathbb{F})) = \mathbf{L}_n(\mathbb{F})$ of $\mathbb{P}_n(\mathbb{F})$ is the graph whose vertices are the non-intersecting line-hyperline pairs of $\mathbb{P}_n(\mathbb{F})$ and in which a vertex (a, A) is adjacent to another vertex (b, B) (in symbols, $(a, A) \perp (b, B)$) if and only if $a \subseteq B$ and $b \subseteq A$.

For a vertex $\mathbf{x}$ of $\mathbf{L}_n(\mathbb{F})$ let $\mathbf{x}^\perp$ denote the set of all vertices at distance one from $\mathbf{x}$; for a set X of vertices, define the **perp of** X as $X^\perp := \bigcap_{\mathbf{x} \in X} \mathbf{x}^\perp$ and the **double perp of** X as $X^{\perp\perp} := (X^\perp)^\perp$. Sometimes $\mathbf{L}_n$ is used to denote $\mathbf{L}_n(\mathbb{F})$ if $\mathbb{F}$ is obvious or not important.

The projective space $\mathbb{P}_n(\mathbb{F})$ induces a Grassmann space of lines on $\mathbf{L}_n(\mathbb{F})$ whose points are of the form $v_l = \{(a, A) \in \mathbf{L}_n(\mathbb{F}) \mid a = l\}$ for a projective line l while its lines are of the form $v_{p,\pi} = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l \in \pi\}$ for an incident point-plane pair (p, π) of $\mathbb{P}_n(\mathbb{F})$. The sets of vertices v_l are called **exterior lines**. The dual construction yields **exterior hyperlines**. A point-line geometry on $\mathbf{L}_n(\mathbb{F})$ isomorphic to $\mathbb{P}_n(\mathbb{F})$ is defined as follows. Its lines are the exterior lines and its points are the full line pencils of exterior lines, i.e., a point is of the form $v_p = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l\}$ for a point p of $\mathbb{P}_n(\mathbb{F})$. A point v_p is called an **exterior point**, the resulting point-line geometry the **exterior projective space**. Dually, define **exterior hyperplanes** and the resulting **dual exterior projective space**. Besides the above geometries one can also induce root subgroup geometries of $SL_{n+1}(\mathbb{F})$ on $\mathbf{L}_n(\mathbb{F})$. The set $v_{p,H} = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l, L \subseteq H\}$, for a fixed point p and a fixed hyperplane $H \ni p$ of $\mathbb{P}_n(\mathbb{F})$, is called an **exterior root point** of $\mathbf{L}_n(\mathbb{F})$. Likewise, an **exterior root line** is defined as the union $v_{l,H} = \bigcup_{p \in l} v_{p,H}$, for a fixed line l and a fixed hyperplane $H \supseteq l$, or as the union $v_{p,L} = \bigcup_{H \supseteq L} v_{p,H}$, for a fixed hyperline L and a fixed point $p \in L$. The geometry of the exterior root points and the exterior root lines of $\mathbf{L}_n(\mathbb{F})$ is isomorphic to the root group geometry of $SL_{n+1}(\mathbb{F})$ and called the **exterior root group geometry** on $\mathbf{L}_n(\mathbb{F})$. Similarly, consider the geometry on the exterior root points of $\mathbf{L}_n(\mathbb{F})$ as points and the vertices of $\mathbf{L}_n(\mathbb{F})$ as lines. That geometry is isomorphic to the hyperbolic root group geometry of $SL_{n+1}(\mathbb{F})$ and is called the **exterior hyperbolic root group geometry** on $\mathbf{L}_n(\mathbb{F})$.

Proposition 2.2
Let $n \geq 3$. The graph $\mathbf{L}_n(\mathbb{F})$ is locally $\mathbf{L}_{n-2}(\mathbb{F})$.

Proof. Let $\mathbf{x} = (x, X)$ be a vertex of $\mathbf{L}_n(\mathbb{F})$. Then $X \cong \mathbb{P}_{n-2}(\mathbb{F})$. Identifying X with $\mathbb{P}_{n-2}(\mathbb{F})$ by means of this isomorphism, we establish an isomorphism $\mathbf{x}^\perp \cong \mathbf{L}(X)$. For any vertex $\mathbf{y} = (y, Y)$ adjacent to $\mathbf{x}$, we have $x \subseteq Y$, $y \subseteq X \setminus (X \cap Y)$, and $\dim(X \cap Y) = n-4$, so $(y, X \cap Y)$ belongs to $\mathbf{L}(X) \cong \mathbf{L}_{n-2}(\mathbb{F})$. Conversely, for any vertex of $\mathbf{L}(X)$, i.e., for any non-intersecting pair (z, Z) consisting of a line and an $(n-4)$-space of $\mathbb{P}_n(\mathbb{F})$ with $z \subseteq X$, $Z \subseteq X$, the pair $(z, \langle Z, x\rangle)$ is a vertex of $\mathbf{x}^\perp$. (Indeed, $z \cap \langle Z, x\rangle = \emptyset$, since $x \cap X = \emptyset$.) Clearly, the maps $(y, Y) \mapsto (y, X \cap Y)$ and $(z, Z) \mapsto (z, \langle Z, x\rangle)$ are each other's inverses. Moreover, these maps preserve adjacency, whence the claim. $\square$

Proposition 2.3
$\mathbf{L}_1$ consists of precisely one point; $\mathbf{L}_2$ is the disjoint union of singletons; $\mathbf{L}_3$ is the disjoint union of cliques of size two; the graphs $\mathbf{L}_4$, $\mathbf{L}_5$, and $\mathbf{L}_6$ are connected; the diameter of $\mathbf{L}_n$, $n \geq 7$, equals two.

Proof. The first four statements are obvious. For $n \geq 7$ let (x, X), (y, Y) be non-adjacent vertices of $\mathbf{L}_n$. The intersection $X \cap Y$ has dimension at least three. Since $x \cap X = \emptyset$ and $y \cap Y = \emptyset$, the intersection $\langle x, y\rangle \cap X \cap Y$ has at most dimension one, whence we can find a line $z \subseteq (X \cap Y) \setminus \langle x, y\rangle$. Moreover the dimension of $\langle x, y\rangle$ is at most three, and there is a hyperline $Z \supseteq \langle x, y\rangle$ with $z \cap Z = \emptyset$. □

Lemma 2.4
Let $n \geq 4$. Let $\mathbf{x} = (x, X)$, $\mathbf{y} = (y, Y)$ be vertices of $\mathbf{L}_n$ with $\{\mathbf{x}, \mathbf{y}\}^{\perp} \neq \emptyset$. Then the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ equals the set of vertices $\mathbf{z} = (z, Z)$ of $\mathbf{L}_n$ with $z \subseteq \langle x, y\rangle$ and $Z \supseteq X \cap Y$.

Proof. The vertices of $\{\mathbf{x}, \mathbf{y}\}^{\perp}$ are precisely the non-intersecting line-hyperline pairs (a, A) with $a \subseteq X \cap Y$ and $A \supseteq \langle x, y\rangle$. Let $\{(a_i, A_i) \in \{\mathbf{x}, \mathbf{y}\}^{\perp} \mid i \in I\}$ be the set of all these vertices, indexed by some set I. If $\{\mathbf{x}, \mathbf{y}\}^{\perp} \neq \emptyset$, then $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ consists of precisely those vertices (z, Z) with $z \subseteq \bigcap_{i\in I} A_i$ and $Z \supseteq \langle (a_i)_{i\in I}\rangle$. But obviously $\bigcap_{i\in I} A_i = \langle x, y\rangle$ and $\langle (a_i)_{i\in I}\rangle = X \cap Y$. □

The rest of this section is dedicated to the development of means to recover the projective spaces from graphs Γ isomorphic to $\mathbf{L}_n$ without making use of a particular isomorphism and coordinization. Recall that the **projective codimension** of a subspace X of $\mathbb{P}_n(\mathbb{F})$ is defined as the length of a maximal chain of proper subspaces of $\mathbb{P}_n(\mathbb{F})$ strictly containing X and strictly containing each other.

Definition 2.5 Vertices $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$ of $\mathbf{L}_n$ are **in relative position** (i, j), if $i = \dim\langle x, y\rangle$ and $j = \mathrm{codim}(\mathrm{X} \cap \mathrm{Y})$, where dim denotes projective dimension and codim projective codimension. Let $\mathbf{x}$, $\mathbf{y}$ be distinct vertices of $\mathbf{L}_n$ with $\{\mathbf{x}, \mathbf{y}\}^{\perp} \neq \emptyset$. The double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is called n**th minimal** if there exist vertices $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{a}_i \neq \mathbf{b}_i$, $1 \leq i \leq n$, with $\{\mathbf{a}_i, \mathbf{b}_i\}^{\perp} \neq \emptyset$ for all i and $\{\mathbf{a}_1, \mathbf{b}_1\}^{\perp\perp} \subsetneq \cdots \subsetneq \{\mathbf{a}_n, \mathbf{b}_n\}^{\perp\perp} = \{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ and there does not exist a longer chain of strict inclusions.

Clearly, vertices $\mathbf{x}$ and $\mathbf{y}$ of $\mathbf{L}_n$ have to be in relative positions $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 2)$, or $(3, 3)$. The following three lemmas will distinguish those cases up to duality.

Lemma 2.6
Let $n \geq 4$, and let $\mathbf{x}$, $\mathbf{y}$ be vertices of $\mathbf{L}_n$.

(i) *$\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1, 1)$ if and only if they are equal.*

(ii) *$\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1, 2)$ or $(2, 1)$ if and only if they are distinct, the perp $\{\mathbf{x}, \mathbf{y}\}^{\perp}$ is non-empty, and the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is first minimal.*

(iii) $\mathbf{x}$ *and* $\mathbf{y}$ *are in relative position* $(1,3)$, $(3,1)$, $(2,2)$, $(2,3)$, $(3,2)$, *or* $(3,3)$ *if and only if they are distinct and the perp* $\{\mathbf{x},\mathbf{y}\}^{\perp}$ *is empty or the double perp* $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ *is not first minimal.*

Proof. The first statement is obvious. Let the relative position of $\mathbf{x}$ and $\mathbf{y}$ be $(1,2)$ or $(2,1)$. Then $\{\mathbf{x},\mathbf{y}\}^{\perp} \neq \emptyset$. Indeed, up to duality we can assume that $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$ are in relative position $(1,2)$, so $x = y$, since x and y span a line. The intersection $X \cap Y$ contains a space of codimension 2, which is at least a line since $n \geq 4$. Hence there exists a common neighbor of $\mathbf{x}$ and $\mathbf{y}$. If $\mathbf{a}$, $\mathbf{b}$ are distinct vertices contained in $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$, then, by Lemma 2.4, $\mathbf{a}$ and $\mathbf{b}$ are in relative position $(1,2)$ or $(2,1)$ and, thus, $\{\mathbf{a},\mathbf{b}\}^{\perp} \neq \emptyset$. Again by Lemma 2.4, the double perps $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ and $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}$ coincide. If $\mathbf{x}$ and $\mathbf{y}$ are in any other relative position and $\{\mathbf{x},\mathbf{y}\}^{\perp}$ is empty, then there is nothing to prove. So let us assume $\{\mathbf{x},\mathbf{y}\}^{\perp} \neq \emptyset$. Then the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is given by Lemma 2.4 and it follows immediately that it contains vertices $\mathbf{a}$ and $\mathbf{b}$ in relative position $(1,2)$ or $(2,1)$. But, again by Lemma 2.4, this gives rise to a strictly smaller double perp. Hence $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is not minimal. Statements (ii) and (iii) follow. □

Lemma 2.7
Let $n \geq 5$, *and let* $\mathbf{x}$ *and* $\mathbf{y}$ *be vertices of* $\mathbf{L}_n$ *in relative position* $(1,3)$ *or* $(3,1)$. *Then* $\{\mathbf{x},\mathbf{y}\}^{\perp} \neq \emptyset$.

