Current groups for non-compact manifolds and their central extensions

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Dedicated to Karl Heinrich Hofmann on the occasion of his 70th birthday

Abstract. In this paper we study two types of groups of smooth maps from a non-compact manifold M into a Lie group K which may be infinite-dimensional: the group $C^\infty_c(M,K)$ of compactly supported maps and for a compact manifold M and a closed subset S the group $C^\infty(M,S;K)$ of those maps which vanish on S, together with all their derivatives. We study central extensions of these groups associated to Lie algebra cocycles of the form $\omega(\xi,\eta)=[\kappa(\xi,d\eta)]$, where $\kappa:\mathfrak{k}\times\mathfrak{k}\to Y$ is a symmetric invariant bilinear map on the Lie algebra $\mathfrak{k}$ of K and the values of ω lie in $\Omega^1(M;Y)/dC^\infty(M;Y)$. For such cocycles we show that a corresponding central Lie group extension exists if and only if this is the case for $M=\mathbb{S}^1$. If K is finite-dimensional semisimple, this implies the existence of a universal central Lie group extension of the identity component of the current groups.

Introduction

If M is a compact manifold and K a Lie group (which may be infinite-dimensional), then the so called current groups $C^\infty(M;K)$, endowed with the group structure given by pointwise multiplication, are interesting infinite-dimensional Lie groups arising in many circumstances. If M is a non-compact manifold, the full group $C^\infty(M;K)$ seems to be far too large to carry a Lie group structure compatible with its natural group topology, so that it is natural to study subgroups of maps $f: M \to K$ that either vanish outside a compact subset or decay fast enough at infinity. In the present paper we investigate the following two types of current groups on a non-compact manifold M. The first class consists of the groups $C^\infty_c(M;K)$ of compactly supported smooth maps and the second class of the groups $C^\infty(M,S;K)$ of maps on a compact manifold M for which all partial derivatives vanish on the closed subset $S \subseteq M$. The groups $C^\infty(M,S;K)$ have the advantage that they are Fréchet–Lie groups if K is a Fréchet–Lie group, the Lie algebra is given by $C^\infty(M,S;\mathfrak{k})$. We consider them as groups of smooth maps on the non-compact manifold $M \setminus S$ vanishing at infinity. The groups $C^\infty_c(M;K)$ are modeled on the space $C^\infty_c(M;\mathfrak{k})$ which is not metrizable in its natural direct limit topology, not even for $K = \mathbb{R}$.

The goal of the present paper is to understand central extensions of current groups G which are identity components of groups of the type $C^\infty_c(M;K)$ or $C^\infty(M,S;K)$. For an infinite-dimensional Lie group G not every Lie algebra cocycle $\omega: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ defines a central extension of $\mathfrak{g}$ by $\mathfrak{z}$ which can be integrated to a Lie group. In [Ne02a] we show that there are two kinds of obstructions. The first one is an element of $\mathrm{Hom}(\pi_1(G), \mathrm{Lin}(\mathfrak{g},\mathfrak{z}))$, and we will see in Theorem V.8 that it always vanishes for current groups. The second obstruction is that the image of a certain "period map" $\mathrm{per}_\omega: \pi_2(G) \to \mathfrak{z}$ need not be discrete. To illuminate the obstructions for the class of current groups, we need a good deal of information on the abelian group $\pi_2(G)$. This information is obtained in Appendix A where we show that the computation of the homotopy groups of G can be reduced to the computation of those of groups $C(X;K)$ of continuous maps, where X is a compact manifold with boundary.

The Lie algebra cocycles we are interested in are those of *product type*, i.e., cocycles $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ for which there exists a sequentially complete locally convex space Y and an invariant continuous symmetric bilinear form $\kappa\colon \mathfrak{k} \times \mathfrak{k} \to Y$ such that $\omega(\xi,\eta) = [\kappa(\xi, d\eta)]$ defines a cocycle with values in $\mathfrak{z} := \mathfrak{z}_{M,c}(Y) := \Omega^1_c(M;Y)/dC^\infty_c(M;Y)$ for $\mathfrak{g} = C^\infty_c(M;\mathfrak{k})$, and $\mathfrak{z} := \mathfrak{z}_{(M,S)}(Y) := \Omega^1(M,S;Y)/dC^\infty(M,S;Y)$ for $\mathfrak{g} = C^\infty(M,S;\mathfrak{k})$. We systematically use forms with values in an infinite-dimensional vector space to incorporate in particular the universal invariant symmetric bilinear form $\kappa\colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$.

The main steps in our analysis of these cocycles and their period maps are as follows. In Section IV we show that the image of the period map always lies in the subspace of $\mathfrak{z}$ coming from the closed 1-forms. Then the problem is to determine the period group $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega) \subseteq \mathfrak{z}$ and to see if it is discrete. For the case $\mathfrak{g} = C_c(M;\mathfrak{k})$ it is quite hard to get information on the discreteness of a subgroup of $\mathfrak{z} = \mathfrak{z}_{M,c}(Y)$, resp., $H^1_{\mathrm{dR,c}}(M;Y)$ because $\mathfrak{z}$ is a direct limit of spaces on which the topology is given by explicit seminorms. We address this problem by approximating the non-compact manifold M by suitably chosen submanifolds X_n with boundary in such a way that

$$H^1_{\mathrm{dR,c}}(M;Y) = \varinjlim H^1_{\mathrm{dR}}(X_n, \partial X_n; Y)$$

(Section III). From this relation we then derive the existence of a countable set B so that

$$H^1_{\mathrm{dR,c}}(M;Y) \cong Y^{(B)},$$

is a locally convex direct sum, where the projections are given by integrals over singular cycles or over piecewise smooth proper maps $\mathbb{R} \to M$. In Section IV this information permits us to see that Π_ω is discrete for each M if and only if this is the case for the circle $M = \mathbb{S}^1$. In the latter case $\pi_2(C^\infty(\mathbb{S}^1, K)) \cong \pi_2(K) \times \pi_3(K)$, the period map vanishes on $\pi_2(K)$, and $\mathfrak{z}_{\mathbb{S}^1}(Y) \cong Y$, so that we arrive at a map $\pi_3(K) \to Y$ which depends only on the bilinear form κ. For finite-dimensional groups K we can now use information from [MN02] to see that the period group is discrete if κ is the universal invariant symmetric bilinear form. This is used in Section VI to construct for a finite-dimensional reductive Lie group K with simply connected center a universal central extension of the groups $C^\infty_c(M;K)_e$ and $C^\infty(M,S;K)_e$. In both cases there are many examples where the period group has infinite rank. A simple example with $M = \mathbb{S}^2$ and S a sequence with limit point is discussed in detail in Example II.12. All the concrete examples of central extensions of infinite-dimensional Lie groups which have been dealt with so far in the literature have finitely generated period groups. In this sense we provide new and concrete examples, where this is not the case.

The class of current groups most extensively studied is the class of loop groups ($M = \mathbb{S}^1$ and K compact) which is completely covered by Pressley and Segal's monograph [PS86]. The main point of the present paper is to see which Lie algebra cocycles of product type can be integrated to a central Lie group extension. These central extensions occur naturally in mathematical physics, where the problem to integrate projective representations of groups to representations of central extensions is at the heart of quantum mechanics ([Mic87], [LMNS98], [Wu01]). The central extensions of current groups are often constructed via representations by pulling back central extensions of certain operator groups ([Mic89]). It is our philosophy that one should try to understand the central extensions of a Lie group G first, and then construct representations of these central extensions. In this context certain discreteness conditions for Lie algebra cocycles appear naturally because they ensure that the corresponding central Lie algebra extension integrates to a central Lie group extension ([Ne02a]). We think of these discreteness conditions as an abstract version of the discreteness of quantum numbers in quantum physics. As an outcome of our analysis, we will see that for our general results we do not have to impose any restriction on the group K. It may be any infinite-dimensional Lie group. This permits in particular iterative constructions based on relations like $C^\infty(M \times N; K) \cong C^\infty(M, C^\infty(N;K))$ for compact manifolds M and N.

The content of the paper is as follows. In Section I we introduce the two kinds of Lie groups we are dealing with: $C^\infty_c(M;K)$ for M non-compact, and $C^\infty(M,S;K)$ for M compact and $S \subseteq M$ closed.

The main result of Section II is that the group $H^1_{\rm dR}(M,S;\Gamma)$ of all de Rham cohomology classes modulo S for which all integrals over singular cycles modulo S are contained in a discrete subgroup Γ of Y is discrete (Theorem II.7). In Section V this is used to prove the discreteness of period groups for cocycles of product type for the groups $C^\infty(M,S;K)$.

Our strategy to get a description of the spaces $\mathfrak{z}_{M,c}(Y)$ and $H^1_{\rm dR,c}(M;Y)$ for a non-compact manifold M is to describe M as a union of certain compact submanifolds with boundary $(X_n)_{n\in\mathbb{N}}$ with $X_n \subseteq X^0_{n+1}$. To get information on the space $H^1_{\rm dR,c}(M;Y)$, we will need detailed information on the natural maps $H^1_{\rm dR}(X_n,\partial X_n;Y) \to H^1_{\rm dR}(X_{n+1},\partial X_{n+1};Y)$ which is obtained in Theorem III.6. This result is used in Theorem IV.7 to obtain the isomorphism $H^1_{\rm dR,c}(M;Y) \cong Y^{(B)}$ mentioned above. As a corollary, we show that if Γ is discrete, then $H^1_{\rm dR,c}(M;\Gamma)$ is discrete.

In Section V we first explain the general setup for central extensions of Lie groups. The main question arising in the integration of Lie algebra cocycles ω to central extensions of Lie groups is whether the corresponding period group Π_ω is discrete. We then show that for cocycles of product type for the groups $C^\infty_c(M;K)_e$ and $C^\infty(M,S;K)_e$ the period group $\Pi_{M,\kappa}$ is discrete if and only if this is the case for $\Pi_{\mathbb{S}^1,\kappa}$. This reduces the discreteness problem to the case of loop groups, which is known for K compact, and therefore for all finite-dimensional Lie groups (cf. [PS86], [MN02]). We further show that $\Pi_{M,\kappa} = H^1_{\rm dR,c}(M;\Pi_{\mathbb{S}^1,\kappa})$ for each non-compact manifold M and each κ.

In Section VI we finally turn to universal central extensions. For the special class of finite-dimensional semisimple Lie groups K, each Lie algebra cocycle $\omega \in Z^2_c(C^\infty_c(M,\mathfrak{k}),\mathfrak{z})$ is equivalent to a cocycle of product type ([Ma02], [Fe88]). This observation permits us to construct a universal central extension of the Lie algebra $\mathfrak{g} := C^\infty_c(M;\mathfrak{k})$, and we show that this construction can be globalized in our context, providing a universal central extension of the connected Lie group $C^\infty_c(M;K)_e$.

In Appendix A we address the topology of the groups $C^\infty_c(M;K)$ and $C^\infty(M,S;K)$. For our purposes it is of particular importance to know their homotopy groups. We write $C_0(M;K)$ for the group of continuous functions vanishing at infinity, endowed with the topology of uniform convergence. Information on homotopy groups is obtained by several approximation arguments showing that the inclusion maps

$$C^\infty_c(M;K) \hookrightarrow C_0(M;K) \quad \text{and} \quad C^\infty(M,S;K) \hookrightarrow C_0(M\setminus S;K)$$

are weak homotopy equivalences, i.e., induce isomorphisms of all homotopy groups. These results are motivated by the fact that it is usually much easier to deal with spaces of continuous maps than with spaces of differentiable maps. We also note that if K is a Banach-, resp., Fréchet–Lie group, then the same holds for the groups $C_0(M;K)$ and $C_0(M\setminus S;K)$.

Appendix B contains several results on direct limits of locally convex spaces. These are needed to deal with the spaces of compactly supported smooth functions or differential forms on a non-compact manifold. The difficulties with these spaces arise from the fact that they are not metrizable, which makes it harder to prove that a subgroup is discrete.

This paper contributes in particular to the program dealing with Lie groups G whose Lie algebras $\mathfrak{g}$ are *root graded* in the sense that there exists a finite irreducible root system Δ such that $\mathfrak{g}$ has a Δ-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$, it contains the split simple Lie algebra $\mathfrak{k}$ corresponding to Δ as a graded subalgebra, and is generated, topologically, by the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Delta$. All Lie groups of the type $C^\infty_c(M;K)$, K simple complex, are of this type, and the same holds for their central extension. A different but related class of groups arising in this context are the Lie groups $\mathrm{SL}_n(A)$ and their central extensions, where A is a continuous inverse algebra, i.e., a locally convex unital associative algebra with open unit group and continuous inversion ([Gl01c], [Ne03]).

In [Ne02b] we discuss the universal central extensions of the groups $\mathrm{SL}_n(A)$, which are Lie group versions of the Steinberg groups $\mathrm{St}_n(A)$. In [MN02, Rem. II.12] we have shown that for $K = \mathrm{SL}_n(A)$, A a commutative continuous inverse algebra, the form $\kappa\colon \mathfrak{k}\times\mathfrak{k} \to A$, $\kappa(x,y) = \mathrm{tr}(xy)$ is universal, and that the image of the corresponding period map is discrete for the corresponding product type cocycle on the Lie algebra $C^\infty(M;\mathfrak{k})$ of the group $C^\infty(M;K)$. For non-commutative algebras the image of the period map is not always discrete ([Ne02b]).

Throughout this paper we will use the concept of an infinite-dimensional Lie group described in detail in [Mil83] (see also [Gl01a] for arguments showing that the completeness requirements made in [Mil83] are not necessary to define the concept). This means that a *Lie group* G is a smooth manifold modeled on a locally convex space $\mathfrak{g}$ for which the group multiplication and the inversion are smooth maps. We write $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G. Let $e \in G$ be the identity element. Then each $X \in T_e(G)$ corresponds to a unique left invariant vector field X_l with $X_l(g) := d\lambda_g(\mathbf{1}).X, g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_e(G)$ a continuous Lie bracket which is uniquely determined by $[X,Y]_l = [X_l, Y_l]$.

All finite-dimensional manifolds M are assumed to be σ-compact which for connected manifolds is equivalent to requiring that M is paracompact or a second countable topological space. This excludes pathologies such as “long lines” which are one-dimensional smooth manifolds constructed from sets of countable ordinal numbers ([SS78, p.72]).

All topological vector spaces in this paper are assumed to be Hausdorff.

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I. Current groups on non-compact manifolds

In this section we introduce two classes of Lie groups of smooth maps: the group $C^\infty_c(M;K)$ of smooth maps with compact support on a non-compact manifold and the group $C^\infty(M,S;K)$ of smooth maps on a compact manifold M that together with all higher partial derivatives vanish on the closed subset S.

Compactly supported smooth maps

Definition I.1. For two topological spaces M and Y we write $C(M;Y)_c$ for the space $C(M;Y)$ of all continuous maps $M \to Y$ endowed with the *compact open topology.* The topology on this space is generated by the sets

$$W(C,O) := \{f \in C(M;Y) \colon f(C) \subseteq O\},$$

where $C \subseteq M$ is compact and $O \subseteq Y$ is open.

(a) If M is locally compact and K is a topological group, then $C(M;K)_c$ is a topological group with respect to pointwise multiplication, and the topology coincides with the topology of uniform convergence on compact subsets of M ([Sch75, Satz II.4.5]). In particular the sets $W(C,U)$, where $C \subseteq M$ is compact and $U \subseteq K$ is an open identity neighborhood, form a basis of identity neighborhoods in $C(M;K)_c$.

For a function $f\colon M \to K$ let $\mathrm{supp}(f) := \overline{\{x \in M \colon f(x) \neq e\}}$ denote its *support.* Then for each compact subset $X \subseteq M$ the subset

$$C_X(M;K) := \{f \in C(M;K) \colon \mathrm{supp}(f) \subseteq X\}$$

is a closed subgroup of $C(M;K)_c$ on which the subspace topology coincides with the topology of uniform convergence.

If M is a discrete set, then $C(M;K)_c \cong K^M$ as a topological group.

(b) If M is a locally compact space and Y is a locally convex space, then (a) implies that $C(M;Y)_c$ is a locally convex space, where the topology is defined by the seminorms

$$p_{X,q}(f) := \sup_{x\in X} q(f(x)),$$

where q is a continuous seminorm on Y and $X \subseteq M$ a compact subset.

If Y is a Fréchet space and M is σ-compact, then the topology is defined by a countable family of seminorms turning $C(M;Y)_c$ into a Fréchet space.
(c) If M is locally compact, $X \subseteq M$ compact, and Y is a locally convex space, then for each open 0-neighborhood $U \subseteq Y$ the subset

$$\{f \in C_X(M;Y)_c : f(M) \subseteq U\} = W(X,U) \cap C_X(M;Y)$$

is open in $C_X(M;Y)_c$. ∎

Definition I.2. Let M be a smooth finite-dimensional σ-compact manifold. If Y is a locally convex space, then each smooth map $f: M \to Y$ defines a sequence of maps

$$d^n f: T^n M \to Y, \quad n \in \mathbb{N}.$$

We endow $C^\infty(M;Y)$ with the topology obtained from the embedding

$$C^\infty(M;Y) \hookrightarrow \prod_{n \in \mathbb{N}_0} C(T^n M, Y)_c$$

turning $C^\infty(M;Y)$ into a locally convex space. If $X \subseteq M$ is a compact subset, we consider on $C^\infty_X(M;Y) \subseteq C^\infty(M;Y)$ the subspace topology.
(a) If K is a Lie group, then $C^\infty_X(M;K)$ is a group with respect to pointwise multiplication. It is shown in [Gl01b, 3.18] that it carries a Lie group structure which is uniquely determined by the property that for each open identity neighborhood $U \subseteq K$ and each chart $\varphi: U \to \mathfrak{k}$ with $\varphi(e) = 0$ there exists an open identity neighborhood $U_0 \subseteq U$ such that the map

$$\{f \in C^\infty_X(M;K) : f(M) \subseteq U_0\} \to \{h \in C^\infty_X(M;\mathfrak{k}) : h(M) \subseteq \varphi(U_0)\}, \quad f \mapsto \varphi \circ f$$

is a diffeomorphism onto an open subset of the locally convex space $C^\infty_X(M;\mathfrak{k})$. The Lie algebra of this group is the locally convex space $C^\infty_X(M;\mathfrak{k})$ with the pointwise Lie bracket, where $\mathfrak{k}$ is the Lie algebra of K ([Gl01b, 3.19]).
(b) For a locally convex space Y we endow the space

$$C^\infty_c(M;Y) := \{f \in C^\infty(M;Y) : \mathrm{supp}(f) \text{ compact}\} = \bigcup_X C^\infty_X(M;Y),$$

where X runs through all compact subsets of M, with the locally convex direct limit topology. This means that a seminorm on $C^\infty_c(M;Y)$ is continuous if and only if its restrictions to all the subspaces $C^\infty_X(M;Y)$ are continuous with respect to the topology defined above.

In M there exists an increasing sequence $(X_n)_{n \in \mathbb{N}}$ of compact subsets X_n with $X_n \subseteq X^0_{n+1}$ and $M = \bigcup_n X_n$. Then each compact subset $X \subseteq M$ is contained in some X_n, and each space $C^\infty_{X_n}(M;Y)$ is a closed subspace of $C^\infty_{X_{n+1}}(M;Y)$. Therefore

$$C^\infty_c(M;Y) = \varinjlim C^\infty_{X_n}(M;Y)$$

is a *strict inductive limit* of the locally convex spaces $C^\infty_{X_n}(M;Y)$ in the sense of [He89, Prop. 1.5.3]. In particular each bounded subset of $C^\infty_c(M;Y)$ is contained in one of the subspaces $C^\infty_{X_n}(M;Y)$. Moreover, $C^\infty_c(M;Y)$ is Hausdorff and the continuous maps $C^\infty_{X_n}(M;Y) \hookrightarrow C^\infty_c(M;Y)$ are embeddings, which in turn implies that all the inclusions

$$C^\infty_X(M;Y) \hookrightarrow C^\infty_c(M;Y)$$

are embeddings (cf. [Kö69, p.222]).

If Y is a Fréchet space, this topology turns $C^\infty_c(M;Y)$ into an LF-space ([Gl01b, 4.6]). It is shown in [Gl01b, 4.18] that for each Lie group K the group $C^\infty_c(M;K)$ carries a Lie group structure, hence in particular the structure of a Hausdorff topological group. In the same way as for the groups $C^\infty_X(M;K)$, the Lie group structure is uniquely determined by the property that for each open identity neighborhood $U \subseteq K$ and each chart $\varphi: U \to \mathfrak{k}$ with $\varphi(e) = 0$ there exists an open identity neighborhood $U_0 \subseteq U$ such that the map

$$\{f \in C^\infty_c(M;K) : f(M) \subseteq U_0\} \to \{h \in C^\infty_c(M;\mathfrak{k}) : h(M) \subseteq \varphi(U_0)\}, \quad f \mapsto \varphi \circ f$$

is a diffeomorphism onto an open subset of the locally convex space $C^\infty_c(M;\mathfrak{k})$. The Lie algebra of this group is the locally convex space $C^\infty_c(M;\mathfrak{k})$ with the pointwise Lie bracket. ∎

Remark I.3. From the fact that $C^\infty_c(M;\mathfrak{k})$ is a strict inductive limit of spaces $C^\infty_X(M;\mathfrak{k})$ and the description of the natural charts of the Lie group $C^\infty_c(M;K)$, we see that for each compact subset $X \subseteq M$ the inclusion map $C^\infty_X(M;K) \hookrightarrow C^\infty_c(M;K)$ is a topological embedding. ∎

Remark I.4. If K is a Lie group with Lie algebra $\mathfrak{k}$, then the tangent bundle of K is a Lie group isomorphic to $\mathfrak{k} \rtimes K$, where K acts on $\mathfrak{k}$ by the adjoint representation (cf. [Ne01b]). Iterating this procedure, we obtain a Lie group structure on all iterated higher tangent bundles $T^n K$ which are diffeomorphic to $\mathfrak{k}^{2^n-1} \times K$.

It follows in particular that for each finite-dimensional manifold M and each $n \in \mathbb{N}_0$ we obtain topological groups $C(T^n M, T^n K)_c$ (Definition I.1(a)). Therefore the canonical inclusion map

$$C^\infty(M;K) \hookrightarrow \prod_{n\in\mathbb{N}} C(T^n M, T^n K)_c$$

leads to a natural topology on $C^\infty(M;K)$ turning it into a topological group.

If M is compact, then it is not hard to see that this procedure leads to the same topology as the Lie group structure defined in Definition I.2. A similar statement holds for $C^\infty_X(M;K)$ if $X \subseteq M$ is a compact subset.

We cannot expect for a general non-compact manifold M that $C^\infty(M;K)$ carries a natural Lie group structure. In the example $M = \mathbb{N}$ the group $C^\infty(\mathbb{N};K) = C(\mathbb{N};\mathbb{K}) \cong K^{\mathbb{N}}$ is the topological direct product group. As the example $K = \mathbb{T}$ already shows, the groups $K^{\mathbb{N}}$ need not be manifolds because they need not be locally contractible.

If M is connected, then the situation seems to be much better, but this needs to be investigated ([NW03]). One can show in particular that for each Banach–Lie group K the group $C^\infty(\mathbb{R},K)$ is a Fréchet–Lie group with respect to its natural topology of uniform convergence of all derivatives on compact subsets of $\mathbb{R}$. Likewise, for each simply connected non-compact complex curve Σ and each complex Banach–Lie group K the group $\mathrm{Hol}(\Sigma,K)$ of all holomorphic maps $\Sigma \to K$ is a Lie group.

∎

Fréchet current groups defined by vanishing conditions

In this subsection M denotes a connected finite-dimensional manifold and $S \subseteq M$ a closed subset. Mostly we will assume that M is compact.

Remark I.5. Let U be an open subset of a locally convex space X and Y another locally convex space. If for a smooth function $f: U \to Y$ its value together with all derivatives up to order k vanish in a point $p \in U$, then the formula for the Taylor expansion of compositions trivially implies that the same holds for all compositions $f \circ \varphi$ in q, where $\varphi: V \to U$ is a C^k-map with $\varphi(q) = p$. It follows in particular that for a smooth function on a manifold it makes sense to say that *all partial derivatives up to order k vanish in a point p*. ∎

Definition I.6. Let M be a manifold with boundary and $S \subseteq M$ a closed subset. For a Lie group K we write $C^\infty(M,S;K)$ for the group of all those smooth maps for which their value together with all derivatives vanish on S. It clearly suffices that for each point $s \in S$ there exists one chart in which all partial derivatives vanish in s.

If M is compact and K is a (Fréchet-)Lie group, then also $C^\infty(M,S;K)$ is a Fréchet–Lie group, where we use the same charts as for $C^\infty(M;K)$ and observe that they restrict to charts of the subgroup $C^\infty(M,S;K)$. In particular $C^\infty(M,S;\mathbb{R})$ is a real Fréchet algebra. For non-compact M we consider $C^\infty(M,S;K)$ only as a topological subgroup of $C^\infty(M;K)$ in the sense of Remark I.4. ∎

Remark I.7. Let us consider the category $\mathcal{P}$ whose objects are pairs (M,S), where M is a (finite-dimensional) manifold and S is a closed subset. A morphism $(M,S) \to (M',S')$

is a smooth map $\varphi: M \to M'$ with $\varphi(S) \subseteq S'$. Remark I.5 implies that the assignment $(M,S) \mapsto C^\infty(M,S;K)$ defines a contravariant functor from $\mathcal{P}$ to the category of topological groups. Here we use that for a morphism $\varphi:(M,S) \to (M',S')$ the corresponding group homomorphism $C^\infty(M',S';K) \to C^\infty(M,S;K), f \mapsto f \circ \varphi$ is continuous, which is an easy consequence of the definitions (cf. Lemma A.1.6). ∎

Lemma I.8. *Let M be a finite-dimensional manifold, $X \subseteq M$ be a smooth submanifold with boundary,* $\dim X = \dim M$, *and Y a locally convex space. For a smooth function $f: X \to Y$ the extension by $f(M \setminus X) = \{0\}$ defines a smooth function $M \to Y$ if and only if f and all its derivatives vanish on ∂X.*

Proof. It clearly is a necessary condition that all derivatives of f vanish on ∂X. Suppose, conversely, that this condition is satisfied and extend f by 0 on $M \setminus X$ to a function $f_M: M \to Y$.

As the smoothness of f_M is equivalent to its weak smoothness (for this result of Grothendieck see [Wa72] or [KM97]), we may w.l.o.g. assume that $Y = \mathbb{R}$. Moreover, we may assume that $M = \mathbb{R}^n$ and that $X = \{x \in \mathbb{R}^n : x_n \leq 0\}$. Then it is clear that all partial derivatives of f extended by 0 on $M \setminus X$ yield continuous functions. Moreover, all partial derivatives of the extended function f_M exist and coincide with the extensions of the partial derivatives of f. This proves that f_M is a C^1-function. Iterating the argument shows that f_M is a C^k-function for each k, hence smooth. ∎

Examples I.9. (a) Let X be a compact manifold with boundary and X^d the *double of* X. This is, by definition, a compact manifold without boundary containing X and a diffeomorphic copy $X^\sharp$ of X such that $X \cap X^\sharp = \partial X = \partial X^\sharp$ and $X \cup X^\sharp = X^d$. Then Lemma I.8 implies that

$$C^\infty(X, \partial X; K) \cong C^\infty_X(X^d; K)$$

and

$$C^\infty(X^d, \partial X; K) \cong C^\infty(X, \partial X; K) \times C^\infty(X^\sharp, \partial X; K) \cong C^\infty(X, \partial X; K)^2.$$

(b) We think of $C^\infty(M,S;K)$ as a group of smooth maps on the non-compact manifold $M \setminus S$. For $M = \mathbb{S}^n$ and $S = \{p\}$ have $M \setminus S \cong \mathbb{R}^n$, and hence a natural Lie group of smooth maps $\mathbb{R}^n \to K$ with a certain decay at infinity.

(c) Let $M = \mathbb{S}^1$. Then $M \setminus S$ is a countable union of intervals I_j, $j \in J$, and we thus obtain an inclusion

$$C^\infty(M,S;K) \hookrightarrow \prod_{j \in J} C^\infty(I_j, \partial I_j; K) \cong C^\infty(I, \partial I; K)^J,$$

where the right hand side does not carry the product topology but the l^∞-topology of uniform convergence of all derivatives uniformly in all components. ∎

II. Relative de Rham cohomology

If M is a compact manifold, $S \subseteq M$ a compact subset, and Y a sequentially complete locally convex space (an s.c.l.c. space), then we consider the space $Z^1_{\mathrm{dR}}(M,S;Y)$ of all Y-valued closed smooth 1-forms that vanish, together with all their derivatives, on S. Integration of 1-forms with this property over singular cycles in M modulo S lead to the subgroup $Z^1_{\mathrm{dR}}(M,S;\Gamma)$ of those closed 1-forms for which all integrals over cycles have values in a subgroup Γ of Y. The main result of this section is Theorem II.7, saying that the image $H^1_{\mathrm{dR}}(M,S;\Gamma)$ of $Z^1_{\mathrm{dR}}(M,S;\Gamma)$ in $H^1_{\mathrm{dR}}(M,S;Y)$ is a discrete subgroup if Γ is discrete. In Examples II.11 and II.12 we see that these subgroups may have infinite rank, even for $Y = \mathbb{R}$.

