A Phan-type theorem for $Sp(2n, q)$

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1 Introduction

In 1977 Kok-Wee Phan published a theorem (see [4]) on generation of the special unitary group $SU(n+1, q^2)$ by a system of its subgroups isomorphic to $SU(3, q^2)$. This theorem is similar in spirit to the famous Curtis-Tits theorem. In fact, both the Curtis-Tits theorem and Phan's theorem were used as principal identification tools in the classification of finite simple groups.

The proof of Phan's theorem given in his 1977 paper is somewhat incomplete. This motivated Bennett and Shpectorov [1] to revise Phan's paper and provide a new and complete proof of his theorem. They used an approach based on the concepts of diagram geometries and amalgams of groups. It turned out that Phan's configuration arises as the amalgam of rank two parabolics in the flag-transitive action of $SU(n+1, q^2)$ on the geometry of nondegenerate subspaces of the underlying unitary space. This point of view leads to a twofold interpretation of Phan's theorem: its complete proof must include (1) a classification of related amalgams; and (2) a verification that—apart from some small exceptional cases—the above geometry is simply connected. These two parts are tied together by a lemma due to Tits, that implies that if a group G acts flag-transitively on a simply connected

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geometry then the corresponding amalgam of maximal parabolics provides a presentation for G, see Proposition 7.1.

The Curtis-Tits theorem can also be restated in similar geometric terms. Let G be a Chevalley group. Then G acts on a spherical building B and also on the corresponding twin building $\mathcal{B} = (B_+, B_-, d^*)$. (Here $B_+ \cong B \cong B_-$ and d^* is a *codistance* between B_+ and B_-, taking values in the Weyl group W of B.) Now one can form the so-called *opposites geometry* Γ_{op} of $\mathcal{B}$. As a chamber system, Γ_{op} can be described as follows: its chambers are all the pairs (C_+, C_-) such that $C_+ \in B_+$, $C_- \in B_-$ and $d^*(C_+, C_-) = 1_W$. It turns out that the Curtis-Tits theorem is equivalent to the statement that Γ_{op} is simply connected for every spherical building B of rank at least three. This approach allows for a short proof of the Curtis-Tits theorem (cf. [3]).

Furthermore, the geometric interpretation of the Curtis-Tits theorem and Phan's theorem allows to relate them. Let $G = SL(n+1, q^2)$ and let σ be the product of the contragredient automorphism and the involutory field automorphism. Then σ can be considered as an "automorphism" of the twin building $\mathcal{B}$ corresponding to G. Unlike the ordinary automorphisms, σ will interchange (rather than stabilize) B_+ and B_-, while preserving distance and codistance between chambers. Let $G_\sigma = C_G(\sigma)$ and

$$\Gamma_\sigma = \{(C_+, C_-) \in \Gamma_{op} \mid C_+^\sigma = C_-\}.$$

Notice that σ is an involution and hence also $C_-^\sigma = C_+$; so, in a sense, Γ_σ consists of all chambers of Γ_{op} that are stabilized (in fact, flipped) by σ. It turns out that $G_\sigma \cong SU(n+1, q^2)$ acts flag- (chamber-) transitively on Γ_σ and Γ_σ is exactly the geometry used in [1] to re-prove Phan's theorem.

Clearly, this construction can be generalized to other types of spherical twin buildings $\mathcal{B}$ and "flips" σ. The chamber system Γ_σ associated with $\mathcal{B}$ and σ, will be refered to as the *flipflop* geometry associated with $\mathcal{B}$ and σ. (Notice that it is unclear in general whether Γ_σ is a geometry; however, it is true for all examples known to us.) The "flipflop" construction becomes a source of Phan-type theorems. In particular, we conjecture that Phan's results on diagrams D_n and E_n from his second paper [5] can be interpreted in this way. It is an interesting open problem to try to determine a complete list of pairs $(\mathcal{B}, \sigma)$ for which Γ_σ is non-empty.

In this paper we take up the case where $\mathcal{B}$ is the twin building for the group $Sp(2n, q^2)$ and σ is a particular flip as defined in Section 3.

Main Theorem.
The following hold.

(1) Γ_σ *is a rank* n *geometry admitting a flag-transitive group of automorphisms* $G_\sigma \cong Sp(2n, q)$.

(2) Γ_σ *is connected unless* $n = 2$ *and* $q = 2$*; it is residually connected if* $q > 2$.

(3) Γ_σ *is simply connected if* $n \geq 5$*, or* $n = 4$ *and* $q \geq 3$*, or* $n = 3$ *and* $q \geq 8$.

In particular, Γ_σ is 2-simply connected if $q \geq 8$; 3-simply connected if $q \geq 3$; and 4-simply connected if $q = 2$ (cf. Section 7 and [1]).

The present paper is organized as follows. Section 2 provides several important group-theoretic corollaries of the Main Theorem. In fact, those corollaries were our motivation to prove the Main Theorem. In Section 3 we introduce and study a class of flips σ on a $2n$-dimensional symplectic space V over $\mathbb{F}_{q^2}$. In section 4 we discuss Γ_σ; we establish that it is a flag-transitive geometry and then study its connectivity properties. In Sections 5 and 6 we prove that Γ_σ is simply connected in all but few cases. Finally, in Section 7 we derive group-theoretic consequences of the simple connectedness of Γ_σ.

Remark.

1. Although we only consider finite fields, we would expect that infinite-field versions of our results hold as well. To be precise, our constructions in Section 3 are completely independent of the field. However, our strategy of proof heavily relies on Lemma 4.3 which only works for finite fields. The crucial point of proving infinite-field versions of our results is therefore to suitably replace Lemma 4.3.

2. The exception $n = 2$, $q = 2$ in part (2) of the Main Theorem is a true exception, see Lemma 4.6. The exception $n = 3$, $q = 2$ in part (3) of the Main Theorem is a true exception as well, see the discussion after Theorem 6.8. It is currently unknown to us whether this is true for the other exceptions of part (3) as well.

3. The case $n = 3$, $q = 2$ shows that any proof of our results has to contain a counting argument of some kind. Requiring that the lines of Γ be thick might be worth trying.

2 Application to group theory

The Main Theorem has some group theoretic implications along the lines of Phan's theorem. Let F be a chamber (maximal flag) of Γ_σ. For $2 \leq s \leq n-1$, let $\mathcal{A}_{(s)}$ be the amalgam of all rank s parabolics, *i.e.*, stabilizers in G_σ of subflags of F of corank s.

Theorem 1. *The following hold.*

(1) *If $q \geq 8$ and $n \geq 3$ then G_σ is the universal completion of $\mathcal{A}_{(2)}$.*

(2) *If $3 \leq q \leq 7$ and $n \geq 4$ then G_σ is the universal completion of $\mathcal{A}_{(3)}$.*

(3) *If $q = 2$ and $n \geq 5$ then G_σ is the universal completion of $\mathcal{A}_{(4)}$.*

The maximal parabolics M_i with respect to F are semisimple subgroups of $G_\sigma \cong Sp(2n,q)$ of the form $GU(i,q^2) \times Sp(2n-2i,q)$, $i = 1,\ldots,n$. Each M_i stabilizes a $2i$-dimensional nondegenerate subspace U_i of the natural symplectic module U of G_σ. It induces $GU(i,q^2)$ on U_i and $Sp(2n-2i,q)$ on $U_i^\perp$. The intersection of all M_i (also known as the Borel subgroup arising from the action of G_σ on Γ_σ) is a maximal torus T of G_σ of order $(q+1)^n$. Let M_i^0 be the subgroup $SU(i,q^2) \times Sp(2n-2i,q)$ of M_i. For an arbitrary parabolic $M_J = \cap_{i \in J} M_i$ define $M_J^0 = \cap_{i\in J} M_i^0$. Here J is a subset of the type set $I = \{1,\ldots,n\}$ of Γ_σ. It can be shown that $M_J = M_J^0 T$.

In case of a minimal parabolic $M_{I\setminus\{i\}}$, we have that $L_i := M^0_{I\setminus\{i\}} \cong SL(2,q)$. In fact, if $1 \leq i \leq n-1$ then L_i arises as $SU(2,q^2) \cong SL(2,q)$, while L_n arises as $Sp(2,q) \cong SL(2,q)$. Notice that $T_i = L_i \cap T$ is a torus in L_i of size $q+1$. Notice also that the subgroups T_i generate T.

If $q \neq 2$ then $\langle L_i, L_j\rangle = M^0_{I\setminus\{i,j\}}$. In particular, the subgroups L_i have the following properties:

(1) $L_i \cong SU(2,q^2)$, if $i = 1,\ldots,n-1$; $L_n \cong Sp(2,q)$;

(2) $\langle L_i, L_j\rangle \cong \begin{cases} L_i \times L_j, & \text{if } |i-j| > 1; \\ SU(3,q^2), & \text{if } |i-j| = 1 \text{ and } \{i,j\} \neq \{n-1,n\}; \\ Sp(4,q), & \text{if } \{i,j\} = \{n-1,n\}. \end{cases}$

These properties are similar to Phan's original description of his configurations.

Define $\mathcal{A}^0_{(s)}$ to be the amalgam formed by the subgroups M^0_J for all parabolics M_J of rank s. The following is a "stripped-of-T" (Phan-type) version of Theorem 1.

Theorem 2. *The following hold.*

(1) *If $q \geq 8$ and $n \geq 3$ then G_σ is the universal completion of $\mathcal{A}^0_{(2)}$.*

(2) *If $3 \leq q \leq 7$ and $n \geq 4$ then G_σ is the universal completion of $\mathcal{A}^0_{(3)}$.*

(3) *If $q = 2$ and $n \geq 5$ then G_σ is the universal completion of $\mathcal{A}^0_{(4)}$.*

Note that M_i, $i \neq n$, is not a maximal semisimple subgroup of G_σ. Namely, M_i is contained in the full stabilizer H_i of the decomposition $U = U_i \oplus U_i^\perp$. The subgroup H_i is isomorphic to $Sp(2i, q) \times Sp(2n-2i, q)$. It is a maximal parabolic with respect to the action of G_σ on the rank $n-1$ pregeometry Δ of all proper nondegenerate subspaces of U. It can be shown that if $n \geq 3$ then Δ is a residually connected geometry on which G_σ acts flag-transitively. Furthermore, $\{U_i \mid 1 \leq i \leq n-1\}$ is a maximal flag of Δ, and H_i's are the corresponding maximal parabolics.

The following results will be derived from Theorem 1 and the results from [1].

Theorem 3. *Let $n \geq 4$. Then Δ is simply connected provided that $(n, q) \notin \{(4, 2), (4, 3)\}$.*

Inductively, Δ is 2-simply connected if $q \geq 4$ and 3-simply connected if $q = 2$ or 3.

As a corollary, we prove the following.

Theorem 4. *If $q \geq 4$ then the amalgam of any three subgroups H_i has G_σ as its universal completion. If $q = 2$ or 3 then the same holds for the amalgam of any four subgroups H_i.*

Notice that if $n \geq 5$ and $q = 2$ or 3 then G_σ can still be recovered from some triples of subgroups H_i. Namely, among others, every amalgam $H_1 \cup H_i \cup H_{n-1}$, $1 < i < n-1$, has G_σ as its universal completion, see Section 7.

3 Flips and forms

Let V be a $2n$-dimensional nondegenerate symplectic space over $\mathbb{F}_{q^2}$ and let $(\cdot, \cdot)$ be the corresponding alternating bilinear form. Let the bar denote the involutive automorphism of $\mathbb{F}_{q^2}$. In this section we study semilinear transformations σ of V satisfying

(T1) $(\lambda v)^\sigma = \bar{\lambda} v^\sigma$;

(T2) $(u^\sigma, v^\sigma) = \overline{(u,v)}$; and

(T3) $\sigma^2 = -Id$.