Proof. Let $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$. Up to duality we have $x = y$. The intersection $X \cap Y$ contains a space of codimension 3, which is at least a line, as $n \geq 5$. So there exists a common neighbor of $\mathbf{x}$ and $\mathbf{y}$. □

Lemma 2.8
Let $n \geq 5$, *and let* $\mathbf{x}$ *and* $\mathbf{y}$ *be vertices of* $\mathbf{L}_n$. *The property '*$\mathbf{x}$ *and* $\mathbf{y}$ *are in relative position* $(1,3)$ *or* $(3,1)$*' is characterized by*

- *the perp* $\{\mathbf{x},\mathbf{y}\}^{\perp}$ *is non-empty,*
- *the double perp* $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ *is second minimal, and*
- *there do not exist vertices* $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ *with* $\mathbf{a} \neq \mathbf{b}$ *and* $\mathbf{c} \neq \mathbf{d}$ *such that* $\{\mathbf{a},\mathbf{b}\}^{\perp\perp} \cap \{\mathbf{c},\mathbf{d}\}^{\perp\perp} = \emptyset$.

Proof. Up to duality we can assume that $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(3,1)$. Then, by Lemma 2.7, the perp $\{\mathbf{x},\mathbf{y}\}^{\perp}$ is non-empty. The double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is described by Lemma 2.4. From that description it is obvious that $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is second minimal. Now let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be vertices as stated in the hypothesis. By Lemma 2.4, the vertices $\mathbf{a}$ and $\mathbf{b}$, respectively $\mathbf{c}$ and $\mathbf{d}$, can only be in relative positions $(2,1)$ or $(3,1)$. But then Lemma 2.6 and Lemma 2.7 imply $\{\mathbf{a},\mathbf{b}\}^{\perp} \neq \emptyset$ and $\{\mathbf{c},\mathbf{d}\}^{\perp} \neq \emptyset$. There is a common vertex in $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}$ and $\{\mathbf{c},\mathbf{d}\}^{\perp\perp}$ if one pair is in relative position $(3,1)$. So suppose both pairs are in relative position $(2,1)$. Let $\mathbf{a}=(a,A)$, $\mathbf{b}=(b,B)$, $\mathbf{c}=(c,C)$, $\mathbf{d}=(d,D)$. We have $A=B=C=D$, since $\mathbf{x}$ and $\mathbf{y}$ are at relative position $(3,1)$, cf. Lemma 2.4. Moreover, both a, b and c, d span planes inside a 3-space, by Lemma 2.4. Two planes in a 3-space have to intersect in at least a line, and we have found a common vertex of $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}$ and $\{\mathbf{c},\mathbf{d}\}^{\perp\perp}$.

Conversely, let $\mathbf{x}$ and $\mathbf{y}$ be in arbitrary relative position. Suppose $\{\mathbf{x},\mathbf{y}\}^{\perp}\neq\emptyset$. Then another application of Lemma 2.4 shows, that $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ only can be second minimal if $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1,3)$, $(3,1)$, or $(2,2)$. But if they are in relative position $(2,2)$, then we can find vertices $\mathbf{a}=(a,A)$, $\mathbf{b}=(b,B)$ in relative position $(1,2)$ and $\mathbf{c}=(c,C)$, $\mathbf{d}=(d,D)$ in relative position $(1,2)$ contained in $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ and such that $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}\cap\{\mathbf{c},\mathbf{d}\}^{\perp\perp}=\emptyset$. (Note that $\{\mathbf{a},\mathbf{b}\}^{\perp}\neq\emptyset$ and $\{\mathbf{c},\mathbf{d}\}^{\perp}\neq\emptyset$ by Lemma 2.6.) Indeed, we have $a=b$ and $c=d$. But since we can choose both $a=b$ and $c=d$ freely in a plane, they only have to intersect in a point, and we have $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}\cap\{\mathbf{c},\mathbf{d}\}^{\perp\perp}=\emptyset$. □

Lemma 2.9
Let $n\geq 5$. Let k, l, and m be distinct exterior lines of $\mathbf{L}_n(\mathbb{F})$. They intersect in a common exterior point (i.e., they are contained in a line pencil), if there exist vertices $\mathbf{a}\in k$, $\mathbf{b}\in l$, $\mathbf{c}\in m$ that are pairwise in relative position $(2,1)$ such that $\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$ contains vertices $\mathbf{x}$, $\mathbf{y}$ in relative position $(3,1)$ with $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$.

Proof. Suppose $\mathbf{a}=(k,K)$, $\mathbf{b}=(l,L)$, $\mathbf{c}=(m,M)$ with $K=L=M$. The lines k, l, m mutually intersect, since (k,K), (l,L), and (m,M) are in mutual relative position $(2,1)$. But, by Lemma 2.4, the lines k, l, and m together span a projective 3-space, because $\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$ contains vertices $\mathbf{x}$, $\mathbf{y}$ in relative position $(3,1)$ with $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$. The claim follows, because three mutually intersecting lines spanning a 3-space necessarily intersect in one point. □

Lemma 2.10
Let $n\geq 5$, and let $\mathbf{a}$ and $\mathbf{b}$ be vertices in relative position $(2,1)$. Then there exists a third vertex $\mathbf{c}$ in relative position $(2,1)$ to both $\mathbf{a}$ and $\mathbf{b}$ such that $\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$ contains vertices $\mathbf{x}$, $\mathbf{y}$ in relative position $(3,1)$ with $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$.

Proof. Suppose $\mathbf{a}=(a,A)$, $\mathbf{b}=(b,B)$ with $A=B$. The lines a and b intersect in a point, p say. Let q be a point outside the plane $\langle a,b\rangle$ such that the line pq does not intersect the hyperline A. Then $\mathbf{c}=(pq,A)$ is a vertex with the required properties. □

3 The interior projective space

The purpose of this section is to reconstruct the projective space $\mathbb{P}_n(\mathbb{F})$ from an arbitrary graph isomorphic to $\mathbf{L}_n(\mathbb{F})$, and to provide a proof of Theorem 3. The notation introduced in Section 2 will be used freely and without reference. Throughout this section let $n \in \mathbb{N}$ and let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$.

Definition 3.1 Let $n \geq 5$. Define a reflexive relation $\approx$ on the vertex set of a graph Γ isomorphic to $\mathbf{L}_n$ where for two distinct vertices $\mathbf{x}$, $\mathbf{y}$ with $\{\mathbf{x},\mathbf{y}\}^\perp \neq \emptyset$ we have $\mathbf{x} \approx \mathbf{y}$ if

- the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is first or second minimal, and
- there do not exist vertices $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ with $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{c} \neq \mathbf{d}$ such that $\{\mathbf{a},\mathbf{b}\}^{\perp\perp} \cap \{\mathbf{c},\mathbf{d}\}^{\perp\perp} = \emptyset$.

Lemma 3.2
Let $n \geq 5$. On the vertex set of Γ there are unique equivalence relations $\approx^l$ and $\approx^h$ such that $\approx = \approx^l \cup \approx^h$, and $\approx^l \cap \approx^h$ is the identity relation. Moreover, for a fixed isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$, we either have

- *$\approx^l$ is the relation 'equal, in relative position $(1,2)$, or in relative position $(1,3)$' and $\approx^h$ is the relation 'equal, in relative position $(2,1)$, or in relative position $(3,1)$', or*
- *$\approx^l$ is the relation 'equal, in relative position $(2,1)$, or in relative position $(3,1)$' and $\approx^h$ is the relation 'equal, in relative position $(1,2)$, or in relative position $(1,3)$'.*

Proof. Vertices $\mathbf{x}$, $\mathbf{y}$ of Γ are in relation $\approx$ if and only if their images (x,X) and (y,Y) in $\mathbf{L}_n(\mathbb{F})$ under some isomorphism $\Gamma \to \mathbf{L}_n(\mathbb{F})$ are in relative position $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, or $(3,1)$. Let us consider equivalence relations that are subrelations of $\approx$. Obviously, the identity relation is such an equivalence relation. Moreover, the relations 'equal, in relative position $(1,2)$, or in relative position $(1,3)$' and 'equal, in relative position $(2,1)$, or in relative position $(3,1)$' are equivalence relations. If we have vertices $\mathbf{x} = (x,X)$, $\mathbf{y} = (y,Y)$, $\mathbf{z} = (z,Z)$ of $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ such that $\mathbf{x}$, $\mathbf{y}$ are in relative position $(1,\cdot)$ and $\mathbf{x}$, $\mathbf{z}$ are in relative position $(\cdot,1)$, then $y \neq z$ and $Y \neq Z$ and $\mathbf{y}$, $\mathbf{z}$ cannot be in relative position $(1,\cdot)$ or $(\cdot,1)$. Consequently, if we want to find two sub-equivalence relations $\approx^l$ and $\approx^h$ of $\approx$ whose union equals $\approx$, then either of $\approx^l$ and $\approx^h$ has to be a subrelation of 'equal, in relative position $(1,2)$, or in relative position $(1,3)$' or of 'equal, in relative position $(2,1)$, or in relative position $(3,1)$'. It follows that the equivalence relations $\approx^l$ and $\approx^h$ have to be of the form as given in the lemma. $\square$

Definition 3.3 Let $n \geq 5$, and let $\mathbf{x}$ be a vertex of Γ. With $\approx^l$ and $\approx^h$ on Γ as in Lemma 3.2, write $[\mathbf{x}]^l$ to denote the equivalence class of $\approx^l$ containing $\mathbf{x}$ and $[\mathbf{x}]^h$ to denote the equivalence class of $\approx^h$ containing $\mathbf{x}$. Refer to $[\mathbf{x}]^l$ as the **interior line** on $\mathbf{x}$ and to $[\mathbf{x}]^h$ as the **interior hyperline** on $\mathbf{x}$ of Γ.

Proposition 3.4
Let $n \geq 5$. There is a one-to-one correspondence between interior lines of Γ and exterior lines of $\mathbf{L}_n$. In particular, any isomorphism $\phi : \Gamma \to \mathbf{L}_n$ induces such a correspondence up

to duality; the image under ϕ of an interior line is an exterior line. A similar statement is true for interior hyperlines of Γ.

Proof. This follows directly from Lemma 3.2. $\square$

Definition 3.5 Let $n \geq 5$. For distinct vertices $\mathbf{x}$, $\mathbf{y}$ of Γ denote by $\mathbf{x} \approx_1^h \mathbf{y}$ that $\mathbf{x} \approx^h \mathbf{y}$ and $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is minimal, and by $\mathbf{x} \approx_2^h \mathbf{y}$ that $\mathbf{x} \approx^h \mathbf{y}$ and $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is second minimal.

Definition 3.6 Let $n \geq 5$. A set S of interior lines of Γ is called **full** if

(i) $|S| \geq 2$;

(ii) for distinct interior lines $k, l \in S$ there exist vertices $\mathbf{a} \in k$, $\mathbf{b} \in l$ with $\mathbf{a} \approx_1^h \mathbf{b}$;

(iii) for vertices $\mathbf{a}$, $\mathbf{b}$ with $[\mathbf{a}]^l$, $[\mathbf{b}]^l \in S$ and $\mathbf{a} \approx_1^h \mathbf{b}$, there exists a vertex $\mathbf{c}$ satisfying $\mathbf{a} \approx_1^h \mathbf{c}$ and $\mathbf{b} \approx_1^h \mathbf{c}$ such that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}^{\perp\perp}$ contains vertices $\mathbf{x}$, $\mathbf{y}$ with $\mathbf{x} \approx_2^h \mathbf{y}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}^{\perp\perp} = \{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$; and

(iv) any interior line $[\mathbf{c}]^l$ containing a vertex $\mathbf{c}$ as in (iii) is also contained in S.