We write $I := [0,1]$ and assume that $S \neq \emptyset$ and that M is connected. Further Y denotes an s.c.l.c. space, Γ is a subgroup of Y, and $T_\Gamma := Y/\Gamma$ the corresponding quotient group. If Γ is discrete, then the quotient topology turns T_Γ into a Lie group with Lie algebra Y. For some statements we do not have to assume that M is compact. If we assume compactness, we will mention it explicitly.

We write $\Omega^1(M;Y)$ for the space of smooth 1-forms on M with values in Y and endow this space with the natural topology corresponding in each chart to the uniform convergence of all derivatives on compact subsets mapping into coordinate charts (cf. [Gl01d]). For a subset $X \subseteq M$ we write $\Omega^1_X(M;Y)$ for the closed subspace of $\Omega^1(M;Y)$ consisting of those forms supported in X. We endow the space $\Omega^1_c(M;Y)$ with the locally convex direct limit topology with respect to the subspaces $\Omega^1_X(M;Y)$, where $X \subseteq M$ is a compact subset. For a closed subset $S \subseteq M$ we write $\Omega^1(M,S;Y) \subseteq \Omega^1(M;Y)$ for the subspace of all forms vanishing with all their partial derivatives on S.

The Lie group $C^\infty(M,S;T_\Gamma)$

Definition II.1. Let M be a smooth manifold and K a Lie group. For an element $f \in C^\infty(M;K)$ we write

$$\delta^l(f)(m) := d\lambda_{f(m)^{-1}}(f(m))df(m)\colon T_m(M) \to \mathfrak{k} \cong T_e(K)$$

for the *left logarithmic derivative of* f. This derivative can be viewed as a $\mathfrak{k}$-valued 1-form on M which we also write simply as $\delta^l(f) = f^{-1}.df$. We thus obtain a map

$$\delta^l\colon C^\infty(M;K) \to \Omega^1(M;\mathfrak{k})$$

satisfying the cocycle condition

$$\delta^l(f_1f_2) = \mathrm{Ad}(f_2)^{-1}.\delta^l(f_1) + \delta^l(f_2).$$

We also have the *right logarithmic derivative* $\delta^r(f) = df.f^{-1}$ satisfying

$$\delta^r(f_1f_2) = \delta^r(f_1) + \mathrm{Ad}(f_1).\delta^r(f_2).$$

(cf. [KM97, 38.1]). If K is abelian, then the cocycle condition shows that $\delta := \delta^l$ is a group homomorphism whose kernel consists of the locally constant maps. ∎

In Section V we will need the following continuity result for the logarithmic derivatives.

Lemma II.2. *For any Lie group K the maps $\delta^l, \delta^r\colon C^\infty_c(M;K) \to \Omega^1_c(M;\mathfrak{k})$ are smooth.*

Proof. In view of the cocycle relations

$$\delta^l(f_1f_2) = \mathrm{Ad}(f_2)^{-1}.\delta^l(f_1) + \delta^l(f_2) \quad \text{and} \quad \delta^r(f_1f_2) = \delta^r(f_1) + \mathrm{Ad}(f_1).\delta^r(f_2),$$

it suffices to prove the smoothness of δ^l and δ^r in an open identity neighborhood U of $C^\infty_c(M;K)$. Here we use that addition is continuous in $\Omega^1_c(M;\mathfrak{k})$, and that the continuity of the linear map $\mathrm{Ad}(f_1)$ on $\Omega^1_c(M;\mathfrak{k})$ follows from its continuity on the subspaces $\Omega^1_X(M;\mathfrak{k})$, $X \subseteq M$ compact. According to the definition of the Lie group structure on $C^\infty_c(M;K)$, we may assume that

$$U = \{f \in C^\infty_c(M;K)\colon f(M) \subseteq V_K\},$$

where $V_K \subseteq K$ is an open identity neighborhood for which there exists a diffeomorphism $\varphi\colon V_{\mathfrak{k}} \to V_K$, where $V_{\mathfrak{k}}$ is an open subset of the locally convex space $\mathfrak{k}$. We now have to show that the map

$$D\colon C^\infty_c(M;V_{\mathfrak{k}}) \to \Omega^1_c(M;\mathfrak{k}), \quad f \mapsto \delta^l(\varphi \circ f)$$

is smooth.

We think of D as a map between spaces of sections of vector bundles over M. Then the values of $D(f)$ in an open subset $O \subseteq M$ only depend on $f|_O$. This implies in particular that D is *local* in the sense of [Gl02, Def. 3.1]. Moreover, for each compact subset $X \subseteq M$ the map

$$D_X := D|_{C^\infty_X(M;K)}\colon C^\infty_X(M;K) \to \Omega^1_X(M;\mathfrak{k})$$

is smooth because the map

$$\delta^l\colon C^\infty_X(M;K) \to \Omega^1_X(M;\mathfrak{k})$$

is obviously smooth. Therefore the Smoothness Theorem 3.2 in [Gl02] implies that D is a smooth map and hence that δ^l is smooth. The smoothness of δ^r is shown similarly. ∎

Lemma II.3. *If Γ is discrete, then*

$$\delta(C^\infty(M,S;T_\Gamma)) = \Big\{\beta \in \Omega^1(M,S;Y)\colon \big(\forall \alpha \in C^\infty((I,\partial I),(M,S))\big) \int_\alpha \beta \in \Gamma\Big\}.$$

Proof. If $\beta = \delta(f)$ for some $f \in C^\infty(M,S;T_\Gamma)$ and $\alpha \in C^\infty((I,\partial I),(M,S))$, then

$$f(\alpha(1)) - f(\alpha(0)) = \int_\alpha \beta + \Gamma$$

vanishes in $T_\Gamma = Y/\Gamma$, so that $\int_\alpha \beta \in \Gamma$.

Suppose, conversely, that $\beta \in \Omega^1(M,S;Y)$ satisfies

$$\int_\alpha \beta \in \Gamma \quad \text{for all} \quad \alpha \in C^\infty((I,\partial I),(M,S)).$$

Pick $s_0 \in S$. Then all integrals of β over smooth loops based in s_0 are contained in Γ (here we need that Y is sequentially complete to ensure the existence of Y-valued Riemann integrals over curves), so that there exists a smooth function $f\colon M \to T_\Gamma$ with $\beta = \delta(f)$ and $f(s_0) = 0$ ([Ne02a, Prop. 3.9]). For each $s \in S$ there exists a smooth path $\alpha \in C^\infty((I,\partial I),(M,S))$ from s_0 to s, and we obtain

$$f(s) = f(s) - f(s_0) = \int_\alpha \beta + \Gamma \in \Gamma.$$

This means that $f\,|_S = 0$. As $\beta = \delta(f)$, all higher derivatives of f vanish on S, so that $f \in C^\infty(M,S;T_\Gamma)$. ■

Corollary II.4. *For each s.c.l.c. space Y we have*

$$dC^\infty(M,S;Y) = \Big\{\beta \in \Omega^1(M,S;Y)\colon \big(\forall \alpha \in C^\infty((I,\partial I),(M,S))\big) \int_\alpha \beta = 0\Big\}.$$

In particular $dC^\infty(M,S;Y)$ is closed in $\Omega^1(M,S;Y)$. ■

Definition II.5. (a) In view of the closedness assertion in Corollary II.4, the quotient

$$\mathfrak{z}_{(M,S)}(Y) := \Omega^1(M,S;Y)/dC^\infty(M,S;Y)$$

carries a natural (Hausdorff) locally convex topology. Moreover, the subspace $Z^1_{\rm dR}(M,S;Y)$ of closed forms in $\Omega^1(M,S;Y)$ is closed, which implies that

$$H^1_{\rm dR}(M,S;Y) := Z^1_{\rm dR}(M,S;Y)/dC^\infty(M,S;Y)$$

is a closed subspace of $\mathfrak{z}_{(M,S)}(Y)$. Let $q\colon \Omega^1(M,S;Y) \to \mathfrak{z}_{(M,S)}(Y)$ denote the quotient map.

We want to relate $H^1_{\rm dR}(M,S;Y)$ to the singular Y-valued cohomology of M modulo S. The abelian group $Z_1(M,S)$ of singular 1-cycles modulo S is generated by those given by continuous maps $(I,\partial I) \to (M,S)$. Therefore $H_1(M,S)$ is generated by the image of the set $\pi_1(M,S) := [(I,\partial I),(M,S)]$ of homotopy classes of maps of pairs (see [Br93, VII.4.10] for more details on Hurewicz maps from homotopy groups to homology groups). Let $\beta \in Z^1_{\rm dR}(M,S;Y)$. Then we can define for each singular 1-chain α the integral $\int_\alpha \beta$. According to Stoke's formula, these integrals vanish on boundaries and also on chains supported by S. We thus obtain a map

$$Z^1_{\rm dR}(M,S;Y) \to H^1(M,S;Y) := \operatorname{Hom}(H_1(M,S);Y),$$

where $H_1(M,S)$ denotes the singular homology group with coefficients in $\mathbb{Z}$ and $H^1(M,S;Y)$ a relative singular cohomology group (cf. [Br93, V.7.2]).

The kernel of this map consists of all closed 1-forms β for which all the integrals of cycles in $Z_1(M,S)$ vanish, which means that $\beta = df$ for some $f \in C^\infty(M,S;Y)$ (Corollary II.4). Hence we obtain an embedding

$$\eta\colon H^1_{\mathrm{dR}}(M,S;Y) \hookrightarrow H^1(M,S;Y). \tag{2.1}$$

As we will see in Example II.12 below, this map is not always surjective.

(b) For a subgroup $\Gamma \subseteq Y$ we define

$$Z^1_{\mathrm{dR}}(M,S;\Gamma) := \Big\{\beta \in Z^1_{\mathrm{dR}}(M,S;Y)\colon (\forall \alpha \in C^\infty((I,\partial I),(M,S)))\ \int_\alpha \beta \in \Gamma\Big\}.$$

Applying Corollary II.4, we see that $dC^\infty(M,S;Y)$ is a closed subspace of $Z^1_{\mathrm{dR}}(M,S;\Gamma)$, so that

$$H^1_{\mathrm{dR}}(M,S;\Gamma) := Z^1_{\mathrm{dR}}(M,S;\Gamma)/dC^\infty(M,S;Y)$$

carries a natural Hausdorff locally convex topology. We also define

$$Z^1_{\mathrm{dR}}(M;\Gamma) := \Big\{\beta \in Z^1_{\mathrm{dR}}(M;Y)\colon (\forall \alpha \in C^\infty(\mathbb{S}^1,M))\ \int_\alpha \beta \in \Gamma\Big\}$$

and $H^1_{\mathrm{dR}}(M;\Gamma) := Z^1_{\mathrm{dR}}(M;\Gamma)/dC^\infty(M;Y)$. ■

Remark II.6. Let M be a connected manifold.

(a) Assume that $\Gamma \subseteq Y$ is a discrete subgroup and let $T_\Gamma := Y/\Gamma$ denote the corresponding quotient Lie group and $q_\Gamma\colon Y \to T_\Gamma$ the quotient map. We consider the abelian topological group $G := C^\infty(M;T_\Gamma)$, the space $\mathfrak{g} := C^\infty(M;Y)$, and the exponential function

$$\exp_G\colon \mathfrak{g} \to G, \quad \xi \mapsto q_\Gamma \circ \xi.$$

The map

$$\delta\colon G = C^\infty(M;T_\Gamma) \to Z^1_{\mathrm{dR}}(M;Y), \quad f \mapsto \delta(f) = f^{-1}df$$

is a continuous group homomorphism whose kernel consists of the locally constant functions on M. If M is connected, then $\ker\delta$ consists only of the constant functions.

According to [Ne02a, Prop. 3.9], a closed 1-form in $Z^1_{\mathrm{dR}}(M;Y)$ can be written as $\delta(f)$ for some $f \in C^\infty(M;T_\Gamma)$ if and only if all integrals over closed piecewise smooth paths are contained in Γ. This means that

$$\mathrm{im}(\delta) = Z^1_{\mathrm{dR}}(M;\Gamma).$$

Using the decomposition $G \cong G_* \times T_\Gamma$ with $G_* := \{f \in G\colon f(x_M) = 0\}$, where $x_M \in M$ is a base point, it follows that

$$\delta\colon G_* \to Z^1_{\mathrm{dR}}(M;\Gamma)$$

is an isomorphism of groups. Here the subgroup $B^1_{\mathrm{dR}}(M;Y) \subseteq Z^1_{\mathrm{dR}}(M;\Gamma)$ corresponds to $\mathrm{im}(\exp_G)$, so that

$$G/\exp_G(\mathfrak{g}) \cong Z^1_{\mathrm{dR}}(M;\Gamma)/B^1_{\mathrm{dR}}(M;Y) = H^1_{\mathrm{dR}}(M;\Gamma).$$

If, in addition, M is compact, then G is a Lie group with Lie algebra $\mathfrak{g}$, $\exp_G$ is the universal covering map of G_e, and $\delta\colon G_* \to Z^1_{\mathrm{dR}}(M;\Gamma)$ is an isomorphism of Lie groups. This leads to

$$\pi_0(G) \cong G/\exp_G(\mathfrak{g}) \cong Z^1_{\mathrm{dR}}(M;\Gamma)/B^1_{\mathrm{dR}}(M;Y) = H^1_{\mathrm{dR}}(M;\Gamma).$$

(b) If M is compact and $S \subseteq M$ a non-empty closed subset, then we obtain with similar arguments as in (a) that the group $G := C^\infty(M,S;T_Y)$ is a Lie group and that $\exp_G$ is the universal covering map of the identity component G_e of G. The connectedness of M and $S \neq \emptyset$ imply $\ker \exp_G = \{0\}$. Therefore the exponential function $\exp_G$ induces a diffeomorphism

$$\exp_G\colon \mathfrak{g} = C^\infty(M,S;Y) \to G_e.$$

Moreover, δ is an injective homomorphism of Lie groups with $\delta(G) = Z^1_{\rm dR}(M,S;\Gamma)$ (Lemma II.3), where G_e corresponds to the subspace $dC^\infty(M,S;Y)$, so that

$$\pi_0(G) \cong H^1_{\rm dR}(M,S;\Gamma).$$

(c) The set $M \setminus S$ is an open subset of M, hence a non-compact manifold. We have inclusions

$$\Omega^1_c(M \setminus S;Y) \hookrightarrow \Omega^1(M,S;Y) \quad \text{and} \quad Z^1_{\rm dR,c}(M\setminus S;Y) \hookrightarrow Z^1_{\rm dR}(M,S;Y).$$

Moreover,

$$dC^\infty_c(M\setminus S;Y) \subseteq Z^1_{\rm dR,c}(M\setminus S;Y) \cap dC^\infty(M,S;Y)$$

and if, conversely, $\zeta = df \in \Omega_c(M\setminus S;Y)$ with $f \in C^\infty(M,S;Y)$, then df vanishes in a neighborhood of S, so that $f^{-1}(0)$ is an open neighborhood of S. If M is compact, then it follows that f has compact support, and therefore that

$$dC^\infty_c(M\setminus S;Y) = Z^1_{\rm dR,c}(M\setminus S;Y) \cap dC^\infty(M,S;Y).$$

This means that we also obtain an inclusion

$$\varphi\colon H^1_{\rm dR,c}(M\setminus S;Y) \hookrightarrow H^1_{\rm dR}(M,S;Y).$$

If X is a compact manifold with boundary, $M = X \cup X^\sharp$ as in Example I.9, and $\mathrm{int}(X) = M \setminus S$, we claim that

$$H^1_{\rm dR,c}(\mathrm{int}(X);Y) \cong H^1_{\rm dR}(X,\partial X;Y) := H^1_{\rm dR}(M, M\setminus \mathrm{int}(X);Y). \tag{2.2}$$

In fact, if $\zeta \in Z^1_{\rm dR}(X,\partial X;Y)$, then the restriction of ζ to ∂X vanishes. Moreover, there exists a tubular neighborhood U of ∂X diffeomorphic to $\partial X \times I$, so that the inclusion $\partial X \hookrightarrow U$ induces an isomorphism $\pi_1(\partial X) \to \pi_1(U)$. We conclude that all periods of $\zeta\,|_U$ vanish, and hence that there exists a smooth function $f \in C^\infty(U,\partial X;Y)$ with $df = \zeta|_U$. Let $\chi \in C^\infty(X;\mathbb{R})$ be constant 1 in a neighborhood of ∂X and 0 on $X \setminus U$. Then $\zeta - d(\chi f) \in Z^1_{\rm dR,c}(\mathrm{int}(X);Y)$ has the same cohomology class as ζ. This proves (2.2).

From [Br97, Prop. II.12.3, Th. III.1.1, Cor. III.4.12] applied to the paracompactifying family Φ of closed subsets of $X \setminus \partial X$, we derive that for singular cohomology we have

$$H^1(X,\partial X;Y) \cong H^1_c(\mathrm{int}(X);Y).$$

Further the general version of de Rham's Theorem with values in sheaves ([Br97, §III.3]) yields an isomorphism

$$H^1_c(\mathrm{int}(X);Y) \cong H^1_{\rm dR,c}(\mathrm{int}(X);Y).$$

Therefore

$$H^1_{\rm dR}(X,\partial X;Y) \cong H^1_{\rm dR,c}(\mathrm{int}(X);Y) \cong H^1_c(\mathrm{int}(X);Y) \cong H^1(X,\partial X;Y) \cong \mathrm{Hom}(H_1(X,\partial X);Y).$$

■

The following theorem on the discreteness of the group $H^1_{\rm dR}(M,S;\Gamma)$ is the main result of the present section.

Theorem II.7. *Let S be a non-empty closed subset of the compact manifold M and $\Gamma \subseteq Y$ a discrete subgroup. Then the subgroup*

$$H^1_{\mathrm{dR}}(M,S;\Gamma) = \Big\{[\beta] \in \mathfrak{z}_{(M,S)}(Y) \colon (\forall \alpha \in C^\infty((I,\partial I),(M,S)))\ \int_\alpha \beta \in \Gamma\Big\}$$

of $\mathfrak{z}_{(M,S)}(Y)$ is discrete.

Proof. Let $Z^1_{\mathrm{dR}}(M;Y) \subseteq \Omega^1(M;Y)$ denote the closed subspace of closed 1-forms. As $\pi_1(M)$ is finitely generated (cf. Proposition III.1 below) and Γ is discrete,

$$dC^\infty(M;Y) = \Big\{\beta \in Z^1_{\mathrm{dR}}(M;Y) \colon (\forall [\alpha] \in \pi_1(M))\ \int_\alpha \beta = 0\Big\}$$

is an open subgroup of

$$Z^1_{\mathrm{dR}}(M;\Gamma) = \Big\{\beta \in Z^1_{\mathrm{dR}}(M;Y) \colon (\forall \alpha \in C^\infty(\mathbb{S}^1,M))\ \int_\alpha \beta \in \Gamma\Big\}.$$

That $H^1_{\mathrm{dR}}(M,S;\Gamma)$ is a discrete subgroup of the quotient space $\mathfrak{z}_{(M,S)}(Y)$ is equivalent to $dC^\infty(M,S;Y)$ being an open subgroup of $Z^1_{\mathrm{dR}}(M,S;\Gamma)$. As a consequence of what we have just seen, the group $Z^1_{\mathrm{dR}}(M,S;\Gamma) \cap dC^\infty(M;Y)$ is open in $Z^1_{\mathrm{dR}}(M,S;\Gamma)$. Therefore it suffices to verify that $dC^\infty(M,S;Y)$ is an open subgroup of $Z^1_{\mathrm{dR}}(M,S;\Gamma) \cap dC^\infty(M;Y)$.

Fix a point $x_M \in S$. We consider the map

$$\Phi \colon Z^1_{\mathrm{dR}}(M,S;\Gamma) \to C(M;T_\Gamma), \quad \Phi(\beta)(x) := \int_{x_M}^x \beta + \Gamma \in T_\Gamma.$$

Then

$$\Phi(Z^1_{\mathrm{dR}}(M,S;\Gamma)) \subseteq C^\infty(M;T_\Gamma), \quad d\big(\Phi(\beta)\big) = \beta, \quad \Phi(\beta)|_S = 0,$$

and Φ is continuous with respect to the topology of uniform convergence on compact subsets of M. Hence

$$\Phi^{-1}(C(M;T_\Gamma)_e) = \Phi^{-1}(\exp(C(M;Y))) = dC^\infty(M,S;Y)$$

is an open subgroup of $Z^1_{\mathrm{dR}}(M,S;\Gamma)$ because $C(M;T_\Gamma)$ is a Lie group (Remark II.6). ■

Lemma II.8. *Let $I = [0,1]$. The integration maps*

(2.3) $$I_{\mathbb{R}} \colon \Omega^1_c(\mathbb{R};Y) = Z^1_{\mathrm{dR,c}}(\mathbb{R};Y) \to Y, \quad \beta \mapsto \int_{\mathbb{R}} \beta,$$

(2.4) $$I_I \colon \Omega^1(I,\partial I;Y) = Z^1_{\mathrm{dR}}(I,\partial I;Y) \to Y, \quad \beta \mapsto \int_I \beta,$$

and

(2.5) $$I_{\mathbb{S}^1} \colon \Omega^1(\mathbb{S}^1;Y) = Z^1_{\mathrm{dR}}(\mathbb{S}^1;Y) \to Y, \quad \beta \mapsto \int_{\mathbb{S}^1} \beta$$

induce topological isomorphisms

$$H^1_{\mathrm{dR,c}}(\mathbb{R};Y) \to Y, \quad H^1_{\mathrm{dR}}(I,\partial I;Y) \to Y \quad \text{and} \quad H^1_{\mathrm{dR}}(\mathbb{S}^1;Y) \to Y.$$

Proof. We have a continuous map $\Omega^1_c(\mathbb{R};Y) \to Y, \beta \mapsto \int_{\mathbb{R}} \beta$, and it is easy to see that this map is surjective because there exists a smooth real-valued 1-form γ with compact support and $\int_{\mathbb{R}} \gamma = 1$. Since the map $Y \to \Omega^1_c(\mathbb{R};Y), v \mapsto \gamma \cdot v$ is continuous, the integration map splits linearly. Further its kernel coincides with the space of exact forms, which proves (2.3). The other two assertions follow by similar arguments. ■

Remark II.9. (a) For each smooth map $\alpha\colon (I,\partial I)\to (M,S)$ of pairs we obtain a natural map

$$I_\alpha : \mathfrak{z}_{(M,S)}(Y)\to Y\cong \mathfrak{z}_{(I,\partial I)}(Y)$$

which is given on the equivalence class of a Y-valued 1-form β by

$$I_\alpha([\beta]) = \int_\alpha \beta := \int_I \alpha^*\beta$$

(cf. Lemma II.8). The description of $dC^\infty(M,S;Y)$ in Lemma II.3 implies that the maps $I_\alpha\colon \mathfrak{z}_{(M,S)}(Y)\to Y$ separate points.
(b) For $(M,S) = (I,\partial I)$ the set $\pi_1(I,\partial I)$ consists of 4 elements. In fact, if $f\colon I\to I$ is a continuous function with $f(\partial I)\subseteq \partial I$, then the convexity of I implies that f is homotopy equivalent to the affine interpolation of the restriction $f|_{\partial I}$, and there are precisely four different maps $\partial I\to \partial I$. ∎

Lemma II.10. *The subspace $H^1_{\rm dR}(M,S;Y)$ of $\mathfrak{z}_{(M,S)}(Y)$ coincides with those elements $[\beta]$ for which all the integrals $I_\alpha([\beta])$ only depend on the homotopy class of $\alpha\in C^\infty((I,\partial I),(M,S))$ in $\pi_1(M,S)$. In particular*
(1) *$H^1_{\rm dR}(M,S;Y)$ is a closed subspace of $\mathfrak{z}_{(M,S)}(Y)$, and*
(2) *if Γ is discrete, then*

$$Z^1_{\rm dR}(M,S;\Gamma) = \Big\{\beta\in\Omega^1(M,S;Y)\colon (\forall \alpha\in C^\infty((I,\partial I),(M,S))\ \int_\alpha \beta\in\Gamma\Big\}.$$

Proof. Fix a point $x_M\in S$. Then we have a natural inclusion $C((I,\partial I),(M,x_M))\to C((I,\partial I),(M,S))$ inducing the map $\pi_1(M,x_M)\to\pi_1(M,S)$.

Let $\beta\in\Omega^1(M,S;Y)$ and suppose first that the integrals $\int_\alpha\beta$ for $\alpha\in C^\infty((I,\partial I),(M,S))$ only depend on the homotopy class. This implies in particular that the integrals over loops in $C^\infty((I,\partial I),(M,x_M))\subseteq C^\infty_*(\mathbb{S}^1,M)$ in x_M only depend on the homotopy class. Let $q_M\colon \widetilde M\to M$ denote the universal covering manifold. That the integrals of β over loops in x_M only depend on the homotopy class implies that there exists a smooth function $f\colon\widetilde M\to Y$ with $df = q_M^*\beta$, hence in particular that $d\beta = 0$, and therefore that $[\beta]\in H^1_{\rm dR}(M,S;Y)$.

Suppose, conversely, that $[\beta]\in H^1_{\rm dR}(M,S;Y)$, i.e., that β is closed. Then integrals over continuous maps $I\to M$ are well-defined. Then $q_M^*\beta$ is exact ([Ne02a, Th. 3.6]), and there exists a smooth function $f\in C^\infty(\widetilde M;Y)$ with $df = q_M^*\beta$. It follows in particular that all integrals of β over contractible loops vanish. Let $\alpha\colon I\times I\to M$ be a continuous map such that the maps $\alpha_t := \alpha(t,\cdot)\colon I\to M$ satisfy $\alpha_t(\{0,1\})\subseteq S$. We have to show that $\int_{\alpha_0}\beta = \int_{\alpha_1}\beta$. We define

$$\widetilde\alpha\colon I\times[0,3]\to M,\quad \widetilde\alpha(t,s) := \begin{cases} \alpha(st,0) & \text{for } 0\le s\le 1\\ \alpha(t,s-1) & \text{for } 1\le s\le 2\\ \alpha((3-s)t,1) & \text{for } 2\le s\le 3\end{cases}$$

and observe that $\widetilde\alpha$ is continuous and that the curves $\widetilde\alpha_t := \widetilde\alpha(t,\cdot)$ start in $\alpha_0(0)$ and end in $\alpha_0(1)$, where $s\mapsto\widetilde\alpha_0(3s)$ is homotopic to α_0. We conclude that for each $t\in I$ we have

$$0 = \int_{\widetilde\alpha_t}\beta - \int_{\widetilde\alpha_0}\beta = \int_{\widetilde\alpha_t}\beta - \int_{\alpha_0}\beta = \int_1^2 \widetilde\alpha_t^*\beta - \int_{\alpha_0}\beta = \int_{\alpha_t}\beta - \int_{\alpha_0}\beta.$$

Here we use that the vanishing of β on S implies that the integrals $\int_0^1\widetilde\alpha_t^*\beta$ and $\int_2^3\widetilde\alpha_t^*\beta$ vanish. For $t=1$ we obtain $\int_{\alpha_0}\beta = \int_{\alpha_1}\beta$, and hence the homotopy characterization of the subspace $H^1_{\rm dR}(M,S;Y)$ of $\mathfrak{z}_{(M,S)}(Y)$.

This implies in particular that $H^1_{\rm dR}(M,S;Y)$ is closed, because it is defined as the intersection of the kernels of the continuous linear maps

$$[\beta]\mapsto \int_{\alpha_1}\beta - \int_{\alpha_0}\beta,\quad \alpha_i\in C^\infty((I,\partial I),(M,S))$$

from above.

Assume now that $\Gamma \subseteq Y$ is a discrete subgroup. Then the requirement $\int_\alpha \beta \in \Gamma$ for each map $\alpha \in C^\infty((I,\partial I),(M,S))$ together with the continuous dependence of the integral from α implies that $\int_\alpha \beta$ only depends on the homotopy class of α in $\pi_1(M,S)$. If all these integrals are contained in the discrete subgroup Γ, it follows from the first part of the proof that β is closed. ■

Example II.11. We consider the closed subset

$$S = \{0\} \cup \left\{\tfrac{1}{n} : n \in \mathbb{N}\right\} \subseteq \mathbb{R}.$$

We claim that

$$H^1_{\mathrm{dR}}(\mathbb{R}, S; \mathbb{R}) \cong E := \{(\lambda_n)_{n\in\mathbb{N}} : (\forall k \in \mathbb{N}_0)\ \lim_{n\to\infty} n^k \lambda_n = 0\}$$

as Fréchet spaces, where the topology on E is given by the seminorms $p_k(\lambda) := \sup_{n\in\mathbb{N}} n^k |\lambda_n|$ for $k \in \mathbb{N}$.

As $\dim \mathbb{R} = 1$, we have $Z^1_{\mathrm{dR}}(M,S;\mathbb{R}) = \Omega^1(\mathbb{R},S;\mathbb{R})$, and each element on this space can be written as the differential of a unique function $f \in C^\infty(\mathbb{R};\mathbb{R})$ with $f(0) = 0$. We have to study the possible restrictions $f|_S$ because they give as the values of $[df]$ on the relative 1-cycles in $Z_1(\mathbb{R},S)$.