We will call such a σ a *flip*. An example of a flip can be constructed as follows. Choose a basis $\mathcal{B} = \{e_1, \dots, e_n, f_1, \dots, f_n\}$ in V such that, for $1 \le i, j \le n$, we have that $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$. This corresponds to the Gram matrix

$$A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Here I is the identity matrix of size $n \times n$, whereas 0 stands for the all-zero matrix of the same size. (A basis like that is called a *hyperbolic* basis.) Let ϕ be the linear transformation of V whose matrix with respect to the basis $\mathcal{B}$ coincides with A and let ψ be the semilinear transformation of V that applies the bar automorphism to the $\mathcal{B}$-coordinates of every vector. If $\sigma_0 = \phi \circ \psi$, then for a vector

$$u = \sum_{i=1}^n x_i e_i + \sum_{i=1}^n y_i f_i$$

we compute that

$$u^{\sigma_0} = -\sum_{i=1}^n \bar{y}_i e_i + \sum_{i=1}^n \bar{x}_i f_i.$$

One easily verifies that (T1) and (T3) are satisfied for σ_0. To check (T2), consider

$$v = \sum_{i=1}^n x'_i e_i + \sum_{i=1}^n y'_i f_i.$$

Then

$$(u^{\sigma_0}, v^{\sigma_0}) = \sum_{i=1}^n (-\bar{y}_i)\bar{x}'_i - \bar{x}_i(-\bar{y}'_i) = \overline{(u,v)},$$

yielding (T2). Thus, σ_0 is a flip. Notice that $\sigma = \sigma_0$ can be characterized as the unique semilinear transformation such that (T1) holds and

$$e_i^\sigma = f_i, \quad f_i^\sigma = -e_i, \quad \text{for } 1 \le i \le n.$$

Whenever these latter conditions are satisfied for a flip σ and a hyperbolic basis $\mathcal{B} = \{e_1, \dots, f_n\}$, we will say that $\mathcal{B}$ is a *canonical* basis for σ.

Let $G \cong Sp(2n,q^2)$ be the group of all linear transformations of V preserving the form $(\cdot,\cdot)$. One of the principal results of this section is the following.

Proposition 3.1. *Every flip admits a canonical basis.*

In other words, every flip σ is conjugate to σ_0 by an element of G.

We start by discussing the general properties of flips. Let σ be a flip. Define

$$((x,y)) = (x,y^\sigma).$$

Lemma 3.2. *The form $((\cdot,\cdot))$ is a nondegenerate Hermitian form. Furthermore, $((u^\sigma,v^\sigma)) = \overline{((u,v))}$ for $u,v \in V$.*

Proof. Clearly, $((\cdot,\cdot))$ is a sesquilinear form. Also, $((v,u)) = (v,u^\sigma) = -(u^\sigma,v) = -\overline{(u^{\sigma^2},v^\sigma)} = -\overline{(-u,v^\sigma)} = \overline{(u,v^\sigma)} = \overline{((u,v))}$. Thus, $((\cdot,\cdot))$ is Hermitian. If u is in the radical of $((\cdot,\cdot))$ then for any $v \in V$, $0 = ((u,v^\sigma)) = ((u,v^{\sigma^2})) = -(u,v)$. Therefore, $u = 0$, as $(\cdot,\cdot)$ is nondegenerate. Finally, $((u^\sigma,v^\sigma)) = (u^\sigma,-v) = (v,u^\sigma) = ((v,u)) = \overline{((u,v))}$. $\square$

In what follows we will work with both $(\cdot,\cdot)$ and $((\cdot,\cdot))$. This calls for two different perpendicularity symbols. We will use $\perp$ for the form $(\cdot,\cdot)$, while $\perp\!\!\!\perp$ will be used for $((\cdot,\cdot))$.

Proof of Proposition 3.1. Let σ be a flip. Pick a vector $u \in V$ such that $((u,u)) = 1$. Such a vector exists since $((\cdot,\cdot))$ is nondegenerate by Lemma 3.2. Set $e_n = u$ and $f_n = u^\sigma$. Since $(\cdot,\cdot)$ is an alternating form we have $(e_n,e_n) = (f_n,f_n) = 0$. Furthermore, $(e_n,f_n) = ((e_n,f_n^{\sigma^{-1}})) = ((e_n,e_n)) = 1$. In particular, the subspace $U = \langle e_n,f_n\rangle$ is nondegenerate with respect to $(\cdot,\cdot)$. Consider now $V' = U^\perp$. Notice that U is invariant under σ. Together with (T2), this implies that V' is also invariant under σ. It is easy to see that the restriction of σ to V' is a flip of V'. By induction, there exists a hyperbolic basis $e_1,\ldots,e_{n-1},f_1,\ldots,f_{n-1}$ in V', such that $e_i^\sigma = f_i$ for $1 \le i \le n-1$. (Since $\sigma^2 = -Id$, this automatically implies $f_i^\sigma = -e_i$.)

Clearly, $\{e_1,\ldots,e_n,f_1,\ldots,f_n\}$ is a canonical basis for σ. $\square$

Next, we discuss the behavior of σ, $(\cdot,\cdot)$, and $((\cdot,\cdot))$ with respect to the subspaces $U \subset V$.

Lemma 3.3. *For a subspace $U \subset V$, we have $U^{\perp\!\!\!\perp} = (U^\sigma)^\perp = (U^\perp)^\sigma$. Similarly, $U^\perp = (U^\sigma)^{\perp\!\!\!\perp} = (U^{\perp\!\!\!\perp})^\sigma$.*

Proof. The first equality in the first claim immediately follows from the definition of $((\cdot,\cdot))$. If $u \in (U^\perp)^\sigma$ (say, $u = (u')^\sigma$ for $u' \in U^\perp$) and $v \in U$ then $((u,v)) = ((u')^\sigma, v^\sigma) = \overline{(u',v)} = 0$. The second claim follows by an application of σ to the equalities in the first claim. $\square$

Lemma 3.4. *The form $(\cdot,\cdot)$ has the same rank on U and U^σ; likewise, it has the same rank on $U^\perp$ and $U^{\perp\!\!\perp} = (U^\perp)^\sigma$. The same statements hold for $((\cdot,\cdot))$.*

Proof. The first claim follows from (T2) for $(\cdot,\cdot)$, and from Lemma 3.2 for $((\cdot,\cdot))$. The second claim follows from the first one and Lemma 3.3. $\square$

If U is σ-invariant then we can say more. It follows from Lemma 3.3 that $U^\perp = U^{\perp\!\!\perp}$. In other words, for a σ-invariant subspace U, the orthogonal complement (and hence also the radical) of U is the same with respect to $(\cdot,\cdot)$ and $((\cdot,\cdot))$. It also follows from Lemma 3.3 that both the orthogonal complement and the radical of U are σ-invariant.

It was noticed in the proof of Proposition 3.1 that the properties (T1)–(T3) are inherited by the restrictions of σ to all σ-invariant subspaces $U \subset V$. If U is nondegenerate—it does not matter with respect to which form—then the restriction of σ to U is a flip of U. We should now discuss what happens when U has a nontrivial radical. First of all, by the above comment, the radical of U is σ-invariant.

Lemma 3.5. *If U is σ-invariant then the radical of U has a σ-invariant complement in U.*

Proof. The proof is analogous to that of Proposition 3.1. If U is totally singular then there is nothing to prove. Otherwise, choose $u \in U$ such that $((u,u)) = 1$. Then $W = \langle u, u^\sigma \rangle$ is a σ-invariant nondegenerate subspace. Hence $U = (U \cap W^\perp) \oplus W$ and the radical of U coincides with the radical of $U_0 = U \cap W^\perp$. Clearly, U_0 is σ-invariant, and so induction applies. $\square$

Notice that the σ-invariant complement in the above lemma is automatically nondegenerate.

Next, let us study the "eigenspaces" of σ on V. For $\lambda \in \mathbb{F}_{q^2}$, define $V_\lambda = \{u \in V \,|\, u^\sigma = \lambda u\}$. Note that V_λ is not a true eigenspace, because σ is not linear.

Lemma 3.6. *The following hold.*

(1) *For* $0 \neq \mu \in \mathbb{F}_{q^2}$*, we have* $\mu V_\lambda = V_{\lambda'}$*, where* $\lambda' = \frac{\bar{\mu}}{\mu}\lambda$*; in particular,* V_λ *is an* $\mathbb{F}_q$*-subspace of* V*.*

(2) $V_\lambda \neq 0$ *if and only if* $\lambda\bar{\lambda} = -1$*; furthermore, if* $V_\lambda \neq 0$ *then* V_λ *contains a basis of* V*.*

Proof. Suppose $u \in V_\lambda$. Then $(\mu u)^\sigma = \bar{\mu} u^\sigma = \bar{\mu}\lambda u = \frac{\bar{\mu}}{\mu}\lambda(\mu u)$. This proves (1). Also, $-u = u^{\sigma^2} = \bar{\lambda}\lambda u$. Thus, if $u \neq 0$ then $\lambda\bar{\lambda} = -1$. This proves the 'only if' part of (2). To prove the 'if' part, choose a canonical basis $\{e_1, \ldots, f_n\}$ for σ. Fix a $\lambda \in \mathbb{F}_{q^2}$ such that $\lambda\bar{\lambda} = -1$. Define $u_i = e_i - \bar{\lambda} f_i$ and $v_i = \bar{\lambda} e_i + f_i$ for $1 \leq i \leq n$. A simple check shows that u_i and v_i are in V_λ. This shows that $V_\lambda \neq 0$. Furthermore, u_i and v_i are not proportional unless $\bar{\lambda} = \lambda$, that is, $\lambda \in \mathbb{F}_q$. Thus, if $\lambda \notin \mathbb{F}_q$ then $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is a basis of V. If $\lambda \in \mathbb{F}_q$ then consider $\lambda' = \frac{\bar{\mu}}{\mu}\lambda$, where μ is chosen so that $\frac{\bar{\mu}}{\mu} \notin \mathbb{F}_q$. By (1), $V_{\lambda'} = \mu V_\lambda$. Also, since $\lambda' \notin \mathbb{F}_q$, we have that $V_{\lambda'}$ contains a basis of V, and hence so does V_λ. □

Consider an $\mathbb{F}_q$-linear map $\phi : v \mapsto v - \bar{\lambda} v^\sigma$, where $\lambda \in \mathbb{F}_{q^2}$ and $\lambda\bar{\lambda} = -1$. It can be checked that ϕ maps V onto V_λ, and its kernel is $V_{\bar{\lambda}}$. The above vectors u_i and v_i are obtained by applying ϕ to the vectors in the canonical basis $\{e_1, \ldots, f_n\}$.

Now fix a $\lambda \in \mathbb{F}_{q^2}$ such that $\lambda\bar{\lambda} = -1$. Also, fix a $\mu \in \mathbb{F}_{q^2}$ with $\bar{\mu} = -\mu$.

Lemma 3.7. *The restriction of* $\mu\lambda(\cdot, \cdot)$ *to* V_λ *is a nondegenerate alternating* $\mathbb{F}_q$*-bilinear form.*

Proof. Clearly, the form $\mu\lambda(\cdot, \cdot)$ is $\mathbb{F}_q$-bilinear and alternating. Since V_λ contains a basis of V by Lemma 3.6 (2), the form is nondegenerate. It remains to see that it takes values in $\mathbb{F}_q$. However, if $u, v \in V_\lambda$, then $\overline{\mu\lambda(u, v)} = \bar{\mu}\bar{\lambda}(u^\sigma, v^\sigma) = -\mu\bar{\lambda}\lambda^2(u, v) = \mu\lambda(u, v)$. □

Observe that the conjugation by σ is an automorphism of G. Let G_σ be the centralizer of σ in G. The above setup gives us means to identify G_σ. Let $H \cong Sp(2n, q)$ be the group of all linear transformations of V_λ preserving the (restriction of the) form $\mu\lambda(\cdot, \cdot)$. Since V_λ contains a basis of V, we can use $\mathbb{F}_{q^2}$-linearity to extend the action of the elements of H to the entire V. This allows us to identify H with a subgroup of G. Clearly, since $h \in H$ preserves $\mu\lambda(\cdot, \cdot)$, it must also preserve $(\cdot, \cdot)$.