Proposition 3.7
Let $n \geq 5$. Up to duality, any isomorphism $\Gamma \to \mathbf{L}_n$ maps a full set of interior lines of Γ onto a full line pencil of exterior lines of $\mathbf{L}_n$ and vice versa.

Proof. By Lemma 2.9 and up to interchange of $\approx^l$ and $\approx^h$, the image $\phi(S)$, where ϕ denotes an isomorphism $\Gamma \to \mathbf{L}_n$, of a full set S of interior lines of Γ is contained in a pencil of exterior lines of $\mathbf{L}_n$, through some exterior point p, say. Let l be an exterior line of $\mathbf{L}_n$ incident with p. The full set S contains distinct lines a and b. If $\phi(a)$, $\phi(b)$, and l span a 3-space, then $\phi^{-1}(l)$ is contained in the full set by definition. So suppose l lies in the plane $\langle \phi(a), \phi(b) \rangle$. Then the full set contain a some line c such that $\phi(a)$, $\phi(b)$, $\phi(c)$ span a 3-space, by Lemma 2.10 and the definition of a full set. But then also l, $\phi(b)$ and $\phi(c)$ span a 3-space, and $\phi^{-1}(l)$ is contained in the full set. $\square$

Definition 3.8 Let $n \geq 5$. Let S be a full set of interior lines of Γ. The **interior point** $p(S)$ of Γ is the union $\bigcup_{l \in S} l$ over all interior lines in the full set S. The geometry of interior points and interior lines with symmetrized containment as incidence is called the **interior projective space on** Γ. Dually, define **interior hyperplanes** and the **dual interior projective space**.

Proposition 3.9
Let $n \geq 5$. Up to duality there exists an isomorphism between the interior projective space on Γ and the exterior projective space on $\mathbf{L}_n(\mathbb{F})$. The same statement holds true for the dual interior projective space. $\square$

Theorem 3 follows immediately from Proposition 3.9.

4 Geometries on interior root points

This section provides constructions of the exterior (hyperbolic) root group geometries on $\mathbf{L}_n(\mathbb{F})$ in terms of exterior points, exterior lines, exterior hyperplanes, and exterior hyperlines. Then Propositions 3.4 and 3.7 will give us means to describe these geometries intrinsically on an arbitrary graph Γ isomorphic to $\mathbf{L}_n(\mathbb{F})$. Throughout this section let $n \in \mathbb{N}$ and let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$.

Lemma 4.1
Let $n \geq 5$, and let $v_{p,H}$ and $v_{q,I}$ be two distinct exterior root points of $\mathbf{L}_n(\mathbb{F})$. Then we have $|v_{p,H} \cap v_{q,I}| \leq 1$. More precisely, the points p and q and the hyperplanes H and I are distinct and the line pq does not intersect the hyperline $H \cap I$ if and only if $|v_{p,H} \cap v_{q,I}| = 1$.

Proof. Suppose $|v_{p,H} \cap v_{q,I}| \neq \emptyset$, i.e., in $\mathbb{P}_n(\mathbb{F})$ there exist a line l and a hyperline L with $l \supseteq \langle p, q \rangle$, $L \subseteq H \cap I$, and $l \cap L = \emptyset$. Assume there exists another line-hyperline pair (m, M) satisfying these conditions. If it is distinct from (l, L), then $l \neq m$ or $L \neq M$. Up to duality, we may assume $L \neq M$. Then immediately $H = I$, whence $p \neq q$. But then $l = m = pq$ is contained in $H = I$, which have hyperplanes L and M. Hence $l \cap L \neq \emptyset$, a contradiction.

If $p \neq q$, $H \neq I$ and $pq \cap H \cap I = \emptyset$, then $(pq, H \cap I)$ is a vertex of $\mathbf{L}_n(\mathbb{F})$ contained in $v_{p,H} \cap v_{q,I}$. Conversely, suppose there exists such a vertex. This implies $p \neq q$ and $H \neq I$. But then the only candidate for being contained in $v_{p,H} \cap v_{q,I}$ is $(pq, H \cap I)$, whence $pq \cap H \cap I = \emptyset$. □

For the next lemma notice that an exterior hyperplane of $\mathbf{L}_n(\mathbb{F})$ is not a hyperplane of the exterior projective space on $\mathbf{L}_n(\mathbb{F})$. However, there is an obvious one-to-one correspondence between exterior hyperplanes and hyperplanes of the exterior projective space, by the map

$$v_H = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid L \subseteq H\} \mapsto \bigcup_{p \in H} v_p = \bigcup_{p \in H} \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l\}.$$

Therefore there is no harm done if one speaks of incidence between exterior points and exterior hyperplanes and rather means incidence between exterior points and the images of exterior hyperplanes by means of this map.

Lemma 4.2
Let $n \geq 5$. An exterior point v_p and an exterior hyperplane v_H of $\mathbf{L}_n(\mathbb{F})$ are non-incident if and only if any exterior line v_l incident with v_p contains a vertex contained in an exterior hyperline v_L incident with v_H and vice versa.

Proof. Suppose v_p and v_H are non-incident and let v_l be an exterior line incident with v_p. The set v_l consists of all vertices of $\mathbf{L}_n(\mathbb{F})$ having l as the first coordinate. The second coordinate ranges over all hyperlines L that do not intersect l. By the isomorphism between the exterior projective space on $\mathbf{L}_n(\mathbb{F})$ and $\mathbb{P}_n(\mathbb{F})$ that maps v_p onto p, v_l onto l, v_H onto H, also p is incident with l and non-incident with H. Hence l intersects H in a unique point. But then there exists a hyperline M non-intersecting l that is contained in H. The vertex (l, M) is contained in the exterior hyperline v_M, which is incident with v_H. Similarly, any exterior hyperline incident with v_H contains a vertex contained in an exterior line incident with v_p. Conversely, suppose v_p and v_H are incident. Choose an exterior line v_l through v_p such that l is contained in H. Now, a hyperline that does not intersect l cannot be contained in H. $\square$

In view of the preceding lemma, in a graph Γ isomorphic to $\mathbf{L}_n(\mathbb{F})$, an interior point p and an interior hyperplane H are called **non-incident** if and only if any interior line l incident with p contains a vertex of Γ contained in an interior hyperline L incident with H and vice versa. Conversely, an interior point and an interior hyperplane are **incident** if they are not non-incident.

Definition 4.3 Let $n \geq 5$. An **interior root point** of Γ is the intersection of an interior point with an incident interior hyperplane. An **interior root line** is of the form

$$\bigcup_{\text{interior point } p \in l} p \cap H$$

for a fixed interior line l contained in the fixed interior hyperplane H or

$$\bigcup_{\text{interior hyperplane } H \supseteq L} p \cap H$$

for a fixed interior hyperline L containing the fixed interior point p.

Proposition 4.4
Let $n \geq 5$. The following hold.

(i) *The geometry of exterior root points and exterior root lines on $\mathbf{L}_n(\mathbb{F})$ with symmetrized containment as incidence is isomorphic to the root group geometry of $SL_{n+1}(\mathbb{F})$.*

(ii) *The geometry of exterior root points and vertices of $\mathbf{L}_n(\mathbb{F})$ with symmetrized containment as incidence is isomorphic to the hyperbolic root group geometry of $SL_{n+1}(\mathbb{F})$.*

(iii) *The geometry of interior root points and interior root lines on Γ with symmetrized containment as incidence is isomorphic to the root group geometry of $SL_{n+1}(\mathbb{F})$.*

(iv) *The geometry of interior root points and vertices of Γ with symmetrized containment as incidence is isomorphic to the hyperbolic root group geometry of $SL_{n+1}(\mathbb{F})$.*

In particular, there is a one-to-one correspondence between the vertices of Γ and the lines of the hyperbolic root group geometry of $SL_{n+1}(\mathbb{F})$. The interior root points of Γ correspond to the full line pencils of the (hyperbolic) root group geometry.

Proof. Notice that by Propositions 3.4 and 3.7 plus the Definition 4.3 of interior root points and interior root lines, Statement (i) is equivalent to Statement (iii) and Statement (ii) is equivalent to Statement (iv). However, the first two statements follow the definition of exterior root points and exterior root lines in Section 2.

The additional claim about the vertices of Γ follows from the isomorphism between Γ and $\mathbf{L}_n(\mathbb{F})$ and the fact that lines of the hyperbolic root group geometry of $SL_{n+1}(\mathbb{F})$ correspond to non-intersecting line-hyperline pairs of the projective space $\mathbb{P}_n(\mathbb{F})$ by assigning the fundamental SL_2 to the pair consisting of its commutator and its centralizer on the natural module of $SL_{n+1}(\mathbb{F})$. The claim about the line pencils follows from the fact that interior points correspond to full line pencils of interior lines (cf. Proposition 3.7), similarly interior hyperplanes correspond to full pencils of interior hyperlines; intersecting an interior point with an incident interior hyperplane, we get a full line pencil of the hyperbolic root group geometry. □

The geometries of interior objects as in the theorem are called the **interior (hyperbolic) root group geometry on** Γ, respectively.

5 Locally line-hyperline graphs

In this section we prove Theorem 1. Throughout the whole section let $n \geq 7$ and let $\mathbb{F}$ be a division ring, or let $n \geq 6$ and let $\mathbb{F}$ be a division ring of order at least three; let Γ be a simply connected, locally $\mathbf{L}_n(\mathbb{F})$ graph. We recall that a graph is called **simply connected** if it is connected and any cycle in Γ can be decomposed into triangles.

Notice that interior points and interior lines only exist on the perps of Γ and may differ on different perps; it is one task of this section to show that there exist well-defined notions of global points and global lines. To avoid confusion, we will index each interior point and each interior line by the vertex whose perp it belongs to. We call those points and lines **local points** and **local lines**; similarly, we use the notion of **local equivalence relations**.

Lemma 5.1
Let $\mathbf{x}$ and $\mathbf{y}$ be adjacent vertices of Γ. Then there exists a choice of local equivalence relations $\approx^l_{\mathbf{x}}$ and $\approx^l_{\mathbf{y}}$ such that the intersections of $\approx^l_{\mathbf{x}}$ and $\approx^l_{\mathbf{y}}$ to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ coincide.

Proof. Choose a local equivalence relation $\approx^l_{\mathbf{x}}$ on $\mathbf{x}^\perp$. This induces coordinates on $\mathbf{x}^\perp$, so we can identify $\mathbf{x}^\perp$ with $\mathbf{L}_n(\mathbb{F})$, inducing coordinates on $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cong \mathbf{L}_{n-2}(\mathbb{F})$, which in turn induces coordinates of $\mathbf{y}^\perp \cong \mathbf{L}_n(\mathbb{F})$ using Lemma 2.6(ii). A choice of $\approx^l_{\mathbf{y}}$ in accordance with the coordinates on $\mathbf{y}^\perp$ finishes the proof. □

Lemma 5.2
There is a choice of local equivalence relations $(\approx^l_{\mathbf{x}})_{\mathbf{x}\in\Gamma}$ *such that for any two adjacent vertices* $\mathbf{x}$ *and* $\mathbf{y}$ *the restrictions of* $\approx^l_{\mathbf{x}}$ *and of* $\approx^l_{\mathbf{y}}$ *to* $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ *coincide.*

Proof. Suppose that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ is a triangle of Γ. In view of Lemma 5.1 we may assume that $\approx^l_{\mathbf{x}}$ and $\approx^l_{\mathbf{y}}$ have the same restriction to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and that $\approx^l_{\mathbf{x}}$ and $\approx^l_{\mathbf{z}}$ have the same restriction to $\mathbf{x}^\perp \cap \mathbf{z}^\perp$. Let $l_{\mathbf{x}}$ be an interior line of $\mathbf{x}^\perp$ such that $l_{\mathbf{x}} \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp \neq \emptyset$. By analysis of $\mathbf{x}^\perp$ we find two vertices, say $\mathbf{u}$ and $\mathbf{v}$, of $l_{\mathbf{x}} \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$. The above choices of local equivalence relations imply that $\mathbf{u}$ and $\mathbf{v}$ belong to both $\approx^l_{\mathbf{y}}$ and $\approx^l_{\mathbf{z}}$, which implies that $\approx^l_{\mathbf{y}}$ and $\approx^l_{\mathbf{z}}$ have the same restriction to $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ by Lemma 2.6(ii). Since Γ is assumed to be simply connected, the lemma follows immediately from the triangle analysis. □

Definition 5.3 Fix a choice of $(\approx^l_{\mathbf{x}})_{\mathbf{x}\in\Gamma}$ as given in Lemma 5.2 and define $\approx^l := \bigcup_{\mathbf{x}\in\Gamma} \approx^l_{\mathbf{x}}$.