First we derive necessary conditions. As $f^{(k)}(0) = 0$ for each $k \in \mathbb{N}$ and

$$f^{(k)}(0) = \lim_{x\to 0} \frac{k! f(x)}{x^k} = \lim_{n\to\infty} k! f(\tfrac{1}{n}) n^k, \tag{2.6}$$

we obtain for each $k \in \mathbb{N}$ the condition $\lim_{n\to\infty} f(\frac{1}{n}) n^k = 0$.

Let $(\lambda_n)_{n\in\mathbb{N}}$ satisfy $\lim_{n\to\infty} n^k \lambda_n = 0$ for each $k \in \mathbb{N}_0$. We are looking for a smooth function f in $C^\infty(\mathbb{R};\mathbb{R})$ with $f' \in C^\infty(\mathbb{R},S;\mathbb{R})$ and $f(\frac{1}{n}) = \lambda_n$ for each n. Let $\psi \in C^\infty_c(\mathbb{R};\mathbb{R})$ be a function with $\mathrm{supp}(\psi) = [-1,1]$, $\mathrm{im}(\psi) \subseteq [0,1]$ and equal to 1 on a neighborhood of 0. Then we obtain for each $a \in \mathbb{R}$ and $\varepsilon > 0$ a smooth function $\psi_{a,\varepsilon}(x) := \psi(\varepsilon^{-1}(x-a))$ supported by $[a-\varepsilon, a+\varepsilon]$ which is constant 1 in a neighborhood of a. We define $\psi_n := \psi_{\frac{1}{n}, \frac{1}{4n(n+1)}}$. Then ψ_n is a function constant 1 in a neighborhood of $\frac{1}{n}$ with support contained in

$$\left]\tfrac{1}{2}\left(\tfrac{1}{n} + \tfrac{1}{n+1}\right), \tfrac{1}{2}\left(\tfrac{1}{n} + \tfrac{1}{n-1}\right)\right[.$$

In particular the supports of the functions ψ_n are pairwise disjoint. We claim that

$$f := \sum_{n=1}^\infty \lambda_n \psi_n$$

defines a function in $C^\infty(\mathbb{R};\mathbb{R})$ with $f' \in C^\infty(\mathbb{R},S;\mathbb{R})$. This will be achieved by showing that all derivatives of the sequence defining f are uniformly convergent. In fact, for $k \in \mathbb{N}_0$ we have

$$\|\psi_n^{(k)}\|_\infty \leq \left(4n(n+1)\right)^k \|\psi^{(k)}\|_\infty \leq c_k n^{2k}$$

for some positive constant c_k. Therefore

$$\sum_n |\lambda_n| \|\psi_n^{(k)}\|_\infty \leq \sum_n |\lambda_n| c_k n^{2k} \leq c_k \sum_n |\lambda_n| n^{2k} < \infty.$$

We conclude that the series $f = \sum_n \lambda_n \psi_n$ defines a smooth function. It follows directly from the construction that f is constant λ_n in a neighborhood of $\frac{1}{n+1}$ and that all derivatives of f vanish in 0 because f vanishes on $]-\infty, 0[$.

This proves that the map

$$\Phi: Z^1_{\mathrm{dR}}(\mathbb{R},S;\mathbb{R}) \to E, \quad h(t)dt \mapsto \left(\int_0^{\frac{1}{n}} h(\tau)\, d\tau\right)_{n\in\mathbb{N}},$$

is surjective. Formula (2.6) easily implies thatg Φ is continuous, hence a quotient map by the Open Mapping Theorem. This proves that the induces map $H^1_{\mathrm{dR}}(\mathbb{R},S;\mathbb{R}) \to E$ is a topological isomorphism. ■

In the next example we take a convergent sequence out of the sphere. This aims at an example of a Fréchet–Lie group $C^\infty(M,S;K)$ where the period group $\Pi_{(M,S)}$ (cf. Definition III.7) is discrete but not finitely generated (see Proposition VII.16).

Example II.12. (Removing a convergent sequence from the sphere) Let $M := \mathbb{S}^2 \subseteq \mathbb{R}^3$ and $S = \{x_n : n \in \mathbb{N}\} \cup \{(0,0,1)\}$, where

$$x_n = \big(\tfrac{1}{n}, 0, \sqrt{1 - \tfrac{1}{n^2}}\big).$$

As $\pi_1(M)$ is trivial, there exists for each $n \in \mathbb{N}$ a path $\gamma_n : [0,1] \to M$ from $x_0 := (0,0,1)$ to x_n such that the group $H_1(M,S)$ is generated by the classes $[\gamma_n]$, $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ there exists a smooth function $f_n \in C^\infty(M,S;\mathbb{R})$ which is constant 1 in a neighborhood of x_n and vanishes in a neighborhood of $S \setminus \{x_n\}$. Then $df_n \in \Omega^1(M,S;\mathbb{R})$ and we have

$$\int_{\gamma_m} df_n = f_n(\gamma_m(1)) - f_n(\gamma_m(0)) = \delta_{mn}.$$

It follows in particular that the classes $[\gamma_n]$ are linearly independent over $\mathbb{Z}$, so that we obtain

$$H_1(M,S) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}[\gamma_n] \cong \mathbb{Z}^{(\mathbb{N})}$$

and therefore that the map

$$H^1(M,S;\mathbb{R}) \to \mathbb{R}^{\mathbb{N}}, \quad f \mapsto (f([\gamma_n]))_{n \in \mathbb{N}}$$

is bijective.

We want to determine the subgroup $H^1_{\rm dR}(M,S;\mathbb{R})$ in $H^1(M,S;\mathbb{R})$. Let $\zeta \in Z^1_{\rm dR}(M,S;\mathbb{R})$. Since $H^1_{\rm dR}(\mathbb{S}^2;\mathbb{R})$ is trivial, there exists a smooth function $f : \mathbb{S}^2 \to \mathbb{R}$ with $f(x_0) = 0$ and $df = \zeta$. Then

$$\int_{\gamma_n} \zeta = \int_{\gamma_n} df = f(x_n) - f(x_0) = f(x_n),$$

and the question is how to characterize those sequences in $\mathbb{R}^{\mathbb{N}}$ which arise as $(f(x_n))_{n \in \mathbb{N}}$ for such a function f. We obtain a natural chart around x_0 via

$$\varphi : U := \{x \in \mathbb{R}^2 : \|x\|_2 < 1\} \to \mathbb{S}^2, \quad \varphi(x) = \Big(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\Big).$$

Each of the functions constructed in Example II.11 may be extended to a smooth compactly supported function on a neighborhood of S in $\mathbb{R}^2$ in such a way that it does not depend on the second variable x_2 in a neighborhood of S. Then we may use the chart φ to obtain a function in $C^\infty(M,S;\mathbb{R})$. We thus obtain

$$H^1_{\rm dR}(M,S;\mathbb{R}) \cong \{(\lambda_n)_{n \in \mathbb{N}} : (\forall k \in \mathbb{N})\, \lambda_n n^k \to 0\} \subseteq H^1(M,S;\mathbb{R}) \cong \mathbb{R}^{\mathbb{N}},$$

i.e., that $H^1_{\rm dR}(M,S;\mathbb{R})$ corresponds to the space of rapidly decreasing sequences with its usual topology.

A function f yields an element in the group $H^1_{\rm dR}(M,S;\mathbb{Z})$ if and only if all its values in the x_n are integral, so that $H^1_{\rm dR}(M,S;\mathbb{Z}) \cong \mathbb{Z}^{(\mathbb{N})}$ corresponds to the integer-valued functions with finite support. In particular $H^1_{\rm dR}(M,S;\mathbb{Z})$ is a discrete subgroup of $H^1_{\rm dR}(M,S;\mathbb{R})$ (cf. Theorem IV.7). ∎

We conclude this section with some additional remarks on the relation between the two spaces $H^1_{\rm dR}(M,S;Y)$ and $H^1_{\rm dR,c}(M \setminus S;Y)$.

Remark II.13. We recall from Remark II.6(c) the injection

$$\varphi : H^1_{\rm dR,c}(M \setminus S;Y) \hookrightarrow H^1_{\rm dR}(M,S;Y).$$

(a) If S is a compact submanifold of M, then φ is surjective. In fact, if $\zeta \in Z^1_{\rm dR}(M,S;Y)$, then $\zeta|_S = 0$. Let U be a tubular neighborhood of S diffeomorphic to $S \times \mathbb{R}$. Then $\zeta|_U$ is exact, and there exists $f \in C^\infty(U;Y)$ with $df = \zeta|_U$. Now there exists a function $f_1 \in C^\infty(M;Y)$ which coincides with f on a neighborhood of S, and then $\zeta - df$ vanishes in a neighborhood of S. This proves that $[\zeta] = [\zeta - df_1] \in \mathrm{im}(\varphi)$.

(b) If $H^1_{\mathrm{dR,c}}(M \setminus S;\mathbb{R})$ is infinite-dimensional, then φ is not surjective. In fact, then the space $H^1_{\mathrm{dR,c}}(M \setminus S;\mathbb{R})$ is a countable direct limit of finite-dimensional spaces, hence of countable dimension (cf. Theorem IV.16). On the other hand $H^1_{\mathrm{dR}}(M,S;\mathbb{R})$ is a quotient of the Fréchet space $Z^1_{\mathrm{dR}}(M,S;\mathbb{R})$ by a closed subspace, hence a Fréchet space. As φ is injective, this space is infinite-dimensional, so that the Baire property implies that it is not countably dimensional. Hence φ is not surjective.

(c) If $H_0(S)$ is finitely generated, i.e., S has only finitely many arc-components, then the exact homology sequence of the pair (M,S) implies that $H_1(M,S)$ is finitely generated, which in turn implies that

$$H^1(M,S;\mathbb{R}) \cong \mathrm{Hom}(H_1(M,S),\mathbb{R})$$

is finite-dimensional. Therefore $H^1_{\mathrm{dR}}(M,S;\mathbb{R})$ is also finite-dimensional (cf. Definition II.5).

Conversely, every locally constant function $S \to \mathbb{Z}$ can be extended to a smooth function $f\colon M \to \mathbb{R}$ (it suffices to consider functions $S \to \{0,1\}$) which is locally constant in a neighborhood of the compact set S. Then $df \in Z^1_{\mathrm{dR,c}}(M \setminus S;\mathbb{Z})$. The class of $[df]$ in $H^1_{\mathrm{dR,c}}(M \setminus S;\mathbb{Z})$ is non-zero if $f|_S$ is not constant. Therefore $H^1_{\mathrm{dR,c}}(M \setminus S;\mathbb{Z})$ has infinite rank if $C(S,\mathbb{Z})$ has infinite rank. Note that this condition is weaker than the requirement that S has only finitely many arc-components. ■

III. Compact manifolds with boundary

Our strategy to get a better description of the spaces $\mathfrak{z}_M(Y)$ and $H^1_{\mathrm{dR,c}}(M;Y)$ for a non-compact manifold is to describe M as a union of certain compact submanifolds with boundary $(X_n)_{n\in\mathbb{N}}$ with $X_n \subseteq X^0_{n+1}$ (cf. Section IV). To get information on the space $H^1_{\mathrm{dR,c}}(M;Y)$, we will need detailed information on the natural maps $H^1_{\mathrm{dR}}(X_n,\partial X_n;Y) \to H^1_{\mathrm{dR}}(X_{n+1},\partial X_{n+1};Y)$. To obtain this information is the main goal of the present section (Theorem III.6). In this section we only deal with compact manifolds with boundary, and in Section IV we describe the approximation of non-compact manifolds.

In the following we write for a topological space X simply $H_*(X) := H_{\mathrm{sing},*}(X;\mathbb{Z})$ for the *singular homology groups* with coefficients in $\mathbb{Z}$. We likewise write $H_*(X,A)$ for the singular homology groups for space pairs (X,A).

Proposition III.1. *Let X be a compact manifold with boundary ∂X. Then the following assertions hold:*

(i) *The singular homology groups $H_*(X)$ are finitely generated.*

(ii) *All homotopy groups $\pi_k(X)$, $k \in \mathbb{N}_0$, are finitely generated.*

(iii) *For each commutative ring R the cohomology groups $H^*(X,R)$ are finitely generated R-modules.*

(iv) *The relative homology groups $H_*(X,\partial X)$ are finitely generated.*

(v) *The inclusion $\mathrm{int}(X) \hookrightarrow X$ is a homotopy equivalence.*

Proof. There exists a compact manifold X^d, the *double of* X, in which X embeds. In particular Whitney's Embedding Theorem implies that X^d and hence X embeds smoothly into $\mathbb{R}^{2d+1}$, where $d = \dim X$. From the proof of Corollary E.5 in [Br93] we derive that there exists a finite CW-complex $K \subseteq \mathbb{R}^{2n+1}$ such that K is a neighborhood of X and there exists a retraction $r\colon K \to X$. The inclusion $j\colon X \hookrightarrow K$ satisfies $r \circ j = \mathrm{id}_X$.

(i) We immediately derive that the spaces $H_*(X)$ are direct summands in $H_*(K)$, hence in particular finitely generated abelian groups.

(ii) We likewise see that for each $k \in \mathbb{N}_0$ we have $\pi_k(K) \cong \ker \pi_k(r) \rtimes \pi_k(X)$. As $\pi_k(K)$ is finitely generated, the same holds for the group $\pi_k(X) \cong \pi_k(K)/\ker \pi_k(r)$.

(iii) In view of [Fu70, Th. 52.2], we have for abelian groups A and C_j, $j \in J$:

$$\mathrm{Ext}(\oplus_{j\in J} C_j, A) = \prod_{j\in J} \mathrm{Ext}(C_j, A).$$

As $\mathrm{Ext}(\mathbb{Z}, A) \cong \mathbf{0}$ and $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$, we conclude that for every commutative ring R and every finitely generated abelian group Γ the group $\mathrm{Ext}(\Gamma, R)$ is a finitely generated R-module. Therefore the Universal Coefficient Theorem implies that for every compact manifold with boundary the groups $H^*(X, R)$ are finitely generated R-modules.
(iv) In view of [Br93, Th. IV.6.15], we further have an exact sequence

$$H_*(\partial X) \to H_*(X) \to H_*(X, \partial X) \to H_{*-1}(\partial X).$$

The fact that $H_{*-1}(\partial X)$ and $H_*(X)$ are finitely generated groups implies that the groups $H_*(X, \partial X)$ are finitely generated.
(v) Using the collar construction for a compact manifold with boundary, we obtain inclusions $\mathrm{int}(X) \hookrightarrow X \hookrightarrow \mathrm{int}(X) \hookrightarrow X$, where the compositions of two successive ones are homotopic to the identity on $\mathrm{int}(X)$, resp., X. Therefore the inclusion $\mathrm{int}(X) \hookrightarrow X$ is a homotopy equivalence.■

Lemma III.2. *For each compact manifold X with boundary the space $H^1_{\mathrm{dR}}(X, \partial X; \mathbb{R})$ is finite-dimensional.*

Proof. In Definition II.5 we have described an embedding

$$H^1_{\mathrm{dR}}(X, \partial X; \mathbb{R}) \hookrightarrow H^1(X, \partial X; \mathbb{R}).$$

Hence the assertion follows from Proposition III.1 which implies that $H^1(X, \partial X; \mathbb{R})$ is finite-dimensional. ■

We take a closer look at the embedding

$$H^1_{\mathrm{dR}}(X, \partial X; \mathbb{R}) \hookrightarrow H^1(X, \partial X; \mathbb{R}) \cong \mathrm{Hom}(H_1(X, \partial X); \mathbb{R})$$

introduced in Definition II.5. The injectivity of this embedding implies that the integration maps

$$I_\gamma : H^1_{\mathrm{dR}}(X, \partial X; \mathbb{R}) \to \mathbb{R}, \quad [\zeta] \mapsto \int_\gamma \zeta$$

for singular cycles $\gamma \in Z_1(X, \partial X)$ separate points. We are interested in a nice set of such cycles for which the integration maps form a basis of the dual space of the finite-dimensional vector space $H^1_{\mathrm{dR}}(X, \partial X; \mathbb{R})$.

We recall the part

$$H_1(\partial X) \to H_1(X) \xrightarrow{\iota} H_1(X, \partial X) \to H_0(\partial X) \xrightarrow{s} H_0(X)$$

of the long exact homology sequence of the pair $(X, \partial X)$ ([Br93, Th. IV.6.15]). Let $\iota : H_1(X) \to H_1(X, \partial X)$ be the natural map and choose piecewise smooth loops $\alpha_1, \ldots, \alpha_a$ in X for which the images $\iota([\alpha_i]) \in H_1(X, \partial X)$ form a $\mathbb{Z}$-basis of the image $\iota(H_1(X))$ modulo torsion. Let $b := \mathrm{rk}\, H_0(\partial X) - 1$ and choose a minimal system of piecewise smooth arcs $\beta_1, \ldots, \beta_b$ in $Z_1(X, \partial X)$ connecting the boundary components of ∂X. Since there are $b+1$ boundary components, b arcs suffice and less would not be enough. Then the images of the classes $[\beta_i]$ in $H_0(\partial X)$ form a $\mathbb{Z}$-basis of the kernel of the summation map $s : H_0(\partial X) \cong \mathbb{Z}^{b+1} \to H_0(X) \cong \mathbb{Z}$.

Since the classes $[\beta_j]$ form a basis of the image of $H_1(X, \partial X)$ in $H_0(\partial X)$, and the classes $\iota([\alpha_i])$ generate the kernel of the map $H_1(X, \partial X) \to H_0(\partial X)$ modulo torsion, the classes $\iota([\alpha_i])$ and $[\beta_j]$ form a $\mathbb{Z}$-basis of the abelian group $H_1(X, \partial X)$ modulo torsion.

The bijectivity of the map η in the following proposition (see also (2.1)) can alternatively be derived from the discussion in Remark II.6(c), which implies that the real vector spaces $H^1_{\mathrm{dR}}(X, \partial X; \mathbb{R})$ and $\mathrm{Hom}(H_1(X, \partial X); \mathbb{R})$ have the same dimension, so that the injectivity of η implies that it is bijective. We will see that Proposition III.3 provides more concrete information which is needed later on.

Proposition III.3. *The integration functionals I_{α_i}, $i = 1,\ldots,a$ and I_{β_j}, $j = 1,\ldots,b$ form a basis of the dual space of $H^1_{\rm dR}(X,\partial X;\mathbb{R})$. In particular, the natural homomorphism*

$$\eta\colon H^1_{\rm dR}(X,\partial X;\mathbb{R}) \to \mathrm{Hom}(H_1(X,\partial X);\mathbb{R}), \quad \eta([\zeta])([\gamma]) = \int_\gamma \zeta$$

from Definition II.5(2.1) *is bijective.*

Proof. Since the classes $\iota([\alpha_i])$ and $[\beta_j]$ generate $H_1(X,\partial X)$ modulo torsion and η is injective (Definition II.5), the integration maps I_{α_i} and I_{β_j} separate the points of $H^1_{\rm dR}(X,\partial X;\mathbb{R})$, hence span its dual space.

Let $\chi_0\colon H_1(X,\partial X) \to \mathbb{R}$ be a homomorphism and $\chi\colon H_1(X) \to \mathbb{R}$ its pull-back to $H_1(X)$. Then χ vanishes on the image of $H_1(\partial X)$ in $H_1(X)$, so that there exists a closed 1-form α on X with

$$\int_\gamma \alpha = \chi(\gamma), \quad \gamma \in H_1(X).$$

This can be proved as [Ne02a, Prop. 3.8]. The main idea is to associate to χ, viewed as a homomorphism $\pi_1(X) \to \mathbb{R}$, an affine $\mathbb{R}$-bundle over X and then to use partitions of unity to obtain a smooth global section s, whose differential can be taken as α. Since χ vanishes on the image of $H_1(\partial X)$ in $H_1(X)$, we can think of it as a homomorphism of the image $\iota(H_1(X))$ of $H_1(X)$ in $H_1(X,\partial X)$ to $\mathbb{R}$.

Let C be a connected component of ∂X, $I := [0,1]$ and $\widehat{C}$ be a neighborhood of C in X diffeomorphic to $I \times C$ in such a way that $\{0\} \times C$ corresponds to C. Then the homomorphism $H_1(C) \to H_1(\partial X) \to \mathbb{R}$ induced by the 1-form α vanishes, so that there exists a smooth function $g_0\colon \widehat{C} \to \mathbb{R}$ with $\alpha|_{\widehat{C}} = dg$. If $\varphi\colon I \to \mathbb{R}$ is smooth with $\varphi = 1$ in a neighborhood of 0 and 0 in a neighborhood of 1, then $\widehat{\varphi}\colon (t,x) \mapsto \varphi(t)$ yields a smooth function on X vanishing in a neighborhood of $X \setminus \widehat{C}$ and taking the value 1 on a neighborhood of C. Hence $\widehat{\varphi}\cdot g$ can be viewed as a smooth function $X \to \mathbb{R}$ whose differential coincides with dg in a neighborhood of C. Now $\alpha - d(\widehat{\varphi}\cdot g_0)$ defines the same homomorphism $\pi_1(X) \to \mathbb{R}$ but, in addition, this 1-form vanishes in a neighborhood of C. Repeating this construction for the other connected components of ∂X yields a closed 1-form $\alpha' \in \Omega^1(X,\partial X;\mathbb{R})$ vanishing in a neighborhood of ∂X for which α' represents χ on $H_1(X)$. We conclude that $\chi_0 - \eta([\alpha'])$ vanishes on $\iota(H_1(X))$ in $H_1(X,\partial X)$, so that it remains to see that each homomorphisms $\chi\colon H_1(X,\partial X) \to \mathbb{R}$ vanishing on the image of $H_1(X)$ is contained in $\mathrm{im}(\eta)$. Let $r\colon H_1(X,\partial X) \to H_0(X)$ denote the boundary map. Then $\chi_0 = \chi' \circ r$ for some $\chi'\colon H_0(\partial X) \cong \mathbb{Z}^{b+1} \to \mathbb{R}$.

Let $C \subseteq \partial X$ be a connected component. Using the collar construction, we obtain a smooth function $f_C\colon X \to \mathbb{R}$ which is 1 in a neighborhood of C and 0 in a neighborhood of all other connected components of ∂X. Then $df_C \in Z^1_{\rm dR}(X,\partial X;\mathbb{R})$ and because the form df_C is exact, it vanishes on all cycles in $\iota(H_1(X))$. Moreover, the function f_C defines a homomorphism

$$F_C\colon H_0(\partial X) \to \mathbb{Z}, \quad C' \mapsto f_C(C') = \delta_{C,C'},$$

and, as a homomorphism $H_1(X,\partial X) \to \mathbb{R}$, the integration of df_C over cycles modulo ∂X is obtained by pulling F_C back via the natural map $H_1(X,\partial X) \to H_0(\partial X)$. As the F_C form a $\mathbb{Z}$-basis of $\mathrm{Hom}(H_0(\partial X),\mathbb{R})$, we conclude that χ' lies in the span of the $\eta([df_C])$, hence is contained in the image of η. This completes the proof of the surjectivity of η. ∎

Lemma III.4. *For any s.c.l.c. space Y the exactness of a closed 1-form $\zeta \in \Omega^1(X,\partial X;Y)$ is equivalent to the vanishing of all integrals $\int_{\alpha_i}\zeta$ and $\int_{\beta_j}\zeta$.*

Proof. If $\zeta \in \Omega^1(X,\partial X;Y)$ is exact, then clearly all integrals $\int_\gamma \zeta$ vanish for $\gamma \in Z_1(X,\partial X)$. Suppose, conversely, that all integrals $\int_{\alpha_i}\zeta$ and $\int_{\beta_j}\zeta$ vanish. For each continuous linear functional $\lambda \in Y'$ we then obtain

$$\int_{\alpha_i} \lambda\circ\zeta = \lambda\Big(\int_{\alpha_i}\zeta\Big) = \lambda\Big(\int_{\beta_j}\zeta\Big) = \int_{\beta_j}\lambda\circ\zeta = 0$$

for each i and j. Since Y' separates points of Y, all integrals of ζ on $Z_1(X,\partial X)$ are trivial, and therefore ζ is exact. ∎

Remark III.5. Let $[\alpha_i^*], [\beta_i^*] \in H^1_{\rm dR}(X, \partial X; \mathbb{R})$ be a basis dual to the integrals I_{α_i} and I_{β_j} from above. Then the map

$$\Phi_X \colon H^1_{\rm dR}(X, \partial X; Y) \to Y^{a+b}, \quad \Phi_X([\zeta]) := \Big(\int_{\alpha_i} \zeta, \int_{\beta_j} \zeta\Big)_{i=1,\ldots,a; j=1,\ldots,b}$$

is continuous and injective (Lemma III.4). Moreover, it is surjective and its inverse is given by

$$\Phi_X^{-1}(y_1, \ldots, y_{a+b}) := \sum_{i=1}^a [\alpha_i^* \cdot y_i] + \sum_{j=1}^b [\beta_j^* \cdot y_{a+j}].$$

It follows in particular that Φ_X^{-1} is continuous, and therefore that Φ_X is an isomorphism of topological vector spaces. The extension of Φ_X to a map

$$\widetilde{\Phi}_X \colon \mathfrak{z}_{(X,\partial X)}(Y) \to Y^{a+b}, \quad \Phi_X([\zeta]) := \Big(\int_{\alpha_i} \zeta, \int_{\beta_j} \zeta\Big)_{i=1,\ldots,a; j=1,\ldots,b}$$

is continuous and surjective. Therefore its kernel is a closed complement to $H^1_{\rm dR}(X, \partial X; Y)$ and the corresponding projection onto $H^1_{\rm dR}(X, \partial X; Y)$ is given by

$$p_X \colon [\zeta] \mapsto \sum_{i=1}^a \Big[\alpha_i^* \cdot \int_{\alpha_i} \zeta\Big] + \sum_{j=1}^b \Big[\beta_j^* \cdot \int_{\beta_j} \zeta\Big]. \qquad \blacksquare$$

Theorem III.6. *Let Z be a compact connected manifold with boundary and $X \subseteq \mathrm{int}(Z)$ a compact connected equidimensional submanifold with boundary. We assume that each connected component of $Z \setminus X$ intersects ∂Z. Then the following assertions hold:*

(1) *The inclusion $Z^1_{\rm dR}(X, \partial X; Y) \hookrightarrow Z^1_{\rm dR}(Z, \partial Z; Y)$ obtained by extension by 0 on $Z \setminus X$ induces an injective map*
$$H^1_{\rm dR}(X, \partial X; Y) \hookrightarrow H^1_{\rm dR}(Z, \partial Z; Y).$$

(2) *The continuous projection p_X extends to a continuous projection p_Z, so that we obtain the commutative diagram*
$$\begin{array}{ccc} \mathfrak{z}_{(X,\partial X)}(Y) & \xrightarrow{p_X} & H^1_{\rm dR}(X, \partial X; Y) \\ \downarrow & & \downarrow \\ \mathfrak{z}_{(Z,\partial Z)}(Y) & \xrightarrow{p_Z} & H^1_{\rm dR}(Z, \partial Z; Y). \end{array}$$

Proof. Let α_i, $i = 1, \ldots, a$ and β_j, $j = 1, \ldots, b$ be as in Proposition III.3. Then the integration functionals $I_{\alpha_1}, \ldots, I_{\alpha_a}, I_{\beta_1}, \ldots, I_{\beta_b}$ form a basis of the dual space of $H^1_{\rm dR}(X, \partial X; \mathbb{R})$.
(1) We claim that

$$dC^\infty(Z, \partial Z; Y) \cap Z^1_{\rm dR}(X, \partial X; Y) = dC^\infty(X, \partial X; Y).$$

The inclusion "$\supseteq$" is trivial. Conversely, let $f \in C^\infty(Z, \partial Z; Y)$ and suppose that $df \in Z^1_{\rm dR}(X, \partial X; Y)$, i.e., that df vanishes on $Z \setminus X$. Then f is constant on all connected components of $Z \setminus X$. By our initial assumptions, all connected components of $Z \setminus X$ intersect ∂Z, which implies that f vanishes on all these components, hence that $f \in C^\infty(X, \partial X; Y)$. This proves (1).
(2) Next we want to choose integration maps $H^1_{\rm dR}(Z, \partial Z; Y) \to Y$ in such a way that those which are additional to the ones needed for X are supported by $Z \setminus \mathrm{int}(X)$, hence vanish on $Z^1_{\rm dR}(X, \partial X; Y)$.