Proposition 3.8. $G_\sigma = H$*.*

Proof. Choose a basis $\{w_1, \ldots, w_{2n}\}$ in V_λ. Then this set is also a basis of V. Let $h \in H$. If $u = \sum_{i=1}^{2n} x_i w_i \in V$ then $u^{\sigma h} = (\sum_{i=1}^{2n} \bar{x}_i \lambda w_i)^h = \lambda \sum_{i=1}^{2n} \bar{x}_i w_i^h$. On the other hand, $u^{h\sigma} = (\sum_{i=1}^{2n} x_i w_i^h)^\sigma = \sum_{i=1}^{2n} \bar{x}_i \lambda w_i^h$. Therefore, $H \leq G_\sigma$. Now take $g \in G_\sigma$. If $u \in V_\lambda$ then $(u^g)^\sigma = (u^\sigma)^g = (\lambda u)^g = \lambda u^g$. This proves that g leaves V_λ invariant. It remains to see that g preserves $\mu\lambda(\cdot,\cdot)$. However, this is clear, because g is $\mathbb{F}_{q^2}$-linear and it preserves $(\cdot,\cdot)$. $\square$

4 The flipflop geometry Γ

We will be using the notation from the previous section. In particular, V is a nondegenerate symplectic $\mathbb{F}_{q^2}$-space of dimension $2n$ with a form $(\cdot,\cdot)$, σ a flip and $((\cdot,\cdot))$ is the corresponding Hermitian form. Also, $G \cong Sp(2n, q^2)$ is the group of linear transformations preserving $(\cdot,\cdot)$ and $G_\sigma = C_G(\sigma)$. Throughout this section, we assume $n \geq 2$. Let B be the building geometry associated with G. Its elements are all the $(\cdot,\cdot)$-totally singular subspaces of V.

Two elements U and U' of B are *opposite* whenever $V = U' \oplus U^\perp$, *i.e.*, U, U' have the same dimension and $U' \cap U^\perp = 0$. Two chambers (maximal flags) F and F' are opposite whenever for each subspace $U \in F$ there is a $U' \in F'$ such that U and U' are opposite. Using this, it can be shown that the opposites geometry Γ_{op} related to B is indeed a geometry and its elements are all pairs (U, U') are opposite totally singular subspaces of V.

Turning to Γ_σ, let F be a maximal flag of B such that F and F^σ are opposite. Then, for every $U \in F$, the space U^σ must be the element of F^σ that is opposite U. Indeed, this follows from the fact that opposite elements have the same dimension. Thus, $(F, F^\sigma) \in \Gamma_\sigma$ if and only if U^σ is opposite U for every element $U \in F$ (that is, $(U, U^\sigma) \in \Gamma_{op}$).

Our first goal is to show that Γ_σ is a geometry, that is to say, its chambers arise as maximal flags of a suitable geometry. The natural candidate for this geometry is the following subset of Γ_{op}:

$$\{(U, U') \in \Gamma_{op} \mid U' = U^\sigma\}.$$

(For convenience, we will refer to this set as Γ_σ, anticipating that correctness of this will be shown later.)

It suffices to show that Γ_σ is a full rank (that is, rank n) subgeometry of Γ_{op}. In order to avoid cumbersome notation, let us project every pair

$(U, U') \in \Gamma_\sigma$ to its first coordinate U. Since $U' = U^\sigma$, this establishes a bijection (in fact, an isomorphism of pregeometries) between Γ_σ and the following subset of B:

$$\Gamma = \{U \in B \mid U^\sigma \text{ is opposite } U\}.$$

The definition of Γ can be nicely restated in terms of the forms $(\cdot, \cdot)$ and $((\cdot, \cdot))$.

Proposition 4.1. *The elements of* Γ *are all subspaces* $U \subset V$ *which are totally isotropic with respect to* $(\cdot, \cdot)$ *and nondegenerate with respect to* $((\cdot, \cdot))$.

Proof. By Lemma 3.3, $U^{\perp\!\perp} = (U^\sigma)^\perp$. Hence U and U^σ are opposite if and only if $U \cap U^{\perp\!\perp} = 0$. $\square$

As in the introduction, we use $\{1, \ldots, n\}$ as the type set of B. In particular, the type function is given by the linear (rather than projective) dimension. We will use the customary geometric terminology. In particular, *points*, *lines* and *planes* are elements of type 1, 2 and 3, respectively.

We stress again that we will mostly work with Γ, using the fact that Γ and Γ_σ are isomorphic. We also notice that the isomorphism between Γ and Γ_σ commutes with the action of $H = G_\sigma$.

Proposition 4.2. *The pregeometry* Γ *is a geometry. Moreover,* H *acts flag-transitively on* Γ.

Proof. Let $V_1 \leq V_2 \leq \cdots \leq V_k$ be a maximal flag. Let $\mathcal{B} = \{e_1, \ldots, e_t\}$ be an orthonormal basis of V_k with respect to $((\cdot, \cdot))$. (This exists since V_k is nondegenerate with respect to $((\cdot, \cdot))$.) Then $\mathcal{B} \cup \mathcal{B}^\sigma$ forms a canonical basis of $V_k \oplus V_k^\sigma$. If V_k is not a maximal totally isotropic subspace of V with respect to $(\cdot, \cdot)$, there exists a nontrivial $u \in (V_k \oplus V_k^\sigma)^\perp = (V_k \oplus V_k^\sigma)^{\perp\!\perp}$ such that $((u, u)) = 1$. Then $\langle V_k, u \rangle$ is totally isotropic for $(\cdot, \cdot)$ and nondegenerate with respect to $((\cdot, \cdot))$, contradicting maximality of the flag. Hence we can assume V_k is a maximal totally isotropic subspace with respect to $(\cdot, \cdot)$. Induction shows that V_{i-1} is a codimension 1 subspace in V_i for $2 \leq i \leq k$, proving that the maximal flag is a chamber.

Let $V_1 \leq V_2 \leq \cdots \leq V_n$ and $V_1' \leq V_2' \leq \cdots \leq V_n'$ be two chambers. Choose bases $\mathcal{B} = \{e_1, \ldots, e_n\}$, $\mathcal{B}' = \{e_1', \ldots, e_n'\}$ for V_n, respectively V_n' such that they are orthonormal with respect to $((\cdot, \cdot))$ and $V_i = \langle e_1, \ldots, e_i \rangle$, $V_i' = \langle e_1', \ldots, e_i' \rangle$. Define $g \in G$ such that $e_i^g = e_i'$ and $(e_i^\sigma)^g = (e_i')^\sigma$. Such a g

obviously exists, since $G \cong Sp(2n, q^2)$ acts flag-transitively on the symplectic polar space $(V, (\cdot,\cdot))$. It is also clear that g maps one chamber onto the other. Moreover notice that $\sigma \circ g = g \circ \sigma$ on the basis $\mathcal{B} \cup \mathcal{B}^\sigma$. Therefore $g \in G_\sigma$. □

The following lemma will prove to be very useful throughout the whole article.

Lemma 4.3. *Let P be a point of Γ and $\Pi \supset P$ be a 3-dimensional subspace of V of rank at least two with respect to $((\cdot,\cdot))$ such that P is in the radical of Π with respect to $(\cdot,\cdot)$. Then any 2-dimensional subspace of Π not containing P is incident with at least $q^2 - q - 1$ (respectively, $q^2 - 2q - 1$) points of Γ collinear to P if its rank is one (respectively, two) with respect to $((\cdot,\cdot))$.*

Proof. Since P is in the radical of Π with respect to $(\cdot,\cdot)$, all lines passing through P will be totally isotropic with respect to $(\cdot,\cdot)$ so we only need to consider $((\cdot,\cdot))$. Notice that if L is a 2-dimensional subspace of V that is not totally isotropic with repect to $((\cdot,\cdot))$ then L contains at least $q^2 - q$ points of Γ. (If the rank of L is one then the radical is the only nontrivial isotropic subspace of L and if the rank of L is two then L contains $q + 1$ distinct nontrivial isotropic subspaces.)

Consider $L_1 = P^{\perp\!\!\perp} \cap \Pi$. Then by the above, there are at least $q^2 - q$ lines of Γ through to P that intersect L_1 in a point of Γ. If L is any other not totally isotropic 2-dimensional subspace of Π not containing P, at most 1, respectively $q + 1$ of the these $q^2 - q$ lines will intersect L in isotropic subspaces. Hence the lemma follows. □

Actually, we also showed the following:

Corollary 4.4. *Let P be a point of Γ and $\Pi \supset P$ be a 3-dimensional subspace of V of rank at least two with respect to $((\cdot,\cdot))$. Then any 2-dimensional subspace of Π not containing P is incident with at least q^2-q-1 (respectively, $q^2 - 2q - 1$) points of Γ that generate a $((\cdot,\cdot))$-nondegenerate two space with P if its $((\cdot,\cdot))$-rank is one (respectively, two).* □

We need to prove that the geometry is connected. This is equivalent to proving connectivity of the point shadow space of Γ which in turn is equivalent with connectivity of the collinearity graph of Γ.

Lemma 4.5. *Suppose $n \geq 3$. Then if $(n, q) \neq (3, 2)$ then the collinearity graph of the geometry Γ has diameter two. If $(n, q) = (3, 2)$ then the collinearity graph of Γ has diameter three. In particular, Γ is connected in all cases.*

Proof. If $(n,q)=(3,2)$ then the claim can be checked computationally (say, in GAP). So suppose $(n,q)\neq(3,2)$. Let P_1,P_2 be two points in the geometry. Consider $W_i := P_i^{\perp}\cap P_i^{\perp\perp}$, $i\in\{1,2\}$. Then $\dim W_i = 2n-2$ so $\dim W_1\cap W_2 \geq 2n-4$. If $2n-4>n-1$ then the space $W_1\cap W_2$ cannot be totally isotropic for $((\cdot,\cdot))$ (it lies inside the $(2n-1)$-dimensional nondegenerate space $P_1^{\perp\perp}$). Therefore if $n>3$ we can find a point Q in the geometry lying in $W_1\cap W_2$. In this case Q connects P_1 and P_2.

If $n=3$, the space $U=P_1^{\perp}\cap P_2^{\perp}\cap P_2^{\perp\perp}$ is at least 3-dimensional inside the 4-dimensional space $P_2^{\perp}\cap P_2^{\perp\perp}$, which is nondegenerate with respect to both forms. Actually, U has rank at least two with respect to $((\cdot,\cdot))$, because if it had a 2-dimensional radical, this radical would be a maximal totally isotropic subspace of $P_2^{\perp}\cap P_2^{\perp\perp}$ and had to be equal to its own polar in $P_2^{\perp}\cap P_2^{\perp\perp}$ with respect to $((\cdot,\cdot))$. Hence we can find a $((\cdot,\cdot))$-nondegenerate 2-dimensional subspace L of U, all points of which actually are collinear to P_2. Applying Lemma 4.3 to the plane $\langle P_1, L\rangle$, we find a common neighbor of P_1 and P_2. $\square$

Lemma 4.6. *If $n=2$ and $q\neq 2$, then Γ is connected. If $n=2$ and $q=2$, then Γ is not connected.*

Proof. Fix a point P of Γ. Then P is collinear to $(q^2-q)(q^2-q-1)$ points of Γ (there are q^2-q lines through P, each of which contains q^2-q-1 points of Γ except P). Now let us estimate the number of points at distance two to P. Each point Q at distance one to P is incident with q^2-q-1 lines that do not contain P. Each of these lines contains q^2-q-1 points other than Q. Moreover, if R is a point at distance two from P, then there are at most q^2 common neighbors of P and R (indeed, $\langle P,R\rangle^{\perp}$ is a 2-dimensional space which is not totally isotropic with respect to $((\cdot,\cdot))$, whence containing either q^2 or q^2-q points of Γ). Hence there are at least $\frac{(q^2-q)(q^2-q-1)^3}{q^2}$ points at distance two from P.

On the other hand, Γ contains $\frac{q^8-1}{q^2-1}-(q^2+1)(q^3+1)$ points (the number of points of the projective space minus the number of points of the unitary generalized quadrangle).

By Proposition 4.2 and Proposition 3.8, the group $G_\sigma \cong Sp(4,q)$ acts flag-transitively on Γ. In particular, it permutes the connected components of Γ. More precisely, the number of connected components is equal to the index of the stabilizer of one component in G_σ. By [2], Table 5.2.A the index of a maximal subgroup of $Sp(4,q)$ is at least 27, if $q>2$. Hence, to show connectivity, it is enough to prove that $1+(q^2-q)(q^2-q-1)+\frac{(q^2-q)(q^2-q-1)^3}{q^2}$ is greater than $\frac{1}{27}\left(\frac{q^8-1}{q^2-1}-(q^2+1)(q^3+1)\right)$, which is true for all $q \geq 3$.