Lemma 5.4
Let $\mathbf{x}$ *be a vertex of* Γ. *Then the restriction of* $\approx^l$ *to* $\mathbf{x}^\perp$ *coincides with* $\approx^l_{\mathbf{x}}$. *In particular, suppose that* $\mathbf{x}$ *and* $\mathbf{y}$ *are vertices of* Γ *such that* $\mathbf{x} \approx^l_{\mathbf{u}} \mathbf{y}$ *for some vertex* $\mathbf{u} \in \{\mathbf{x},\mathbf{y}\}^\perp$. *Then* $\mathbf{x} \approx^l_{\mathbf{v}} \mathbf{y}$ *for every vertex* $\mathbf{v} \in \{\mathbf{x},\mathbf{y}\}^\perp$.

Proof. Obviously, $\approx^l_{|\mathbf{v}^\perp\times\mathbf{v}^\perp} \supseteq \approx^l_{\mathbf{v}}$. Conversely assume there exist $\mathbf{x},\mathbf{y} \in \mathbf{v}^\perp$ with $\mathbf{x} \approx^l \mathbf{y}$. Since $\approx^l := \bigcup_{\mathbf{x}\in\Gamma} \approx^l_{\mathbf{x}}$, there exists a vertex $\mathbf{u} \in \{\mathbf{x},\mathbf{y}\}^\perp$ with $\mathbf{x} \approx^l_{\mathbf{u}} \mathbf{y}$. In $\mathbf{y}^\perp$ denote the intersection of $\langle u_{\mathbf{y}}, v_{\mathbf{y}}\rangle$ with $U_{\mathbf{y}}$ by $b_{\mathbf{y}}$; similarly, denote in $\mathbf{x}^\perp$ the intersection of $\langle u_{\mathbf{x}}, v_{\mathbf{x}}\rangle$ with $U_{\mathbf{x}}$ by $c_{\mathbf{x}}$. First we will prove that we can assume that $b_{\mathbf{y}} \cap V_{\mathbf{y}} = \emptyset$ and $c_{\mathbf{x}} \cap V_{\mathbf{x}} = \emptyset$, or, equivalently, $\langle v_{\mathbf{x}}, u_{\mathbf{x}}\rangle \cap V_{\mathbf{x}} \cap U_{\mathbf{x}} = \emptyset$ and $\langle v_{\mathbf{y}}, u_{\mathbf{y}}\rangle \cap V_{\mathbf{y}} \cap U_{\mathbf{y}} = \emptyset$. Indeed, consider $\mathbf{u}^\perp$; the spaces $b_{\mathbf{y}}$ and $c_{\mathbf{x}}$ arise as $b_{\mathbf{u}}$, respectively $c_{\mathbf{u}}$. Choose some line $u'_{\mathbf{u}}$ in $X_{\mathbf{u}} \cap Y_{\mathbf{u}}$. This line occurs as the line $u'_{\mathbf{x}}$ in $\mathbf{x}^\perp$ and as the line $u'_{\mathbf{y}}$ in $\mathbf{y}^\perp$. Define $c'_{\mathbf{x}} := \langle v_{\mathbf{x}}, u'_{\mathbf{x}}\rangle \cap V_{\mathbf{x}}$ and $c''_{\mathbf{x}} := \langle u_{\mathbf{x}}, c'_{\mathbf{x}}\rangle \cap U_{\mathbf{x}}$. Similary, let $b'_{\mathbf{y}} := \left\langle v_{\mathbf{y}}, u'_{\mathbf{y}}\right\rangle \cap V_{\mathbf{y}}$ and $b''_{\mathbf{y}} := \left\langle u_{\mathbf{y}}, b'_{\mathbf{y}}\right\rangle \cap U_{\mathbf{y}}$. The spaces $b''_{\mathbf{y}}$ and $c''_{\mathbf{x}}$ arise as $b''_{\mathbf{u}}$, respectively $c''_{\mathbf{u}}$ in $\mathbf{u}^\perp$. Let $b'''_{\mathbf{u}} := \langle x_{\mathbf{u}}, b''_{\mathbf{u}}\rangle \cap X_{\mathbf{u}}$. Choose a hyperline $U'_{\mathbf{u}}$ that contains $x_{\mathbf{u}}$ and does not intersect $c''_{\mathbf{u}}$, $b'''_{\mathbf{u}}$, or $u'_{\mathbf{u}}$, which is possible if $\mathbb{F} \neq \mathbb{F}_2$. Then the vertex $\mathbf{u}_0 := (u'_{\mathbf{u}}, U'_{\mathbf{u}})$ is adjacent to $\mathbf{x}$, $\mathbf{u}$, $\mathbf{y}$, and we have $\langle v_{\mathbf{x}}, u'_{\mathbf{x}}\rangle \cap V_{\mathbf{x}} \cap U'_{\mathbf{x}} = \emptyset$ and $\left\langle v_{\mathbf{y}}, u'_{\mathbf{y}}\right\rangle \cap V_{\mathbf{y}} \cap U'_{\mathbf{y}} = \emptyset$.

So now assume that $b_{\mathbf{y}} \cap V_{\mathbf{y}} = \emptyset$ and $c_{\mathbf{x}} \cap V_{\mathbf{x}} = \emptyset$. Choose a hyperline $W_{\mathbf{u}}$ that contains $b_{\mathbf{u}}$ and $c_{\mathbf{u}}$, but not $x_{\mathbf{u}}$, if such a choice is possible. Then let $\mathbf{w} := (x_{\mathbf{u}}, W_{\mathbf{u}})$. There exists a hyperline $W^1_{\mathbf{u}}$ containing $b_{\mathbf{u}}$ but not $x_{\mathbf{u}}$ that intersects both $W_{\mathbf{u}}$ and $Y_{\mathbf{u}}$ in a hyperplane of either. Let $\mathbf{w}_1 := (x_{\mathbf{u}}, W^1_{\mathbf{u}})$. In $\mathbf{y}^\perp$ denote $U_{\mathbf{y}} \cap V_{\mathbf{y}}$ by $UV_{\mathbf{y}}$. This space arises as a space $UV_{\mathbf{u}}$ in $\mathbf{u}^\perp$. The intersection $UV_{\mathbf{u}} \cap W^1_{\mathbf{u}}$ contains a projective line $l_{\mathbf{u}}$. The span $\langle x_{\mathbf{u}}, b_{\mathbf{u}}\rangle$ does not intersect $l_{\mathbf{u}}$, so we can find a hyperline $L_{\mathbf{u}}$ that does not intersect $l_{\mathbf{u}}$ but contains $\langle x_{\mathbf{u}}, b_{\mathbf{u}}\rangle$. The vertex $\mathbf{l} := (l_{\mathbf{u}}, L_{\mathbf{u}})$ is adjacent to $\mathbf{u}$, $\mathbf{w}_1$, $\mathbf{y}$, and $\mathbf{v}$. Local analysis of $\mathbf{l}^\perp$ shows that $\mathbf{w}_1 \perp \mathbf{v}$. The fact $\mathbf{w}_1 \approx^l_{\mathbf{u}} \mathbf{y}$ implies $\mathbf{w}_1 \approx^l_{\mathbf{l}} \mathbf{y}$, which implies $\mathbf{w}_1 \approx^l_{\mathbf{v}} \mathbf{y}$. Similarly, we find a vertex $\mathbf{l}'$ adjacent to $\mathbf{w}$, $\mathbf{u}$, $\mathbf{w}_1$, $\mathbf{v}$, and establish $\mathbf{w} \perp \mathbf{v}$. This implies $\mathbf{w} \approx^l_{\mathbf{v}} \mathbf{y}$. By symmetry we also have $\mathbf{w} \approx^l_{\mathbf{v}} \mathbf{x}$, so transitivity of $\approx^l_{\mathbf{v}}$ yields $\mathbf{x} \approx^l_{\mathbf{v}} \mathbf{y}$. If the choice of the hyperline $W_{\mathbf{u}}$ containing $b_{\mathbf{u}}$ and $c_{\mathbf{u}}$ but not $x_{\mathbf{u}}$ is not possible, then there exists a hyperline $T_{\mathbf{u}}$ that contains $b_{\mathbf{u}}$ and $c_{\mathbf{u}}$ as well as $x_{\mathbf{u}}$. Let $t_{\mathbf{u}} \subset UV_{\mathbf{u}}$ be a line that does not intersect $T_{\mathbf{u}}$. Then continue the above argument with vertices $\mathbf{w} := (x_{\mathbf{u}}, W_{\mathbf{u}})$ and $\mathbf{w}' := (x_{\mathbf{u}}, W'_{\mathbf{u}})$ such that $W_{\mathbf{u}}$ contains $t_{\mathbf{u}}$ and $b_{\mathbf{u}}$ and $W'_{\mathbf{u}}$ contains $t_{\mathbf{u}}$ and $c_{\mathbf{u}}$.

If $\mathbb{F} = \mathbb{F}_2$ we can assume $n \geq 7$, and the proof of the lemma is straightforward; cf. Lemma 4.7 of [1] or Lemma 1.3.9 of [5]. □

Lemma 5.5
The relation $\approx^l$ is an equivalence relation. In particular, distinct vertices $\mathbf{x} \approx^l \mathbf{y}$ are at distance two in Γ.