We have to modify the curves β_i so that they represent elements on $Z_1(Z, \partial Z)$. Since every connected component of $Z \setminus X$ meets ∂Z, we can extend every piecewise smooth curve β_i to a piecewise smooth curve $\widetilde{\beta}_i$ connecting two boundary components of Z. For this we may w.l.o.g.

assume that we have parametrizations $\beta_j\colon [0,1] \to X$ and $\widetilde{\beta}_j\colon [-1,2] \to Z$ with $\widetilde{\beta}_j\,|_{[0,1]} = \beta_j$ and $[0,1] = \widetilde{\beta}_j^{-1}(X)$. In particular we have for each 1-form ζ supported by X the relation

$$\int_{\beta_i} \zeta = \int_{\widetilde{\beta}_i} \zeta.$$

Next we choose piecewise smooth closed curves $\gamma_1, \ldots, \gamma_c$ in $Z \setminus X$ connecting those connected components of ∂Z lying in the same connected component of $Z \setminus X$. We further need closed curves on $\delta_1, \ldots, \delta_d$ in $Z \setminus \mathrm{int}(X)$ whose homology classes generate $H_1(Z \setminus \mathrm{int}(X); \mathbb{R})$ modulo the image of $H_1(\partial Z; \mathbb{R})$. We will show below that the classes of $\alpha_i, \widetilde{\beta}_j, \gamma_k$ and δ_l generate $H_1(Z, \partial Z)$ modulo torsion by showing that the corresponding integrals separate points on $H^1_{\mathrm{dR}}(Z, \partial Z; \mathbb{R})$.

Let $\zeta \in Z^1_{\mathrm{dR}}(Z, \partial Z; Y)$ be such that all integrals over the $\alpha_i, \widetilde{\beta}_j, \gamma_k$ and δ_l vanish. We claim that ζ is exact. In particular all integrals coming from $H_1(\partial X)$ vanish, so that there exists an open neighborhood $U \cong I \times \partial X$ of ∂X on which ζ is exact. Let $f \in C^\infty(U; Y)$ with $df = \zeta|_U$. Multiplying f with a smooth function χ on U of the form $(t,x) \mapsto \varphi(t)$, where $\varphi \in C^\infty(I; \mathbb{R})$ is 1 on a neighborhood of 0 and vanishes outside some interval $[-\varepsilon, \varepsilon]$, we obtain a smooth function $\widetilde{f} := \chi \cdot f \in C^\infty(Z, \partial Z; Y)$ with $d\widetilde{f} = \zeta$ in a neighborhood of ∂X. Replacing ζ by $\zeta - d\widetilde{f}$, we may assume that ζ vanishes on a neighborhood of ∂X. Then $\zeta|_X \in Z^1_{\mathrm{dR}}(X, \partial X; Y)$ is exact because the integrals over the α_i vanish. Likewise $\zeta|_{Z\setminus X}$ is exact because all integrals over the δ_i vanish. Let $f_1 \in C^\infty(X; Y)$ with $df_1 = \zeta\,|_X$ and $f_2 \in C^\infty(Z \setminus \mathrm{int}(X); Y)$ with $df_2 = \zeta\,|_{Z\setminus X}$. We normalize f_2 by the condition that it vanishes on ∂Z. That this is possible follows from the vanishing of all integrals of ζ over the γ_i. We further normalize f_1 such that on one boundary point $x \in \partial X$ we have $f_1(x) = f_2(x)$. In a neighborhood of ∂X both functions f_1 and f_2 are locally constant, hence constant on all connected components of ∂X. It remains to show that $f_1\,|_{\partial X} = f_2\,|_{\partial X}$, so that both combine to a function $f \in C^\infty(Z, \partial Z; Y)$ with $df = \zeta$.

Let β_i be such that either its end or starting point lies in the same connected component of ∂X as x. We recall the parametrizations $\beta_i\colon [0,1] \to X$ from above. We further observe that $f_1(x) = f_1(\beta_i(0)) = f_2(x) = f_2(\beta_i(0))$ because $f_1 = f_2$ is constant on the whole component of X containing x. We also recall the parameterization of $\widetilde{\beta}_j$ on $[-1,2]$ from above and put $y := \beta_i(1) \in \partial X$. Let $p := \widetilde{\beta}_i(-1)$ and $q := \widetilde{\beta}_i(2)$. Then

$$\begin{aligned} f_1(y) - f_2(y) &= \Big(f_1(x) + \int_{\beta_i} \zeta\Big) + \underbrace{f_2(q)}_{=0} - f_2(y) = f_2(x) + \int_{\beta_i} \zeta + f_2(q) - f_2(y) \\ &= \int_{-1}^0 \widetilde{\beta}_i^* \zeta + \int_0^1 \widetilde{\beta}_i^* \zeta + \int_1^2 \widetilde{\beta}_i^* \zeta = \int_{\widetilde{\beta}_i} \zeta = 0. \end{aligned}$$

This proves $f_1(y) = f_2(y)$. Using the other paths $\widetilde{\beta}_i$, we conclude inductively that $f_1 = f_2$ holds on all connected components of ∂X, and this completes the proof of the exactness of ζ.

Therefore the integration maps $I_{\alpha_i}, I_{\widetilde{\beta}_j}, I_{\gamma_k}$ and I_{δ_l} separate points on $H^1_{\mathrm{dR}}(Z, \partial Z; \mathbb{R})$. Since the maps I_{α_i}, $i = 1, \ldots, a$, and $I_{\widetilde{\beta}_j}$, $j = 1, \ldots, b$, are linearly independent on the subspace $H^1_{\mathrm{dR}}(X, \partial X; \mathbb{R})$, by omitting some of the γ_k and δ_l, we may w.l.o.g. assume that the whole collection is linearly independent.

We recall the maps Φ_X and p_X from Remark III.5. Then we see that

$$\Phi_Z\colon H^1_{\mathrm{dR}}(Z, \partial Z; Y) \to Y^{a+b+c+d},$$

$$\Phi_Z([\zeta]) := \Big(\int_{\alpha_i} \zeta, \int_{\beta_j} \zeta, \int_{\gamma_k} \zeta, \int_{\delta_l} \zeta\Big)_{i=1,\ldots,a;j=1,\ldots,b;k=1,\ldots,c;l=1,\ldots,d}$$

is a topological isomorphism. The corresponding projection $p_Z\colon \mathfrak{z}_{(Z,\partial Z)}(Y) \to H^1_{\mathrm{dR}}(Z, \partial Z; Y)$ is given by

$$p_Z\colon [\zeta] \mapsto \sum_{i=1}^a \Big[\alpha_i^* \cdot \int_{\alpha_i} \zeta\Big] + \sum_{j=1}^b \Big[\beta_j^* \cdot \int_{\widetilde{\beta}_j} \zeta\Big] + \sum_{k=1}^c \Big[\gamma_k^* \cdot \int_{\gamma_k} \zeta\Big] + \sum_{l=1}^d \Big[\delta_l^* \cdot \int_{\delta_l} \zeta\Big].$$

Since the integrals over the γ_k and δ_l vanish for $\zeta \in \Omega^1(X, \partial X; Y)$, and the integrals over β_j and $\widetilde{\beta}_j$ are the same for these 1-forms, we obtain $p_Z\,|_{\mathfrak{z}(X,\partial X)(Y)} = p_X$. ∎

Example III.7. (Oriented surfaces) Let X be an oriented compact connected surface with boundary. All the boundary components are diffeomorphic to the circle. Collapsing each boundary component to a point leads to an oriented compact surface Σ. Let $g := g(X) := g(\Sigma)$ denote the *genus* of Σ and $p := p(X)$ be the number of boundary components.

We recall the part

$$\ldots \to H_2(X) \to H_2(X, \partial X) \to H_1(\partial X) \xrightarrow{\alpha} H_1(X) \to H_1(X, \partial X) \to H_0(\partial X) \to H_0(X)$$

of the long exact homology sequence of the pair $(X, \partial X)$. Then $H_0(\partial X) \cong H_1(\partial X) \cong \mathbb{Z}^p$. According to Proposition III.1(v), the inclusion $\mathrm{int}(X) \hookrightarrow X$ is a homotopy equivalence, so that $H_1(X) \cong H_1(\mathrm{int}(X))$. On the other hand $\mathrm{int}(X) \cong \Sigma \setminus P$, where P is the image of ∂X in Σ.

Let $\widehat{P}$ be a disjoint union of open discs in Σ around each point of P. Then $\Sigma = \mathrm{int}(X) \cup \widehat{P}$ is a union of two open subsets, and the exact Mayer–Vietoris Sequence ([Br93, Th. IV.18.1]) yields an exact sequence

$$\begin{aligned}\ldots \to H_2(\mathrm{int}(X)) \oplus H_2(\widehat{P}) \to H_2(\Sigma) \to H_1(\mathrm{int}(X) \cap \widehat{P}) \to H_1(\mathrm{int}(X)) \oplus H_1(\widehat{P}) \to H_1(\Sigma)\\ \to H_0(\mathrm{int}(X) \cap \widehat{P}) \to H_0(\mathrm{int}(X)) \oplus H_0(\widehat{P}) \to H_0(\Sigma).\end{aligned}$$

We have $H_0(\widehat{P}) \cong \mathbb{Z}^P$, $H_1(\widehat{P}) = H_2(\widehat{P}) = \mathbf{0}$, $H_0(\mathrm{int}(X)) \cong \mathbb{Z}$, $H_0(\mathrm{int}(X) \cap \widehat{P}) \cong \mathbb{Z}^P$, $H_1(\mathrm{int}(X) \cap \widehat{P}) \cong \mathbb{Z}^P$, and $H_2(\mathrm{int}(X)) = \mathbf{0}$ because $\mathrm{int}(X)$ is not compact. Therefore we obtain an exact sequence

$$H_2(\Sigma) \cong \mathbb{Z} \hookrightarrow \mathbb{Z}^P \to H_1(\mathrm{int}(X)) \to H_1(\Sigma) \cong \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z}^P \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}^P \to \mathbb{Z}.$$

The vanishing of the homomorphism in the middle follows from the injectivity of the map $H_0(\mathrm{int}(X) \cap \widehat{P}) \to H_0(\widehat{P})$. This implies that the sequence

$$\mathbb{Z} \hookrightarrow \mathbb{Z}^P \to H_1(\mathrm{int}(X)) \to \mathbb{Z}^{2g} \to \mathbf{0}$$

is exact. As $\pi_1(\mathrm{int}(X))$ is a free group [tD00, Satz II.8.8], the homology group $H_1(\mathrm{int}(X)) \cong \pi_1(\mathrm{int}(X))/[\pi_1(\mathrm{int}(X)), \pi_1(\mathrm{int}(X))]$ is a free abelian group, which leads to

$$H_1(X) \cong H_1(\mathrm{int}(X)) \cong \mathbb{Z}^{2g(X)+p(X)-1}.$$

Now we obtain with $H_2(X) \cong H_2(\mathrm{int}(X)) = \mathbf{0}$ for $H_1(X, \partial X)$ the exact sequence

$$H_2(X, \partial X) \hookrightarrow H_1(\partial X) \cong \mathbb{Z}^p \xrightarrow{\alpha} H_1(X) \cong \mathbb{Z}^{2g+p-1} \to H_1(X, \partial X) \to \mathbb{Z}^p \to \mathbb{Z}.$$

The image of α in $H_1(X)$ corresponds to the image of $H_1(\mathrm{int}(X) \cap \widehat{P})$ in $H_1(\mathrm{int}(X))$ in the exact Mayer–Vietoris Sequence, and is isomorphic to $\mathbb{Z}^{p-1}$. The cokernel of α is isomorphic to $\mathbb{Z}^{2g}$. The map $H_0(\partial X) \cong \mathbb{Z}^p \to H_0(X) \cong \mathbb{Z}$ is the summation map, so that its kernel is isomorphic to $\mathbb{Z}^{p-1}$. We thus obtain a short exact sequence

$$\mathrm{coker}(\alpha) \cong \mathbb{Z}^{2g} \hookrightarrow H_1(X, \partial X) \twoheadrightarrow \mathbb{Z}^{p-1},$$

and finally

$$H_1(X, \partial X) \cong \mathbb{Z}^{2g(X)+p(X)-1}. \qquad \blacksquare$$

Example III.8. (Non-orientable surfaces) Let X be a non-orientable compact connected surface with boundary and proceed as in Example III.7. Then Σ is non-orientable. We define $g(X)$ and $p(X)$ as in Example III.8.

For the finite subset $P \subseteq \Sigma$ we now obtain with the exact Mayer-Vietoris sequence:

$$\begin{aligned}\ldots \to H_2(\Sigma) = \mathbf{0} \to H_1(\mathrm{int}(X) \cap \widehat{P}) \cong \mathbb{Z}^p \to H_1(\mathrm{int}(X)) \oplus H_1(\widehat{P}) \to H_1(\Sigma) \cong \mathbb{Z}^g \oplus \mathbb{Z}_2\\ \to H_0(\mathrm{int}(X) \cap \widehat{P}) \cong \mathbb{Z}^p \to H_0(\mathrm{int}(X)) \oplus H_0(\widehat{P}) \cong \mathbb{Z}^{p+1} \to H_0(\Sigma) \cong \mathbb{Z}.\end{aligned}$$

This leads to an exact sequence

$$\mathbb{Z}^p \hookrightarrow H_1(\mathrm{int}(X)) \to H_1(\Sigma) \cong \mathbb{Z}^g \oplus \mathbb{Z}_2 \to \mathbb{Z}^p \hookrightarrow \mathbb{Z}^{p+1},$$

and further to

$$\mathbb{Z}^p \hookrightarrow H_1(\mathrm{int}(X)) \twoheadrightarrow \mathbb{Z}^g \oplus \mathbb{Z}_2.$$

As $H_1(\mathrm{int}(X))$ is a free abelian group, it follows that

$$H_1(X) \cong H_1(\mathrm{int}(X)) \cong \mathbb{Z}^{g(X)+p(X)}.$$

Now we obtain with the long exact homology sequence of the pair $(X, \partial X)$:

$$\ldots \to H_2(X) \to H_2(X, \partial X) \to H_1(\partial X) \xrightarrow{\alpha} H_1(X) \to H_1(X, \partial X) \to H_0(\partial X) \to H_0(X)$$

and hence

$$\mathbb{Z}^p \xrightarrow{\alpha} \mathbb{Z}^{g+p} \to H_1(X, \partial X) \to \mathbb{Z}^p \xrightarrow{s} \mathbb{Z}.$$

The image of α in $H_1(X)$ corresponds to the image of $H_1(\mathrm{int}(X) \cap P)$ in $H_1(\mathrm{int}(X))$, hence is isomorphic to $\mathbb{Z}^p$, and $\mathrm{coker}(\alpha) \cong \mathbb{Z}^g$. Here $s\colon H_0(\partial X) \cong \mathbb{Z}^p \to H_0(X) \cong \mathbb{Z}$ is the summation map, so that its kernel is isomorphic to $\mathbb{Z}^{p-1}$. We thus obtain a short exact sequence

$$\mathrm{coker}(\alpha) \cong \mathbb{Z}^g \hookrightarrow H_1(X, \partial X) \twoheadrightarrow \mathbb{Z}^{p-1} = \ker s,$$

which leads to

$$H_1(X, \partial X) \cong \mathbb{Z}^{g(X)+p(X)-1}.$$ ■

IV. Approximating non-compact manifolds by compact ones

In this section M denotes a connected σ-compact finite-dimensional manifold. We call a submanifold X of M *equidimensional* if $\dim X = \dim M$. In this section we first prove the existence of well behaved sequences $(X_n)_{n\in\mathbb{N}}$ of equidimensional compact submanifolds with boundary exhausting M (Lemma IV.4). The main result of this section is Theorem IV.16 providing a topological isomorphism

$$\Phi_M\colon H^1_{\mathrm{dR,c}}(M; Y) \to Y^{(B)}$$

for a certain set B which might be infinite. The components of Φ_M are given by integration over singular cycles in M or over curves obtained from proper maps $\mathbb{R} \to M$. Here we make heavy use of Theorem III.6 about the cohomology of compact manifolds with boundary to construct the set B in such a way that Φ_M becomes an isomorphism. As a corollary, we show that if Γ is discrete, then $H^1_{\mathrm{dR,c}}(M;\Gamma) \cong \Gamma^{(B)}$ is discrete.

Saturated exhaustive sequences

Lemma IV.1. *For each compact equidimensional submanifold $X \subseteq M$ with boundary the number of connected components of $M \setminus X$ is finite.*

Proof. As every connected component of $M \setminus X$ contains some component of ∂X in its closure, and the number of components of the compact manifold ∂X is finite, the assertion follows. ■

Definition IV.2. Let $X \subseteq M$ be an equidimensional compact submanifold with boundary. We observe that each connected component of ∂X is contained in the closure of exactly one connected component of $M \setminus X$. We write $\widehat{X}$ for the union of X with all those components of $M \setminus X$ which are relatively compact. As the number of these components is finite (Lemma IV.1), $\widehat{X}$ is compact, because for each component $C \subseteq M \setminus X$ the boundary ∂C is a union of connected components of ∂X. This argument further shows that $\widehat{X}$ is a compact submanifold with boundary in M. ■

Lemma IV.3. *For two equidimensional submanifolds with boundary $X_1, X_2 \subseteq M$ with $X_1 \subseteq X_2^0$ we have $\widehat{X}_1 \subseteq \widehat{X}_2^0$.*

Proof. Let $C \subseteq M \setminus X_1$ be a relatively compact connected component. Then $C \setminus X_2$ is also relatively compact in M, hence contained in $\widehat{X}_2$. Therefore $\widehat{X}_1 \subseteq \widehat{X}_2$. If $p \in \partial \widehat{X}_2$ is a boundary point, then it is in particular a boundary point of X_2, hence not contained in X_1, and therefore not in ∂X_1. If the connected component of $M \setminus X_2^0$ containing p is non-compact, then this is likewise true for the connected component of $M \setminus X_1$ containing p, which shows that it is not contained in $\widehat{X}_1$. This proves $\widehat{X}_1 \subseteq \widehat{X}_2^0$. ■

For the case of surfaces the following lemma can also be found in [tD00, Satz 7.3].

Lemma IV.4. *There exists a sequence X_n of compact connected manifolds with boundary in M such that*
(E1) $X_n \subseteq X_{n+1}^0$,
(E2) $\bigcup_n X_n = M$,
(E3) $\widehat{X}_n = X_n$, *i.e., each connected component of $M \setminus X_n$ is not relatively compact in M.*

Proof. Let $\varphi \colon M \to \mathbb{R}$ be a proper smooth function which is bounded from below. Such a function can be obtained from an embedding $\iota \colon M \hookrightarrow \mathbb{R}^n$ as $\varphi(x) := \|x\|_2^2$. Then Sard's Theorem implies that there exists an increasing sequence $(r_n)_{n\in\mathbb{N}}$ of regular values of φ with $r_n \to \infty$. Then each $Y_n := \{x \in M \colon \varphi(x) \leq r_n\}$ is a compact equidimensional submanifold with boundary. Pick $x_0 \in Y_1$. We define Z_n to be the connected component of Y_n containing x_0 and $X_n := \widehat{Z}_n$. From $r_n < r_{n+1}$ we derive $Y_n \subseteq Y_{n+1}^0$, so that $Z_n \subseteq Z_{n+1}^0$, and Lemma IV.3 implies (E1). From $r_n \to \infty$ we get $\bigcup_n Y_n = M$. Each $x \in M$ can be connected to x_0 by an arc, which lies in some Y_n, whence $x \in Z_n$, and (E2) follows. Eventually (E3) follows from the definition of $\widehat{Z}_n$. ■

We call a sequence $(X_n)_{n\in\mathbb{N}}$ as in Lemma IV.4 a *saturated exhaustive sequence* of M.

Lemma IV.5. *For each $x \in M$ there exists a proper smooth map $\gamma \colon \mathbb{R}^+ := [0, \infty[\to M$ with $\gamma(0) = x$. If $X = \widehat{X}$ is an equidimensional compact submanifold with boundary and $x \in \partial X$, then there exists a γ as above with $\gamma(]0, \infty[) \subseteq M \setminus X$.*

Proof. Pick a saturated exhaustive sequence $(X_n)_{n\in\mathbb{N}}$ of M and choose points $x_n \in \partial X_n$ such that x_{n+1} lies in the connected component of $M \setminus X_n$ containing x_n in its boundary. Since this component is not relatively compact in M, it intersects ∂X_{n+1}. Then there exists a smooth curve $\gamma \colon \mathbb{R}^+ \to M$ with $\gamma(0) = x$, $\gamma(n) = x_n$ for all $n \in \mathbb{N}$, and $\gamma([n, n+1]) \subseteq X_{n+1} \setminus X_n^0$. The latter condition implies that γ is proper.

If $x \in \partial X$ holds for an equidimensional compact submanifold with boundary X, then $X \subseteq X_N$ for N sufficiently large, and we can proceed as above by connecting first x in $X_N \setminus X_1$ to a point in the boundary of X_N, then to a point in X_{N+1} etc. We thus obtain γ with the required properties. ■

Lemma IV.6. *For $x, y \in M$ there exists a proper smooth map $\gamma \colon \mathbb{R} \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

Proof. Using Lemma IV.5, we find a smooth map $\gamma \colon \mathbb{R} \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ such that the restrictions to $[1, \infty[$ and $]-\infty, 0]$ are proper. This implies that γ itself is proper. ■

The following lemma is obvious.

Lemma IV.7. *Let M be a topological space and $(M_j)_{j\in J}$ a directed family of open subsets of M with $M = \bigcup_j M_j$. Then $M = \varinjlim M_j$ holds in the category of topological spaces, each compact subset of M is contained in some M_j, and for each $x_M \in M$ and $k \in \mathbb{N}_0$ we have*

$$\pi_k(M, x_M) \cong \varinjlim \pi_k(M_j, x_M),$$

where $\{j \in J : x_M \in M_j\}$ is cofinal in J. ■

Remark IV.8. The preceding lemma applies in particular to saturated exhaustions $(X_n)_{n\in\mathbb{N}}$ of a non-compact manifold M with $M_n = X_n^0$. Then we obtain with Proposition III.1(v):

$$\pi_k(M) \cong \varinjlim \pi_k(X_n^0) \cong \varinjlim \pi_k(X_n)$$ ■

Proposition IV.9. *For each σ-compact connected finite-dimensional manifold M all homotopy groups are countable.*

Proof. This is a direct consequence of Lemma IV.7, Remark IV.8 and Proposition III.1(ii). ■

De Rham cohomology with compact supports is a direct sum

If Y is a s.c.l.c. space and $(X_n)_{n\in\mathbb{N}}$ is a saturated exhaustive sequence of M, then $\Omega^1_c(M;Y)$ carries the locally convex direct limit topology of the spaces $\Omega^1_{X_n}(M;Y) \subseteq \Omega^1(M;Y)$ (cf. Section II). The differential $d: C^\infty_c(M;Y) \to \Omega^1_c(M;Y)$ is a continuous linear map because $C^\infty_c(M;Y)$ carries the locally convex direct limit topology of the subspaces $C^\infty_{X_n}(M;Y)$ on which d is continuous.

Lemma IV.10. *Let $X = \widehat{X}$ be an equidimensional compact submanifold with boundary. Then $\Omega^1_X(M;Y) \cong \Omega^1(X, \partial X; Y)$ and*

$$\Omega^1_X(M;Y) \cap dC^\infty_c(M;Y) = dC^\infty_X(M;Y).$$

Proof. (cf. Step 1 in the proof of Theorem III.6) It is clear that $dC^\infty_X(M;Y)$ is contained in $\Omega^1_X(M;Y) \cap dC^\infty_c(M;Y)$. To prove the converse inclusion, let $\beta \in \Omega^1_X(M;Y)$ and $f \in C^\infty_c(M;Y)$ with $\beta = df$. Then f is constant on all connected components of $M \setminus X$. Since all these components are not relatively compact in M and f has compact support, it follows that $f(M \setminus X) = \{0\}$, and therefore $f \in C^\infty_X(M;Y)$. ■

From the isomorphisms

$$\Omega^1_X(M;Y) \cong \Omega^1(X, \partial X; Y) \quad \text{and} \quad C^\infty_X(M;Y) \cong C^\infty(X, \partial X; Y)$$

obtained by extension on $M \setminus X$ by 0, we now derive

$$\Omega^1_X(M;Y)/\big(dC^\infty_c(M;Y) \cap \Omega^1_X(M;Y)\big) \cong \Omega^1(X, \partial X; Y)/dC^\infty(X, \partial X; Y) = \mathfrak{z}_{(X,\partial X)}(Y).$$

Lemma IV.11. *For each s.c.l.c. space Y the subspace $B^1_{\mathrm{dR,c}}(M;Y) = dC^\infty_c(M;Y)$ of $\Omega^1_c(M;Y)$ is closed.*

Proof. For each equidimensional compact submanifold $X = \widehat{X}$ with boundary, Lemma IV.10 implies that $\Omega^1_X(M;Y) \cap dC^\infty_c(M;Y) = dC^\infty_X(M;Y)$, which corresponds to the subspace

$$dC^\infty(X, \partial X; Y) \subseteq \Omega^1(X, \partial X; Y)$$

whose closedness follows from Corollary II.4 which also applies to the pair $(X, \partial X)$, as it has the same space of smooth functions as the pair $(X^d, X^\sharp)$ (cf. Example I.9(a)).

For each saturated exhaustive sequence $(X_n)_{n\in\mathbb{N}}$, the space $\Omega^1_c(M;Y)$ is the locally convex direct limit of the subspaces $\Omega^1_{X_n}(M;Y)$, so that the closedness of $dC^\infty_c(M;Y)$ follows from the closedness of the intersections with the spaces $\Omega^1_{X_n}(M;Y)$ (Lemma B.4(ii)). ■

Definition IV.12. As a consequence of Lemma IV.11, the space

$$\mathfrak{z}_{M,c}(Y) := \Omega^1_c(M;Y)/dC^\infty_c(M;Y)$$

carries a natural (Hausdorff) locally convex topology. It is isomorphic to

$$\begin{aligned}\varinjlim \Omega^1_{X_n}(M;Y)/\big(\Omega^1_{X_n}(M;Y) \cap dC^\infty_c(M;Y)\big) &\cong \varinjlim \Omega^1_{X_n}(M;Y)/dC^\infty_{X_n}(M;Y)\\ &= \varinjlim \mathfrak{z}_{(X_n,\partial X_n)}(Y)\end{aligned}$$

(Lemmas B.4 and IV.10). We write $q\colon \Omega^1_c(M;Y) \to \mathfrak{z}_{M,c}(Y)$ for the quotient map. The cohomology space

$$H^1_{\rm dR,c}(M;Y) := Z^1_{\rm dR,c}(M;Y)/dC^\infty_c(M;Y)$$

is a closed subspace of $\mathfrak{z}_{M,c}(Y)$. For a compact subset $X \subseteq M$ we define

$$H^1_{{\rm dR},X}(M;Y) := Z^1_{{\rm dR},X}(M;Y)/\big(Z^1_{{\rm dR},X}(M;Y) \cap dC^\infty_c(M;Y)\big)$$

and observe that $H^1_{\rm dR,c}(M;Y)$ is the union of the subspaces $H^1_{{\rm dR},X_n}(M;Y)$. ■

Remark IV.13. For each compact equidimensional submanifold $X \subseteq M$ with $X = \widehat{X}$, Lemma IV.10 implies that

$$\begin{aligned}H^1_{{\rm dR},X}(M;Y) &= Z^1_{{\rm dR},X}(M;Y)/dC^\infty_X(M;Y) \cong Z^1_{\rm dR}(X,\partial X;Y)/dC^\infty(X,\partial X;Y)\\ &= H^1_{\rm dR}(X,\partial X;Y).\end{aligned}$$

Therefore Lemma III.2 implies that for $\dim Y < \infty$ these spaces are finite-dimensional[1]. ■

Lemma IV.14. *Let M be a non-compact finite-dimensional manifold, $(X_n)_{n\in\mathbb{N}}$ a saturated exhaustion of M and Y a Fréchet space. Then the following assertions hold:*
(i) $\Omega^1_c(M;\mathbb{R})$ *is a nuclear LF-space.*
(ii) $H^1_{\rm dR,c}(M;Y)$ *is the locally convex direct limit of the subspaces* $H^1_{\rm dR}(X_n,\partial X_n;Y)$.

Proof. (i) $\Omega^1_c(M;\mathbb{R})$ is the direct limit of the Fréchet spaces $\Omega^1_{X_n}(M;\mathbb{R})$. Each space $\Omega^1_{X_n}(M;\mathbb{R})$ can be embedded into a product of finitely many spaces of the form $\Omega^1(U;\mathbb{R})$, where U is an open subset of $\mathbb{R}^d$, $d = \dim M$. As the spaces $\Omega^1(U;\mathbb{R})$ are nuclear, the spaces $\Omega^1_{X_n}(M;\mathbb{R})$ are nuclear, and the assertion follows ([Tr67, Prop. 50.1]).
(ii) First we verify that the pairs $X_n \subseteq X_{n+1}$ satisfy the assumptions of Theorem III.6. Let C be a connected component of $X_{n+1} \setminus X_n$. If C does not intersect ∂X_{n+1}, then it also is a connected component of $M \setminus X_n$. Further it is contained in the compact set X_{n+1}, so that $X_{n+1} = \widehat{X}_{n+1}$ leads to a contradiction. Therefore all connected components of $X_{n+1} \setminus X_n$ are non-compact, Theorem III.6 applies, and we obtain inductively continuous projections

$$p_n\colon \mathfrak{z}_n := \mathfrak{z}_{(X_n,\partial X_n)}(Y) \to H^1_n := H^1_{\rm dR}(X_n,\partial X_n;Y)$$

which are compatible in the sense that $p_{n+1}|_{\mathfrak{z}_n} = p_n$. Since $\mathfrak{z}_{(M,c)}$ is the locally convex direct limit of the subspaces $\mathfrak{z}_n$ (Definition IV.12), there exists a continuous projection

$$p\colon \mathfrak{z}_{(M,c)}(Y) \to H^1_{\rm dR,c}(M;Y)$$

with $p|_{\mathfrak{z}_n} = p_n$ for each $n \in \mathbb{N}$.