To deal with the case $n=2$, $q=2$, notice first that in this case two points P, Q of Γ are collinear if and only if they are perpendicular with respect to both forms. Therefore, for any point P, the set of points collinear to P is contained in the nondegenerate (with respect to both forms) 2-dimensional space $P^\perp \cap P^{\perp\!\!\perp}$, which contains precisely two points of Γ. Considering those two points inside $P^\perp \cap P^{\perp\!\!\perp}$, we see that their neighbors coincide (they are precisely the points P and P^σ). Therefore the connected component of the point P consists of precisely four points. On the other hand Γ consists of $40=\frac{2^8-1}{2^2-1}-(2^2+1)(2^3+1)$ points, so Γ is not connected. □

We summarize Lemmas 4.5 and 4.6 in the following

Theorem 4.7. *Suppose $n \geq 2$. Then Γ is connected unless $(n,q)=(2,2)$.* □

Combined with the results of [1], this yields

Corollary 4.8. *If $q \neq 2$ then Γ is residually connected.* □

Finally, let us discuss the diagram of the geometry Γ_σ. Notice that it is a linear (string) diagram. Furthermore, it follows from Proposition 4.1 that the residue of an element of maximal type $n-1$ is the geometry of all nondegenerate subspaces of a nondegenerate n-dimensional unitary space. The residue of a point is a geometry similar to Γ but with rank $n-1$. This leads to the diagram

$$\begin{array}{ccccccccccc} & U & & U & & & & U & & S & \\ \circ & \text{———} & \circ & \text{———} & \circ & \cdots & \circ & \text{———} & \circ & \text{———} & \circ . \\ q^2-q & & q^2-q & & q^2-q & & q^2-q & & q^2-q & & q^2-q \end{array}$$

The exact meaning of the edges $\circ\overset{U}{\text{——}}\circ$ and $\circ\overset{S}{\text{——}}\circ$ is as follows. The first one represents the geometry of all 1- and 2-dimensional nondegenerate subspaces of a 3-dimensional unitary space. It appears in [1]. The second edge represents our flipflop geometry in the case of rank two. We note that both geometries are disconnected for $q=2$ and connected for $q \geq 3$. See [1] for $\circ\overset{U}{\text{——}}\circ$ and Lemma 4.6 for $\circ\overset{S}{\text{——}}\circ$.

5 Simple connectedness, Part I

In this and the next section we will prove that, apart from a few exceptional cases, the geometry Γ is simply connected. Here we collect some general statements and then complete the case $n \geq 4$. The next section handles the case $n = 3$, which is somewhat more complicated.

Recall the definition of the fundamental group of a connected geometry Δ. A path of length k in the geometry is a sequence of elements $x_0, \ldots, x_k$ such that x_i and x_{i+1} are incident, $0 \leq i \leq k-1$. We do not allow repetitions; hence $x_i \neq x_{i+1}$. A *cycle* based at an element x is a path in which $x_0 = x_k = x$. Two paths are *homotopically equivalent* if one can be obtained from the other via the following operations (called *elementary homotopies*): inserting or deleting a return (*i.e.*, a cycle of length 2) or a triangle (*i.e.*, a cycle of length 3). The equivalence classes of cycles based at an element x form a group under the operation induced by concatenation of cycles. This group is called the *fundamental group* of Δ and denoted by $\pi_1(\Delta, x)$. A geometry is called simply connected if its fundamental group is trivial.

Notice that in order to prove that Δ is simply connected it is enough to prove that any cycle based at x is homotopically equivalent to the cycle of length 0. A cycle with this property is called *null homotopic*, or *homotopically trivial*.

Let us go back to the flipflop geometry Γ. We pick the base element x to be a point of Γ.

Lemma 5.1. *Unless $n = 3$, $q = 2$, every cycle based at x is homotopically equivalent to a cycle passing only through points and lines.*

Proof. We will induct on the number of elements of the path that are not points or lines. If this number is zero there is nothing to prove. Take an arbitrary cycle $\gamma := xx_1 \ldots x_{k-1}x$. Let x_i be the first element that is not a point or a line. Clearly $i \notin \{0, k\}$. There are two cases to consider:

If the type of x_{i+1} is bigger than the type of x_i then x_{i-1} and x_{i+1} are incident and γ is homotopically equivalent to the cycle $xx_1 \ldots x_{i-1}x_{i+1} \ldots x$.

Suppose the type of x_{i+1} is smaller than the type of x_i. Let y be an element of type n which is incident to x_i (in particular, take x_i, if the type of x_i is n), then y is incident to both x_{i-1} and x_{i+1} (the type of x_{i-1} is clearly smaller than the type of x_i). Therefore γ is homotopically equivalent to the path $xx_1 \ldots x_{i-1}yx_{i+1} \ldots x$. Now pick two points z, w such that z

is incident to x_{i+1} and w is x_{i-1}, if x_{i-1} is a point, or a point incident to x_{i-1}, otherwise. Using Lemma 4.5 and Lemma 4.6 we can connect w and z with a path $ww_1 \dots w_t z$ of only points and lines incident to y. Then γ is homotopically equivalent to $xx_1 \dots x_{i-1} w_1 \dots w_t z x_{i+2} \dots x$ which contains fewer elements that are not points and lines. □

We can therefore restrict our attention to the point-line incidence graph of Γ and, thus, to the collinearity graph of Γ.

The first step is the analysis of triangles (*i.e.*, 3-cycles in the collinearity graph). We will call a triangle (P, Q, R) a *good triangle* if P, Q, and R are incident to a common plane of the geometry. Conversely, all triangles that are not good are called *bad*.

Now we are to prove that all bad triangles are homotopically trivial, *i.e.*, they can be decomposed into good triangles or are contained in objects of Γ of higher rank.

Lemma 5.2. *Let (P, Q, R) be a bad triangle. Then the plane $\langle P, Q, R\rangle$ contains a 1-dimensional radical with respect to $((\cdot, \cdot))$.*

Proof. It is clear that the plane $\Pi = \langle P, Q, R\rangle$ is totally isotropic with respect to $(\cdot, \cdot)$. Since P, Q, R is a bad triangle, Π is degenerate with repect to $((\cdot, \cdot))$. Also, the rank of Π with respect to $((\cdot, \cdot))$ is at least two (it contains the nondegenerate projective line $\langle P, Q\rangle$), so the radical is obviously 1-dimensional. □

Lemma 5.3. *Every bad triangle (P, Q, R) can be decomposed as a product of two (bad) triangles in which two of the vertices are perpendicular with respect to $((\cdot, \cdot))$.*

Proof. If two of P, Q and R are already perpendicular with respect to $((\cdot, \cdot))$, then there is nothing to show. So assume that no two of P, Q and R are perpendicular with respect to $((\cdot, \cdot))$. Let X be the radical of the plane $\langle P, Q, R\rangle$. Consider the unique projective point S of the line $\langle P, Q\rangle$ such that $R \perp\!\!\!\perp S$. It is sufficient to prove that S is a point of Γ. Suppose it is not, then $\langle R, S\rangle = S^{\perp\!\!\!\perp} \cap \langle P, Q, R\rangle$ and so it contains X. Since $\langle P, Q\rangle$ is a line of Γ, X is not contained in $\langle P, Q\rangle$, yielding that $X \neq S$. Therefore $\langle R, S\rangle = \langle X, S\rangle$ is a totally isotropic space with respect to $((\cdot, \cdot))$ containing R, contradicting the fact that R is a point of Γ. Hence (P, R, S) and (Q, R, S) are triangles as required. □

Lemma 5.4. *Let (P, Q, R) be a bad triangle with $P \perp\!\!\perp Q$ and let X be the radical of the plane $\langle P, Q, R\rangle$. If $X^\sigma = X$, then we can find a canonical basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V for σ such that (P, Q, R) equals $(\langle e_1\rangle, \langle e_2\rangle, \langle xe_1 + ye_2 + (ce_3 + f_3)\rangle)$ with $c\bar{c} = -1$ and $xy \neq 0$ and $x\bar{x} + y\bar{y} \neq 0$.*

Proof. Choose a canonical basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that $P = \langle e_1\rangle$, $Q = \langle e_2\rangle$. Then $X \in U := \langle e_1, e_2\rangle^\perp \cap \langle e_1, e_2\rangle^{\perp\!\!\perp} = \langle e_1, e_2, f_1, f_2\rangle^{\perp\!\!\perp}$, which is a nondegenerate space with respect to both forms. Pick a, a' such that $a\bar{a} = -1 = a'\bar{a}'$ and $a \neq a'$. Then the vectors $e_3 + af_3, e_3 + a'f_3, \ldots, e_n + af_n, e_n + a'f_n$ are isotropic with respect to $((\cdot,\cdot))$ and they form a basis of U. Furthermore, every vector from this basis spans a σ-invariant 1-dimensional subspace. The radical X cannot be orthogonal to all of these vectors, so there exists one vector u in this basis such that $((u, X)) \neq 0$. The space $\langle u, X\rangle$ is nondegenerate and σ-invariant so it will contain a vector e such that $((e, e)) = 1$ and therefore $\langle u, X\rangle = \langle e, e^\sigma\rangle$. Chosing a new canonical basis of U starting with e we can assume that the bad triangle is contained in the space $\langle e_1, e_2, e_3, f_3\rangle$ and $X = \langle ce_3 + f_3\rangle$. The conditions on x, y and c as in the statement of the lemma can now be verified. □

For the rest of this section assume $n \geq 4$.

Proposition 5.5. *Let (P, Q, R) be a bad triangle. Then the triangle is homotopically trivial.*

Proof. By Lemma 5.2, the plane $\langle P, Q, R\rangle$ has a 1-dimensional radical X (with respect to $((\cdot,\cdot))$).

Suppose $X = X^\sigma$. By Lemma 5.3 and Lemma 5.4 we can assume that our triangle has the form $P = \langle e_1\rangle$, $Q = \langle e_2\rangle$, $R = \langle xe_1 + ye_2 + (ce_3 + f_3)\rangle$ where $c\bar{c} = -1$ and $x\bar{x} + y\bar{y} \neq 0$. (Here, as usual, $e_1, \ldots, f_n$ is a canonical basis.) Now one can add to P, Q and R the point $\langle e_4\rangle$ and form a tetrahedron in which all triangles but the initial one are good.

If $X \neq X^\sigma$, then consider the line $L = PQ$ of Γ. Let $V' = L^{\perp\!\!\perp} \cap (L^\sigma)^{\perp\!\!\perp}$. Then V' is a nondegenerate σ-invariant subspace of dimension $2n-4$. Moreover, $X \in V'$. Nondegeneracy of V' and the fact that $X \neq X^\sigma$ imply the existence of a vector $v \in V'$ with $(v, X) = 0$ and $((v, X)) = (v, X^\sigma) = 1$. Hence $\langle X, v\rangle$ is a line of Γ, and $\langle P, Q, X, v\rangle$ is totally isotropic with respect to $(\cdot,\cdot)$ and nondegenerate with respect to $((\cdot,\cdot))$, whence it is an object of Γ containing the triangle (P, Q, R). □

The next task is proving that all quadrangles are homotopically trivial. Recall that if a subspace U is σ-invariant then $U^\perp = U^{\perp\!\!\perp}$ and, in particular,

U is nondegenerate with respect to $(\cdot,\cdot)$ if and only if it is nondegenerate with respect to $((\cdot,\cdot))$.

Lemma 5.6. *If U is a σ-invariant nondegenerate subspace of V of dimension $2k \geq 4$ and P is a point of Γ, then P is collinear with a point of U or $2k = 4$ and $q = 2$.*

Proof. Consider the decomposition $V = U \oplus U^\perp$. Let $P_1 \in U$ be the projection of P onto U (with respect to this decomposition). If we find a point Q of Γ in $P_1^\perp \cap P_1^{\perp\!\!\perp} \cap U$, then we are done. Indeed, $Q \perp P_1$, $Q {\perp\!\!\perp} P_1$ implies $Q \perp P$, $Q {\perp\!\!\perp} P$ by our choice of the projection. In particular this holds, if $k > 2$; then $2k-2 > k$ and $P_1^\perp \cap P_1^{\perp\!\!\perp} \cap U$ cannot be totally isotropic. (Notice, that we are also done, if P_1 itself is nonsingular with respect to $((\cdot,\cdot))$.)

Thus, consider the case $k = 2$. The space $U \cap P_1^\perp$ is 3-dimensional and has rank at least two with respect to $((\cdot,\cdot))$. Choose a projective line L of $((\cdot,\cdot))$-rank two in $U \cap P_1^\perp$. Notice that $P \perp L$, whence by Lemma 4.3, the projective line L contains $q^2 - 2q - 1$ points of Γ collinear to P, giving at least one, if $q > 2$. $\square$

A pair P, Q of points of Γ will a be called *solid* if the space $P^\perp \cap P^{\perp\!\!\perp} \cap Q^\perp \cap Q^{\perp\!\!\perp}$ is nondegenerate.