Proof. Reflexivity and symmetry follow from reflexivity and symmetry of each $\approx^l_{\mathbf{x}}$. In order to prove transitivity suppose $\mathbf{x} \approx^l_{\mathbf{u}} \mathbf{y}$ and $\mathbf{y} \approx^l_{\mathbf{v}} \mathbf{z}$. In $\mathbf{y}^\perp$ we can assume that $v_{\mathbf{y}} \not\subset U_{\mathbf{y}}$ and $u_{\mathbf{y}} \not\subset V_{\mathbf{y}}$. Fix a hyperline $W_{\mathbf{y}}$ that contains $U_{\mathbf{y}} \cap V_{\mathbf{y}}$, that intersects both $U_{\mathbf{y}}$ and $V_{\mathbf{y}}$ in a hyperplane of $W_{\mathbf{y}}$, and that does not intersect $u_{\mathbf{y}}$ or $v_{\mathbf{y}}$. The line-hyperline pairs $(u_{\mathbf{y}}, W_{\mathbf{y}})$ and $(v_{\mathbf{y}}, W_{\mathbf{y}})$ give rise to vertices $\mathbf{u}'$, respectively $\mathbf{v}'$ that are both adjacent to $\mathbf{y}$. Let us now study the path $\mathbf{x} \perp \mathbf{u} \perp \mathbf{y} \perp \mathbf{u}'$. In $\mathbf{u}^\perp$ denote the intersection $X_{\mathbf{u}} \cap Y_{\mathbf{u}}$ by $XY_{\mathbf{u}}$. This space induces a space $XY_{\mathbf{y}}$ of $\mathbf{y}^\perp$. The intersection $U_{\mathbf{y}} \cap W_{\mathbf{y}}$ is a hyperplane of $U_{\mathbf{y}}$, so $U_{\mathbf{y}} \cap W_{\mathbf{y}} \cap XY_{\mathbf{y}}$ contains a projective line, $l_{\mathbf{y}}$ say. Choosing any hyperline $L_{\mathbf{y}} \supset u_{\mathbf{y}}$ that does not intersect $l_{\mathbf{y}}$, we find a vertex $\mathbf{a} = (l_{\mathbf{y}}, L_{\mathbf{y}})$ that is adjacent to $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{y}$, and $\mathbf{x}$. In $\mathbf{a}^\perp$ we see that $\mathbf{x} \perp \mathbf{u}'$; moreover $\mathbf{x} \approx^l_{\mathbf{a}} \mathbf{y}$, and, thus, $\mathbf{x} \approx^l_{\mathbf{u}'} \mathbf{y}$. Similarly one establishes $\mathbf{z} \perp \mathbf{v}'$ and $\mathbf{y} \approx^l_{\mathbf{v}'} \mathbf{z}$. In $\mathbf{v}'^\perp$ we can find a vertex $\mathbf{z}'$ in the same $\approx^l_{\mathbf{v}'}$ equivalence class as $\mathbf{y}$ and $\mathbf{z}$ such that the hyperline of $\mathbf{z}'$ intersects the hyperlines of $\mathbf{y}$ and $\mathbf{z}$ in hyperplanes of the hyperline. Denote the intersection $Y_{\mathbf{v}'} \cap Z'_{\mathbf{v}'}$ by $YZ'_{\mathbf{v}'}$ and the intersection $X_{\mathbf{u}'} \cap Y_{\mathbf{u}'}$ by $XY_{\mathbf{u}'}$. Both spaces induces subspaces $YZ'_{\mathbf{y}}$, respectively $XY_{\mathbf{y}}$ of $W_{\mathbf{y}}$ in $\mathbf{y}^\perp$. Inside $\mathbf{y}^\perp$, the intersection $YZ'_{\mathbf{y}} \cap XY_{\mathbf{y}}$ contains a projective line, $m_{\mathbf{y}}$ say. If $\langle u_{\mathbf{y}}, v_{\mathbf{y}} \rangle$ does not intersect $m_{\mathbf{y}}$, then we can find a hyperline $M_{\mathbf{y}}$ that together with $m_{\mathbf{y}}$ forms a vertex $\mathbf{m}$ of Γ that is adjacent to $\mathbf{y}$, $\mathbf{u}'$, $\mathbf{x}$, $\mathbf{v}'$, and $\mathbf{z}'$. If $\langle u_{\mathbf{y}}, v_{\mathbf{y}} \rangle$ does intersect $m_{\mathbf{y}}$, then we can choose a line $n_{\mathbf{y}} \subset YZ'_{\mathbf{y}} \backslash m_{\mathbf{y}}$ and a non-intersecting hyperline $N_{\mathbf{y}} \supset \langle v_{\mathbf{y}}, m_{\mathbf{y}} \rangle$, which gives rise to a vertex $\mathbf{n} = (n_{\mathbf{y}}, N_{\mathbf{y}})$ adjacent to $\mathbf{y}$, $\mathbf{v}'$, and $\mathbf{z}'$. Moreover, the space $\langle n_{\mathbf{y}}, u_{\mathbf{y}} \rangle$ does not intersect $m_{\mathbf{y}}$, and as above we find a vertex $\mathbf{m}$ of Γ that is adjacent to $\mathbf{x}$, $\mathbf{u}'$, $\mathbf{y}$, $\mathbf{n}$, and $\mathbf{z}'$. The facts $\mathbf{x} \approx^l_{\mathbf{u}'} \mathbf{y}$ and $\mathbf{y} \approx^l_{\mathbf{v}'} \mathbf{z}'$ imply $\mathbf{x} \approx^l_{\mathbf{m}} \mathbf{y}$ and $\mathbf{y} \approx^l_{\mathbf{m}} \mathbf{z}'$, because $\mathbf{m} \perp \mathbf{u}'$ and either $\mathbf{m} \perp \mathbf{v}'$ or $\mathbf{m} \perp \mathbf{n} \perp \mathbf{v}'$, so transitivity of $\approx^l_{\mathbf{m}}$ yields $\mathbf{x} \approx^l_{\mathbf{m}} \mathbf{z}'$. We have reduced the problem to the path $\mathbf{x} \perp \mathbf{m} \perp \mathbf{z}' \perp \mathbf{v}' \perp \mathbf{z}$. However, in $\mathbf{v}'^\perp$ the hyperlines of $\mathbf{z}'$ and $\mathbf{z}$ intersect in a hyperplane of either hyperline, so by considerations as above we can find a vertex $\mathbf{m}'$ adjacent to $\mathbf{x}$, $\mathbf{z}'$, and $\mathbf{z}$, yielding $\mathbf{x} \approx^l_{\mathbf{m}'} \mathbf{z}$ and, thus, $\mathbf{x} \approx^l \mathbf{z}$.

The second statement follows from the above considerations and Lemma 5.4. □

Definition 5.6 A **global line** of Γ is an equivalence class of $\approx^l$. Dually, a **global hyperline** is an equivalence class of $\approx^h$. By Lemma 5.4 the local intersection of a global line is either empty or a local line.

Lemma 5.7
Let l be a global line of Γ. Then $L \cap \mathbf{x}^\perp \neq \emptyset$ if and only if $L \cap \mathbf{y}^\perp \neq \emptyset$ for an arbitrary global hyperline L and $\mathbf{x}, \mathbf{y} \in l$. The dual statement holds as well.

Proof. By Lemma 5.5 there exists a vertex $\mathbf{z}$ adjacent to $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{h}_1$ be some neighbor of $\mathbf{y}$. We have to prove that there exists a neighbor of $\mathbf{x}$ contained in the same $\approx^h$ equivalence class as $\mathbf{h}_1$. Denote this $\approx^h$ equivalence class by H. First of all, we can assume that in $\mathbf{z}^\perp$ the intersection $X_\mathbf{z} \cap Y_\mathbf{z}$ is a hyperplane of $X_\mathbf{z}$ and $Y_\mathbf{z}$. Indeed, there always exists a neighbor $\mathbf{y}'$ of $\mathbf{z}$ with $\mathbf{y} \approx^l \mathbf{y}'$ such that $X_\mathbf{z} \cap Y'_\mathbf{z}$ and $Y'_\mathbf{z} \cap Y_\mathbf{z}$ are hyperplanes of $Y'_\mathbf{z}$, so we can first consider the path $\mathbf{y}' \perp \mathbf{z} \perp \mathbf{y} \perp \mathbf{h}_1$ to establish a neighbor $\mathbf{h}'_1$ of $\mathbf{y}'$ with $\mathbf{h}_1 \approx^h \mathbf{h}'_1$, and subsequently consider the path $\mathbf{x} \perp \mathbf{z} \perp \mathbf{y}' \perp \mathbf{h}'_1$.

So now assume that in $\mathbf{z}^\perp$ the intersection $X_\mathbf{z} \cap Y_\mathbf{z}$ is a hyperplane of $X_\mathbf{z}$ and $Y_\mathbf{z}$. Denote this intersection $X_\mathbf{z} \cap Y_\mathbf{z}$ by $XY_\mathbf{z}$, which induces a subspace $XY_\mathbf{y}$ of $Z_\mathbf{y}$ in $\mathbf{y}^\perp$. The intersection $XY_\mathbf{y} \cap H_\mathbf{y}$ contains a projective line, $v_\mathbf{y}$ say. Choose a hyperline $V_\mathbf{y} \supset z_\mathbf{y}$ that does not intersect $v_\mathbf{y}$, which yields a vertex $\mathbf{v} = (v_\mathbf{y}, V_\mathbf{y})$ that is adjacent to $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{x}$. Since $v_\mathbf{y} \subset H_\mathbf{y}$, there exists a neighbor $\mathbf{h}_2$ of $\mathbf{y}$ with $\mathbf{h}_1 \approx^h \mathbf{h}_2$ and $\mathbf{h}_2 \perp \mathbf{v}$. In $\mathbf{v}^\perp$ we have $l_\mathbf{v} \subset H_\mathbf{v}$. Choosing a line in $X_\mathbf{v} \backslash H_\mathbf{v}$ we have found a vertex $\mathbf{h}_3$ adjacent to $\mathbf{v}$ and $\mathbf{x}$ with $\mathbf{h}_3 \approx^h \mathbf{h}_2$. Therefore $\mathbf{h}_3 \approx^h \mathbf{h}_1$ by Lemma 5.5, and the lemma is proved. □

Lemma 5.8
Let $\mathbf{l} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{m}$ be a path of vertices with $\mathbf{l} \approx^l \mathbf{m}$. Then there exists a vertex $\mathbf{k} \in \{\mathbf{x}, \mathbf{y}\}^\perp$ with $\mathbf{l} \approx^l \mathbf{k} \approx^l \mathbf{m}$.

Proof. Consider $\mathbf{x}^\perp$. If $l_\mathbf{x} \subset Y_\mathbf{x}$, then there is nothing to prove. If $l_\mathbf{x} \cap Y_\mathbf{x} = \emptyset$, then let $\mathbf{y}' := (l_\mathbf{x}, Y_\mathbf{x})$. Note that $\mathbf{l} \approx^l_\mathbf{x} \mathbf{y}'$. The fact $\mathbf{y}' \approx^h_\mathbf{x} \mathbf{y}$ implies, by Lemma 5.7, the existence of a vertex $\mathbf{m}'$ adjacent to $\mathbf{y}'$ with $\mathbf{m}' \approx^l \mathbf{m}$. But $\mathbf{m}' \approx^l \mathbf{m} \approx^l \mathbf{l} \approx^l \mathbf{y}'$ yields $\mathbf{m}' \approx^l \mathbf{y}'$, a contradiction to $\mathbf{m}' \perp \mathbf{y}'$ and Lemma 5.4.