Now let $f_n\colon H^1_n \to E$ be continuous linear functions into a locally convex space E with

$$f_{n+1}|_{H^1_n} = f_n, \quad \text{for} \quad n \in \mathbb{N}.$$

Then the functions $f_n \circ p_n\colon \mathfrak{z}_n \to E$ are continuous linear maps with $f_{n+1} \circ p_{n+1}|_{\mathfrak{z}_n} = f_n \circ p_n$, so that there exists a continuous linear map $F\colon \mathfrak{z}_{(M,c)}(Y) \to E$ with $F|_{\mathfrak{z}_n} = f_n \circ p_n$ for each $n \in \mathbb{N}$, and therefore the restriction $f := F|_{H^1_{\rm dR,c}(M;Y)}$ is continuous. This proves the universal direct limit property of the locally convex space $H^1_{\rm dR,c}(M;Y)$. ■

---

[1] There is some subtle point that one has to observe here. In general a closed subspace Y of an LF-space $X = \varinjlim X_n$ does not have to carry the LF-space topology defined by the subspaces $Y \cap X_n$ (cf. [Tr67, Rem. 13.2]).

Lemma IV.15. *If Y is a locally convex space and $\Gamma \subseteq Y$ a discrete subgroup, then the subgroup $\Gamma^{(\mathbb{N})}$ is discrete in the space $Y^{(\mathbb{N})}$ endowed with the locally convex direct limit topology of the finite products $Y^n = Y^{\{1,\ldots,n\}}$, $n \in \mathbb{N}$.*

Proof. Let $U \subseteq Y$ be a convex 0-neighborhood with $U \cap \Gamma = \{0\}$. Then $U^{(\mathbb{N})}$ is a convex 0-neighborhood in $Y^{(\mathbb{N})}$ with $U^{(\mathbb{N})} \cap \Gamma^{(\mathbb{N})} = \{0\}$. ∎

Theorem IV.16. *Let Y be a s.c.l.c. space and M a non-compact connected manifold with a saturated exhaustion $(X_n)_{n\in\mathbb{N}}$. Then there exists a set $B = \bigcup_n B_n$ consisting of piecewise smooth cycles and of piecewise smooth proper maps $\mathbb{R} \to M$ such that:*

(1) *For each $n \in \mathbb{N}$ the subset B_n is finite, and the integration map*

$$\Phi_{X_n}: H^1_{\mathrm{dR}}(X_n, \partial X_n; Y) \to Y^{B_n}, \quad [\zeta] \mapsto \Big(\int_b \zeta\Big)_{b\in B_n}$$

is a topological isomorphism.

(2) *The integration map*

$$\Phi_M: H^1_{\mathrm{dR,c}}(M; Y) \to Y^{(B)} \cong \varinjlim Y^{B_n}, \quad [\zeta] \mapsto \Big(\int_b \zeta\Big)_{b\in B}$$

is a topological isomorphism.

Proof. Using the construction in the proof of Theorem III.6, we inductively obtain finite sets B_n of piecewise smooth cycles in X_n modulo ∂X_n such that $B_n \subseteq B_{n+1}$ holds in the sense that those cycles in B_n which are not cycles in X_{n+1} are "extended" to relative cycles modulo ∂X_{n+1} in X_{n+1}, and the set $B_{n+1} \setminus B_n$ consists of cycles supported in $X_{n+1} \setminus X_n$. Moreover, for each $n \in \mathbb{N}$ the integration map Φ_{X_n} is a topological isomorphism (Remark III.5) which, in addition, satisfies

$$\Phi_{X_{n+1}}|_{H^1_{\mathrm{dR}}(X_n,\partial X_n;Y)} = \Phi_{X_n}.$$

Therefore Lemma IV.14(ii) leads to a topological isomorphism

$$\Phi: H^1_{\mathrm{dR,c}}(M; Y) \to \varinjlim Y^{B_n} \cong Y^{(B)},$$

where $B := \bigcup_n B_n$, and the space $Y^{(B)} = \bigcup_n Y^{B_n}$ carries the locally convex direct limit topology. ∎

Discrete subgroups of de Rham cohomology

Remark IV.17. In the following we write $C^\infty_p(N, M)$ for the set of proper smooth maps from the manifold N to the manifold M.

Every smooth loop in $C^\infty(\mathbb{S}^1, M)$ is homotopic to a smooth loop α for which all derivatives vanish in the base point $1 \in \mathbb{S}^1$, where we consider $\mathbb{S}^1$ as a subset of $\mathbb{C}$. Then we can view it as a smooth map $[0,1] \to M$ which extends to a proper smooth map $\widetilde{\alpha}: \mathbb{R} \to M$ by using a smooth proper map $\gamma: \mathbb{R}^+ \to M$ with $\gamma(0) = \alpha(1)$ for which all derivatives vanish in 0 and then define $\alpha(t) := \gamma(t-1)$ for $t \geq 1$ and $\alpha(t) := \gamma(-t)$ for $t \leq 0$ (cf. Lemma IV.5). For each compactly supported 1-form β we then have

$$\int_\alpha \beta = \int_{\widetilde{\alpha}} \beta - \int_\gamma \beta + \int_\gamma \beta = \int_{\widetilde{\alpha}} \beta. \qquad ∎$$

Lemma IV.18. *Let $X = \widehat{X} \subseteq M$ be an equidimensional compact submanifold with boundary. Then the following assertions hold:*

(i) *For $x, y \in \partial X$ there exists a smooth proper curve $\alpha: \mathbb{R} \to M$ with $\alpha(0) = x$, $\alpha(1) = y$, and $[0,1] = \alpha^{-1}(X)$. For $\zeta \in \Omega^1(X, \partial X; Y)$ we then have*

$$\int_\alpha \zeta = \int_{\alpha|_{[0,1]}} \zeta.$$

(ii) *For $\zeta \in Z^1_{\mathrm{dR}}(X, \partial X; Y)$ the subgroup of Y generated by the set of all integrals $\int_\alpha \zeta$, $\alpha \in C^\infty_p(\mathbb{R}, M)$, coincides with the set of all integrals over elements in $Z_1(X, \partial X)$.*

Proof. (i) This follows from Lemma IV.6 and its proof.

(ii) From (i), Remark IV.17 and Proposition III.3 it follows that each integral over a cycle in $Z_1(X, \partial X)$ can also be written as a sum of integrals over proper smooth maps $\mathbb{R} \to M$.

Suppose, conversely, that $\alpha: \mathbb{R} \to M$ is smooth and proper. Then α is smoothly homotopic to a proper curve γ which is transversal to the compact submanifold ∂X of M ([BJ73, Satz 14.7; p.158]). Therefore $\gamma^{-1}(X)$ is a finite union of compact intervals $I_1, \ldots, I_m$, because it is locally connected and compact. Then

$$\int_\alpha \zeta = \int_\gamma \zeta = \sum_j \int_{\gamma|_{I_j}} \zeta,$$

and the restrictions $\gamma|_{I_j}$ can be interpreted as cycles in $Z_1(X, \partial X)$. ∎

We conclude from Lemma IV.18 that for the sake of testing integrality conditions of 1-forms supported by X, we could either work with 1-cycles in X modulo ∂X or with proper smooth maps $\mathbb{R} \to M$. The latter approach has the advantage of being independent of X.

Definition IV.19. For a subgroup $\Gamma \subseteq Y$ let

$$Z^1_{\mathrm{dR,c}}(M; \Gamma) := \Big\{\beta \in Z^1_{\mathrm{dR,c}}(M; Y): (\forall \alpha \in C^\infty_p(\mathbb{R}, M)) \int_\alpha \beta \in \Gamma\Big\}$$

and observe that this equals $\{\beta \in \Omega^1_c(M; Y): (\forall \alpha \in C^\infty_p(\mathbb{R}, M)) \int_\alpha \beta \in \Gamma\}$ if Γ is discrete (cf. Lemma II.10(2)). We also define

$$H^1_{\mathrm{dR,c}}(M; \Gamma) := Z^1_{\mathrm{dR,c}}(M, \Gamma)/dC^\infty_c(M; Y).$$ ∎

Proposition IV.20. *Let $\Gamma \subseteq Y$ be a discrete subgroup and $T_\Gamma := Y/\Gamma$. Then $\delta(C^\infty_c(M; T_\Gamma))$ consists of those 1-forms whose integrals over all elements of $C^\infty_p(\mathbb{R}, M)$ are contained in Γ. In particular,*

$$H^1_{\mathrm{dR,c}}(M; \Gamma) = \delta(C^\infty_c(M; T_\Gamma))/d(C^\infty_c(M; Y)).$$

Proof. For each closed 1-form $\delta(f)$, $f \in C^\infty_c(M; T_\Gamma)$, the integrals over elements of $C^\infty_p(\mathbb{R}, M)$ are obviously contained in Γ. If, conversely, $\zeta \in \Omega^1_c(M; Y)$ has this property, then we pick an equidimensional compact manifold $X = \widehat{X}$ with boundary containing the support of ζ. Then Lemmas II.3 and IV.18 imply the existence of $f \in C^\infty(X, \partial X; T_\Gamma) \subseteq C^\infty_c(M; T_\Gamma)$ with $\beta = \delta(f)$. This proves that $\delta(C^\infty_c(M; T_\Gamma))$ consists of those 1-forms whose integrals over all elements of $C^\infty_p(\mathbb{R}, M)$ are contained in Γ. ∎

For the following corollary we recall the set B from Theorem IV.16. For the case where Y is finite-dimensional, the following discreteness result can also be obtained from Proposition B.3, combined with Theorem II.7.

Corollary IV.21. *We have $\Phi_M\big(H^1_{\mathrm{dR,c}}(M; \Gamma)\big) = \Gamma^{(B)}$ and in particular*

$$H^1_{\mathrm{dR,c}}(M; \Gamma) \cong \Gamma^{(B)} \subseteq Y^{(B)} \cong H^1_{\mathrm{dR,c}}(M; Y).$$

Moreover, for $H^1_{\mathrm{dR,c}}(M; \mathbb{R}) \neq \{0\}$ the group Γ is discrete if and only if $H^1_{\mathrm{dR,c}}(M; \Gamma)$ is discrete.

Proof. In view of Lemma IV.18(ii), we have

$$Z^1_{\mathrm{dR,c}}(M;\Gamma) = \bigcup_{n\in\mathbb{N}} Z^1_{\mathrm{dR}}(X_n, \partial X_n;\Gamma),$$

and therefore $\Phi_M(H^1_{\mathrm{dR,c}}(M;\Gamma)) \subseteq \Gamma^{(B)}$. On the other hand, we have for each n the restriction isomorphism

$$\Phi_{X_n} = \Phi_M|_{H^1_{\mathrm{dR}}(X_n,\partial X_n;Y)} \colon H^1_{\mathrm{dR}}(X_n, \partial X_n; Y) \to Y^{B_n} \subseteq Y^{(B)}.$$

Let $x_M \in X_1$ be a base point. If $\Phi_{X_n}([\zeta]) \in \Gamma^{B_n}$, then the construction of the set B_n (cf. Theorem III.6) implies that all integrals of ζ over cycles in $Z_1(X_n, \partial X_n)$ lie in Γ, and hence that all integrals over curves in $C^\infty_p(\mathbb{R}, M)$ lie in Γ (Lemma IV.18(ii)). Therefore $\zeta \in Z^1_{\mathrm{dR,c}}(M;\Gamma)$ and $\Phi_M([\zeta]) = \Phi_{X_n}([\zeta])$. We conclude that $\Phi_M(H^1_{\mathrm{dR,c}}(M;\Gamma)) = \Gamma^{(B)}$.

Now we use Lemma IV.15 to see that for a non-empty set B the subgroup $\Gamma^{(B)}$ of the locally convex direct sum $Y^{(B)}$ is discrete if and only if Γ is discrete in Y. ■

For the following, we observe that we have a natural continuous multiplication map

$$\Omega^1(M;\mathbb{R}) \times Y \to \Omega^1(M;Y), \quad (\zeta, y) \mapsto \zeta \cdot y$$

which induces continuous bilinear maps

$$H^1_{\mathrm{dR}}(M;\mathbb{R}) \times Y \to H^1_{\mathrm{dR}}(M;Y) \quad \text{and} \quad H^1_{\mathrm{dR,c}}(M;\mathbb{R}) \times Y \to H^1_{\mathrm{dR,c}}(M;Y).$$

Corollary IV.22. *For each subgroup Γ of Y we have $H^1_{\mathrm{dR,c}}(M;\mathbb{Z}) \cdot \Gamma = H^1_{\mathrm{dR,c}}(M;\Gamma)$.*

Proof. The inclusion $H^1_{\mathrm{dR,c}}(M;\mathbb{Z}) \cdot \Gamma \subseteq H^1_{\mathrm{dR,c}}(M;\Gamma)$ is trivial. For the converse, let $\zeta \in Z^1_{\mathrm{dR}}(X_n, \partial X_n;\Gamma)$. Then $\Phi_{X_n}([\zeta]) \in \Gamma^{B_n}$ (Lemma IV.18). Suppose that $B_n = \{b_1, \ldots, b_m\}$. Let $b_i^* \in Z^1_{\mathrm{dR}}(X_n, \partial X_n;\mathbb{R})$ be elements with $I_{b_i} b_j^* = \delta_{ij}$. Then $\int_b \zeta = 0$ for $b \in B \setminus B_n$ implies that $\zeta - \sum_{i=1}^m b_i^* \cdot \int_{b_i} \zeta$ is exact, so that

$$[\zeta] = \sum_{j=1}^n [b_i^*] \cdot \int_{b_i} \zeta \in H^1_{\mathrm{dR,c}}(M;\mathbb{Z}) \cdot \Gamma$$

holds in $H^1_{\mathrm{dR,c}}(M;Y)$. As B_n generates $Z_1(X_n, \partial X_n)$ modulo torsion, we get $b_i^* \in H^1_{\mathrm{dR,c}}(M;\mathbb{Z})$ (Lemma IV.18). ■

The following proposition will be helpful in understanding the assertion of Proposition V.12 below.

Proposition IV.23. *If S is a closed subset of the compact manifold M, then for each discrete subgroup $\Gamma \subseteq Y$ we have*

$$H^1_{\mathrm{dR}}(M, S;\mathbb{Z}) \cdot \Gamma = H^1_{\mathrm{dR}}(M, S;\Gamma).$$

Proof. The inclusion "$\subseteq$" is clear. It remains to show the converse. So let $\zeta \in Z^1_{\mathrm{dR}}(M, S;\Gamma)$. First we show that the group $\langle \zeta, H_1(M, S)\rangle \subseteq \Gamma$ is finitely generated.

Since $H_1(M)$ is finitely generated, $\Gamma_0 := \langle \zeta, H_1(M)\rangle$ is a finitely generated subgroup of Γ. Let $p\colon Y \to Y/\Gamma_0$ denote the quotient map. Then all periods of the 1-form $\zeta_0 := p \circ \zeta$ are trivial, and there exists a smooth function $f_0\colon M \to Y/\Gamma_0$ with $df_0 = \zeta_1$ and $f_0(S) \subseteq \Gamma/\Gamma_0$. Moreover, the function f_0 lifts to a smooth function $f_1\colon \widetilde{M} \to Y$, with $f_1(q_M^{-1}(S)) \subseteq \Gamma$, where $q_M\colon \widetilde{M} \to M$ is a universal covering of M. As Γ is discrete, the function f_1 is locally constant on $q_M^{-1}(S)$, and therefore f_0 is locally constant on S. Therefore $f_0(S)$ is finite. As $\langle \zeta, H_1(M, S)\rangle/\Gamma_0 \subseteq \langle f_0(S)\rangle$, it follows that $\langle \zeta, H_1(M, S)\rangle$ is finitely generated.

Moreover, there exists a smooth function $f_2\colon M \to Y$ locally constant on a neighborhood of S such that for each $s \in S$ we have $f_2(s) + \Gamma_0 = f_0(s)$. Then $df_2 \in H^1_{\mathrm{dR}}(M, S;\mathbb{Z})$ lies in the image of

$$H^1_{\mathrm{dR,c}}(M \setminus S;\Gamma) \cong H^1_{\mathrm{dR,c}}(M \setminus S;\mathbb{Z}) \cdot \Gamma$$

(Corollary IV.22). For $\zeta_1 := \zeta - df_2$ we now have

$$\Gamma_0 = \langle \zeta, H_1(M)\rangle = \langle \zeta_1, H_1(M)\rangle = \langle \zeta_1, H_1(M,S)\rangle,$$

so that there exists some $f_3 \in C^\infty(M,S;Y/\Gamma_0)$ with $df_3 = \zeta_1$.

As Γ_0 is finitely generated, it spans a finite-dimensional subspace $Y_0 \subseteq Y$. Extending the identity map $Y_0 \to Y_0$ to a continuous linear map $Y \to Y_0$ using the Hahn–Banach Extension Theorem, we obtain a topological direct sum decomposition $Y \cong Y_0 \oplus Y_1$, where Y_1 is the kernel of the extension. Then $Y/\Gamma_0 \cong (Y_0/\Gamma_0) \times Y_1$ as Lie groups. Moreover, $\zeta_1 = \alpha_1 + \alpha_2$ with $\alpha_j \in Z^1_{\rm dR}(M,S;Y_j)$, $j = 1,2$, and $f_3 = h_1 + h_2$ with $h_1 \in C^\infty(M,S;Y_0/\Gamma_0)$, $h_2 \in C^\infty(M,S;Y_1)$, $\delta(h_1) = \alpha_1$ and $dh_2 = \alpha_2$. This proves that $[\zeta_1] = [\alpha_1]$. As Y_0/Γ_0 is a finite-dimensional torus, we can write it as $\mathbb{R}^d/\mathbb{Z}^d$ with $Y_0 \cong \mathbb{R}^d$ and $\mathbb{Z}^d \cong \Gamma_0$. This means that h_1 is a finite product of the d component functions $l_1, \ldots, l_d \in C^\infty(M,S;\mathbb{T})$. If $e_1, \ldots, e_d$ denote the canonical basis vectors in $\mathbb{R}^d$, this leads to

$$[\alpha_1] = \sum_{j=1}^d [dl_j]\cdot e_i \in H^1_{\rm dR}(M,S;\mathbb{Z})\cdot\Gamma.$$

Summing up, we obtain

$$\begin{aligned} H^1_{\rm dR}(M,S;\Gamma) &\subseteq H^1_{\rm dR,c}(M\setminus S;\Gamma) + H^1_{\rm dR}(M,S;\mathbb{Z})\cdot\Gamma \\ &= H^1_{\rm dR,c}(M\setminus S;\mathbb{Z})\cdot\Gamma + H^1_{\rm dR}(M,S;\mathbb{Z})\cdot\Gamma \subseteq H^1_{\rm dR}(M,S;\mathbb{Z})\cdot\Gamma. \end{aligned}$$

■

Example IV.24. Let $M := \mathbb{R}^2 \setminus P$, where P is a subset without cluster points. We want to get an explicit picture of $H^1_{\rm dR,c}(M;\mathbb{R})$.
(a) First we consider on $\mathbb{R}^2 \setminus \{(0,0)\}$ in polar coordinates the 1-form

$$\alpha(re^{i\varphi}) := f(r)dr,$$

where $f\colon]0,\infty[\to \mathbb{R}$ has compact support and satisfies $\int_0^\infty f(r)\,dr = 1$. Then

$$d\alpha = f'(r)dr\wedge dr + \frac{\partial f}{\partial\varphi}d\varphi\wedge dr = 0,$$

and for each proper map $\gamma\colon \mathbb{R}\to\mathbb{R}^2$ with $\lim_{t\to-\infty}\gamma(t) = (0,0)$ and $\lim_{t\to\infty}\gamma(t) = \infty$ we have

$$\int_\gamma \alpha = 1.$$

(b) To calculate $H^1_{\rm dR,c}(M;\mathbb{R})$, we approximate M by compact submanifolds X_n which are obtained from closed discs D_n with $\partial D_n \cap P = \emptyset$ by removing open discs around the finitely many points in $D_n \cap P$. Note that the set P is countable, so that there exist arbitrarily large discs D_n whose boundaries do not intersect P.

Assume that $D := D_n$ contains k elements of P and put $X := X_n$. Then $\pi_1(X) \cong \pi_1(\mathrm{int}(X))$ is a free group of k generators. For each closed 1-form ζ with compact support in X^0 the integrals over the loops in X are trivial (make them very small around the points in P). Hence every such 1-form is exact. Let $\zeta = df$ with $f \in C^\infty(X;\mathbb{R})$. As ζ has compact support, f is constant on the connected complement of D, so that we may w.l.o.g. assume that $f = 0$ on the outer circle $\partial D \subseteq \partial X$. Then we connect ∂D by arcs $\gamma_1, \ldots, \gamma_k$ to the other boundary components. If all integrals of ζ over the γ_j vanish, then $\zeta \in dC^\infty(X,\partial X;\mathbb{R})$. If $\alpha_1, \ldots, \alpha_k \in Z^1_{\rm dR}(X,\partial X;\mathbb{R})$ are the 1-forms supported close to the elements of $P\cap D$ as in (a), we see that $\int_{\gamma_i}\alpha_j = \delta_{ij}$ for an appropriate normalization, so that $[\zeta] = \sum_j \int_{\gamma_j}\zeta\cdot[\alpha_j]$. Therefore

$$H^1_{\rm dR}(X,\partial X;\mathbb{R}) = \bigoplus_{p\in P\cap D}\mathbb{R}[\alpha_p],$$

and further
$$H^1_{\mathrm{dR,c}}(M;\mathbb{R}) = \lim_{\longrightarrow} H^1_{\mathrm{dR}}(X_n,\partial X_n;\mathbb{R}) = \bigoplus_{p\in P}\mathbb{R}[\alpha_p] \cong \mathbb{R}^{(P)}.$$

The subgroup $H^1_{\mathrm{dR,c}}(M;\mathbb{Z})$ of integral elements in $H^1_{\mathrm{dR,c}}(M;\mathbb{R})$ consists of those cohomology classes whose integrals over all paths between elements of P are integers. For $p,q\in P$ we write $\gamma_{p,q}$ for an arc from p to q. Then
$$\int_{\gamma_{p,q}} \alpha_r = \delta_{p,r} - \delta_{q,r}.$$
This means that $\sum_r \lambda_r\alpha_r$ is integral if and only if all differences $\lambda_r - \lambda_s$ are integral. As only finitely many coefficients λ_r are non-zero, it follows that
$$H^1_{\mathrm{dR,c}}(M;\mathbb{Z}) = \sum_p \mathbb{Z}[\alpha_p] \cong \mathbb{Z}^{(P)}.$$
■

V. Central extensions of Lie groups and period maps

In this section we first explain the general setup for central extensions of infinite-dimensional Lie groups. The main question arising in the integration process of Lie algebra cocycles ω to central extensions of Lie groups is whether the corresponding period group Π_ω is discrete. In this section we show that for cocycles of product type for the groups $C^\infty_c(M;K)_e$ and $C^\infty(M,S;K)_e$ the period group is discrete for any M if and only if this is the case for $M=\mathbb{S}^1$. This reduces the discreteness problem to the case of loop groups, which is known for K compact, and therefore for all finite-dimensional Lie groups K.

Generalities on central Lie group extensions

Definition V.1. (a) Let $\mathfrak{z}$ be a topological vector space and $\mathfrak{g}$ a topological Lie algebra. A *continuous $\mathfrak{z}$-valued 2-cocycle* is a continuous skew-symmetric function $\omega\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{z}$ with
$$\omega([x,y],z)+\omega([y,z],x)+\omega([z,x],y)=0.$$
It is called a *coboundary* if there exists a continuous linear map $\alpha\in\mathrm{Lin}(\mathfrak{g},\mathfrak{z})$ with $\omega(x,y)=\alpha([x,y])$ for all $x,y\in\mathfrak{g}$. We write $Z^2_c(\mathfrak{g},\mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles and $B^2_c(\mathfrak{g},\mathfrak{z})$ for the subspace of coboundaries. We define the *second continuous Lie algebra cohomology space*
$$H^2_c(\mathfrak{g},\mathfrak{z}) := Z^2_c(\mathfrak{g},\mathfrak{z})/B^2_c(\mathfrak{g},\mathfrak{z}).$$
(b) If ω is a continuous $\mathfrak{z}$-valued cocycle on $\mathfrak{g}$, then we write $\mathfrak{g}\oplus_\omega\mathfrak{z}$ for the topological Lie algebra whose underlying topological vector space is the product space $\mathfrak{g}\times\mathfrak{z}$, and the bracket is defined by
$$[(x,z),(x',z')] = \big([x,x'],\omega(x,x')\big).$$
Then $q\colon\mathfrak{g}\oplus_\omega\mathfrak{z}\to\mathfrak{g}, (x,z)\mapsto x$ is a central extension and $\sigma\colon\mathfrak{g}\to\mathfrak{g}\oplus_\omega\mathfrak{z}, x\mapsto(x,0)$ is a continuous linear section of q.

If, conversely, a central Lie algebra extension $q\colon\widehat{\mathfrak{g}}\to\mathfrak{g}$ with kernel $\mathfrak{z}$ has a continuous linear section $\sigma\colon\mathfrak{g}\to\widehat{\mathfrak{g}}$, then it can be described by a continuous Lie algebra cocycle $\omega\in Z^2_c(\mathfrak{g},\mathfrak{z})$ defined by $\omega(x,y):=[\sigma(x),\sigma(y)]-\sigma([x,y])$, because the map
$$\mathfrak{g}\oplus_\omega\mathfrak{z}\to\widehat{\mathfrak{g}},\quad (x,z)\mapsto\sigma(x)+z$$
is an isomorphism of topological Lie algebras. As two Lie algebra cocycles define equivalent central extensions if and only if they differ by a coboundary, we obtain an identification of the set of equivalence class of all central $\mathfrak{z}$-extensions of $\mathfrak{g}$ (with a continuous linear section) with the vector space $H^2_c(\mathfrak{g},\mathfrak{z})$. ■

Definition V.2. (a) *Central extensions of Lie groups* are always assumed to have a smooth local section. Let $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ be a central extension of the connected Lie group G by the abelian Lie group Z. We assume that the identity component Z_e of Z can be written as $Z_e = \mathfrak{z}/\pi_1(Z)$, where the Lie algebra $\mathfrak{z}$ of Z is a s.c.l.c. space. The group $(\mathfrak{z}, +)$ can be identified in a natural way with the universal covering group of Z_e, and Z_e is a quotient of $\mathfrak{z}$ modulo a discrete subgroup which can be identified with $\pi_1(Z)$. Since the quotient map $q: \widehat{G} \to G$ has a smooth local section, the corresponding Lie algebra homomorphism $\widehat{\mathfrak{g}} \to \mathfrak{g}$ has a continuous linear section $\sigma: \mathfrak{g} \to \widehat{\mathfrak{g}}$, hence can be described by a continuous Lie algebra cocycle (Definition V.1).
(b) If G is a group and Z an abelian group, then we define the group

$$\begin{aligned} Z^2(G,Z) := \{f: G \times G \to Z: (\forall x,y,z \in G) \\ f(\mathbf{1},x) = f(x,\mathbf{1}) = \mathbf{1},\ f(x,y)f(xy,z) = f(x,yz)f(y,z)\} \end{aligned}$$

of Z-*valued* 2-*cocycles* and the subgroup

$$B^2(G,Z) := \{f: G \times G \to Z: (\exists h: G \to Z)\ h(\mathbf{1}) = \mathbf{1}, (\forall x,y \in G)\ f(x,y) = h(xy)h(x)^{-1}h(y)^{-1}\}$$

of Z-*valued* 2-*coboundaries*. In both cases the group structure is given by pointwise multiplication.