Lemma 5.7. *Let A, B be two distinct points of Γ with $B \notin \langle A, A^\sigma\rangle$. The pair A, B is solid if and only if the projection of B onto $\langle A, A^\sigma\rangle^\perp$ (via the decomposition $V = \langle A, A^\sigma\rangle \oplus \langle A, A^\sigma\rangle^\perp$) is nonsingular.*

Proof. Let $B' = \mathrm{pr}_{\langle A,A^\sigma\rangle^\perp}(B)$ be the projection of B onto $\langle A, A^\sigma\rangle^\perp$. Notice that $B' \neq 0$. We have $\langle A, A^\sigma, B\rangle = \langle A, A^\sigma, B'\rangle$ which is of rank three with respect to $((\cdot,\cdot))$ if and only if B' is nonsingular with respect to $((\cdot,\cdot))$. But if the rank of this space is three, then the rank of $\langle A, A^\sigma, B, B^\sigma\rangle$ has to be four, since its radical with respect to $((\cdot,\cdot))$ equals the radical with respect to $(\cdot,\cdot)$ and it contains a subspace of rank three with respect to $((\cdot,\cdot))$. (Notice that an alternating form always has even rank.) This settles the 'if' part of the lemma.

Now, suppose B' is singular with respect to $((\cdot,\cdot))$. Then $\langle A, A^\sigma, B, B^\sigma\rangle = \langle A, A^\sigma, B', (B')^\sigma\rangle$ and B' is obviously contained in the radical of the latter space. $\square$

Lemma 5.8. *If $n \geq 5$ or $n = 4$ and $q \neq 2$, any quadrangle (P, Q, R, S) with a solid pair P, R is null homotopic.*

Proof. Assume P, R is a solid pair and let $U = P^\perp \cap P^{\perp\perp} \cap R^\perp \cap R^{\perp\perp}$. Then U is a σ-invariant nondegenerate $(2n-4)$-dimensional subspace and all points of Γ in U are collinear to both P and R. By Lemma 5.6, Q and S are collinear to points in U unless $n = 4$ and $q = 2$. Also, because of Lemma 4.5 and Lemma 4.6, the intersection of U with the geometry Γ is connected unless $n = 4$, $q = 2$. This finishes the proof. □

Proposition 5.9. *If $n \geq 5$ or $n = 4$ and $q \neq 2$, then any quadrangle is homotopically trivial.*

To prove this proposition we will need some facts from linear algebra:

Lemma 5.10. *Let $n \geq 2$, $q \geq 3$ and let W be an $\mathbb{F}_{q^2}$-vector space of dimension n. Suppose f_1 and f_2 are two nontrivial Hermitian forms on W. Then there exists a vector of W which is nonsingular with respect to both f_1 and f_2.*

Proof. First suppose that f is a Hermitian form on W and L is a 2-dimensional subspace in W that is not totally singular with respect to f. Then if L is nondegenerate with respect to f then out of the total number of q^2+1 1-dimensional subspaces of L exactly $q+1$ are singular. Similarly, if f has rank one on L then L contains exactly one singular 1-dimensional subspace.

Now, since f_1 is nontrivial, any f_1-singular 1-dimensional subspace of W is contained in a 2-dimensional subspace L which is not totally isotropic with respect to f_1. If L is not totally isotropic with respect to f_2, then it contains at least $q^2+1-q-1-q-1 \geq 2$ 1-dimensional subspaces that are nonsingular with respect to both f_1 and f_2. On the other hand, if any such L is totally isotropic with respect to f_2, then every 1-dimensional subspace that is singular with respect to f_1, is also singular with respect to f_2. But since f_2 is nontrivial on W, there exists a vector that is nonsingular with respect to f_2, and hence with respect to f_1, too. □

Lemma 5.11. *Let $n \geq 3$, $q \geq 3$ and let W be an $\mathbb{F}_{q^2}$-vector space of dimension n. Suppose f_1, f_2 and f_3 are three nontrivial Hermitian forms on W, and, furthermore, assume that f_1 is nondegenerate. Then there exists a vector of W which is nonsingular with respect to all three forms.*

Proof. Since f_1 is nondegenerate and since $n \geq 3$, any 1-dimensional subspace singular with respect to f_1 is contained in a 2-dimensional subspace

L of f_1-rank one. Notice that L contains exactly q^2 1-dimensional subspaces that are nonsingular with respect to f_1. If L is not totally isotropic with respect to both f_2 and f_3, then there are at least $q^2 - q - 1 - q - 1 \geq 1$ 1-dimensional subspaces that are nonsingular with respect to all three forms.

Therefore, suppose that any such subspace L is totally singular with respect to either f_2 or f_3. This means that every f_1-singular 1-dimensional subspace is singular with respect to f_2 or f_3. However, by Lemma 5.10, there is a vector $w \in W$ that is nonsingular with respect to both f_2 and f_3. Consequently, w is also nonsingular with respect to f_1. $\square$

Proof of Proposition 5.9. Let (A, B, C, D) be a quadrangle. In view of Lemma 5.8, we may assume that both pairs A, C and B, D are not solid.

Assume first that $q \neq 2$. The space $U = \langle A, B, A^\sigma, B^\sigma \rangle^\perp$ is nondegenerate of dimension $2n - 4$. We want to find a point X of Γ in U that forms a solid pair with both C and D. Besides $((\cdot, \cdot))$, consider two more forms $f_2(u, v) = ((u', v'))$ and $f_3(u, v) = ((u'', v''))$ where u', v' are the projections of u and v to $\langle C, C^\sigma \rangle^\perp$ and u'', v'' are the projections of u and v to $\langle D, D^\sigma \rangle^\perp$ as in Lemma 5.7. We remark that both f_1 and f_2 are nontrivial. For example, for f_1, it suffices to see that the image of the projection of U to $\langle C, C^\sigma \rangle^\perp$ cannot be totally isotropic. Let pr_1 and pr_2 be the projections to $\langle C, C^\sigma \rangle$ and $\langle C, C^\sigma \rangle^\perp$, respectively. If $\mathrm{pr}_2(U)$ is totally isotropic then pr_1 is isometric on U. In particular, $U \cap \langle C, C^\sigma \rangle^\perp$ is in the radical of U, a contradiction. Thus f_1 and f_2 are nontrivial.

By Lemma 5.11, with $f_1 = ((\cdot, \cdot))$, there exists a point X of Γ such that its projections onto both $\langle C, C^\sigma \rangle^\perp$ and $\langle D, D^\sigma \rangle^\perp$ are nonsingular. Hence, by Lemma 5.7, the point X forms a solid pair with both C and D, as we wanted. Now, let $W = \langle C, D, C^\sigma, D^\sigma \rangle^\perp$, which is also of dimension $2n - 4$ and nondegenerate. By Lemma 5.6, W contains a point Y of Γ collinear to X.

We have accomplished the following: the quadrangle (A, B, C, D) has been decomposed into the triangles (A, B, X), (C, D, Y) and the quadrangles (C, B, X, Y), (A, D, Y, X), both of which contain a solid pair.

It remains to deal with $q = 2$, in which case $n \geq 5$. Recall that for $q = 2$ two points are collinear if and only if they are perpendicular with respect to both forms. In particular, B and D are in $U = \langle A, A^\sigma, C, C^\sigma \rangle^\perp$. Since A, C is not a solid pair, $W = \langle A, A^\sigma, C, C^\sigma \rangle$ is singular and hence it has rank two with respect to either form. If $\dim W = 3$ then U has dimension at least seven and rank at least six. It now follows from Lemma

4.7 that the set of points of Γ contained in U is connected with respect to collinearity. Hence (A, B, C, D) can be decomposed into triangles. Hence, let us assume that W has dimension four and rank two. Then U has dimension six and rank four. By Lemma 3.5, U contains a σ-invariant subspace U_0 complementing the radical of U. Choose points $B' \in U_0 \cap (\langle B, B^\sigma \rangle^\perp)$ and $D' \in U_0 \cap (\langle D, D^\sigma \rangle^\perp)$. Then (A, B, C, D) decomposes as a product of (A, B', C, D') and four triangles. Since B' and D' are contained in U_0, they either form a solid pair or $\langle B', (B')^\sigma, D', (D')^\sigma \rangle$ is 3-dimensional of rank two. In either case, (A, B', C, D') is known to be null-homotopic. □

Finally, the decomposition of pentagons is now easy:

Proposition 5.12. *If $n \geq 5$ or $n = 4$ and $q \neq 2$, then any pentagon is homotopically trivial.*

Proof. Let (A, B, C, D, E) be a pentagon. Consider $U := \langle A, B, A^\sigma, B^\sigma \rangle^\perp$ of dimension $2n - 4$, which is nondegenerate with respect to both forms. By Lemma 5.6, the point D is collinear to a point F of Γ inside U, decomposing the pentagon into triangles and quadrangles. □

We can summarize the results of this section as follows. Suppose $n \geq 4$ and $(n, q) \neq (4, 2)$. Then the diameter of the collinearity graph of Γ is two and 3-, 4- and 5-cycles are null-homotopic. This implies the following theorem.

Theorem 5.13. *If $n \geq 4$ then the geometry Γ is simply connected, unless $(n, q) = (4, 2)$.* □

We remark that it is unknown to us whether the case $(n, q) = (4, 2)$ is a true exception.

6 Simple connectedness, Part II

In this section we assume $n = 3$. We will prove that the geometry Γ is simply connected for $q \geq 8$. As usual $e_1, \dots, f_n$ is a canonical basis.

Lemma 6.1. *Let (P, Q, R) be a bad triangle and let X be the radical of the plane $\langle P, Q, R \rangle$ with respect to $((\cdot, \cdot))$. Then $X^\sigma = X$.*

Proof. Suppose $X^\sigma \neq X$. Then the $(\cdot,\cdot)$-totally isotropic planes $\langle P, Q, R\rangle$ and $\langle P^\sigma, Q^\sigma, R^\sigma\rangle$ do not intersect. Indeed, if they did, then the the radical of $\langle P, Q, R\rangle$ were contained in the intersection. Hence, by symmetry, $\langle P, Q, R\rangle \cap \langle P^\sigma, Q^\sigma, R^\sigma\rangle$ had to contain the two space $\langle X, X^\sigma\rangle$, which on one hand were contained in the radical of $\langle P, Q, R\rangle$ and on the other hand is totally isotropic with respect to $((\cdot,\cdot))$, contradicting the fact that the rank with respect to $((\cdot,\cdot))$ of $\langle P, Q, R\rangle$ equals two. Consequently, $V = \langle P, Q, R, P^\sigma, Q^\sigma, R^\sigma\rangle$, which has a radical with respect to $(\cdot,\cdot)$ containing X, contradicting nondegeneracy of $(\cdot,\cdot)$. $\square$

In view of this lemma and of Lemmas 5.3 and 5.4 the bad triangles we need to deal with are very restricted.

Lemma 6.2. *Let (P, Q, R) be the bad triangle $(\langle e_1\rangle, \langle e_2\rangle, \langle xe_1 + ye_2 + (ce_3 + f_3)\rangle)$ with $c\bar{c} = -1$ and $xy \neq 0$ and $x\bar{x} + y\bar{y} \neq 0$. Furthermore, assume that $x\bar{x} \neq 1$ or 2, $y\bar{y} \neq 1$ or 2, $x\bar{x} + y\bar{y} \neq 1$ or 2, $(x\bar{x} - 1)(x\bar{x} + y\bar{y} - 1) \neq 1$, $(y\bar{y} - 1)(x\bar{x} + y\bar{y} - 1) \neq 1$. Then (P, Q, R) can be decomposed into good triangles.*

Proof. Consider the plane $\langle f_1, f_2, f_3\rangle$ and fix the points $A = \langle f_3\rangle$, $B = \langle -xf_3 + cf_1\rangle$, $C = \langle -yf_3 + cf_2\rangle$. These are uniquely determined by the conditions that $A \perp \langle P, Q\rangle$, $B \perp \langle Q, R\rangle$ and $C \perp \langle P, R\rangle$.

Notice that A, B, C are points of Γ if and only if $x\bar{x} \neq 1$ and $y\bar{y} \neq 1$ which is satisfied by assumption.