The above considerations and symmetry of $\mathbf{x}^\perp$ and $\mathbf{y}^\perp$ leave the following case: In $\mathbf{x}^\perp$ assume that $l_\mathbf{x}$ intersects $Y_\mathbf{x}$ in the point $p_\mathbf{x}$, while in $\mathbf{y}^\perp$ assume that $m_\mathbf{y}$ intersects $X_\mathbf{y}$ in the point $q_\mathbf{y}$. The point $q_\mathbf{y}$ arises as the point $q_\mathbf{x}$ in $\mathbf{x}^\perp$; let $a_\mathbf{y} := \langle x_\mathbf{y}, m_\mathbf{y} \rangle \cap X_\mathbf{y}$, which arises as $a_\mathbf{x}$ in $\mathbf{x}^\perp$. If $\langle l_\mathbf{x}, y_\mathbf{x} \rangle \ni q_\mathbf{x}$, then $\langle a_\mathbf{x}, l_\mathbf{x}, y_\mathbf{x} \rangle$ is contained in a hyperline, so we can find a vertex $\mathbf{u}$ adjacent to $\mathbf{x}$ and $\mathbf{y}$ containing $l_\mathbf{u}$ and $m_\mathbf{u}$. Lemma 5.4 implies $l_\mathbf{u} = m_\mathbf{u}$, and we are done. So assume that $\langle l_\mathbf{x}, y_\mathbf{x} \rangle \not\ni q_\mathbf{x}$. Then there exists a hyperline $W_\mathbf{x}$ containing $\langle l_\mathbf{x}, y_\mathbf{x} \rangle$ and a line $w_\mathbf{x}$ containing $q_\mathbf{x}$ that does not intersect $W_\mathbf{x}$. Let $\mathbf{w} := (w_\mathbf{x}, W_\mathbf{x})$. Then by Lemma 5.7 there exists a vertex $\mathbf{l}'$ adjacent to $\mathbf{w}$ with $\mathbf{l} \approx^l \mathbf{l}'$. Denote $\langle l_\mathbf{w}, y_\mathbf{w} \rangle \cap Y_\mathbf{w}$ by $b_\mathbf{w}$. This space translates to a space $b_\mathbf{y}$ of $\mathbf{y}^\perp$. The local line $w_\mathbf{y}$ intersects the local line $m_\mathbf{y}$, so $\langle b_\mathbf{y}, m_\mathbf{y}, w_\mathbf{y} \rangle$ is contained in a hyperline. Hence we can find a neighbor $\mathbf{v}$ of $\mathbf{w}$ and $\mathbf{y}$ that contains $l_\mathbf{v}$ and $m_\mathbf{v}$, thus yielding $l_\mathbf{v} = m_\mathbf{v}$. However, while $w_\mathbf{v}$ intersects with $m_\mathbf{v}$, the hyperline $W_\mathbf{v}$ contains $l_\mathbf{v}$, a contradiction to $w_\mathbf{v} \cap W_\mathbf{v} = \emptyset$. □

Lemma 5.9
Let $\mathbf{h}_1 \approx^h \mathbf{h}_2$ and $\mathbf{i}_1 \approx^h \mathbf{i}_2$ be such that there exist vertices $\mathbf{x} \in \{\mathbf{h}_1, \mathbf{i}_1\}^\perp$ and $\mathbf{y} \in \{\mathbf{h}_2, \mathbf{i}_2\}^\perp$. Then there exist vertices $\mathbf{h}_3, \mathbf{i}_3 \in \mathbf{y}^\perp$ with $\mathbf{h}_2 \approx^h \mathbf{h}_3$ and $\mathbf{i}_2 \approx^h \mathbf{i}_3$ and a vertex $\mathbf{x}' \in \{\mathbf{h}_3, \mathbf{i}_3\}^\perp$ in the same connected component of $\{\mathbf{h}_3, \mathbf{i}_3\}^\perp$ as $\mathbf{y}$ with $\mathbf{x} \approx^l \mathbf{x}'$.

Proof. Suppose that $H_\mathbf{y} \cap I_\mathbf{y}$ is a hyperline of both $H_\mathbf{y}$ and $I_\mathbf{y}$. Then we can find adjacent vertices $\mathbf{h}_3$ and $\mathbf{i}_3$ in $\mathbf{y}^\perp$ with $\mathbf{h}_3 \approx^h \mathbf{h}_2$ and $\mathbf{i}_3 \approx^h \mathbf{i}_2$. By Lemma 5.7 there exist vertices $\mathbf{x}_1 \perp \mathbf{h}_3$ and $\mathbf{x}_2 \perp \mathbf{i}_3$ with $\mathbf{x}_1 \approx^l \mathbf{x} \approx^l \mathbf{x}_2$. But now by Lemma 5.8 there exists a vertex $\mathbf{x}' \in \{\mathbf{h}_3, \mathbf{i}_3\}^\perp$ with $\mathbf{x}_1 \approx^l \mathbf{x}' \approx^l \mathbf{x}_2$ and, thus, $\mathbf{x} \approx^l \mathbf{x}'$.

So now we can suppose that $H_\mathbf{y} \cap I_\mathbf{y}$ is a hyperplane of both $H_\mathbf{y}$ and $I_\mathbf{y}$. Denote this intersection by $HI_\mathbf{y}$; it induces a space $HI_{\mathbf{h}_2}$ of $\mathbf{h}_2^\perp$. Additionally we can assume that $h^2{}_\mathbf{y} = i^2{}_\mathbf{y}$. By Lemma 5.7 there exist vertices $\mathbf{x}_1 \perp \mathbf{h}_2$ and $\mathbf{x}_2 \perp \mathbf{i}_2$ with $\mathbf{x}_1 \approx^l \mathbf{x} \approx^l \mathbf{x}_2$. In $\mathbf{h}_2^\perp$ we can assume that the hyperline $X^1_{\mathbf{h}_2}$ contains the line $y_{\mathbf{h}_2}$, if $x^1_{\mathbf{h}_2}$ does not intersect $y_{\mathbf{h}_2}$. Otherwise we can assume that the hyperline $X^1_{\mathbf{h}_2}$ intersects the line $y_{\mathbf{h}_2}$. Now in $\mathbf{h}_2^\perp$ the intersection $HI_{\mathbf{h}_2} \cap X^1_{\mathbf{h}_2}$ contains a projective line. Moreover, $\langle x^1_{\mathbf{h}_2}, y_{\mathbf{h}_2} \rangle \cap HI_{\mathbf{h}_2} \cap X^1_{\mathbf{h}_2} = \emptyset$, by the above assumptions on $X^1_{\mathbf{h}_2}$ and $y_{\mathbf{h}_2}$. Therefore there exists a vertex $\mathbf{v}$ adjacent to $\mathbf{x}_1$, $\mathbf{h}_2$, $\mathbf{y}$, and $\mathbf{i}_2$. By Lemma 5.8 there exists a vertex $\mathbf{x}_3 \in \{\mathbf{v}, \mathbf{i}_2\}^\perp$ with $\mathbf{x}_1 \approx^l \mathbf{x}_3 \approx^l \mathbf{x}_2$ and, thus, $\mathbf{x} \approx^l \mathbf{x}_3$. Local analysis of $\mathbf{v}^\perp$ yields a vertex $\mathbf{x}_4 \in \{\mathbf{h}_2, \mathbf{i}_2\}^\perp$ with $\mathbf{x}_4 \approx^l \mathbf{x}_3 \approx^l \mathbf{x}$. □

Lemma 5.10
Let l and m be global lines of Γ. Then there exist vertices of l and m at mutual distance at most two.

Proof. Assume there exist vertices $\mathbf{l} \in l$ and $\mathbf{m} \in m$ at distance three and let $\mathbf{l} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{m}$ be a path from $\mathbf{l}$ to $\mathbf{m}$. In $\mathbf{y}^\perp$ denote $\langle x_\mathbf{y}, m_\mathbf{y} \rangle \cap X_\mathbf{y}$ by $a_\mathbf{y}$. This line induces a line $a_\mathbf{x}$ of $\mathbf{x}^\perp$. If $\langle l_\mathbf{x}, a_\mathbf{x}, y_\mathbf{x} \rangle$ is contained in a hyperline, then we can choose a hyperline $Y'_\mathbf{x} \supset \langle l_\mathbf{x}, a_\mathbf{x}, y_\mathbf{x} \rangle$ and a non-intersecting line $y'_\mathbf{x} \subset Y_\mathbf{x}$. This yields a vertex $\mathbf{y}'$ adjacent to $\mathbf{x}$ and $\mathbf{y}$ which has a neighbor $\mathbf{m}' \approx^l \mathbf{m}$. Moreover, $\mathbf{y}'$ is adjacent to a vertex $\mathbf{l}' \approx^l \mathbf{l}$. By Lemma 5.7 the vertex $\mathbf{l}$ is adjacent to some vertex $\mathbf{z} \approx^h \mathbf{y}'$, which in turn is adjacent to a vertex $\mathbf{m}'' \approx \mathbf{m}$, and we have found $\mathbf{l} \in l$ and $\mathbf{m}'' \in m$ at distance two.

So now suppose that $\langle l_\mathbf{x}, a_\mathbf{x}, y_\mathbf{x} \rangle$ is not contained in a hyperline. Then we can choose a hyperline $Y'_\mathbf{x}$ that contains $\langle a_\mathbf{x}, y_\mathbf{x} \rangle$ and intersects $l_\mathbf{x}$ in a point, and we can choose a non-intersecting line $y'_\mathbf{x}$ in $Y_\mathbf{x}$. If $\langle l_\mathbf{x}, a_\mathbf{x}, y'_\mathbf{x} \rangle$ is contained in a hyperline, then we are in the situation of the above paragraph, so assume that $\langle l_\mathbf{x}, a_\mathbf{x}, y'_\mathbf{x} \rangle$ is not contained in a hyperline. Then we can choose any hyperline $Y''_\mathbf{x}$ containing $\langle a_\mathbf{x}, y'_\mathbf{x} \rangle$ but not $l_\mathbf{x}$ and a non-intersecting line $y''_\mathbf{x}$ inside $Y'_\mathbf{x}$ that intersects $l_\mathbf{x}$. Now $\langle l_\mathbf{x}, a_\mathbf{x}, y''_\mathbf{x} \rangle$ is contained in a hyperline and, by the above paragraph, we can find vertices of l and m at distance two.

The lemma follows by induction on the length of some path from some element of l to some element of m. □

Definition 5.11 Let l and m be distinct global lines of Γ, let $\mathbf{l} \in l$, $\mathbf{m} \in m$ be at distance at most two, and let $\mathbf{x}$ be adjacent to $\mathbf{l}$ and $\mathbf{m}$. Then $\langle l, m \rangle$ consists of those global lines k with $k \cap \mathbf{x}^\perp \neq \emptyset$ such that $k_\mathbf{x}$ lies in the span of $l_\mathbf{x}$ and $m_\mathbf{x}$. If $l_\mathbf{x}$ intersects $m_\mathbf{x}$, then $\langle l, m \rangle$ is called a **global plane**, and otherwise a **global** 3**-space**. Dually one defines intersections of global hyperplanes.

Lemma 5.12
The notion of global planes and global 3*-spaces is well defined. The same holds for the dual statement.*

Proof. We will prove the dual statement instead. Let $\mathbf{h}_1 \approx^h \mathbf{h}_2$ and $\mathbf{i}_1 \approx^h \mathbf{i}_2$ be such that there exists vertices $\mathbf{x} \in \{\mathbf{h}_1, \mathbf{i}_1\}^\perp$ and $\mathbf{y} \in \{\mathbf{h}_2, \mathbf{i}_2\}^\perp$. By Lemma 5.9 there exist vertices $\mathbf{h}_3, \mathbf{i}_3 \in \mathbf{y}^\perp$ with $\mathbf{h}_2 \approx^h \mathbf{h}_3$ and $\mathbf{i}_2 \approx^h \mathbf{i}_3$ and a vertex $\mathbf{w} \in \{\mathbf{h}_3, \mathbf{i}_3\}^\perp$ with $\mathbf{x} \approx^l \mathbf{w}$ and a path from $\mathbf{y}$ to $\mathbf{w}$ in $\{\mathbf{h}_3, \mathbf{i}_3\}^\perp$. By Lemma 5.5 there exists a vertex $\mathbf{z}$ adjacent to $\mathbf{x}$ and $\mathbf{w}$. We can assume that $X_\mathbf{z} \cap W_\mathbf{z}$ is a hyperplane in both $X_\mathbf{z}$ and $W_\mathbf{z}$. For, there exists a vertex $\mathbf{w}' \approx^l \mathbf{x}, \mathbf{w}$ adjacent to $\mathbf{z}$ such that $X_\mathbf{z} \cap W'_\mathbf{z}$ and $W'_\mathbf{z} \cap W_\mathbf{z}$ are hyperplanes of $W'_\mathbf{z}$. By Lemma 5.7 there are vertices $\mathbf{h}'_3, \mathbf{i}'_3 \in \mathbf{w}'^\perp$ with $\mathbf{h}_3 \approx^h \mathbf{h}'_3$ and $\mathbf{i}_3 \approx^h \mathbf{i}'_3$.