If G and Z are Lie groups, we write $Z^2_s(G,Z)$ for the subgroup of $Z^2(G,Z)$ consisting of those cocycles f which are smooth in a neighborhood of (e,e), and $B^2_s(G,Z)$ for the subgroup of all functions of the form $(g,g') \mapsto h(gg')h(g)^{-1}h(g')^{-1}$, where $h: G \to Z$ is smooth in an identity neighborhood. We recall from [Ne02a, Prop. 4.2] that central Lie group extensions as above can always be written as

$$\widehat{G} \cong G \times_f Z \quad \text{with} \quad (g,z)(g',z') = \big(gg', zz'f(g,g')\big),$$

for some $f \in Z^2_s(G,Z)$. Two cocycles f_1, f_2 define equivalent Lie group extensions if and only if $f_1 \cdot f_2^{-1} \in B^2_s(G,Z)$ (for $f_2^{-1}(x,y) := f_2(x,y)^{-1}$), and the quotient group $H^2_s(G,Z) := Z^2_s(G,Z)/B^2_s(G,Z)$ parametrizes the equivalence classes of central Z-extensions of G with smooth local sections ([Ne02a, Remark 4.4]). There is a natural map $H^2_s(G,Z) \to H^2_c(\mathfrak{g},\mathfrak{z})$ induced by the map

$$(5.1) \quad D: Z^2_s(G,Z) \to Z^2_c(\mathfrak{g},\mathfrak{z}), \quad D(f)(x,y) = d^2f(e,e)((x,0),(0,y)) - d^2f(e,e)((y,0),(x,0))$$

([Ne02a, Lemma 4.6]), where $d^2f(e,e)$ is well-defined because $df(e,e)$ vanishes, which follows from $f(g,e) = f(e,g) = \mathbf{1}$. For more details on central extensions of Lie groups we refer to [Ne02a]. ■

Definition V.3. If $\mathfrak{z}$ is a s.c.l.c. space, G a Lie group, and $\Omega \in \Omega^2(G,\mathfrak{z})$ a closed $\mathfrak{z}$-valued 2-form, then we obtain with [Ne02a, Lemma 5.7] a group homomorphism

$$\mathrm{per}_\Omega: \pi_2(G) \to \mathfrak{z}$$

called the *period map*. It is given on smooth representatives $\sigma: \mathbb{S}^2 \to G$ of classes in $\pi_2(G)$ by the integral

$$\mathrm{per}_\Omega([\sigma]) = \int_{\mathbb{S}^2} \sigma^*\Omega = \int_\sigma \Omega.$$

We recall that each homotopy class contains smooth representatives. Here we use the sequential completeness of $\mathfrak{z}$ to ensure that the integrals, which can be obtained as limits of Riemann sums, do exist. If Ω is exact, then the period map is trivial by Stoke's Theorem. The image $\Pi_\Omega := \mathrm{per}_\Omega(\pi_2(G))$ is called the *period group* of Ω. ■

Definition V.4. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ a continuous Lie algebra cocycle with values in the s.c.l.c. space $\mathfrak{z}$. Let $\Gamma \subseteq \mathfrak{z}$ be a discrete subgroup and $Z := \mathfrak{z}/\Gamma$ the corresponding quotient Lie group. Further let Ω be the corresponding left invariant closed $\mathfrak{z}$-valued 2-form on G. Then we define a homomorphism

$$P: H^2_c(\mathfrak{g}, \mathfrak{z}) \to \mathrm{Hom}(\pi_2(G), Z) \times \mathrm{Hom}(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z}))$$

as follows. For the first component we take

$$P_1([\omega]) := q_Z \circ \mathrm{per}_\omega,$$

where $q_Z: \mathfrak{z} \to Z$ is the quotient map and $\mathrm{per}_\omega := \mathrm{per}_\Omega: \pi_2(G) \to \mathfrak{z}$ is the period map of ω. To define the second component, for each $X \in \mathfrak{g}$ we write X_r for the corresponding right invariant vector field on G. Then $i_{X_r}\Omega$ is a closed $\mathfrak{z}$-valued 1-form ([Ne02a, Lemma 3.11]) to which we associate a homomorphism $\pi_1(G) \to \mathfrak{z}$ via

$$P_2([\omega])([\gamma])(X) := \int_\gamma i_{X_r}\Omega.$$

We refer to [Ne02a, Sect. 7] for arguments showing that P is well-defined, i.e., that the right hand sides only depend on the Lie algebra cohomology class of ω. ■

The following theorem completely describes the obstructions for a Lie algebra cocycle to integrate to a central Lie group extension. It is the main result of [Ne02a].

Theorem V.5. *Let $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ be a continuous Lie algebra cocycle. Then the central Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z} \twoheadrightarrow \mathfrak{g}$ integrates to a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ if and only if $P([\omega]) = 0$.*

Proof. [Ne02a, Th. 7.12]. ■

Applications to current groups

Now we turn to central extensions of the two classes of current Lie groups given as the identity components of $C^\infty_c(M; K)$ and $C^\infty(M, S; K)$. The methods developed in this paper are well suited for the study of Lie algebra cocycles of product type introduced below. Here the main problem is to decide for a given cocycle if its period group is discrete (cf. Theorem V.5).

Definition V.6. Let $\mathfrak{k}$ be a locally convex topological Lie algebra, M a manifold and $\mathfrak{g} := C^\infty(M; \mathfrak{k})$. We consider a continuous invariant symmetric bilinear map $\kappa: \mathfrak{k} \times \mathfrak{k} \to Y$, where Y is a s.c.l.c. space. We then obtain a continuous $\mathfrak{z}_M(Y)$-valued cocycle on $\mathfrak{g}$ by

$$\omega_M(\xi, \eta) := \omega_{M,\kappa}(\xi, \eta) := [\kappa(\xi, d\eta)] \in \mathfrak{z}_M(Y),$$

where we view $\kappa(\xi, d\eta)$ as the element of $\Omega^1(M; Y)$ whose value in a tangent vector $v \in T_p(M)$ is given by $\kappa(\xi(p), d\eta(p)(v))$.
(a) On $C^\infty(M, S; \mathfrak{k})$ we obtain by restriction a continuous $\mathfrak{z}_{(M,S)}(Y)$-valued Lie algebra cocycle $\omega_{(M,S)}$. For a compact manifold M the group $C^\infty(M, S; K)$ has a natural Lie group structure (Definition I.6), so that we can define the period map

$$\mathrm{per}_{\omega_{(M,S)}}: \pi_2(C^\infty(M, S; K)) \to \mathfrak{z}_{(M,S)}(Y)$$

corresponding to the left invariant 2-form $\Omega_{(M,S)}$ on $C^\infty(M, S; K)$ with $\Omega_{(M,S),e} = \omega_{(M,S)}$. We write $\Pi_{(M,S)}$ for the corresponding period group.

(b) If ξ and η have compact support, then the same holds for $\kappa(\xi,\eta)$, so that we also obtain a Lie algebra cocycle

$$\omega_M \in Z^2_c(C^\infty_c(M;\mathfrak{k}), \mathfrak{z}_{M,c}(Y)), \quad \mathfrak{z}_{M,c}(Y) := \Omega^1_c(M;Y)/dC^\infty_c(M;Y).$$

The continuity of this cocycle follows from the continuity of the map

$$C^\infty_c(M;\mathfrak{k}) \times \Omega^1_c(M;\mathfrak{k}) \to \Omega^1_c(M;Y), \quad (f,\xi) \mapsto \kappa(f,\xi),$$

which in turn follows from [Gl01d, Th. 4.7] because it can be interpreted as a map on the level of compactly supported sections of vector bundles induced by the bundle map determined by the continuous map

$$\mathfrak{k} \times \mathrm{Lin}(T_p(M);\mathfrak{k}) \to \mathrm{Lin}(T_p(M);Y), \quad (x,\beta) \mapsto \kappa(x,\beta(\cdot))$$

on the fiber in $p \in M$.

(c) For any Lie group K we define $V(\mathfrak{k})$ as follows. We first endow $\mathfrak{k} \otimes \mathfrak{k}$ with the projective tensor product topology and define $V(\mathfrak{k})$ as the completion of the quotient of $V(\mathfrak{k})$ by the closure of the subspace spanned by all elements of the form

$$x \otimes y - y \otimes x \quad \text{and} \quad [x,y] \otimes z + y \otimes [x,z], \quad x,y,z \in \mathfrak{k}.$$

If $[z]$ denotes the image of $z \in \mathfrak{k} \otimes \mathfrak{k}$ in $V(\mathfrak{k})$, we obtain a continuous invariant bilinear map

$$\kappa\colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k}), \quad \kappa(x,y) := [x \otimes y]$$

which leads to the cocycle $\omega = \omega_{\mathbb{S}^1,\kappa} \in Z^2_c(\mathfrak{g}, V(\mathfrak{k}))$ on $\mathfrak{g} := C^\infty(\mathbb{S}^1;\mathfrak{k})$ given by $\omega(\xi,\eta) := [\kappa(\xi, d\eta)]$. As $\pi_2(C^\infty(\mathbb{S}^1;K)) \cong \pi_3(K)$ (Corollary A.15), the period map per_ω yields a homomorphism

$$\mathrm{per}_K\colon \pi_3(K) \to V(\mathfrak{k}). \qquad \blacksquare$$

Proposition V.7. *Let $\mathfrak{g} := C^\infty_c(M;\mathfrak{k})$ and $\kappa\colon \mathfrak{k} \times \mathfrak{k} \to Y$ be a continuous invariant symmetric bilinear form. Then we obtain for the cocycle $\omega(\xi,\eta) := [\kappa(\xi,d\eta)]$ an automorphic action of the group $C^\infty(M,K)$ on $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z}_M(Y)$ by*

$$f.(\xi,z) := (\mathrm{Ad}(f).\xi, z - [\kappa(\delta^l(f),\xi)]). \tag{5.2}$$

The corresponding derived action is given by

$$\eta.(\xi,z) = [(\eta,0),(\xi,z)] = ([\eta,\xi], \omega(\eta,\xi)). \tag{5.3}$$

Proof. The arguments can be taken over from [MN02, Prop. III.3]. Here we only have to add Lemma II.2 to see that δ^l is smooth. $\blacksquare$

Theorem V.8. *Let K be a connected Lie group, M a connected manifold, $G := C^\infty_c(M,K)_e$ and $\omega_{M,\kappa} \in Z^2_c(\mathfrak{g}, \mathfrak{z}_M(Y))$ as above. Suppose that the period group $\Pi_{M,\kappa} \subseteq \mathfrak{z}_M(Y)$ is discrete. For $Z := \mathfrak{z}_M(Y)/\Pi_{\omega_{M,\kappa}}$ we then obtain a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ corresponding to the cocycle $\omega_{M,\kappa}$.*

Proof. In view of Theorem V.5, we only have to see that $P_2([\omega_{M,\kappa}]) = 0$. According to [Ne02a, Prop. 7.6], this is equivalent to the existence of a smooth linear action of G on $\widehat{\mathfrak{g}}$ whose derived action is given by $\eta.(\xi,z) = ([\eta,\xi], \omega(\eta,\xi))$. Proposition V.7 implies that such a representation exists. $\blacksquare$

For the following theorem we recall that we can use the continuous bilinear form $\kappa\colon \mathfrak{k}\times\mathfrak{k} \to Y$ to define a wedge product

$$\wedge_\kappa\colon \Omega^1(M;\mathfrak{k}) \times \Omega^1(M;\mathfrak{k}) \to \Omega^2(M;Y)$$

by

$$(\alpha \wedge_\kappa \beta)(v,w) := \kappa(\alpha_p(v), \beta_p(w)) - \kappa(\beta_p(v), \alpha_p(w)), \quad v,w \in T_p(M).$$

The following theorem describes a situation where we have a global smooth group cocycle associated to the cocycle obtained by composing a cocycle of product type with the de Rham differential $\mathfrak{z}_{M,c}(Y) \to \Omega^2_c(M;Y)$. The reason behind the existence of the global cocycle lies in the fact that all periods of $\omega_{M,\kappa}$ lie in the kernel $H^1_{\mathrm{dR,c}}(M;Y)$ of d (see [Ne02a, Section 8] for more details on the existence of global smooth cocycles).

Theorem V.9. *Let $G^+ := C^\infty_c(M,K)$. Then the map*

$$h: G^+ \times G^+ \to \Omega^2_c(M;Y), \quad h(f,g) := \delta^l(f) \wedge_\kappa \delta^r(g)$$

defines a a smooth $\Omega^2_c(M;Y)$-valued group 2-cocycle on G^+, so that we obtain a central Lie group extension $\widehat{G}^+ := G^+ \times_h \Omega^2_c(M;Y)$. The corresponding Lie algebra cocycle Dh from (5.1) *is given by*

$$Dh(\xi,\eta) = 2d\xi \wedge_\kappa d\eta \quad \text{for} \quad \xi,\eta \in C^\infty_c(M;\mathfrak{k}).$$

The map $\gamma: \mathfrak{z}_{M,c}(Y) \to \Omega^2_c(M;Y), [\beta] \mapsto 2d\beta$ satisfies $\gamma \circ \omega_{M,\kappa} = Dh$ and induces a Lie algebra homomorphism

$$\gamma_{\mathfrak{g}}: \widehat{\mathfrak{g}} = \mathfrak{g} \oplus_{\omega_{M,\kappa}} \mathfrak{z}_{M,c}(Y) \to \widehat{\mathfrak{g}}^+ := \mathfrak{g} \oplus_{Dh} \Omega^2_c(M;Y), \quad (X,[\beta]) \mapsto (X, 2d\beta).$$

This homomorphism is G^+-equivariant with respect to the action on $\widehat{\mathfrak{g}}^+$ induced by the adjoint action of $\widehat{G}^+$, which is given by

$$\mathrm{Ad}_{\widehat{\mathfrak{g}}^+}(g).(\xi,z) = \big(\mathrm{Ad}(g).\xi, z - d(\kappa(\delta^l(g),\xi))\big).$$

Proof. This follows with the same arguments as in the proof of [MN02, Th. III.9]. For non-compact manifolds we have to use Lemma II.2 for the smoothness of the maps $\delta^l, \delta^r: C^\infty_c(M,K) \to \Omega^1_c(M;\mathfrak{k})$. ■

Period maps for $C^\infty(M,S;K)$

Now we turn to the period groups $\Pi_{(M,S)}$ for the Lie algebra cocycles $\omega_{(M,S)}$ associated to the Lie algebras $C^\infty(M,S;\mathfrak{k})$, where M is compact and $S \subseteq M$ a closed subset.

Lemma V.10. *For each $\alpha \in C^\infty((I,\partial I),(M,S))$ let*

$$\alpha_K: C^\infty(M,S;K) \to C^\infty(I,\partial I;K)$$

denote the corresponding group homomorphism. Then

$$\mathrm{per}_{\omega_{(I,\partial I)}} \circ \pi_2(\alpha_K) = I_\alpha \circ \mathrm{per}_{\omega_{(M,S)}}.$$

Proof. First we recall from Lemma A.16 that the map α_K is a Lie group homomorphism. Let $G := C^\infty(M,S;K)_e$ and $\Omega_{(M,S)} \in \Omega^2(G, \mathfrak{z}_{(M,S)}(Y))$ denote the left invariant 2-form corresponding to $\omega_{(M,S)}$. Then $I_\alpha \circ \Omega_{(M,S)}$ is a Y-valued left invariant 2-form on G whose value in $\mathbf{1}$ is $I_\alpha \circ \omega_{(M,S)}$. Further $\alpha_K^*\Omega_{(I,\partial I)}$ is a left invariant 2-form on G whose value in $\mathbf{1}$ is given by

$$\begin{aligned}(\xi,\eta) \mapsto \omega_{(I,\partial I)}(\xi\circ\alpha, \eta\circ\alpha) &= [\kappa(\xi\circ\alpha, d(\eta\circ\alpha))] \\ &= [\kappa(\alpha^*\xi, \alpha^*(d\eta))] = \int_I \kappa(\alpha^*\xi, \alpha^*(d\eta)) = \int_\alpha \kappa(\xi, d\eta) = I_\alpha\big(\omega_{(M,S)}(\xi,\eta)\big).\end{aligned}$$

This implies

$$\alpha_K^*\Omega_{(I,\partial I)} = I_\alpha \circ \Omega_{(M,S)}$$

for each $\alpha \in C^\infty((I,\partial I),(M,S))$, and hence the assertion. ■

Lemma V.11. *If we identify* $\mathfrak{z}_{\mathbb{S}^1}(Y)$, $\mathfrak{z}_{(I,\partial I)}(Y)$, *and* $\mathfrak{z}_{\mathbb{R}}(Y)$ *with* Y *via the integration maps from* Lemma II.8*, then*

$$\Pi_{\mathbb{S}^1} = \Pi_{(I,\partial I)} = \Pi_{\mathbb{R}}.$$

Proof. According to Corollary A.15, the natural inclusion

$$C^\infty((I,\partial I);K) \hookrightarrow C^\infty_*(\mathbb{S}^1;K)$$

induced from the canonical map $\alpha \in C^\infty((I,\partial I),(\mathbb{S}^1,*))$ is a weak homotopy equivalence. Therefore $\pi_2(\alpha_K)$ is an isomorphism, and Lemma V.10, applied to $(M,S) = (\mathbb{S}^1, 1)$, implies that

$$\Pi_{(I,\partial I)} = I_\alpha \circ \Pi_{\mathbb{S}^1} \cong \Pi_{\mathbb{S}^1}$$

because the map $I_\alpha \colon \mathfrak{z}_{\mathbb{S}^1}(Y) \to Y$ is the integration isomorphism which we ignore by identifying $\Pi_{\mathbb{S}^1}$ and $\Pi_{(I,\partial I)}$ as subsets of Y.

To obtain $\Pi_{\mathbb{R}} = \Pi_{(I,\partial I)}$, we first use Theorem A.13 and a diffeomorphism $\alpha \colon \mathbb{R} \to I \setminus \partial I$ to see that the natural embedding

$$\varphi_K \colon C^\infty_c(\mathbb{R};K) \to C^\infty_c(I\setminus\partial I;K) \hookrightarrow C^\infty(I,\partial I;K)$$

is a weak homotopy equivalence. Moreover, $\mathbf{L}(\varphi_K)^*\omega_{(I,\partial I)} = \omega_{\mathbb{R}}$, so that $\varphi_K^*\Omega_{(I,\partial I)} = \Omega_{\mathbb{R}}$, and by integration over $\mathbb{R}$ we obtain $\Pi_{(I,\partial I)} = \Pi_{\mathbb{R}}$. ∎

Proposition V.12. *For each* κ *the period group* $\Pi_{(M,S)}$ *is contained in* $H^1_{\rm dR}(M,S;Y)$*, and we have*

$$H^1_{\rm dR}(M,S;\mathbb{Z}) \cdot \Pi_{\mathbb{S}^1} \subseteq \Pi_{(M,S)} \subseteq H^1_{\rm dR}(M,S;\Pi_{\mathbb{S}^1}).$$

If $\Pi_{\mathbb{S}^1}$ *is discrete, then*

$$\Pi_{(M,S)} = H^1_{\rm dR}(M,S;\Pi_{\mathbb{S}^1}) = H^1_{\rm dR}(M,S;\mathbb{Z}) \cdot \Pi_{\mathbb{S}^1}.$$

Proof. In the situation of Lemma V.10, the homomorphism $\pi_2(\alpha_K)$ only depends on the homotopy class of α (Lemma A.16). Therefore Lemma V.10 implies that the restriction of I_α to $\Pi_{(M,S)}$ depends only on the homotopy class of α, hence $\Pi_{(M,S)} \subseteq H^1_{\rm dR}(M,S;Y)$ by Lemma II.10. From Lemmas V.10 and V.11 we further get

$$\Pi_{(M,S)} \subseteq H^1_{\rm dR}(M,S;\Pi_{(I,\partial I)}) = H^1_{\rm dR}(M,S;\Pi_{\mathbb{S}^1}).$$

To prove the inclusion

$$H^1_{\rm dR}(M,S;\mathbb{Z}) \cdot \Pi_{\mathbb{S}^1} \subseteq \Pi_{(M,S)},$$

let $[\zeta] \in H^1_{\rm dR}(M,S;\mathbb{Z})$. Then Lemma II.3 implies the existence of $f \in C^\infty(M,S;\mathbb{T})$ with $\delta(f) = \zeta$. Let $0 \in \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ denote the identity element in $\mathbb{T}$. The map f induces a smooth group homomorphism

$$f_K \colon C^\infty_*(I,\partial I;K) \to C^\infty(M,S;K), \quad \varphi \mapsto \varphi \circ f$$

(Lemma A.16). We now get from Lemma V.10 for each $\alpha \in C^\infty((I,\partial I),(M,S))$ the relation

$$I_\alpha \circ \mathrm{per}_{\omega_{(M,S)}} \circ \pi_2(f_K) = \mathrm{per}_{\omega_{(I,\partial I)}} \circ \pi_2(\alpha_K) \circ \pi_2(f_K) = \mathrm{per}_{\omega_{(I,\partial I)}} \circ \pi_2((f\circ\alpha)_K),$$

where $f \circ \alpha$ is viewed as a map in $C^\infty((I,\partial I),(\mathbb{T},\{0\}))$. This map factors through a smooth map $I/\partial I \cong \mathbb{T} \to \mathbb{T}$, and $\pi_2((f\circ\alpha)_K)$ is the multiplication with the winding number $\deg(f\circ\alpha)$ of this map ([MN02, Lemma I.10]). For each

$$[\sigma] \in \pi_2(C^\infty(\mathbb{T},\{0\};K)) \cong \pi_2(C^\infty_*(\mathbb{S}^1;K))$$

we then have

$$I_\alpha(\mathrm{per}_{\omega_{(M,S)}}(\pi_2(f_K)[\sigma])) = \deg(f\circ\alpha)\,\mathrm{per}_{\omega_{(I,\partial I)}}([\sigma]) = I_\alpha(\zeta)\cdot \mathrm{per}_{\omega_{(I,\partial I)}}([\sigma]).$$

Since the I_α separate points on $H^1_{\rm dR}(M,S;Y)$, it follows that

$$\mathrm{per}_{\omega_{(M,S)}}(\pi_2(f_K)[\sigma]) = [\zeta]\cdot \mathrm{per}_{\omega_{(I,\partial I)}}([\sigma]),$$

and hence that

$$H^1_{\rm dR}(M,S;\mathbb{Z}) \cdot \Pi_{\mathbb{S}^1} = H^1_{\rm dR}(M,S;\mathbb{Z}) \cdot \Pi_{(I,\partial I)} \subseteq \Pi_{(M,S)}.$$

If $\Pi_{\mathbb{S}^1}$ is discrete, then we apply Proposition IV.23 to obtain the asserted equalities. ∎

Corollary V.13. *If $\Pi_{\mathbb{S}^1}$ is discrete, then $\Pi_{(M,S)}$ is discrete for each pair (M,S).*

Proof. Proposition V.12 implies that $\Pi_{(M,S)} \subseteq H^1_{\rm dR}(M,S;\Pi_{\mathbb{S}^1})$, and the latter group is discrete by Theorem II.7. ■

Remark V.14. In view of the preceding corollary, everything reduces to the study of the period map

$$\mathrm{per}_{\omega_{\mathbb{S}^1}}\colon \pi_3(K) \cong \pi_2(C^\infty(\mathbb{S}^1;K)) \to Y.$$

It is not necessary to know $\pi_2(G)$ explicitly. ■

Proposition V.15. *Suppose that $Y = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, so that $T_\Gamma = \mathbb{T}$. We further assume that $\mathfrak{k}$ is compact and simple and that κ in normalized in such a way that $\kappa(i\check\alpha, i\check\alpha) = -2$, where $\check\alpha \in \mathfrak{k}_{\mathbb{C}}$ is a coroot corresponding to a long root. For $G = C^\infty(M,S;K)_e$ we then have*

$$\Pi_{(M,S)} = H^1_{\rm dR}(M,S;\mathbb{Z}).$$

Proof. We first recall from the calculations in Appendix IIa to Section II in [Ne01a] that under the present assumptions we have $\Pi_{(I,\partial I)} = \Pi_{\mathbb{S}^1} = \mathbb{Z}$ (see also [MN02, Th. II.9]). Therefore Proposition V.12 directly leads to

$$\begin{aligned} H^1_{\rm dR}(M,S;\mathbb{Z})\cdot \Pi_{\mathbb{S}^1} &= \mathbb{Z}\cdot H^1_{\rm dR}(M,S;\mathbb{Z}) \\ &= H^1_{\rm dR}(M,S;\mathbb{Z}) \subseteq \Pi_{(M,S)} \subseteq H^1_{\rm dR}(M,S;\Pi_{\mathbb{S}^1}) = H^1_{\rm dR}(M,S;\mathbb{Z}). \end{aligned}$$

■

Applying Proposition V.15 to the group $C^\infty(M,S;K)$ from Example II.12, we obtain a cocycle on the Lie algebra of a Fréchet–Lie group for which the period group $\Pi_{(M,S)}$ is discrete but not finitely generated.

Period maps for $C^\infty_c(M;K)$

Let M be a connected non-compact manifold and Y a s.c.l.c. space. For a proper smooth map $\alpha\colon \mathbb{R} \to M$ and $\zeta \in Z^1_{\rm dR,c}(M;Y)$ the integral

$$I_\alpha(\zeta) := \int_\alpha \zeta := \int_{\mathbb{R}} \alpha^*\zeta$$

is defined because $\alpha^*\zeta$ has compact support. We thus obtain a linear map

$$I_\alpha\colon Z^1_{\rm dR,c}(M;Y) \to Y$$

which is easily seen to be continuous.

Lemma V.16. *For each $\alpha \in C^\infty_p(\mathbb{R},M)$ let*

$$\alpha_K\colon C^\infty_c(M;K) \to C^\infty_c(\mathbb{R};K), \quad f \mapsto f\circ\alpha$$

denote the corresponding Lie group homomorphism. Then

$$\mathrm{per}_{\omega_{\mathbb{R}}} \circ \pi_2(\alpha_K) = I_\alpha \circ \mathrm{per}_{\omega_M}. \tag{5.4}$$

Proof. From Lemma A.12 we recall that α_K is a Lie group homomorphism. The remaining argument can be copied from Lemma V.10. ■

Proposition V.17. *For each non-compact manifold M and each κ we have*

$$\Pi_M = H^1_{\mathrm{dR,c}}(M;\Pi_{\mathbb{R}}).$$

Proof. In the situation of Lemma V.16, the homomorphism $\pi_2(\alpha_K)$ only depends on the homotopy class of α (Lemma A.16). Therefore Lemma V.10 implies that the restriction of I_α to Π_M depends only on the homotopy class of α, hence $\Pi_M \subseteq H^1_{\mathrm{dR,c}}(M;Y)$ by Lemma II.10. From Lemma V.16 we further get $\Pi_M \subseteq H^1_{\mathrm{dR,c}}(M;\Pi_{\mathbb{R}})$.

To prove the converse inclusion $H^1_{\mathrm{dR,c}}(M;\Pi_{\mathbb{R}}) \subseteq \Pi_M$, we first recall from Corollary IV.22 that

$$H^1_{\mathrm{dR,c}}(M,\Pi_{\mathbb{R}}) = H^1_{\mathrm{dR,c}}(M;\mathbb{Z})\cdot\Pi_{\mathbb{R}}.$$

It therefore suffices to prove $H^1_{\mathrm{dR,c}}(M;\mathbb{Z})\cdot\Pi_{\mathbb{R}} \subseteq \Pi_M$. Let $[\zeta] \in H^1_{\mathrm{dR}}(M;\mathbb{Z})$. Then Proposition IV.20 implies the existence of $f \in C^\infty_c(M,\mathbb{T})$ with $\delta(f) = \zeta$. Let $0 = \mathbb{Z} \in \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ denote the identity element in $\mathbb{T}$. The map f induces a smooth group homomorphism

$$f_K\colon C^\infty_c(\mathbb{T};K) \to C^\infty_c(M;K), \quad f \mapsto f\circ\varphi$$

(Lemma A.12). In view of Lemma V.16, we have for each $\alpha \in C^\infty_p(\mathbb{R},M)$

$$I_\alpha \circ \mathrm{per}_{\omega_M} \circ \pi_2(f_K) = \mathrm{per}_{\omega_{\mathbb{R}}} \circ \pi_2(\alpha_K)\circ\pi_2(f_K) = \mathrm{per}_{\omega_{\mathbb{R}}} \circ \pi_2((f\circ\alpha)_K),$$

where $f\circ\alpha$ is viewed as a map in $C^\infty_c(\mathbb{R},\mathbb{T})$. Viewing $\mathbb{R}$ as $\mathbb{T}\setminus\{0\}$, this map extends to a smooth map $\mathbb{T}\to\mathbb{T}$, and $\pi_2((f\circ\alpha)_K)$ is the multiplication with the winding number

$$\deg(f\circ\alpha) = \int_\alpha \zeta$$

of this map ([MN02, Lemma I.10]). For each $[\sigma] \in \pi_2(C^\infty_c(\mathbb{R};K))$ we then have

$$I_\alpha(\mathrm{per}_{\omega_M}(\pi_2(f_K)[\sigma])) = \deg(f\circ\alpha)\,\mathrm{per}_{\omega_{\mathbb{R}}}([\sigma]) = I_\alpha(\zeta)\,\mathrm{per}_{\omega_{\mathbb{R}}}([\sigma]).$$

Since the I_α separate points on $H^1_{\mathrm{dR,c}}(M;Y)$ (here we need that M is non-compact), it follows that

$$\mathrm{per}_{\omega_M}(\pi_2(f_K)[\sigma]) = [\zeta]\cdot\mathrm{per}_{\omega_{\mathbb{R}}}([\sigma])$$

and hence that $H^1_{\mathrm{dR,c}}(M;\mathbb{Z})\cdot\Pi_{\mathbb{R}} \subseteq \Pi_M$. ∎

Corollary V.18. *If $\Pi_{\mathbb{R}}$ is discrete, then Π_M is discrete for each non-compact connected manifold manifold M.* ∎

For the following proposition we recall the space $V(\mathfrak{k})$ from Definition V.7.