The projective lines AP, AQ, BQ, and CP are lines of Γ because the two points on them are perpendicular with respect to $((\cdot,\cdot))$. Also AB and AC are in fact the projective lines $\langle f_1, f_3\rangle$, respectively $\langle f_2, f_3\rangle$, so they are lines of Γ.

Next we have to investigate the conditions under which the projective lines BC, BR, and CR are lines in Γ. We need to see that $((\cdot,\cdot))$ is nondegenerate on each of these 2-dimensional spaces, so we will investigate the Gram matrices and find their determinants.

In the case of BC we get

$$\det \begin{pmatrix} x\bar{x} - 1 & x\bar{y} \\ \bar{x}y & y\bar{y} - 1 \end{pmatrix} = -x\bar{x} - y\bar{y} + 1.$$

The space BR yields

$$\det \begin{pmatrix} x\bar{x} - 1 & -x \\ -\bar{x} & x\bar{x} + y\bar{y} \end{pmatrix} = (x\bar{x} - 1)(x\bar{x} + y\bar{y} - 1) - 1.$$

In the case of CR we get

$$\det \begin{pmatrix} y\bar{y}-1 & -y \\ -\bar{y} & x\bar{x}+y\bar{y} \end{pmatrix} = (y\bar{y}-1)(x\bar{x}+y\bar{y}-1)-1.$$

Now we compute conditions such that (A,B,C), (A,B,Q), (A,C,P), (A,P,Q), (B,C,R), (B,Q,R), and (C,P,R) are good triangles. Notice that the triangles (A,B,C), (A,B,Q), (A,P,Q), and (A,C,P) are automatically good.

Moreover, the case of (B,Q,R) gives

$$\det \begin{pmatrix} x\bar{x}-1 & 0 & -x \\ 0 & 1 & \bar{y} \\ -\bar{x} & y & x\bar{x}+y\bar{y} \end{pmatrix} = x\bar{x}(x\bar{x}-2).$$

In the case of (B,C,R) we get

$$\det \begin{pmatrix} x\bar{x}-1 & x\bar{y} & -x \\ \bar{x}y & y\bar{y}-1 & -y \\ -\bar{x} & -\bar{y} & x\bar{x}+y\bar{y} \end{pmatrix} = (x\bar{x}+y\bar{y})(2-x\bar{x}-y\bar{y}).$$

Finally, for (C,P,R) we have

$$\det \begin{pmatrix} y\bar{y}-1 & 0 & -y \\ 0 & 1 & \bar{x} \\ -\bar{y} & x & x\bar{x}+y\bar{y} \end{pmatrix} = y\bar{y}(y\bar{y}-2).$$

This gives us exactly the conditions contained in the hypothesis of the lemma. □

Lemma 6.3. *Let $q=p^e$ and let $c,d \in \mathbb{F}_{q^2}$ such that $c\bar{c}=-1$, $d \neq 0$. Then the system of equations $x\bar{x}+y\bar{y}=1$ and $\bar{x}-\bar{y}c=d$ has exactly q solutions.*

Proof. The pair (x,y) is a solution of the first equation if and only if the matrix $A_{x,y} := \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}$ has determinant one, thus the solutions of the first equation are parametrized by the elements of the group $SU(2,q^2)$. Observe that

$$(c,1)A_{x,y} = (xc+y, \bar{x}-\bar{y}c) = \left(c\overline{(\bar{x}-\bar{y}c)}, \bar{x}-\bar{y}c\right).$$

Therefore two pairs (x, y), (x', y') are solutions for the system of equations if and only if the matrix $A_{x,y}A^{-1}_{x',y'}$ stabilizes the vector $(c, 1)$ which is of norm 0 with respect to the unitary form. The stabilizer of such a vector is the p-Sylow subgroup of the unitary group. So, if the above system has a solution, then it has exactly q solutions, for a fixed d. Since the order of $SU(2, q^2)$ is $q(q^2 - 1)$, the above system has q solutions for each $d \neq 0$. (Indeed, there are $q^2 - 1$ possible d's.) $\square$

Proposition 6.4. *Let $q \geq 8$ and let (P, Q, R) be a bad triangle. Then the triangle can be decomposed into good triangles.*

Proof. Let X be the radical of the plane $\langle P, Q, R\rangle$. By the preceding lemma we have $X = X^\sigma$. Now, by Lemmas 5.3 and 5.4, we can assume $(P, Q, R) = (\langle e_1\rangle, \langle e_2\rangle, \langle xe_1 + ye_2 + (ce_3 + f_3)\rangle)$ satisfying $c\bar{c} = -1$ and $xy \neq 0$ and $x\bar{x} + y\bar{y} \neq 0$. It is enough to show that this triangle is conjugate to a triangle satisfying the hypothesis of Lemma 6.2.

Let $g \in G_\sigma$ fixing e_1, e_2, f_1, f_2 pointwise. Then Lemma 6.3 shows that, for any nontrivial $d \in \mathbb{F}_{q^2}$, the element g can be chosen such that $(ce_3 + f_3)^g = d(c\frac{\bar{d}}{d}e_3 + f_3)$, and we have conjugated (P, Q, R) to $(P^g, Q^g, R^g) = (\langle e_1\rangle, \langle e_2\rangle, \langle \frac{x}{d}e_1 + \frac{y}{d}e_2 + (c\frac{\bar{d}}{d}e_3 + f_3)\rangle)$.

It remains to be seen that we can pick d such that $x' = \frac{x}{d}$, $y' = \frac{y}{d}$ satisfy the conditions of 6.2. Then, by that lemma, we can decompose (P^g, Q^g, R^g) (and hence its conjugate (P, Q, R)) into good triangles. Notice that $xy \neq 0$ if and only if $\frac{x}{d}\frac{y}{d} \neq 0$, and $c\bar{c} = -1$ if and only if $c\frac{\bar{d}}{d}\overline{\left(c\frac{\bar{d}}{d}\right)} = -1$. The same holds for the condition $x\bar{x} + y\bar{y} \neq 0$.

If there are seven different non-zero values of $d\bar{d}$ in $\mathbb{F}_q$, then we are able to modify $x\bar{x}$ and $y\bar{y}$ (to $\frac{x\bar{x}}{d\bar{d}}$ respectively $\frac{y\bar{y}}{d\bar{d}}$) such that the conditions $x\bar{x} \neq 1$, $x\bar{x} \neq 2$, $y\bar{y} \neq 1$, $y\bar{y} \neq 2$, $x\bar{x} + y\bar{y} \neq 1$, $x\bar{x} + y\bar{y} \neq 2$ are satisfied for the modified parameters. Furthermore, if there are four more values of $d\bar{d}$, we can additionally modify $x\bar{x}$ and $y\bar{y}$ for $(x\bar{x}-1)(x\bar{x}+y\bar{y}-1) \neq 1$, $(y\bar{y}-1)(x\bar{x}+y\bar{y} - 1) \neq 1$ to hold. This is the case for $q \geq 13$, which leaves $q \in \{8, 9, 11\}$. A straightforward check by hand or in GAP will show that any pair $x\bar{x}$, $y\bar{y}$ can be scaled by $d\bar{d}$ to satisfy all conditions. $\square$

Now we will shift our attention to quadrangles. By the preceding results, it is enough to decompose quadrangles into triangles, regardless whether they are good or bad. Notice that if in a quadrangle (A, B, C, D) we have that A and C (or B and D) are collinear then this quadrangle is immediately decomposed into two triangles. We call (A, B, C, D) *special* if $\langle A, C\rangle$ is nondegenerate with respect to both forms $(\cdot,\cdot)$ and $((\cdot,\cdot))$.

Lemma 6.5. *Let $q \geq 5$. Then any quadrangle can be decomposed into triangles and special quadrangles.*

Proof. Consider an arbitrary quadrangle (A, B, C, D). Without loss of generality we may assume that B and D are noncollinear. Pick an arbitrary point $S \in X = A^{\perp\!\perp} \cap B^{\perp} \cap D^{\perp}$. The point S exists because X is not totally isotropic with respect to $((\cdot,\cdot))$, being a 3-space contained in the nondegenerate 5-space $A^{\perp\!\perp}$. The projective line $L = \langle A, S\rangle$ has rank two with respect to $((\cdot,\cdot))$. Using Corollary 4.4, L contains at least $q^2 - 2q - 1$ points of Γ that are collinear with B, respectively D, and at least $q^2 - 2q - 1$ points of Γ that generate a nondegenerate 2-space with C. Since $q \geq 5$ and since L contains $q^2 - q$ points of Γ, the space L has to contain a point P of Γ that generates a nondegenerate 2-space with C and that is collinear to both B and D. Clearly (A, B, C, D) decomposes as a product of (A, B, P, D) and (C, B, P, D). If $(A, P) = 0$ then $\langle A, P\rangle$ is a line, implying that (A, B, P, D) decomposes into triangles. Otherwise, (A, B, P, D) is special. Similarly, for (C, B, P, D). □

Proposition 6.6. *Let $q \geq 7$. Then any quadrangle can be decomposed into triangles.*

Proof. Denote the quadrangle by (A, B, C, D), as in the proof of the preceding lemma. By that lemma, we can assume that $(A, C) \neq 0$ and that $\langle A, C\rangle$ is nondegenerate with respect to $((\cdot,\cdot))$. Set $W := A^{\perp} \cap C^{\perp}$ and $U_1 := W \cap B^{\perp}$ and $U_2 := W \cap D^{\perp}$.

If $L = U_1 \cap U_2$ is of rank two with respect to $((\cdot,\cdot))$, then we can apply Lemma 4.3 to the planes $\langle A, L\rangle$, $\langle B, L\rangle$, $\langle C, L\rangle$, and $\langle D, L\rangle$ to obtain q^2-5q-4 points of Γ on L collinear to all of A, B, C, D. Notice that this is a positive number for $q \geq 7$.

Suppose now that $L = U_1 \cap U_2$ is of rank one. Then the plane $\langle B, L\rangle$ has rank at least one. However, it cannot have rank one, since it lies inside the $((\cdot,\cdot))$-nondegenerate 4-dimensional space $A^{\perp} \cap B^{\perp} = (A^{\sigma})^{\perp\!\perp} \cap (B^{\sigma})^{\perp\!\perp}$. Indeed, a 2-dimensional radical would be maximal totally isotropic inside $A^{\perp} \cap B^{\perp}$ and could not have a polar of dimension three. Similar arguments hold for the points A, C, D instead of B. Applying Lemma 4.3 as in the above paragraph gives a point of Γ collinear to all of A, B, C, D.

Suppose now L is totally isotropic with respect to $((\cdot,\cdot))$. Then L has to contain the radicals R_1 and R_2 (with respect to $((\cdot,\cdot))$) of the planes U_1 and U_2. These radicals cannot coincide as otherwise we would obtain a radical for the $((\cdot,\cdot))$-nondegenerate space $A^\perp \cap C^\perp$. Notice that $R_2^\perp \cap U_1 = BR_2$. Choose a line of Γ through B inside U_1. (This exists since the rank with respect to $((\cdot,\cdot))$ of U_1 is two.) This line contains a point P collinear to both A and C, by Lemma 4.3. Now $P^\perp \cap W$ intersects U_2 in a line that does not contain R_2. Hence its rank with respect to $((\cdot,\cdot))$ is two. The arguments given in the second paragraph of this proof settle the claim. $\square$

As in the $n \geq 4$ case, pentagons are easy to handle.

Proposition 6.7. *Let $q \geq 5$. Then any pentagon is null homotopic.*

Proof. Let (A, B, C, D, E) be a pentagon. Consider the space $U := \langle A, B, D\rangle^\perp$ of dimension three. Its rank with respect to $((\cdot,\cdot))$ has to be at least two, as the rank of $\langle A, B\rangle$ is two. Choosing a $((\cdot,\cdot))$-nondegenerate projective line L in U and applying Lemma 4.3 in turn on the planes $\langle A, L\rangle$, $\langle B, L\rangle$, $\langle D, L\rangle$, we will find $q^2 - 2q - 1 - q - 1 - q - 1 = q^2 - 4q - 3 > 0$ points on L collinear to all of A, B, D, decomposing the pentagon. $\square$

We summarize the results of this section as follows.