So now assume that $X_\mathbf{z} \cap W_\mathbf{z}$ is a hyperplane in both $X_\mathbf{z}$ and $W_\mathbf{z}$. Denote this intersection by $XW_\mathbf{z}$, which induces a space $XW_\mathbf{w}$. Up to passing to a neighbor of $\mathbf{x}$, $\mathbf{z}$, $\mathbf{w}$ we can assume that in $\mathbf{w}^\perp$ the hyperline $Z_\mathbf{w}$ contains $H_\mathbf{w} \cap I_\mathbf{w}$. Therefore $XW_\mathbf{w} \cap H_\mathbf{w} \cap I_\mathbf{w}$ contains a projective line, and we can find a neighbor $\mathbf{v}$ of $\mathbf{x}$, $\mathbf{z}$, $\mathbf{w}$ whose line in $\mathbf{w}^\perp$ is contained in $H_\mathbf{w} \cap I_\mathbf{w}$. Therefore there exist vertices $\mathbf{h}_4, \mathbf{i}_4 \in \mathbf{v}^\perp$ with $\mathbf{h}_3 \approx^h \mathbf{h}_4$ and $\mathbf{i}_3 \approx^h \mathbf{i}_4$. Four applications of Lemma 5.8 finish the proof. $\square$

Definition 5.13 Denote by $\mathcal{L}_\Gamma$ the set of global lines of Γ. A **global pre-point** is defined as the union $\bigcup_{l \in S} l$ of global lines contained in some set S such that any pair l, m of distinct global lines contained in S spans a global plane and for any triple k, l, m of distinct global lines contained in S there exists a global line n in S that spans a global plane with any of k, l, or m distinct from n and a global 3-space with any pair of k, l, or m not containing n. A maximal global pre-point is called a **global point**. Denote the set of global points by $\mathcal{P}_\Gamma$.

Proposition 5.14
The point-line geometry $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ *is a projective space.*

Proof. Let p and q be points of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$. Then there exists lines l through p and m through q. By Lemma 5.10 there exists vertices $\mathbf{l} \in l$ and $\mathbf{m} \in m$ at distance at most two, so there exists a vertex $\mathbf{x}$ adjacent to both $\mathbf{l}$ and $\mathbf{m}$. The intersections $p \cap \mathbf{x}^\perp$ and $q \cap \mathbf{x}^\perp$ are local points of $\mathbf{x}$, so there exists a local line $k_\mathbf{x}$ connecting $p_\mathbf{x}$ and $q_\mathbf{x}$. Hence we have found a global line k joining p and q, so $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ is a linear space.

It remains to prove Pasch's axiom. Let l and m be intersecting lines. By Lemma 5.10 there exists a vertex $\mathbf{z}$ adjacent to some vertices $\mathbf{l} \in l$ and $\mathbf{m} \in m$. In $\mathbf{z}^\perp$ the plane $\langle l_\mathbf{z}, m_\mathbf{z} \rangle$

is a projective plane, hence so is the global plane spanned by l and m, and the proof is finished. □

Proposition 5.15
The graph Γ is isomorphic to the line-hyperline graph of $\mathbb{P}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$.

Proof. Denote by $\langle \mathbf{x}^\perp \rangle$ the set of global lines of Γ that have a non-empty intersection with $\mathbf{x}^\perp$. This set obviously is a hyperline of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$. Therefore the map $\Gamma \to \mathbf{L}(\mathbb{P}_\Gamma)$ defined by $\mathbf{x} \mapsto ([\mathbf{x}]_{\approx^l}, \langle \mathbf{x}^\perp \rangle)$ defines an isomorphism between Γ and the line-hyperline graph of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$. □

Proof of Theorem 1. Assuming Γ to be simply connected, Proposition 5.15 implies that Γ is isomorphic to $\mathbf{L}_{n+2}(\mathbb{F})$. Therefore, if Γ is not necessarily simply connected, then Γ is isomorphic to a quotient of $\mathbf{L}_{n+2}(\mathbb{F})$. However, by Lemma 2.3 the diameter of $\mathbf{L}_{n+2}(\mathbb{F})$ equals two, so $\mathbf{L}_{n+2}(\mathbb{F})$ does not admit any proper quotients that are locally $\mathbf{L}_n(\mathbb{F})$, and the theorem follows. □

6 Hyperbolic root group geometries

This section gives a proof of Theorem 2. Throughout the whole section, let Γ be a connected, locally $\mathbf{L}_{n-1}(\mathbb{F})$ graph for a division ring $\mathbb{F}$ and $n \geq 6$.

Definition 6.1 Let $\Gamma = (\mathcal{V}, \perp)$ be a connected, locally $\mathbf{L}_{n-1}(\mathbb{F})$ graph. Γ is **geometrizable** if there exists a family $\mathcal{S}$ of subsets of $\mathcal{V}$ such that

- for any $S \in \mathcal{S}$ and any vertex $\mathbf{x} \in \mathcal{V}$ the intersection $S \cap \mathbf{x}^\perp$ is either empty or an interior root point of $\mathbf{x}^\perp$, and
- for any interior root point $p_\mathbf{x}$ of $\mathbf{x}^\perp$, $\mathbf{x} \in \mathcal{V}$, there exists a unique set $S \in \mathcal{S}$ containing $p_\mathbf{x}$.

The point-line geometry $(\mathcal{S}, \mathcal{V})$ with symmetrized containment as incidence is called a **geometrization of** Γ. An element of $\mathcal{S}$ is called a **global root point**.

Lemma 6.2
Let Γ be geometrizable and let $\mathbf{x}$ and $\mathbf{y}$ be two vertices of Γ. If $\mathbf{p}$, $\mathbf{q}$ are two vertices adjacent to both $\mathbf{x}$ and $\mathbf{y}$ that belong to a common interior root point of $\mathbf{x}^\perp$, then they also belong to a common interior root point of $\mathbf{y}^\perp$.

Proof. Let $(\mathcal{S}, \mathcal{V})$ be a geometry on Γ. Then there is a unique $S \in \mathcal{S}$ containing $\mathbf{p}$ and $\mathbf{q}$. But since $\mathbf{p}, \mathbf{q} \in S \cap \mathbf{y}^\perp$, they also belong to an interior root point of $\mathbf{y}^\perp$. □

Lemma 6.3
Let Γ be geometrizable. Then, up to isomorphism, there is at most one geometrization of Γ with the property that any two vertices contained in the same global point are at distance two in Γ.

Proof. Suppose such a geometry on Γ exists. Fix a vertex $\mathbf{x}$ and consider the interior hyperbolic root group geometry on $\mathbf{x}^\perp \cong \mathbf{L}_{n-1}(\mathbb{F})$. Let $\mathbf{a}$, $\mathbf{b}$ be two distinct vertices of an interior root point p of $\mathbf{x}^\perp$. Note that $\mathbf{a}$ and $\mathbf{b}$ uniquely determine this interior root point, by Lemma 4.1. Now let $\mathbf{y}$ be an arbitrary vertex of Γ. The proposition is proved, if it is determined whether $\mathbf{y}$ belongs to the set $S \in \mathcal{S}$ that contains p or not.

We may assume that there exists a vertex $\mathbf{z}$ adjacent to $\mathbf{y}$ and $\mathbf{a}$, since otherwise $\mathbf{y}$ cannot be contained in S by hypothesis. By Proposition 2.3 there exists a chain of vertices in $\mathbf{a}^\perp \cong \mathbf{L}_{n-1}(\mathbb{F})$ connecting $\mathbf{x}$ and $\mathbf{z}$. Denote the vertex closest to $\mathbf{x}$ by $\mathbf{w}$. By local analysis of $\mathbf{x}^\perp$ we can find another vertex $\mathbf{c}$ in $\mathbf{x}^\perp \cap \mathbf{w}^\perp$ belonging to the interior root point p aside from $\mathbf{a}$. By Lemma 6.2 the vertices $\mathbf{a}$ and $\mathbf{c}$ are contained in a common interior root point q of $\mathbf{w}^\perp$. Obviously the interior root point q of $\mathbf{w}^\perp$ has also to be contained in S. Using induction, we see that it is determined whether $\mathbf{y}$ is contained in the set S or not. □

Proposition 6.4
Let $n \geq 6$, let $\mathbb{F}$ be a division ring, and let $(\mathcal{P}, \mathcal{L}, \perp)$ be a partial linear space endowed with a symmetric relation $\perp$ on the point set such that $x \perp p$ and $x \perp q$ for distinct points p, q on some line l and any point x implies $x \perp y$ for all points y of l. Moreover suppose for any line $k \in \mathcal{L}$ the space $k^\perp$ is isomorphic to the hyperbolic root group geometry of $PSL_n(\mathbb{F})$ with $l \perp m$ if and only if $[l, m] = 1$ for lines l, m inside $k^\perp$.

(i) *If any two intersecting lines of $(\mathcal{P}, \mathcal{L})$ are at distance two in $(\mathcal{L}, \perp)$, then $(\mathcal{L}, \perp)$ is geometrizable, a geometrization of $(\mathcal{L}, \perp)$ as given in Lemma 6.3 exists, and $(\mathcal{P}, \mathcal{L})$ is isomorphic to that geometry.*

(ii) *If the graph $(\mathcal{L}, \perp)$ is isomorphic to $\mathbf{L}_{n+1}(\mathbb{F})$, then $(\mathcal{P}, \mathcal{L})$ is isomorphic to the hyperbolic root group geometry of $PSL_{n+2}(\mathbb{F})$.*

Proof. Let us start with a proof of Statement (i). The graph $(\mathcal{L}, \perp)$ is locally $\mathbf{L}_{n-1}(\mathbb{F})$. Consider the family of all full line pencils of $(\mathcal{P}, \mathcal{L})$. This family gives rise to a geometry on $(\mathcal{L}, \perp)$ in the sense of Definition 6.1. Indeed, any intersection of a full line pencil with $k^\perp$ for an arbitrary line k is either empty or a full line pencil of the subspace $k^\perp$. But by Proposition 4.4 a full line pencil of $k^\perp$ corresponds to an interior root point. Conversely, any interior root point of a perp of a line corresponds to a full line pencil of this perp, which is contained in a unique full line pencil of the whole geometry. Hence $(\mathcal{L}, \perp)$ is geometrizable. Moreover, since any two intersecting lines are demanded to be at distance two in $(\mathcal{L}, \perp)$, the global geometry on $(\mathcal{L}, \perp)$ we just have constructed satisfies the hypothesis of Proposition 6.3. The last claim follows from the fact that $(\mathcal{P}, \mathcal{L})$ is isomorphic to the geometry on the full line pencils as points and the line set $\mathcal{L}$. Statement (ii) follows from Proposition 2.3, Proposition 4.4, and Statement (i). □

Theorem 2 follows from Theorem 1 and Statement (ii) of Proposition 6.4. Notice that the restriction of n to be greater or equal to 7 in Theorem 2 is crucial. In case $n = 6$, the point-line geometry on the long root subgroups and the fundamental SL_2's of the group $E_6(\mathbb{F})$ with $\perp$ being the commutation relation satisfies the hypothesis of the theorem.

7 Group-theoretic consequences

Finally, we will prove Theorem 4 and Theorem 5.