Proposition V.19. *If $\dim K < \infty$, and $\kappa\colon\mathfrak{k}\times\mathfrak{k}\to V(\mathfrak{k})$ is the universal symmetric invariant bilinear map, then there exists for $Z := V(\mathfrak{k})/\Pi_{M,\kappa}$ a central Lie group extension*

$$Z \hookrightarrow \widehat{G} \twoheadrightarrow G = C^\infty_c(M,K)_e.$$

Proof. In view of [MN02, Th. II.9], the period group $\Pi_{\mathbb{S}^1,\kappa} = \Pi_{\mathbb{R},\kappa}$ is discrete (cf. Lemma V.11), and Corollary V.18 now shows that Π_M is discrete. Therefore Theorem V.5 applies. ∎

Remark V.20. The main idea behind our identification of the period group for current groups is as follows. Let M be a compact manifold, $x_M \in M$, and

$$G := C^\infty_*(M;K) := \{f \in C^\infty(M;K) : f(x_M) = e\}.$$

The evaluation map

$$\mathrm{ev}\colon G \times M \to K, \quad (f,p) \mapsto f(p)$$

induces maps

$$\varphi_{k,l}\colon \pi_k(G) \times \pi_l(M) \to \pi_{k+l}(K)$$

as follows. We view $\pi_k(M)$ as the set of arc-components in the space $C((I^n, \partial I^n), (M, x_M))$ of continuous maps of pairs, where I is the unit interval. Then $\varphi_{k,l}([f],[h])$ is the class defined by the map

$$I^{k+l} \to K, \quad (x,y) \mapsto f(x)(h(y)),$$

vanishing on the boundary

$$\partial I^{k+l} = (\partial I^k \times I^l) \cup (I^k \times \partial I^l).$$

In particular we obtain a map

$$\varphi_{2,1}\colon \pi_2(G) \times \pi_1(M) \to \pi_3(K),$$

and our analysis of the period map is based on the commutative diagram

$$\begin{array}{ccccc}
\pi_2(G) & \times & \pi_1(M) & \to & \pi_3(K) \\
\big\downarrow{\scriptstyle \mathrm{per}_{\omega_M}} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{per}_{\mathbb{S}^1}} \\
H^1_{\mathrm{dR}}(M;Y) & \times & \pi_1(M) & \to & H^1_{\mathrm{dR}}(\mathbb{S}^1;Y) \cong Y
\end{array}$$

The effectiveness of this picture comes from the fact that the natural pairing

$$H^1_{\mathrm{dR}}(M;Y) \times \pi_1(M) \to Y$$

defined by integration over loops is non-degenerate in the sense that the integrals separate points in $H^1_{\mathrm{dR}}(M;Y)$.

The arguments for non-compact manifolds essentially follow the same line, where we have to take smooth proper curves instead of loops. ■

VI. Universal central extensions of current groups

For the special class of finite-dimensional semisimple Lie groups K, each Lie algebra cocycle $\omega \in Z^2_c(C^\infty_c(M,\mathfrak{k}),\mathfrak{z})$ is equivalent to a cocycle of product type ([Ma02]). This observation permits us to construct a universal central extension of the Lie algebra $\mathfrak{g} := C^\infty_c(M;\mathfrak{k})$. In the present section we show that this construction can be globalized in the sense that we construct a universal central extension of the connected Lie group $C^\infty_c(M;K)_e$.

First cyclic homology of function spaces

Definition VI.1. Let E, F and G be locally convex spaces over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Then the *projective topology* on the tensor product $E \otimes F$ is defined by the seminorms

$$(p \otimes q)(x) = \inf\Big\{\sum_{j=1}^n p(y_j)q(z_j)\colon x = \sum_j y_j \otimes z_j\Big\},$$

where p, resp., q is a continuous seminorm on E, resp., F (cf. [Tr67, Prop. 43.4]). We write $E \otimes_\pi F$ for the locally convex space obtained by endowing $E \otimes F$ with the locally convex topology defined by this family of seminorms. It is called the *projective tensor product of E and F*. It has the universal property that the continuous bilinear maps $E \times F \to G$ are in one-to-one correspondence with the continuous linear maps $E \otimes_\pi F \to G$ (here we need that G is locally convex). We write $E \widehat{\otimes}_\pi F$ for the completion of the projective tensor product of E and F. ■

Definition VI.2. Let A be a unital locally convex topological algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
(a) We recall that the first *Hochschild homology space* $HH_1(A)$ is defined as

$$HH_1(A) := Z_1(A)/\overline{B_1(A)},$$

where

$$Z_1(A) := \ker b_A \subseteq A \otimes A, \quad b_A(a \otimes b) = [a,b] = ab - ba$$

and

$$B_1(A) := \operatorname{span}\{xy \otimes z - x \otimes yz + zx \otimes y : x, y, z \in A\}.$$

Here we endow $A \otimes A$ with the projective tensor product topology.

Suppose that A is commutative. Then $Z_1(A) = A \otimes A$. Let M be a *continuous A-module*, i.e., M is a locally convex space with an A-module structure given by a continuous bilinear map $A \times M \to M$. For a linear map $D: A \to M$ the bilinear map

$$A \otimes A \to M, \quad x \otimes y \mapsto x.Dy$$

annihilates $B_1(A)$ if and only if D is a derivation. Hence $HH_1(A)$ has the universal property of the universal differential module $\Omega^1(A)$ with respect to the differential

$$d: A \to HH_1(A), \quad a \mapsto [\mathbf{1} \otimes a].$$

This means that for each continuous derivation $D: A \to M$ there exists a unique continuous linear map $\varphi: HH_1(A) \to M$ with $D = \varphi \circ d$ (cf. [Ma02]). Therefore $HH_1(A)$ is isomorphic to the topological module $\Omega^1(A)$ of Kähler differentials on A ([Lo98, Prop. 1.1.10]).
(b) The first *cyclic homology space* of A can be obtained as the quotient

$$HC_1(A) := Z_1^\lambda(A)/\overline{B_1^\lambda(A)},$$

where

$$Z_1^\lambda := \ker b_A \subseteq \Lambda^2(A), \quad b_A(a \wedge b) := [a,b],$$

and

$$B_1^\lambda(A) := \operatorname{span}\{xy \wedge z - x \wedge yz + zx \wedge y : x, y, z \in A\}$$

(cf. [Lo98, Th. 2.15]).

If A is commutative, then $a \otimes b + b \otimes a - 1 \otimes ab \in B_1(A)$ implies that the universal differential $d: A \to HH_1(A)$ satisfies

$$\operatorname{im}(d) = [\mathbf{1} \otimes A] \cong \mathbf{1} \otimes A + B_1(A) = \{a \otimes b + b \otimes a : a, b \in A\} + B_1(A).$$

Hence

$$HH_1(A)/\overline{\operatorname{im} d} \cong \Lambda^2(A)/\overline{B_1^\lambda(A)} \cong HC_1(A)$$

(cf. [Lo98, Prop. 2.1.14]). ∎

Let M be a finite-dimensional manifold and $A := C_c^\infty(M;\mathbb{K})$. According to [Gl01c], the multiplication on $C_c^\infty(M;\mathbb{K})$ is a continuous bilinear map, so that A is a locally convex topological algebra. This is not obvious because the topology on $C_c^\infty(M;\mathbb{K})$ is the locally convex direct limit topology which differs from the direct limit topology with respect to the subspaces $C^\infty_{X_n}(M;\mathbb{K})$, where $(X_n)_{n\in\mathbb{N}}$ is an exhaustive sequence of compact submanifolds with boundary in M. Hence there is no a priori reason for a bilinear map on $C_c^\infty(M;\mathbb{K})$ to be continuous if all the restrictions to the subspaces $C^\infty_{X_n}(M;\mathbb{K})$ are continuous.

Let $A_+ := \mathbb{K}\mathbf{1} + A \subseteq C^\infty(M;\mathbb{K})$. In this section we will show that, as locally convex spaces, we have

$$HH_1(A) := HH_1(A_+) \cong \Omega^1_c(M;\mathbb{K}) \quad \text{and} \quad HC_1(A) \cong \Omega^1_c(M;\mathbb{K})/dA = \mathfrak{z}_{M,c}(\mathbb{K}).$$

Theorem VI.3. (Glöckner's Theorem) *$\Omega^1_c(M;\mathbb{K})$ is a continuous module of $C_c^\infty(M;\mathbb{K})$.*

Proof. This follows from [Gl01d, Th. 5.1] because the module structure is induced by the bundle map given in a point $p \in M$ by the scalar multiplication $\mathbb{K} \times T_p(M)^* \to T_p(M)^*$. ∎

Theorem VI.4. $HH_1(C^\infty_c(M;\mathbb{K})) \cong \Omega^1_c(M;\mathbb{K})$.

Proof. (cf. [Ma02, Th. 11]) We will show that the continuous derivation $d: A = C^\infty_c(M\ \mathbb{K}) \to \Omega^1_c(M;\mathbb{K})$ has the universal property of the universal differential module of A. From this the assertion follows, as $HH_1(A)$ can be viewed as the universal differential module of A (Definition VI.2).

We consider the map

$$\tau: C^\infty(M \times M;\mathbb{K}) \to \Omega^1(M;\mathbb{K}), \quad \tau(F)(x)(v) := dF(x,x)(0,v).$$

Via the natural embedding

$$A_+ \otimes A_+ \to C^\infty(M \times M,\mathbb{K}), \quad (f,g) \mapsto ((x,y) \mapsto f(x)g(y)),$$

we view $A_+ \otimes A_+$ (the algebraic tensor product) as a subalgebra of $C^\infty(M \times M,\mathbb{K})$. This embedding is topological on the subspaces of the form

$$C^\infty_X(M;\mathbb{K}) \otimes_\pi C^\infty_X(M;\mathbb{K})$$

for compact subsets $X \subseteq M$ ([Gr55, Ch. 2, p.81]). Let

$$I := \{F \in A_+ \otimes A_+ : (\forall x \in M) F(x,x) = 0\}.$$

This is an ideal of $A_+ \otimes A_+$ which can also be viewed as the kernel of the multiplication map $\mu: A_+ \otimes A_+ \to A_+$. Note that $\tau(f \otimes g) = f \cdot dg \in \Omega^1_c(M;\mathbb{K})$ for $f,g \in A_+$.

(1) Let $(\varphi_j)_{j\in J}$ be a locally finite partition of unity in A for which $\mathrm{supp}(\varphi_j)$ is contained in a coordinate neighborhood $U_j \subseteq M$ with U_j diffeomorphic to $\mathbb{R}^d$, $d := \dim M$. With this partition of unity we write each $\alpha \in \Omega^1_c(M;\mathbb{K})$ as

$$\alpha = \sum_j \varphi_j \alpha,$$

where the sum is finite because only finitely many of the supports of the functions φ_j intersect the support of α. As $U_j \cong \mathbb{R}^d$ and $\mathrm{supp}(\varphi_j)$ is a compact subset of U_j, there exist functions $\overline{y}^j_1, \ldots, \overline{y}^j_d \in A$ such that on $\mathrm{supp}(\varphi_j)$ the differentials $d\overline{y}^j_i$, $i = 1,\ldots,d$, are linearly independent. Then we write

$$\varphi_j \alpha = \sum_{i=1}^d \alpha^j_i d\overline{y}^j_i$$

with $\alpha^j_i \in A$.

(2) $\tau(A \otimes A) = \tau(A_+ \otimes A_+) = \Omega^1_c(M;\mathbb{K})$: This follows from

$$\alpha = \sum_j \sum_i \alpha^j_i d\overline{y}^j_i = \sum_{j,i} \tau(\alpha^j_i \otimes \overline{y}^j_i).$$

(3) As $\mu(A_+ \otimes 1) = A_+$ and $\tau(A_+ \otimes 1) = 0$, we have $\tau(I) = \tau(A_+ \otimes A_+) = \Omega^1_c(M;\mathbb{K})$ by (2). Let $N := \ker(\tau|_I)$. We claim that $N = \overline{I^2}$. The inclusion $I^2 \subseteq N$ follows directly from

$$\tau(FG) = F\tau(G) + \tau(F)G, \tag{6.1}$$

which also shows that N is an ideal of $A_+ \otimes A_+$. As τ is continuous and I is closed, we also obtain $\overline{I^2} \subseteq N$. Now let $F \in N$. Since F can be written as a finite sum

$$F = \sum_{i,j} (\varphi_i \otimes \varphi_j) F,$$

where each summand is contained in the ideal N, it suffices to assume that $\mathrm{supp}(F) \subseteq U_i \times U_j \cong \mathbb{R}^{2d}$ for some pair $(i,j) \in J^2$. Then we have

$$F(x,y) = \sum_{l=1}^{d} (x_l - y_l) F_l(x,y)$$

with

$$F_l(x,y) := \frac{1}{2} \int_0^1 \frac{\partial F}{\partial x_l}(tx + (1-t)y, y) - \frac{\partial F}{\partial y_l}(x, tx + (1-t)y)\, dt,$$

and it is easy to see that the supports of the functions F_l are compact. From

$$\tau(F)(x) = -\sum_{l=1}^{d} F_l(x,x) dx_l$$

we derive that the functions F_l vanish on the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$, so that Lemma 5 in [Ma02] implies that $F_l \in C^\infty_c(M \times M, \mathbb{K})$ is contained in the closure $\overline{I}$ of the ideal $I \subseteq A_+ \otimes A_+$. Let $C \subseteq \mathbb{R}^d$ be a compact subset such that $C^0 \times C^0$ contains the support of all the functions F_l. We replace the coordinate functions x_j on $\mathbb{R}^d$ by functions $\overline{x}_j \in C^\infty_c(\mathbb{R}^d; \mathbb{K})$ with $\mathrm{supp}(\overline{x}_j) \subseteq C$ and obtain

$$F(x,y) = \sum_{l=1}^{d} (\overline{x}_l - \overline{y}_l) F_l(x,y) \in I \cdot \overline{I} \subseteq \overline{I^2},$$

where the closure is taken in

$$C^\infty_{C \times C}(\mathbb{R}^{2d}, \mathbb{K}) \cong C^\infty_C(\mathbb{R}^d, C^\infty_C(\mathbb{R}^d, \mathbb{K})) \cong C^\infty_C(\mathbb{R}^d; \mathbb{K}) \widehat{\otimes}_\pi C^\infty_C(\mathbb{R}^d, \mathbb{K})$$

(cf. [Gr55, Ch. 2, p.81]).

(4) The derivation $d: A \to \Omega^1_c(M; \mathbb{K})$ has the universal property of the universal topological differential module $\Omega^1(A)$: Let E be a topological A-module and $d_E: A \to E$ a continuous derivation. We will complete the proof by showing that there exists a continuous linear map $\Phi: \Omega^1_c(M; \mathbb{K}) \to E$ with $\Phi(fdg) = f d_E(g)$.

We have seen above that $\ker(\tau|_I) = \overline{I^2} \cap N = \overline{I^2}$ with respect to the relative topology, so that $\tau|_I$ leads to a continuous bijective linear map $I/\overline{I^2} \cong \Omega^1(A) \to \Omega^1_c(M; \mathbb{K})$. Therefore the natural map

$$A_+ \otimes A_+ \supseteq I \to E, \quad f \otimes g \mapsto f d_E(g)$$

yields a linear map

$$\Phi: \Omega^1_c(M; \mathbb{K}) \to E \quad \text{with} \quad \Phi(fdg) = \Phi(\tau(f \otimes g)) = f d_E(g).$$

Hence it only remains to show that Φ is continuous when viewed as a linear map on $\Omega^1_c(M; \mathbb{K})$. As the topology on $\Omega^1_c(M; \mathbb{K})$ is the locally convex direct limit topology with respect to the subspaces $\Omega^1_X(M; \mathbb{K})$, $X \subseteq M$ compact, it suffices to verify that the restrictions $\Phi|_{\Omega^1_X(M;\mathbb{K})}$ are continuous.

The set $J_X := \{ j \in J : \mathrm{supp}(\varphi_j) \cap X \neq \emptyset \}$ is finite, and for each $\alpha \in \Omega^1_X(M; \mathbb{K})$ we have

$$\alpha = \sum_{j \in J_X} \varphi_j \alpha = \sum_{j \in J_X} \sum_i \alpha_i^j d\overline{y}_i^j.$$

Now

$$\Phi(\alpha) = \sum_{j \in J_X} \Phi(\varphi_j \alpha) = \sum_{j \in J_X} \sum_i \alpha_i^j d_E(\overline{y}_i^j)$$

because the sum is finite. The functions $\overline{y}_i^j$ do not depend on α, and the multiplication with φ_j is a continuous endomorphism of $\Omega^1_c(M; \mathbb{K})$. Therefore the maps

$$\Omega^1_c(M; \mathbb{K}) \to A, \quad \alpha \mapsto \alpha_i^j$$

are continuous. Now the continuity of the module structure on E implies that Φ is continuous. ■

Corollary VI.5. *For $A = C^\infty_c(M; \mathbb{K})$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we have*

$$HC_1(A) \cong HH_1(A)/dA \cong \Omega^1_c(M; \mathbb{K})/dC^\infty_c(M; \mathbb{K}).$$ ■

Universal central extensions

In this subsection we turn to the question whether for a finite-dimensional semisimple Lie group K the central extension of $C^\infty_c(M,K)_e$ from Proposition V.19 is universal. This question will be answered affirmatively if $\mathfrak{k}$ is finite-dimensional and semisimple. First we recall some concepts and a result from [Ne01c] on weakly universal central extensions of Lie groups and Lie algebras.

Definition VI.6. (cf. [Ne01c]) Let $\mathfrak{g}$ be a topological Lie algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathfrak{a}$ be a topological vector space considered as a trivial $\mathfrak{g}$-module. We call a central extension $q\colon \widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$ with $\mathfrak{z} = \ker q$ (or simply the Lie algebra $\widehat{\mathfrak{g}}$) *weakly universal for* $\mathfrak{a}$ if the corresponding map $\delta_{\mathfrak{a}}\colon \mathrm{Lin}(\mathfrak{z},\mathfrak{a}) \to H^2_c(\mathfrak{g},\mathfrak{a}), \gamma \mapsto [\gamma \circ \omega]$ is bijective.

We call $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ *universal for* $\mathfrak{a}$ if for every central extension $q_1\colon \widehat{\mathfrak{g}}_1 \to \mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{a}$ with a continuous linear section there exists a unique homomorphism $\varphi\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}_1$ with $q_1 \circ \varphi = q$. Note that this universal property immediately implies that two central extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by $\mathfrak{a}_1$ and $\mathfrak{a}_2$ such that both $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ are universal for $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are isomorphic. A central extension is said to be *(weakly) universal* if it is (weakly) universal for all locally convex spaces $\mathfrak{a}$. ■

Definition VI.7. We call a central extension $\widehat{G} = G \times_f Z$ of the connected Lie group G by the abelian Lie group Z given by $f \in Z^2_s(G,Z)$ *weakly universal for the abelian Lie group* A if the map

$$\delta_A\colon \mathrm{Hom}(Z,A) \to H^2_s(G,A), \quad \gamma \mapsto [\gamma \circ f]$$

is bijective. It is called *universal for the abelian Lie group* A if for every central extension

$$q_1\colon G \times_\varphi A \to G, \quad \varphi \in Z^2_s(G,A),$$

there exists a unique Lie group homomorphism $\psi\colon G \times_f Z \to G \times_\varphi A$ with $q_1 \circ \psi = q$ (cf. Definition V.1). A central extensional is said to be *(weakly) universal* if it is (weakly) universal for all Lie groups A with $A_e \cong \mathfrak{a}/\pi_1(A)$ and $\mathfrak{a}$ s.c.l.c. ■

Definition VI.8. If $\mathfrak{g}$ is a locally convex Lie algebra, then we write $H_1(\mathfrak{g})$ for the completion of the quotient space $\mathfrak{g}/\overline{[\mathfrak{g},\mathfrak{g}]}$. If $\mathfrak{g}$ is a Fréchet space, then $\mathfrak{g}/\overline{[\mathfrak{g},\mathfrak{g}]}$ is also Fréchet, and no completion is necessary.

If G is a connected Lie group with Lie algebra $\mathfrak{g}$ and $\widetilde{G}$ its universal covering group, then we have a natural homomorphism $d_G\colon \widetilde{G} \to H_1(\mathfrak{g})$. Its kernel is denoted by $(\widetilde{G},\widetilde{G})$. If G is finite-dimensional, then $(\widetilde{G},\widetilde{G})$ is the commutator group of $\widetilde{G}$. ■

Theorem VI.9. (Recognition Theorem) *Assume that $q\colon \widehat{G} \to G$ is a central Z-extension of Lie groups over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ for which*

(1) *the corresponding Lie algebra extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is weakly $\mathbb{K}$-universal,*

(2) *$\widehat{G}$ is simply connected, and*

(3) $\pi_1(G) \subseteq (\widetilde{G},\widetilde{G})$.

If $\widehat{\mathfrak{g}}$ is weakly universal for a s.c.l.c. space $\mathfrak{a}$, then $\widehat{G}$ is weakly universal for each abelian Lie group A with $A_e \cong \mathfrak{a}/\pi_1(A)$.

Proof. The original statement of this theorem in [Ne01c, Th. IV.13] is formulated only for Fréchet–Lie groups, but one easily verifies that the proof yields the more general result stated above. ■

Theorem VI.10. *Let K be a finite-dimensional semisimple Lie group and $G := C^\infty_c(M,K)_e$. Further let $\mathfrak{z} := \mathfrak{z}_{M,c}(V(\mathfrak{k}))$ and $\omega = \omega_{M,\kappa} \in Z^2_c(\mathfrak{g},\mathfrak{z})$ be a cocycle of product type given by $\omega(\eta,\xi) = [\kappa(\eta, d\xi)]$. Then the corresponding central Lie algebra extension $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z}$ is universal, and there exists a corresponding central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ with*

$Z \cong \pi_1(G) \times (\mathfrak{z}/\Pi_M)$ which is universal for all Lie groups A with $A_e \cong \mathfrak{a}/\Gamma$, where $\mathfrak{a}$ is a s.c.l.c. space and $\Gamma \subseteq \mathfrak{a}$ a discrete subgroup.

Proof. First we show that $\widehat{\mathfrak{g}}$ is perfect. In fact, for $x, y \in \mathfrak{k}$ and $f, g \in C^\infty(M; \mathbb{K})$ we have in $\widehat{\mathfrak{g}}$ the relation

$$[f \otimes x, g \otimes y] - [g \otimes x, f \otimes y] = \big(fg \otimes [x,y] - gf \otimes [x,y], 2[fdg] \cdot \kappa(x,y)\big) = \big(0, 2[fdg] \cdot \kappa(x,y)\big).$$

Since $V(\mathfrak{k})$ is spanned by $\mathrm{im}(\kappa)$, the fact that $\mathfrak{z}_{M,c}(\mathbb{K})$ is spanned by elements of the form $[f \cdot dg]$ implies that $\widehat{\mathfrak{g}}$ is perfect.

Since $\widehat{\mathfrak{g}}$ is perfect, for each locally convex space $\mathfrak{a}$ the natural map

$$\delta\colon \mathrm{Lin}(\mathfrak{z}, \mathfrak{a}) \to H^2_c(\mathfrak{g}, \mathfrak{a}), \quad \gamma \mapsto [\gamma \circ \omega]$$

is injective ([Ne01c, Rem. I.6]). It has been shown in [Ma02, Thm. 16] that δ is also surjective, so that $\widehat{\mathfrak{g}}$ is weakly universal for all locally convex spaces $\mathfrak{a}$. Since $\widehat{\mathfrak{g}}$ is perfect, it even is a universal central extension of $\mathfrak{g}$ ([Ne01c, Lemma I.12]).

Furthermore, the period map $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ has discrete image Π_ω (Proposition V.19). In view of Theorem V.8, Theorem V.5 now implies the existence of a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ with $Z \cong (\mathfrak{z}/\Pi_\omega) \times \pi_1(G)$ corresponding to the Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \to \mathfrak{g}$ and such that the connecting homomorphism $\pi_1(G) \to \pi_0(Z)$ is an isomorphism.

To prove the universality of $\widehat{G}$, we use the Recognition Theorem VI.9. For that we have to verify that

(1) $\widehat{\mathfrak{g}}$ is weakly universal,

(2) $\pi_1(\widehat{G}) = \mathbf{1}$,

(3) $\pi_1(G) \subseteq (\widetilde{G}, \widetilde{G})$.

Condition (1) has been verified above. Further (3) follows from the perfectness of $\mathfrak{g}$, which implies $(\widetilde{G}, \widetilde{G}) = \widetilde{G}$. It therefore remains to verify (2). For that we consider a part of the long exact homotopy sequence of the Z-principal bundle $q\colon \widehat{G} \to G$:

$$\pi_2(G) \xrightarrow{\delta} \pi_1(Z) \to \pi_1(\widehat{G}) \to \pi_1(G) \to \pi_0(Z). \tag{6.2}$$

According to [Ne02a, Prop. 5.11], we have $\delta = -\mathrm{per}_\omega$, so that $\pi_1(Z) = \Pi_\omega$ (as subsets of $\mathfrak{z}$) implies that δ is surjective. Moreover, the natural homomorphism $\pi_1(G) \to \pi_0(Z)$ is an isomorphism by the construction of $\widehat{G}$, so that the exactness of (6.2) implies that $\widehat{G}$ is simply connected. ∎

Remark VI.11. (a) If K is finite-dimensional and reductive, then $\widetilde{K} \cong \mathfrak{z}(\mathfrak{k}) \times (\widetilde{K}, \widetilde{K})$. Therefore $\pi_1(K)$ is contained in $(\widetilde{K}, \widetilde{K})$ if and only if $K \cong \mathfrak{z}(\mathfrak{k}) \times (K, K)$. In this case we have

$$C^\infty(M, K) \cong C^\infty(M, \mathfrak{z}(\mathfrak{k})) \times C^\infty(M, (K, K))$$

and hence we have for $G = C^\infty(M, K)_e$ the direct product decomposition

$$G = G_D \times G_Z \quad \text{with} \quad G_D := C^\infty(M, (K, K))_e \quad \text{and} \quad G_Z := C^\infty(M, \mathfrak{z}(\mathfrak{k})).$$

In this case the Lie algebra $\mathfrak{g} = C^\infty(M; \mathfrak{k})$ has the direct decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$ with $\mathfrak{g}' = C^\infty(M; \mathfrak{k}')$ and $\mathfrak{z}(\mathfrak{g}) = C^\infty(M; \mathfrak{z}(\mathfrak{k}))$, where $\mathfrak{k}'$, resp., $\mathfrak{g}'$ denote the commutator algebra. It is easy to see that every Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g}; Y)$ vanishes on $\mathfrak{g}' \times \mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g} \times \mathfrak{g}$ because $\mathfrak{g}'$ is perfect. From that one further derives that a weakly universal central extension of $\mathfrak{g}$ can be obtained with

$$\mathfrak{z} := \mathfrak{z}_M(V(\mathfrak{k}')) \oplus \Lambda^2(\mathfrak{z}(\mathfrak{g})),$$

where for a locally convex space E the space $\Lambda^2(E)$ is defined as the quotient of $E \otimes_\pi E$ modulo the closure of the subspace spanned by the elements $e \otimes e$, $e \in E$. To describe the corresponding cocycle, we write $\xi \in \mathfrak{g}$ as $\xi = (\xi', \xi_{\mathfrak{z}})$ with $\xi' \in \mathfrak{g}'$ and $\xi_{\mathfrak{z}} \in \mathfrak{z}(\mathfrak{g})$. Then a weakly universal cocycle is given by

$$\omega(\xi, \eta) = ([\kappa_{\mathfrak{k}'}(\xi', d\eta')], \xi_{\mathfrak{z}} \wedge \eta_{\mathfrak{z}}).$$

Let $\widehat{G}_D$ be the universal central extension of G_D from Theorem VI.10 and define $\widehat{G} := \widehat{G}_D \times \widehat{G}_Z$, where $\widehat{G}_Z$ is the 2-step nilpotent Lie algebra

$$\mathfrak{z}(\mathfrak{g}) \times_{\omega_Z} \Lambda^2(\mathfrak{z}(\mathfrak{g})) \quad \text{with} \quad \omega_Z(\xi, \eta) = \xi \wedge \eta,$$

viewed as a Lie group with the multiplication $x * y := x + y + \frac{1}{2}[x, y]$. Using Theorem VI.9, we see that $\widehat{G}_Z$ is a weakly universal central extension of $G_Z \cong \mathfrak{g}_Z$. Theorems VI.9 and VI.10 now imply that $\widehat{G}$ is a weakly universal central extension of G. ■

Appendix A. Homotopy groups of smooth current groups

In this section we show that the homotopy groups of the Lie groups of smooth maps $C^\infty_c(M; K)$, resp., $C^\infty(M, S; K)$ introduced in Section I coincide with the homotopy groups of the corresponding groups of continuous maps $C_0(M; K)$, resp., $C_0(M \setminus S; K)$. The latter groups are usually better accessible by means of topological methods.

More specifically, for the group $C^\infty_c(M; K)$ of compactly supported smooth functions on a manifold M with values in a Lie group K the main result will be that the inclusion $C^\infty_c(M; K) \hookrightarrow C_0(M; K)$ is a weak homotopy equivalence. For the group $C^\infty(M, S; K)$ of smooth maps on a compact manifold M vanishing with all derivatives on a closed subset S we show that the inclusion $C^\infty(M, S; K) \hookrightarrow C_0(M \setminus S; K)$ is a weak homotopy equivalence.