Theorem 6.8. *If $n = 3$ and $q \geq 8$ then Γ is simply connected.* $\square$

It is easy to see that Γ is *not* simply connected if $(n, q) = (3, 2)$. We do not know whether this is the case for $7 \geq q \geq 3$. In order to prove our claim let P be any point of Γ. Then all points collinear to P are contained in $P^\perp \cap P^{\perp\perp}$, because $q = 2$. The subspace $P^\perp \cap P^{\perp\perp}$ is not connected by Lemma 4.6. Let A and B be points contained in distinct connected components of $P^\perp \cap P^{\perp\perp}$. Consider an arbitrary cycle consisting of A, P, B, and points at distance at least two from P (e.g., A, P, B, P^σ). This cycle is not null-homotopic, because A, P, B do not admit a common neighbor and are not contained in a plane of Γ.

We completed the proof of the Main Theorem. Indeed, part (1) of the Main Theorem follows from Propositions 3.8 and 4.2. Part (2) follows from Theorem 4.7 and Corollary 4.8. Finally, part (3) is proved in Theorems 5.13 and 6.8.

7 Consequences of simple connectedness

In this section we prove Theorems 1, 2, 3, and 4. Throughout this section, $n \geq 3$.

In the present paper an *amalgam* $\mathcal{A}$ of groups is a set with a partial operation of multiplication and a collection of subsets $\{H_i\}_{i\in I}$, for some index set I, such that the following hold:

(1) $\mathcal{A} = \cup_{i\in I} H_i$;

(2) the product ab is defined if and only if $a, b \in H_i$ for some $i \in I$;

(3) the restriction of the multiplication to each H_i turns H_i into a group; and

(4) $H_i \cap H_j$ is a subgroup in both H_i and H_j for all $i, j \in I$.

It follows that the groups H_i share the same identity element, which is then the only identity element in $\mathcal{A}$, and that $a^{-1} \in \mathcal{A}$ is well-defined for every $a \in \mathcal{A}$. We will call the groups H_i the *members* of the amalgam $\mathcal{A}$. Notice that our definition is a special case of the general definition of an amalgam of groups as found, say, in [6].

A group H is called a *completion* of an amalgam $\mathcal{A}$ if there exists a map $\pi : \mathcal{A} \to H$ such that

(1) for all $i \in I$ the restriction of π to H_i is a homomorphism of H_i to H; and

(2) $\pi(\mathcal{A})$ generates H.

Among all completions of $\mathcal{A}$ there is one "largest" which can be defined as the group having the following presentation:

$$U(\mathcal{A}) = \langle t_h \mid h \in \mathcal{A},\, t_x t_y = t_z, \text{ whenever } xy = z \text{ in } \mathcal{A}\rangle.$$

Obviously, $U(\mathcal{A})$ is a completion of $\mathcal{A}$ since one can take π to be the mapping $h \mapsto t_h$. Every completion of $\mathcal{A}$ is isomorphic to a quotient of $U(\mathcal{A})$, and because of that $U(\mathcal{A})$ is called the *universal completion*.

Suppose a group $H \leq \operatorname{Aut}\Gamma$ acts flag-transitively on a geometry Γ. A *rank k parabolic* is the stabilizer in H of a flag of corank k from Γ. Parabolics of rank $n-1$ (where n is the rank of Γ) are called *maximal parabolics*. They are exactly the stabilizers in H of single elements of Γ.

Let F be a maximal flag in Γ, and let H_x denote the stabilizer in H of $x \in \Gamma$. The amalgam $\mathcal{A} = \mathcal{A}(F) = \cup_{x \in F} H_x$ is called the amalgam of maximal parabolics in H. Since the action of H is flag-transitive, this amalgam is defined uniquely up to conjugation in H. For a fixed flag F we can also use the notation M_i for the maximal parabolic H_x, where $x \in F$ is of type i. (We defined this notation in the introduction.) For a subset $J \subset I = \{0, 1, \ldots, n-1\}$, define M_J to be $\cap_{j \in J} M_j$, including $M_\emptyset = H$. Notice that M_J is a parabolic of rank $|I \setminus J|$; indeed, it is the stabilizer of the subflag of F of type J. Similarly to $\mathcal{A}$, we can define the amalgam $\mathcal{A}_{(s)}$ as the union of all rank s parabolics. With this notation we can write $\mathcal{A} = \mathcal{A}_{(n-1)}$. Moreover, according to our definition, $\mathcal{A}_{(n)} = H$.

Now we need to define coverings of geometries. Suppose Γ and $\hat{\Gamma}$ are two geometries over the same type set and suppose $\phi : \hat{\Gamma} \to \Gamma$ is a *morphism* of geometries, *i.e.*, ϕ preserves the type and sends incident elements to incident elements. The morphism ϕ is called a *covering* if and only if for every non-empty flag $\hat{F}$ in $\hat{\Gamma}$ the mapping ϕ induces an isomorphism between the residue of $\hat{F}$ in $\hat{\Gamma}$ and the residue of $F = \phi(\hat{F})$ in Γ. Coverings of a geometry correspond to the usual topological coverings of its flag complex. In particular, a simply connected geometry (as defined in Section 5) admits no nontrivial covering.

The notion of coverings can also be defined in the more broad context of chamber systems. In this context one can define more general notions of k-coverings and k-simple connectedness. A chamber system is k-simply connected if and only if it has no proper k-coverings. Unfortunately, it is conceivable that a k-cover of a geometry is not a geometry. Still the following claims can be made: A morphism $\phi : \hat{\Gamma} \to \Gamma$ of geometries is a *k-covering*, if for any flag $\hat{F}$ of corank at most k of $\hat{\Gamma}$, the induced mapping from the residue of $\hat{F}$ onto the residue of $\phi(\hat{F})$ is an isomorphism. Consequently, if n is the rank of a geometry Γ, then the coverings of Γ are precisely the $(n-1)$-coverings of Γ. If a connected geometry is k-simply connected then it admits no proper k-coverings. Also, every k-covering is a $(k-1)$-covering and $(k-1)$-simple connectedness of a geometry implies k-simple connectedness.

Proposition 7.1 (Tits' Lemma). *Suppose a group H acts flag-transitively on a geometry Γ and let $\mathcal{A}$ be the amalgam of maximal parabolics associated* *with some maximal flag F. Then H is the universal completion of the amalgam $\mathcal{A}$ if and only if Γ is simply connected.*

Proof. Follows from [7], *Corollaire* 1, applied to the flag complex of Γ. $\square$

In case of $\Gamma = \Gamma_\sigma$ and $H = G_\sigma$ (cf. Section 4), the Main Theorem and Tits' Lemma imply that H is the universal completion of $\mathcal{A}$ unless (n, q) one of $(3,2)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,7)$, $(4,2)$.

Recall that the *direct sum* of two geometries Γ_1 and Γ_2 is defined as follows. The type set (respectively, element set) of $\Gamma_1 \oplus \Gamma_2$ is the disjoint union of the type sets (respectively, element sets) of Γ_1 and Γ_2. The incidence relation on $\Gamma_1 \oplus \Gamma_2$ is the combination of the incidence relations on Γ_1 and Γ_2 and the condition that every element of Γ_1 is incident with every element of Γ_2.

To prove Theorem 1 we will need the following lemma.

Lemma 7.2. *Assume that $\Sigma = \Sigma_1 \oplus \Sigma_2$ with Σ_1 connected of rank at least two. Then Σ is simply connected.*

Proof. Certainly, Σ is connected. Choose a base point $x \in \Sigma_1$. We first prove that any cycle originating at x is homotopic to a cycle fully contained in Σ_1. Let $xx_1 \dots x_{n-1}x$ be a cycle. Proceed by induction on the number of elements on the cycle which are not in Σ_1. Suppose x_s is the first element in the cycle which is not in Σ_1. Let $y \in \Sigma_1$ such that $y \neq x_{s+1}$ and y is incident with x_{s+1}. (Recall that Σ_1 has rank at least two.) Notice that y is incident with x_s. Since the residue of x_s contains Σ_1, we can connect x_{s-1} with y via a path $x_{s-1}y_1 \dots y_{k-1}y$ fully contained in Σ_1. Furthermore, this path is homotopic to the path $x_{s-1}x_sy$. Thus, our original path is homotopic to the path $xx_1 \dots x_{s-1}y_1 \dots y_{k-1}yx_{s+1} \dots x_{n-1}x$. This path has fewer elements outside Σ_1, and our claim is proved.

Choosing an element $z \in \Sigma_2$ we see that this z is incident to all elements in Σ_1, so any cycle in Σ_1 is null homotopic. $\square$

Proof of Theorem 1. Let $s \geq 2$ if $q \geq 8$, $s \geq 3$ if $7 \geq q \geq 3$, and $s \geq 4$ if $q = 2$. Suppose that $n \geq s+1$. We will proceed by induction and show that the universal completion of $\mathcal{A}_{(s)}$ coincides with the universal completion of $\mathcal{A}_{(s+1)}$. Denote by $H_{(s)}$ the universal completion of $\mathcal{A}_{(s)}$.

Let $J \subset I$ and $|I \setminus J| = s+1$. Let $F_J \subset F$ be of type J, so that M_J is the stabilizer of F_J in H. Observe that the residue of F_J (denoted by Γ_J)

is connected. Indeed, if $q > 2$ then Γ is residually connected by Corollary 4.8. In particular, Γ_J is connected. If $q = 2$ then either the diagram of Γ_J is disconnected, or the diagram is connected. In the first case, Γ_J is connected, since the incidence on Γ is defined as symmetrized inclusion. In the second case, Γ_J is either our flipflop geometry of rank $s+1$, or the geometry as in [1]. The connectedness follows from Theorem 4.7 and [1].

Observe also that Γ_J is simply connected. Indeed, either the diagram of Γ_J is disconnected, or it is connected. In the first case, the simple connectivity follows from Lemma 7.2. The connectivity assumption in that lemma holds because one of Σ_1 and Σ_2 has sufficient rank (rank at least two, if $q \geq 3$, and rank at least three, if $q = 2$) to be connected. If the diagram of Γ_J is connected then Γ_J is simply connected by the Main Theorem (3) or [1], depending on its diagram.

The universal completion $H_{(s+1)}$ of $\mathcal{A}_{(s+1)}$ is also a completion of $\mathcal{A}_{(s)}$. Indeed, if $n = s+1$, then $H_{(n)} = H = G_\sigma$, which certainly is a completion of $\mathcal{A}_{(n-1)}$. In case $n > s+1$, the amalgam $\mathcal{A}_{(s+1)}$ is the union of all M_J with J of corank $s+1$ and we have a map $\pi : \mathcal{A}_{(s+1)} \to H_{(s+1)}$ such that $\pi_{|M_J} : M_J \to H_{(s+1)}$ is a homomorphism. Consequently, also $\pi_{|M_J \cap M_{J'}} : M_J \cap M_{J'} \to H_{(s+1)}$ is a homomorphism. It remains to show that the set of all images $\pi(M_J \cap M_{J'})$ with $|I \backslash (J \cup J')| = s$ actually generate $H_{(s+1)}$. But since Γ_J is connected, the group $\pi(M_J) \leq H_{(s+1)}$ is generated by all those images for a fixed J (because the $M_J \cap M_{J'}$ are maximal parabolics in M_J). Thus, $H_{(s+1)}$ is a completion of $\mathcal{A}_{(s)}$, as it is generated by the $\pi(M_J)$.

Therefore there is a canonical homomorphism ϕ from $H_{(s)}$ onto $H_{(s+1)}$ whose restriction to $\mathcal{A}_{(s)}$ is the identity. Let ψ be the inverse of the restriction of ϕ to $\mathcal{A}_{(s)}$. Let $J \subset I$ be such that $|I \setminus J| = s+1$ and let $\hat{M}_J$ be defined as $\langle \psi(M_J \cap \mathcal{A}_{(s)}) \rangle$. By simple connectedness of Γ_J and by Tits' Lemma, ϕ induces an isomorphism of $\hat{M}_J$ onto M_J. Therefore, ψ extends to an isomorphism of $\mathcal{A}_{(s+1)} \subset H_{(s+1)}$ onto

$$\hat{\mathcal{A}}_{s+1} = \bigcup_{J \subset I, |I \setminus J| = s+1} \hat{M}_J \qquad \subset H_{(s)}.$$

Hence the universal completion of $\mathcal{A}_{(s)}$ coincides with the universal completion of $\mathcal{A}_{(s+1)}$. The fact $H_{(n)} = G_\sigma$ finishes the proof. □

Notice that we actually proved that Γ is 2-simply connected if $q \geq 8$, 3-simply connected if $q \geq 3$, and 4-simply connected if $q = 2$, as claimed after the statement of the Main Theorem in the introduction.