Proof of Theorem 4. Choose an involution $z \in J \cap K$ that is the central involution of some group isomorphic to $SL_2(\mathbb{F})$ which is a fundamental SL_2 in both J and K. Note that z commutes with x and y. The elements y and z are conjugate in K by an involution, whence they are conjugate in G. Similarly, x and z are conjugate in J by an involution. Therefore the conjugation action of the group G induces an action as the group Sym_3 on the set $\{x, y, z\}$ and as the group Sym_2 on the set $\{x, y\}$. Consider the graph Γ on all conjugates of x in G. A pair a, b of vertices of Γ is adjacent if there exists an element $g \in G$ such that $(x^g, y^g) = (a, b)$. Since G induces the action of Sym_2 on $\{x, y\}$, this definition of adjacency is symmetric, and we have defined an undirected graph. Moreover, the elements x, y, and z are pairwise adjacent and, thus, form a 3-clique of Γ. Define U_1 as the stabilizer in G of the vertex x and U_2 as the stabilizer in G of the edge $\{x, y\}$. The stabilizer of $\{x, y\}$ permutes x and y and therefore interchanges $C_G(x) \geq K$ and $C_G(y) \geq J$. Hence the stabilizer of x together with the stabilizer of $\{x, y\}$ generates G, as $G = \langle J, K \rangle \leq \langle U_1, U_2 \rangle$. Consequently, the graph Γ is connected. Also, Γ is locally $\mathbf{L}_{n-1}(\mathbb{F})$ by construction. To prove this, it is enough to show that any triangle in Γ is a conjugate of (x, y, z). Let (a, b, c) be a triangle, which means there exist vertices d, e, f of Γ such that (a, b, d), (a, c, e), and (b, c, f) are conjugates of (x, y, z) in G. Let $g \in G$ with $(x^g, y^g, z^g) = (a, b, d)$. Notice that $b, d \in K^g$ are commuting central involutions of fundamental SL_2's of K^g. The triangles (a, b, d) and (a, c, e) are conjugate in $C_G(a) = X^g \times K^g$. Choose $h \in C_G(a)$ such that $(a^h, b^h, d^h) = (a, c, e)$. Then $h = h_X h_K$ with $h_X \in X^g$, $h_K \in K^g$. The element h_X centralizes b and d, since $b, d \in K^g$. Therefore $c = b^h = b^{h_K} \in K^g$ is the central involution of a fundamental SL_2 of K^g. The elements x and y commute and so do b and c because the triangle (b, c, f) is conjugate to the triangle (x, y, z). Hence (a, b, d) and (a, b, c) are conjugate in K^g. Therefore (a, b, c) and (x, y, z) are conjugate in G.

By Theorem 1 the graph Γ is isomorphic to $\mathbf{L}_{n+1}(\mathbb{F})$, so, by Theorem 3, the group G modulo the kernel N of its action on Γ can be embedded in $P\Gamma L_{n+2}(\mathbb{F}).2$ or $P\Gamma L_{n+2}(\mathbb{F})$. To determine N choose a $g \in N$. Then g acts trivially on Γ, in particular it centralizes x and y, so we have $g \in X \times K$ and $g \in Y \times J$. Let $g_X \in X$ and $g_K \in K$ be such that

$g = g_X g_K$. The element g_X commutes with K, and therefore also centralizes all neighbors of x. Consequently, also $g_K = g_X^{-1} g$ centralizes all neighbors of x, and hence lies in the center of K. We have proved that g commutes with K. Similarly, g commutes with J. This implies that g commutes with $G = \langle J, K \rangle$, and, thus, $g \in Z(G)$. Certainly, $Z(G)$ acts trivially on Γ, whence $N = Z(G)$. The statement about the isomorphism type of $G/Z(G)$ follows from the isomorphism type of J and K. □

The proof of Theorem 5 consists of a series of lemmas.

Lemma 7.1
Under the assumptions of Theorem 5 any conjugate y^g, $g \in C_G(x)$, of y lies in K; its centralizer $C_G(y^g)$ has a component J^g with $A \leq J^g$. Moreover, $[A, B^g] = 1$. In particular, $[A, K] = 1$.

Proof. Let F be a fundamental SL_2 of J, and let f be a p-element of F. Let V be the natural module of J. Then we can decompose V as $[V, F] \oplus C_V(F) = [V, f] \oplus C_V(f)$. Certainly $C_V(F) \subset C_V(f)$ and $[V, F] \supset [V, f]$. If $C_V(f)$ is strictly larger than $C_V(F)$, then $C_V(f)$ is a hyperplane, so f is an axial collineation. It cannot be a translation, since the order of f is does not divide the order of the field, so it has to be a reflection. However, F does not contain reflections of J, so $C_V(f) = C_V(F)$ and, thus, $[V, f] = [V, F]$, so the p-element f is contained in a unique fundamental SL_2 of J.

We have $x^{jk} = y$, so $C_G(x) \cong C_G(y)$. The group K is characteristic in $C_G(x)$; in particular K is normal in $C_G(x)$. Therefore $K^{jk} = J$ or, equivalently, $K^j = J^k$. We have $B = C^k \subset J^k$ and $A = C^j \subset K^j$. Certainly we also have $A^k \in J^k$ and $B^j \in K^j$. Moreover, $x \in A \cap A^k$ and $y \in B \cap B^j$, so $K^j = J^k$ implies $A = A^k$ and $B = B^j$.

Now let $g \in C_G(x)$ with $z^g = z$. Then y is mapped onto y^g, and B is mapped onto B^g. We have $1 = [A, B] = [A^g, B^g]$. The group J^k is normal in $C_G(z)$, so $J^{kg} = J^k$. Since $A \leq J^k$ and $A^g \leq J^{kg} = J^k$, we have $A = A^g$, so $[A, B^g] = 1$. Moreover, $A = A^g$ and $A^g \leq J^g$ implies $A \leq J^g$. We have established the lemma for any conjugate of y in $C_G(x) \cap C_G(z)$. Connectedness of the graph on the fundamental SL_2's of K with commuting as adjacency finishes the proof; notice that K is generated by the set of fundamental SL_2's of K. □

Lemma 7.2
Under the assumptions of Theorem 5 define Γ to be the set of p-groups generated by conjugates of x in G and define an adjacency relation $\perp$ on Γ where $\langle a \rangle \perp \langle b \rangle$ if and only if there exists a $g \in G$ with $(\langle a \rangle, \langle b \rangle) = (\langle x \rangle^g, \langle y \rangle^g)$. Then $(\Gamma, \perp)$ is an undirected graph in which every triangle is conjugate to $(\langle x \rangle, \langle y \rangle, \langle z \rangle)$. In particular, Γ is locally homogeneous.

Proof. The vertices $\langle x \rangle$, $\langle y \rangle$ are obviously adjacent. Conjugation with k stabilizes $\langle x \rangle$ while interchanging $\langle y \rangle$ and $\langle z \rangle$, so $\langle x \rangle$ and $\langle z \rangle$ are adjacent. Conjugation with j centralizes $\langle y \rangle$ and interchanges $\langle x \rangle$ and $\langle z \rangle$, yielding the adjacency of $\langle z \rangle$ and $\langle y \rangle$.

Conjugation with jkj interchanges $\langle x\rangle$ and $\langle y\rangle$, so Γ is undirected. All neighbors of $\langle x\rangle$ are contained in K by Lemma 7.1. (Note that $C_G(x)$ is normal in $N_G(x)$, so K is normal in $N_G(x)$.) Moreover, if $\langle a\rangle = \langle x\rangle^g$ is a neighbor of $\langle x\rangle$, then $A \leq K^g$. So if $(\langle a\rangle, \langle b\rangle, \langle c\rangle)$ is a triangle of Γ, then there exist $g_1, g_2 \in G$ with $\langle a\rangle = \langle x\rangle^{g_1} = \langle x\rangle^{g_2}$, $\langle b\rangle = \langle y\rangle^{g_1}$, $\langle c\rangle = \langle y\rangle^{g_2}$. Without loss of generality we can assume $\langle a\rangle = \langle x\rangle$ and $\langle b\rangle = \langle y\rangle$. Then $A^{g_1} = A = A^{g_2}$ and $B^{g_1} = B$; moreover $[A, B^{g_2}] = 1$ and $B^{g_2} \leq K$ by Lemma 7.1; similarly $B^{g_2} \leq J$. Therefore $B^{g_2} \leq J \cap K$, actually $B^{g_2} \leq E(J \cap K)$. Hence there exists a $g \in K$ with $\langle y\rangle^g = \langle y\rangle$ and $\langle z\rangle^g = \langle c\rangle$, so the triangle $(\langle x\rangle, \langle y\rangle, \langle c\rangle)$ is conjugate to $(\langle x\rangle, \langle y\rangle, \langle z\rangle)$ in K, whence in G. □

Lemma 7.3
Under the assumptions of Theorem 5 let $(\Gamma, \perp)$ *be the graph defined in Lemma 7.2. Define* $\approx$ *on* Γ *by* $\langle a\rangle \approx \langle b\rangle$ *if and only if* $\langle a\rangle^\perp = \langle b\rangle^\perp$. *Then* $\approx$ *is an equivalence relation of* Γ *and* $\Gamma/\approx$ *is isomorphic to* $\mathbf{L}_{n+1}(\mathbb{F})$.

Proof. The graph Γ is locally homogeneous by Lemma 7.2, so it is enough to investigate the neighbors of $\langle x\rangle$, which are conjugates of $\langle y\rangle$ in K. Since any edge of Γ in K is conjugate to $(\langle y\rangle, \langle z\rangle)$ by Lemma 7.2, the graph $\langle x\rangle^\perp$ is isomorphic to the graph on the p-subgroups of K generated by the conjugates of y in which distinct p-groups are adjacent if and only if the fundamental SL_2's containing them commute. So, if $\approx_{\langle x\rangle}$ is the equivalence relation on $\langle x\rangle^\perp$ with $\langle a\rangle \approx_{\langle x\rangle} \langle b\rangle$ if and only if $\langle a\rangle^\perp = \langle b\rangle^\perp$ for neighbors $\langle a\rangle$, $\langle b\rangle$ of $\langle x\rangle$, then $\langle x\rangle^\perp / \approx_{\langle x\rangle}$ is isomorphic to $\mathbf{L}_{n-1}(\mathbb{F})$. Therefore the lemma follows from Theorem 1 and connectedness of Γ (it is connected, because $G = \langle J, K\rangle$), if $\langle x\rangle^\perp \cap \approx \; = \; \approx_{\langle x\rangle}$. So consider $\langle a\rangle^\perp$, which is connected and contains more than one element. If $\langle b\rangle$ has precisely the same set of neighbors, then we can choose some $\langle c\rangle \in \langle a\rangle^\perp = \langle b\rangle^\perp$, and $\langle c\rangle^\perp \cap \langle a\rangle^\perp$ is equal to $\langle c\rangle^\perp \cap \langle b\rangle^\perp$. Conversely, if $\langle c\rangle$ is adjacent to $\langle a\rangle$ and $\langle b\rangle$ and $\langle c\rangle^\perp \cap \langle a\rangle^\perp$ is equal to $\langle c\rangle^\perp \cap \langle b\rangle^\perp$, then by connectedness of $\langle a\rangle^\perp$ and symmetry it is enough to show that any vertex $\langle d\rangle \in \langle c\rangle^\perp \cap \langle a\rangle^\perp$ satisfies $\langle d\rangle^\perp \cap \langle a\rangle^\perp = \langle d\rangle^\perp \cap \langle b\rangle^\perp$. But inside $\langle c\rangle^\perp$ the vertices $\langle a\rangle$ and $\langle b\rangle$ lie inside the same fundamental SL_2, so they have to lie in the same fundamental SL_2 of $\langle d\rangle^\perp$ as well, and we have established $\langle d\rangle^\perp \cap \langle a\rangle^\perp = \langle d\rangle^\perp \cap \langle b\rangle^\perp$. □

Proof of Theorem 5. The group G acts via conjugation on Γ. In particular, if two vertices have the same set of neighbors, then their images under G also have the same set of neighbors, so G acts on $\Gamma/\approx$ as well. Therefore the group G modulo the kernel N of its action on $\Gamma/\approx$ can be embedded in $P\Gamma L_{n+2}(\mathbb{F})$. However, any element of N centralizes K and J, so it centralizes G. Conversely, any element in the center of N acts trivially on $\Gamma/\approx$, and the theorem is proved. □

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