Let $\{e_1, \ldots, f_n\}$ be a canonical basis for σ. We will use the notation T, L_i, T_i as introduced before Theorem 2 in Section 1. For the purposes of proving that theorem, we will assume that the flag F consists of the subspaces $\langle e_1\rangle$, $\langle e_1, e_2\rangle$, $\ldots$, $\langle e_1, \ldots, e_n\rangle$. With respect to this basis, T consists of all diagonal matrices $diag(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1})$, where each a_i is of order dividing $q+1$. Furthermore, T_i, $1 \leq i < n$, consists of matrices from T, for which $a_i = a_{i+1}^{-1} = a_{n+i}^{-1} = a_{n+i+1}$, with all other a_j equal to one. If $i = n$ then $a_n = a_{2n}^{-1}$ and $a_j = 1$ for all other j. Manifestly, T is the direct product of all T_i's.

Proof of Theorem 2. Let $s = 2$ if $q \geq 8$, $s = 3$ if $7 \geq q \geq 3$, and $s = 4$ if $q = 2$, and suppose that $n \geq s+1$. Let $\hat{H}$ be the universal completion of the amalgam $\mathcal{A}^0_{(s)}$. Let ϕ be the canonical homomorphism of $\hat{H}$ onto H, that exists due to the fact that H is a completion of $\mathcal{A}^0_{(s)}$. Denote by $\hat{\mathcal{A}}^0_{(s)}$ the copy of $\mathcal{A}^0_{(s)}$ in $\hat{H}$, so that ϕ induces an isomorphism of $\hat{\mathcal{A}}^0_{(s)}$ onto $\mathcal{A}^0_{(s)}$. As in the proof of Theorem 1, let $\psi : \mathcal{A}^0_{(s)} \to \hat{\mathcal{A}}^0_{(s)}$ be the inverse of $\phi_{|\hat{\mathcal{A}}^0_{(s)}}$. Additionally, define $\hat{T}_i = \psi(T_i)$ and $\hat{T} = \langle \hat{T}_1, \ldots, \hat{T}_n\rangle$. Observe that $T_i, T_j \leq M^0_{I\setminus\{i,j\}} = \langle L_i, L_j\rangle \subset \mathcal{A}^0_{(s)}$. Since ψ restricted to the latter group is an isomorphism to $\psi(M^0_{I\setminus\{i,j\}})$, the groups $\hat{T}_i$ and $\hat{T}_j$ commute elementwise. Because T is the direct product of T_i's, the map ϕ establishes an isomorphism between $\hat{T}$ and T.

Let J be a subset of I with $|I \setminus J| = s$. Observe that $M_J = M^0_J T$. Accordingly, we would like to define $\hat{M}_J$ as $\hat{M}^0_J \hat{T}$, where $\hat{M}^0_J = \psi(M^0_J)$. For this definition to make sense, we need to show that $\hat{T}$ normalizes $\hat{M}^0_J$. Assume first that $q > 2$. Since M^0_i is normal in M_i and since $T \leq M_i$, we have that T normalizes all M_i and therefore T normalizes every $L_i = \cap_{j\in I\setminus\{i\}} M^0_j$. Observe that $T_j \leq L_j$ and $L_i, L_j \leq M^0_{I\setminus\{i,j\}} = \langle L_i, L_j\rangle$. Since ψ is an isomorphism from $\mathcal{A}^0_{(s)}$ to $\hat{\mathcal{A}}^0_{(s)}$, the group $\hat{T}_j$ normalizes $\hat{L}_i$ for all i and j. It is clear that M^0_J is generated by L_i, $i \in I \setminus J$. The same must be true for $\hat{M}^0_J$ and $\hat{L}_i$'s. Therefore every $\hat{T}_j$ will normalize every $\hat{M}^0_J$ which means that also $\hat{T}$ normalizes $\hat{M}^0_J$. If $q = 2$ the same result can be achieved by using $M^0_{I\setminus\{i,j\}}$'s in place of L_i's; recall that in this case we assume $s = 4$.

Since $\hat{T}$ normalizes M^0_J and since $\hat{T} \cap \hat{M}^0_J = \langle \hat{T}_j \mid j \in I \setminus J\rangle$ is isomorphic (via ϕ) to $T \cap M^0_J$, the map ϕ establishes an isomorphism between $\hat{M}_J$ and

M_J, and, thus, ϕ extends to an isomorphism

$$\hat{\mathcal{A}}_{(s)} = \bigcup_{J \subset I, |I \setminus J| = s} \hat{M}_J \qquad \longrightarrow \qquad \mathcal{A}_{(s)} \quad .$$

Therefore, the universal completions of $\mathcal{A}_{(s)}$ and $\mathcal{A}^0_{(s)}$ are isomorphic, and the claim follows from Theorem 1. □

Recall from the introduction that Δ is the pregeometry on the nondegenerate proper subspaces of a nondegenerate $2n$-dimensional symplectic space V with symmetrized inclusion as incidence where the type of a subspace equals half its dimension.

Lemma 7.3. *Δ is a connected geometry and the action of $G_\sigma \cong Sp(2n,q)$ on it is flag-transitive.*

Proof. Let $U_1 \leq \cdots \leq U_t$ be a maximal flag. If the dimension of U_t is not $2n-2$, then the dimension of $U_t^\perp$ is at least four and we can find a proper nondegenerate 2-dimensional subspace U of $U_t^\perp$. But now $U_t \oplus U$ is still a proper nondegenerate subspace of V and $U_1 \leq \cdots \leq U_t \leq U_t \oplus U$ is a flag of Δ, a contradiction. Hence U_t has dimension $2n-2$. Similarly one can show that U_{i-1} has codimension 2 in U_i for $2 \leq i \leq n-1$. Therefore, Δ is a geometry.

Given any maximal flag $U_1 \leq \cdots \leq U_{n-1}$, we can choose a hyperbolic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V such that $U_i = \langle e_1, \ldots, e_i, f_1, \ldots, f_i \rangle$, $1 \leq i \leq n-1$. Flag-transitivity of the group $Sp(2n,q)$ now follows from transitivity of $Sp(2n,q)$ on the set of hyperbolic bases of V.

It remains to show connectedness of Δ. Let U and U' be two nondegenerate 2-dimensional subspaces of V. If U and U' are orthogonal then $\langle U, U' \rangle$ is nondegenerate and so U and U' are adjacent in the collinearity graph of Δ. If U and U' meet in a 1-dimensional space then $\langle U, U' \rangle$ is of dimension three and rank two. Therefore it is contained in a nondegenerate 4-dimensional space. Thus again U and U' are adjacent. Finally if U and U' are disjoint and not perpendicular, we can find vectors $u \in U$ and $u' \in U'$ such that $\langle u, u' \rangle$ is nondegenerate. Clearly the latter subspace is adjacent to both U and U' so they are at distance two. We have shown that the collinearity graph of Δ has diameter two. In particular, it is connected. □

Corollary 7.4. *Δ is residually connected.* □

Proof of Theorem 3. Suppose that $n \geq 4$ and $n \geq 5$ if $q = 2$ or 3. Let $\mathcal{B} = \cup_{1 \leq i \leq n-1} H_i$. According to Tits' Lemma, the conclusion of the theorem is equivalent to $U(\mathcal{B}) \cong G_\sigma$. Let $\mathcal{A} = \cup_{1 \leq i \leq n} M_i$ be, as before, the amalgam of maximal parabolics related to the action of $H = G_\sigma$ on the flipflop geometry Γ. Let $\mathcal{A}' = \cup_{1 \leq i \leq n-1} M_i$. Then $\mathcal{A}'$ is contained in $\mathcal{B}$, since $M_i \leq H_i$ for $1 \leq i \leq n-1$. The claim of the theorem will follow from the Main Theorem (3) and Tits' Lemma, once we show that $U(\mathcal{B}) \cong U(\mathcal{A}')$ and $U(\mathcal{A}') \cong U(\mathcal{A})$.

We will start with the second isomorphism. Let $\hat{H} = U(\mathcal{A}')$. Let also ψ be the canonical embedding of $\mathcal{A}'$ into $\hat{H}$ and define $\hat{M}_i = \psi(M_i)$, $1 \leq i \leq n-1$, and $\hat{\mathcal{A}}' = \psi(\mathcal{A}')$. Notice that $M_n \cap \mathcal{A}'$ is the amalgam of maximal parabolics in M_n acting on the residue $\Gamma_{\{n\}}$ of Γ. By [1], $\Gamma_{\{n\}}$ is simply connected. Therefore, $\psi(M_n \cap \mathcal{A}')$ generates in $\hat{H}$ a subgroup $\hat{M}_n$ isomorphic to M_n. Clearly, $\hat{\mathcal{A}}' \cup \hat{M}_n$ is isomorphic to $\hat{\mathcal{A}}$ and hence $U(\mathcal{A}') \cong U(\mathcal{A})$.

Turning to the isomorphism $U(\mathcal{B}) \cong U(\mathcal{A}')$, we let $\hat{H} = U(\mathcal{B})$ and let ψ to be the embedding of $\mathcal{B}$ into $\hat{H}$. We claim that $\psi(\mathcal{A}')$ generates $\hat{H}$. Indeed, since Δ is residually connected (cf. the preceding corollary), any two $\psi(H_i)$ generate $\hat{H}$. Take $i = n-1$ or $n-2$. Then $H_i = L \times R$, where $L \cong Sp(2i, q)$ and $R \cong Sp(2n-2i, q)$. Observe that $R \leq M_j$ for $1 \leq j \leq i$ and that $\cup_{1 \leq j \leq i} (L \cap M_j)$ is the amalgam of maximal parabolics for L acting on its corresponding flipflop geometry (of rank i). Since that geometry is connected, $\psi(H_i) \leq \langle \psi(\mathcal{A}') \rangle$. Thus, $\psi(\mathcal{A}')$ indeed generates $\hat{H}$.

Consequently, $\hat{H}$ must be a quotient of $U(\mathcal{A}') \cong U(\mathcal{A}) \cong H$. Since also, H is isomorphic to a quotient of $\hat{H}$, we finally obtain $U(\mathcal{B}) \cong H \cong U(\mathcal{A}')$. □

Proof of Theorem 4. Let $s = 2$ if $q \geq 4$ and $s = 3$ if $q = 2$ or 3. Let $\mathcal{B}_{(s)}$ be the subamalgam of $\mathcal{B}$ (see the proof of Theorem 3) consisting of all rank s parabolics. As in the proof of Theorem 1, we can show that $U(\mathcal{B}_{(s)}) \cong H = G_\sigma$. (Like before, this also implies 2-simple connectedness, respectively 3-simple connectedness of Δ, as claimed after Theorem 3 in the introduction.) Since the union of any three (four, if $q = 2$ or 3) H_i contains $\mathcal{B}_{(s)}$ and since $H_i \cap \mathcal{B}_{(s)}$ generates H_i for all i, we are done. □

Finally, the claim after Theorem 4 can be proven as follows. Let $H_J = \cap_{i \in J} H_i$. By Theorem 4, the amalgam of rank three parabolics (*i.e.*, the amalgam of all subgroups H_J with $|I \setminus J| = 3$) has G_σ as its universal completion. The only rank 3 parabolic that cannot be found inside the amalgam $H_1 \cup H_i \cup H_{n-1}$ is $H_{I \setminus \{1,i,n-1\}}$. Since $n \geq 5$, $i \neq 2$ or $i \neq n-2$. In the first case $H_{I \setminus \{1,i,n-1\}}$ is isomorphic to $H_{I \setminus \{1\}} \times H_{I \setminus \{i,n-1\}}$. In the second case it is isomorphic to $H_{I\setminus\{1,i\}} \times H_{I\setminus\{n-1\}}$. Let us assume we are in the first case. By connectivity (see Lemma 7.3), the rank two parabolic $H_{I\setminus\{i,n-1\}}$ is generated by the two minimal parabolics $H_{I\setminus\{i\}}$ and $H_{I\setminus\{n-1\}}$. It remains to notice that both $H_{I\setminus\{1\}}$ and $H_{I\setminus\{i\}}$ are contained in H_{n-1}, while both $H_{I\setminus\{1\}}$ and $H_{I\setminus\{n-1\}}$ are contained in H_i. So $H_{I\setminus\{1,i,n-1\}}$ does not contain any new relations.

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