In the present paper the results of this section are mainly needed to get information on the second homotopy group which is important for period maps associated to Lie algebra cocycles (cf. Section V). Moreover, the results of this appendix are of independent interest in many other contexts, where they provide valuable information on the topology of current groups.

Groups of compactly supported functions

Lemma A.1. *For each compact subset E of $C^\infty_c(M; K)$ there exists a compact subset $X \subseteq M$ with $E \subseteq C^\infty_X(M; K)$.*

Proof. Let $U \subseteq \mathfrak{k} := \mathbf{L}(K)$ be an open 0-neighborhood and $\varphi: U \to \varphi(U)$ a chart with $\varphi(0) = e$. Then there exists an open 0-neighborhood $U_0 \subseteq U$ such that we obtain a local chart for $G := C^\infty_c(M; K)$ by $\varphi_G(f) := \varphi \circ f$ (Definition I.2(b)). Let $V := \{f \in C^\infty_c(M; \mathfrak{k}) : f(M) \subseteq U_0\}$ and observe that

$$\varphi_G: V \to \varphi_G(V) = \{f \in C^\infty_c(M; K) : f(M) \subseteq \varphi(U_0)\}.$$

Then for each $f \in G$ the set $f\varphi_G(V)$ is an open neighborhood, and the map

$$\varphi_f: V \to f\varphi_G(V), \quad \xi \mapsto f\varphi_G(\xi)$$

is a diffeomorphism. Let $W \subseteq V$ be a closed 0-neighborhood such that $\varphi_G(W)\varphi_G(W) \subseteq \varphi_G(V)$. Since $\overline{\varphi_G(W)}$ is the intersection of all sets $\varphi_G(W)N$, where N is an identity neighborhood in $C^\infty_c(M; K)$, $\overline{\varphi_G(W)} \subseteq \varphi_G(V)$, so that the closedness of W implies that $\varphi_G(W)$ is closed.

Since the compact set E is covered by the open sets $f\varphi_G(W^0)$, $f \in E$, there exist $f_1, \ldots, f_n \in E$ with

$$E \subseteq f_1\varphi_G(W^0) \cup \ldots \cup f_n\varphi_G(W^0).$$

The closedness of $\varphi_G(W)$ implies that each set $E \cap f_j\varphi_G(W)$ is compact, so that for each j the closed set

$$\varphi_{f_j}^{-1}(E \cap f_j\varphi_G(W)) = W \cap \varphi_{f_j}^{-1}(E) \subseteq C^\infty_c(M; \mathfrak{k}) = \varinjlim C^\infty_X(M; \mathfrak{k})$$

is compact, so that there exists a compact subset $X_j \subseteq M$ with $\varphi_{f_j}^{-1}(E \cap f_j\varphi_G(W)) \subseteq C^\infty_{X_j}(M;\mathfrak{k})$ ([He89, Prop. 1.5.3]). Let

$$X := X_1 \cup \ldots \cup X_n \cup \mathrm{supp}(f_1) \cup \ldots \cup \mathrm{supp}(f_n).$$

Then X is compact and $E \subseteq C^\infty_X(M;K)$. ■

Lemma A.2. *Let E be a compact space and $f\colon E \to C^\infty_c(M;K)$ a continuous map. Then there exists a compact subset $X \subseteq M$ and a continuous map $f_X\colon E \to C^\infty_X(M;K)$ such that $f = \eta_X \circ f_X$ holds for the inclusion map $\eta_X\colon C^\infty_X(M;K) \to C^\infty_c(M;K)$.*

Proof. Since $C^\infty_c(M;K)$ is Hausdorff, the set $f(E)$ is compact. In view of Lemma A.1, there exists a compact subset $X \subseteq M$ with $f(E) \subseteq C^\infty_X(M;K)$. Let $f_X\colon E \to C^\infty_X(M;K)$ denote the corestriction of f to $C^\infty_X(M;K)$. Since η_X is a topological embedding (Remark I.3), the map f_X is continuous. It obviously satisfies $f = \eta_X \circ f_X$. ■

Proposition A.3. *Let $X_n \subseteq M$ be compact with $X_n \subseteq X^0_{n+1}$ and $M = \bigcup_n X_n$. Then the map*

$$\varinjlim C^\infty_{X_n}(M;K) \to C^\infty_c(M;K)$$

is a weak homotopy equivalence. In particular $\pi_m(C^\infty_c(M;K)) \cong \varinjlim \pi_m(C^\infty_{X_n}(M;K))$ for each $m \in \mathbb{N}_0$.

Proof. Lemma A.2 first implies that each continuous map $f\colon \mathbb{S}^m \to C^\infty_c(M;K)$ factors through some inclusion $C^\infty_{X_n}(M;K) \to C^\infty_c(M;K)$. If two such maps f_1, f_2 are homotopic, then each homotopy $h\colon \mathbb{S}^m \times [0,1] \to C^\infty_c(M;K)$ also factors through some group $C^\infty_{X_k}(M;K)$. This implies that the natural map

$$\varinjlim \pi_m(C^\infty_{X_n}(M;K)) \cong \pi_m(\varinjlim C^\infty_{X_n}(M;K)) \to \pi_m(C^\infty_c(M;K))$$

is bijective, i.e., that the continuous map $\varinjlim C^\infty_{X_n}(M;K) \to C^\infty_c(M;K)$ is a weak homotopy equivalence. ■

Remark A.4. A similar argument as the one leading to Proposition A.3 shows that the map

$$\varinjlim C_{X_n}(M;K) \to C_c(M;K)$$

is a weak homotopy equivalence. ■

If M and N are topological spaces, we write $[M,N]$ for the set of homotopy classes of continuous maps $f\colon M \to N$. If, in addition, $x_M \in M$ and $x_N \in N$ are base points, then $C_*(M,N) := \{f \in C(M,N)\colon f(x_M) = x_N\}$ denotes the set of base point preserving continuous maps and $[M,N]_*$ denotes the corresponding set of homotopy classes. We recall that if M is locally compact, then homotopy classes correspond to arc components in the compact open topology.

Eventually we want to show that the map

$$C^\infty_c(M;K) \to C_c(M;K)$$

is a weak homotopy equivalence, so that the homotopy groups of $C^\infty_c(M;K)$ are the limits of the corresponding homotopy groups of $C_X(M;K)$. These groups are more approachable since they are isomorphic to $C_*(X/\partial X;K)$, where $X/\partial X$ is a compact space, with the image of ∂X as the base point.

If M is a compact manifold with boundary, then the homotopy groups $\pi_m(C_*(M/\partial M;K))$ might be well accessible. Note that if ∂M is empty, then $C_*(M/\partial M;K)$ should be read as the group $C(M;K)$.

Lemma A.5. *Let $X_1, X_2 \subseteq M$ be compact subsets with $X_1 \subseteq X_2^0$ and $f \in C_{X_1}(M;K)$. Then every neighborhood of f contains a map f' in $C^\infty_{X_2}(M;K)$. The image of the homomorphism*

$$\eta\colon \pi_0(C^\infty_{X_2}(M;K)) \to \pi_0(C_{X_2}(M;K))$$

contains the image of $\pi_0(C_{X_1}(M;K))$. Moreover, if f is contained in $C_{X_1}(M;K)_e$, then we may choose $f' \in C^\infty_{X_2}(M;K)_e$.

Proof. The first assertion follows from [Ne02a, Th. A.3.7]. Since the groups $C_X(M;K)$ and $C^\infty_X(M;K)$ are Lie groups, their connected components are open, so that every connected component of $C_{X_2}(M;K)$ meeting $C_{X_1}(M;K)$ contains a smooth element.

If the map $f \in C_{X_1}(M;K)$ is sufficiently close to e in the sense that $f(M) \subseteq V$ for some chart e-neighborhood $V \subseteq K$ diffeomorphic to an open convex set, we find $f_1 \in C^\infty_{X_2}(M;K)$ with $f_1(M) \subseteq V$. Now any two smooth maps $f_1, f_2 \in C^\infty_{X_2}(M;K)$ with $f_j(M) \subseteq V$ are smoothly homotopic, hence contained in the same connected component of $C^\infty_{X_2}(M;K)$.

If $f \in C_{X_1}(M;K)$ is contained in the identity component, then there exists a continuous curve $\gamma\colon [0,1] \to C_{X_1}(M;K)$ with $\gamma(0) = e$ and $\gamma(1) = f$. For a sufficiently fine subdivision $0 = t_0 < t_1 < \ldots < t_N = 1$ we now find smooth maps $f_j \in C^\infty_{X_2}(M;K)$ close to $\gamma(t_j)$ in the sense that $(f_j^{-1} \cdot \gamma(t_i))(M) \subseteq V$, where for $j < N$ the maps f_j and f_{j+1} are smoothly homotopic. Hence f_N is contained in the identity component of $C^\infty_{X_2}(M;K)$. ■

Lemma A.6. *The map $\iota\colon C^\infty_c(M;K) \to C_c(M;K)$ induces an isomorphism*

$$\pi_0(\iota)\colon \pi_0(C^\infty_c(M;K)) \to \pi_0(C_c(M;K)).$$

Proof. The surjectivity of $\pi_0(\iota)$ follows directly from Lemma A.5. If $f \in C^\infty_c(M;K)$ satisfies $[f] \in \ker \pi_0(\iota)$, then there exists a compact subset $X \subseteq M$ and a continuous map $\gamma\colon [0,1] \to C_X(M;K)$ with $\gamma(0) = e$ and $\gamma(1) = f$ (Lemma A.2). Let $Y \subseteq M$ be a compact subset with $X \subseteq Y^0$. Then Lemma A.5 implies that we can approximate f by smooth functions f' in the identity component of $C^\infty_Y(M;K)$. It follows in particular that f is contained in the identity component of $C^\infty_Y(M;K)$, hence also in the identity component of $C^\infty_c(M;K)$. This shows that $\pi_0(\iota)$ is injective. ■

In M we fix a base point x_M and in any group we consider the unit element e as the base point. We write $C^\infty_*(M;K) \subseteq C^\infty(M;K)$ for the subgroup of base point preserving maps and observe that

$$C^\infty(M;K) \cong C^\infty_*(M;K) \rtimes K$$

as Lie groups, where we identify K with the subgroup of constant maps. This relation already leads to

$$\pi_k(C^\infty(M;K)) \cong \pi_k(C^\infty_*(M;K)) \times \pi_k(K), \quad k \in \mathbb{N}_0. \tag{A.1}$$

In particular we have

$$\pi_0(C^\infty(M;K)) \cong \pi_0(C^\infty_*(M;K))$$

if K is connected.

On the other hand, we have for each topological group G and each $k \in \mathbb{N}$ the relation

$$\pi_k(G) \cong \pi_0(C_*(\mathbb{S}^k, G)) = \pi_0(C_*(\mathbb{S}^k, G_e)) = \pi_0(C(\mathbb{S}^k, G_e)), \tag{A.2}$$

where G_e denotes the arc-component of the identity in G.

The following theorem is one of the two main results of this section. It provides a valuable tool to determine the homotopy groups of groups of smooth maps in terms of the corresponding groups of continuous maps.

Theorem A.7. *If M is a connected σ-compact finite-dimensional manifold and K a Lie group, then the inclusion $C_c^\infty(M;K) \to C_c(M;K)$ is a weak homotopy equivalence. If M is compact and $x_M \in M$ is a base point, then the inclusion*

$$C_*^\infty(M;K) \to C_*(M;K) := \{f \in C(M;K) \colon f(x_M) = e\} \tag{A.3}$$

is a weak homotopy equivalence.

Proof. We have to show that the inclusion induces for each $k \in \mathbb{N}_0$ an isomorphism

$$\pi_k(C_c^\infty(M;K)) \to \pi_k(C_c(M;K)).$$

For $k = 0$ this is Lemma A.6. If M is compact, then

$$\pi_0(C_c^\infty(M;K)) = \pi_0(C^\infty(M;K)) \cong \pi_0(C_*^\infty(M;K)) \times \pi_0(K)$$

and

$$\pi_0(C_c(M;K)) = \pi_0(C(M;K)) \cong \pi_0(C_*(M;K)) \times \pi_0(K),$$

so that (A.3) follows from Lemma A.6. We only observe that if f_t is a homotopy between f_0 and f_1 in $C_c^\infty(M;K)$ and $x_M \in M$ is a base point, then $\widetilde{f}_t(x) f_t(x_M)^{-1}$ is a homotopy between f_0 and f_1 in $C_*^\infty(M;K)$.

Next we assume that $k \geq 1$ and observe that the inclusions

$$\begin{aligned} C_*(\mathbb{S}^k, C_c^\infty(M;K)) = C_*(\mathbb{S}^k, C_c^\infty(M;K)_e) &\hookrightarrow C(\mathbb{S}^k, C_c^\infty(M;K)_e) \hookrightarrow C(\mathbb{S}^k, C_c(M;K)_e) \\ &\hookrightarrow C(\mathbb{S}^k, C_c(M;K)) \cong C_c(\mathbb{S}^k \times M;K) \end{aligned}$$

are continuous homomorphisms of Lie groups, where

$$C(\mathbb{S}^k, C_c(M;K)_e) \hookrightarrow C(\mathbb{S}^k, C_c(M;K))$$

is an open embedding. For the group of connected components, we obtain for $k \geq 1$ with (A.2) the homomorphisms

$$\begin{aligned} \pi_k(C_c^\infty(M;K)) &\cong \pi_0\big(C_*(\mathbb{S}^k, C_c^\infty(M;K))\big) \cong \pi_0\big(C(\mathbb{S}^k, C_c^\infty(M;K)_e)\big) \\ &\to \pi_0\big(C(\mathbb{S}^k, C_c(M;K)_e)\big) \cong \pi_k\big(C_c(M;K)\big). \end{aligned}$$

If $f \colon \mathbb{S}^k \times M \to K$ is a continuous map with compact support corresponding to an element of $C_*(\mathbb{S}^k; C_c(M;K)_e)$, then Lemma A.5 first implies that every neighborhood of f contains a smooth map with compact support. Thus every connected component of $C_c(\mathbb{S}^k \times M;K)$ contains an element of $C(\mathbb{S}^k, C_c^\infty(M;K))_e$ by the openness argument from above. This means that the homomorphism $\pi_k(C_c^\infty(M;K)) \to \pi_k(C_c(M;K))$ is surjective. To see that it is injective, suppose that $\sigma \in C\big(\mathbb{S}^k, C_c^\infty(M;K)_e\big)$ satisfies $\sigma \in C\big(\mathbb{S}^k, C_c(M;K)_e\big)_e \cong C_c(\mathbb{S}^k \times M;K)_e$. From Lemma A.6 we obtain

$$C_c^\infty(\mathbb{S}^k \times M;K) \cap C_c(\mathbb{S}^k \times M;K)_e \subseteq C_c^\infty(\mathbb{S}^k \times M;K)_e,$$

so that approximating σ by elements in $C_c^\infty(\mathbb{S}^k \times M;K)$ (Lemma A.5), we see that we may even approximate it by elements in $C_c^\infty(\mathbb{S}^k \times M;K)_e$, which implies that σ lies in the identity component of $C\big(\mathbb{S}^k, C_c^\infty(M;K)_e\big)$. This proves that the homomorphisms $\pi_k(C_c^\infty(M;K)) \to \pi_k(C_c(M;K))$, $k \in \mathbb{N}_0$, are isomorphisms. ∎

Theorem A.7 can also be extended to non-connected manifolds M as follows. Let $M = \bigcup_{j \in J} M_j$ be the decomposition of M into connected components M_j. Here one can use

$$C_c(M;K) = \bigoplus_{j \in J} C_c(M_j;K),$$

and for each compact subset $X \subseteq M$ we have the finite sum decomposition

$$C_X(M;K) = \bigoplus_{X \cap M_j \neq \emptyset} C_{X \cap M_j}(M_j;K).$$

If M has only finitely many connected components, then there is no problem, but if M has infinitely many connected components, then one has to take the direct sum topology on $C_c(M;\mathfrak{k})$ into account and the corresponding Lie group topology on $C_c(M;K)$.

Lemma A.8 and Proposition A.9 provide additional information on the homotopy type of the topological current groups.

Lemma A.8. *If M is a locally compact space, then the inclusion $\eta\colon C_c(M;K) \to C_0(M;K)$ induces an isomorphism $\pi_0(C_c(M;K)) \to \pi_0(C_0(M;K))$.*

Proof. Let $f \in C_0(M;K)$. Then there exists a compact subset $X \subseteq M$ such that $f(M\setminus X)$ is contained in an identity neighborhood of K which is diffeomorphic to a convex 0-neighborhood U in $\mathfrak{k}$, where 0 corresponds to $e \in K$. Using a continuous function $h \in C_c(M;\mathbb{R})$ which is 1 on X and satisfies $h(M) \subseteq [0,1]$, we define a function $\widetilde{f} \in C_c(M;K)$ by $\widetilde{f} = f$ on X and $\widetilde{f} = hf$ on $M \setminus X$, where we consider $f|_{M\setminus X}$ as a function with values in U. Then

$$F\colon M \times [0,1] \to K, \quad F(x,t) := \begin{cases} f(x) & \text{for } x \in X \\ (t + (1-t)h(x))f(x) & \text{for } x \in M \setminus X \end{cases}$$

is a homotopy between f and $\widetilde{f}$, and we see that $\pi_0(\eta)$ is surjective.

A similar argument shows that for $f, g \in C_c(M;K)$ any path joining f and g in $C_0(M;K)$ can be deformed to a path lying completely inside of $C_X(M;K)$ for a compact subset X of M. Therefore $\pi_0(\eta)$ is injective. ∎

Proposition A.9. *If M is a locally compact space, then the inclusion $\eta\colon C_c(M;K) \to C_0(M;K)$ is a weak homotopy equivalence.*

Proof. Let $M_\infty = M \cup \{\infty\}$ denote the one-point compactification of M. For every compact space X we have an embedding of topological groups

$$C(X, C_0(M;K)) \cong C(X, C_*(M_\infty;K)) \hookrightarrow C(X, C(M_\infty;K)) \cong C(X \times M_\infty;K),$$

which easily leads to the isomorphism

$$C(X, C_0(M;K)) \cong C_0(X \times M;K).$$

In view of Lemma A.8, there exists for each $f \in C_0(X \times M;K)$ some compact subset $Y \subseteq M$ and a continuous map $f_Y \in C(X, C_Y(M;K)) \subseteq C(X \times Y;K)$ homotopic to f. The same argument applies to $[0,1] \times X$ instead of X, so that we see that the inclusion $C_c(M;K) \to C_0(M;K)$ induces a bijection $[X, C_c(M;K)] \to [X, C_0(M;K)]$ on the level of homotopy classes.

Applying this to $X := \mathbb{S}^k$, $k \in \mathbb{N}$, we obtain with Lemma A.8 that the natural map

$$\begin{aligned}\pi_k(C_c(M;K)) &\cong [\mathbb{S}^k, C_c(M;K)]_* \cong [\mathbb{S}^k, C_c(M;K)_e] \to [\mathbb{S}^k, C_0(M;K)_e] \\ &\cong [\mathbb{S}^k, C_0(M;K)]_* \cong \pi_k(C_0(M;K))\end{aligned}$$

is bijective, hence an isomorphism of groups. ∎

Theorem A.10. *For each σ-compact connected finite-dimensional manifold M and each Lie group K the inclusion map*

$$C^\infty_c(M;K) \to C_0(M;K) \cong C_*(M_\infty;K)$$

is a weak homotopy equivalence.

Proof. We only have to combine Proposition A.9 with Theorem A.7. ∎

Example A.11. For $M = \mathbb{R}^n$ we obtain with Theorem A.10 for each $k \in \mathbb{N}_0$:

$$\pi_k(C^\infty_c(\mathbb{R}^n;K)) \cong \pi_k(C_*(\mathbb{R}^n_\infty;K)) \cong \pi_k(C_*(\mathbb{S}^n;K)) \cong \pi_{k+n}(K). \qquad ∎$$

Lemma A.12. *Let $\varphi\colon N \to M$ be a smooth proper map.*

(i) *The map*

$$\varphi_K\colon C^\infty_c(M;K) \to C^\infty_c(N;K), \quad f \mapsto f \circ \varphi$$

is a morphism of Lie groups.

(ii) *Let* $\varphi_\infty : M_\infty \to N_\infty$ *denote the continuous extension of* φ *to the one-point compactifications. Then for each* $k \in \mathbb{N}_0$ *the map*

$$\pi_k(\varphi_K): \pi_k(C_c^\infty(M;K)) \to \pi_k(C_c^\infty(N;K))$$

only depends on the homotopy class of φ_∞ *in the set* $[M_\infty, N_\infty]_*$ *of pointed homotopy classes.*

Proof. (i) It is clear that φ_K maps $C_c^\infty(N;K)$ into $C_c^\infty(M;K)$ and that it is a group homomorphism. It therefore suffices to show smoothness in some identity neighborhood.

Let $U \subseteq K$ be an open identity neighborhood and $\psi: U \to W$ a chart of K where $W \subseteq \mathfrak{k}$ is an open subset and $\psi(e) = 0$. Then there exists an open 0-neighborhood $V \subseteq W$ such that

$$C_c^\infty(N, W) := \{f \in C_c^\infty(N;K): f(N) \subseteq \psi^{-1}(V)\}$$

is an open subset of $C_c^\infty(N;K)$ ([Gl01b]). Now it suffices to see that the map

$$C_c^\infty(M, V) \to C_c^\infty(N, V), \quad f \mapsto f \circ \varphi$$

is smooth. As this map is the restriction of a linear map, we only have to show that it is continuous.

For each compact subset $X \subseteq M$ we have

$$C_X^\infty(M;K) \circ \varphi \subseteq C_{\varphi^{-1}(X)}^\infty(M;K),$$

so that the assertion follows from the observation that for each $n \in \mathbb{N}$ the map $d^n(f \circ \varphi)$ depends continuously on f, when considered as an element of $C(T^n(N), \mathfrak{k})_c$ (cf. Definition I.2).

(ii) Let $\eta_M: C_c^\infty(M;K) \to C_*(M_\infty;K)$ denote the natural inclusion. Then $\eta_N \circ \varphi_K = \widetilde{\varphi}_K \circ \eta_M$ holds with

$$\widetilde{\varphi}_K: C_*(N_\infty;K) \to C_*(M_\infty;K), \quad f \mapsto f \circ \varphi.$$

We know from Theorem A.10 that the maps η_M and η_N are weak homotopy equivalences. Therefore it suffices to show that the maps $\pi_k(\widetilde{\varphi}_K)$ only depend on the homotopy class of φ. If $\varphi, \psi: M \to N$ are proper and smooth such that φ_∞ and ψ_∞ are homotopic, then it is easy to see that the maps $\widetilde{\varphi}_K$ and $\widetilde{\psi}_K$ are homotopic, hence induce the same homomorphisms on homotopy groups. ■

Homotopy groups of groups defined by vanishing conditions

In this subsection we discuss the other major class of groups of smooth maps $C^\infty(M, S; K)$. Theorem A.13 is a variant of Theorem A.7 for this context.

Theorem A.13. *Let* M *be a compact manifold,* $S \subseteq M$ *a closed subset and* $C^\infty(M, S; K)$ *the subgroup of* $C^\infty(M;K)$ *consisting of all smooth maps vanishing together with all their partial derivatives on* S. *Then the inclusion*

$$\eta: C_c^\infty(M \setminus S; K) \to C^\infty(M, S; K)$$

is a weak homotopy equivalence.

Proof. As M is compact, the group $C^\infty(M, S; K)$, when considered as a group of maps $M \setminus S \to K$, is contained in $C_0(M \setminus S; K)$. The inclusion $C_c^\infty(M \setminus S; K) \to C_0(M \setminus S; K)$ is a weak homotopy equivalence by Theorem A.10, so that all the maps $\pi_k(\eta)$, $k \in \mathbb{N}_0$, are injective. It therefore remains to show that they are also surjective.

So let

$$\sigma \in C_*(\mathbb{S}^k, C^\infty(M,S;K)) \subseteq C_*(\mathbb{S}^k, C_0(M \setminus S;K)) \subseteq C_0(\mathbb{S}^k \times (M \setminus S);K).$$

Then there exists a compact subset $X \subseteq M \setminus S$ such that $\sigma(\mathbb{S}^k \times (M \setminus X^0))$ is contained in an identity neighborhood of K which is diffeomorphic to a convex 0-neighborhood U in $\mathfrak{k}$, where 0 corresponds to $e \in K$. Let $\varphi: U \to \varphi(U) \subseteq K$ denote the corresponding chart and $h \in C_c^\infty(M \setminus S;\mathbb{R})$ with $h(X) = \{1\}$ and $h(M) \subseteq [0,1]$. We now define

$$\widetilde{\sigma}: \mathbb{S}^k \times M \to K, \quad \widetilde{\sigma}(t,x) := \begin{cases} \sigma(t,x) & \text{for } x \in X \\ \varphi\big(h(x)\varphi^{-1}(\sigma(t,x))\big) & \text{for } x \notin X. \end{cases}$$

As $\sigma(\mathbb{S}^k \times (M \setminus X^0))$ is a compact subset of $\varphi(U)$, it easily follows that $\widetilde{\sigma}$ is continuous and that $t \mapsto \widetilde{\sigma}(t,\cdot)$ yields a continuous map $\mathbb{S}^k \to C_c^\infty(M \setminus S;K)$. In fact, the support of each map $\widetilde{\sigma}(t,\cdot)$ is contained in the support of h. Moreover,

$$F:[0,1] \times \mathbb{S}^k \times M \to K, \quad F(s,t,x) := \begin{cases} \sigma(t,x) & \text{for } x \in X \\ \varphi\big([sh(x) + (1-s)] \cdot \varphi^{-1}(\sigma(t,x))\big) & \text{for } x \notin X \end{cases}$$

is a homotopy between σ and $\widetilde{\sigma}$ preserving base points. This implies that the map $\pi_k(\eta)$ is surjective. ∎

Note that Theorem A.13 does not imply that $C_c^\infty(M \setminus S;K)$ is dense in $C^\infty(M,S;K)$. This will be shown in Theorem A.18 below.

Corollary A.14. *Let M be a compact manifold and $\emptyset \neq S \subseteq M$ a closed subset. Then the inclusion*

$$\zeta: C^\infty(M,S;K) \to C_0(M \setminus S;K) \cong C_*(M/S;K)$$

is a weak homotopy equivalence.

Proof. According to Theorem A.10, the inclusion $C_c^\infty(M \setminus S;K) \to C_0(M \setminus S;K)$ is a weak homotopy equivalence, and this map is the composition of ζ and the inclusion map η from Theorem A.10. This implies that ζ also is a weak homotopy equivalence. ∎

Corollary A.15. *For a compact manifold M and $k \in \mathbb{N}_0$ we have*

$$\pi_k(C^\infty(M,S;K)) \cong \pi_k(C_*(M/S;K))$$

and in particular

$$\pi_k(C^\infty(I,\partial I;K)) \cong \pi_k(C_*(\mathbb{S}^1;K)) \cong \pi_{k+1}(K).$$

Proof. For $M = I$ and $S = \partial I$ we have $M/S \cong \mathbb{S}^1$ and therefore

$$\pi_k(C^\infty(I,\partial I;K)) \cong \pi_k(C_*(\mathbb{S}^1;K)) \cong \pi_{k+1}(K).$$ ∎

Lemma A.16. *For each $\alpha \in C^\infty((M',S'),(M,S))$ let*

$$\alpha_K: C^\infty(M,S;K) \to C^\infty(M',S';K), \quad f \mapsto f \circ \alpha.$$

Then α_K is a homomorphism of Lie groups and the homomorphisms $\pi_k(\alpha_K)$ only depend on the homotopy class of α in the space $C((M',S'),(M,S))$.

Proof. First we observe that the chain rule for Taylor expansions implies that α_K does indeed map $C^\infty(M,S;K)$ into $C^\infty(M',S';K)$. That α_K is a homomorphism of Lie groups follows by similar arguments as in the proof of Lemma A.12(i).

Viewing α as a continuous map $(M',S') \to (M,S)$ of space pairs, we see that it induces a continuous map

$$\alpha^*: C_*(M/S;K) \to C_*(M'/S';K), \quad f \mapsto f \circ \alpha.$$

Since the inclusion $C^\infty(M,S;K) \to C_*(M/S;K)$ is a weak homotopy equivalence (Corollary A.14), the maps $\pi_k(\alpha_K)$ are conjugate to the maps $\pi_k(\alpha^*)$. It is easy to see that $\pi_k(\alpha^*)$ only depends on the homotopy class of α because for each continuous map $\sigma: \mathbb{S}^k \to C_*(M'/S';K)$ the map $\alpha^* \circ \sigma: \mathbb{S}^k \to C_*(M/S;K)$ depends continuously on α. ∎

Lemma A.17. *For each locally convex space Y the space $C^\infty(M,S;Y)$ is a closed subspace of $C^\infty(M;Y)$ invariant under multiplication with elements of $C^\infty(M;\mathbb{R})$.*

Proof. This follows directly from the Leibniz formula for the higher partial derivatives of a product of two functions. ∎

Theorem A.18. (Approximation Theorem) *If M is compact, then $C^\infty_c(M\setminus S;K)$ is dense in the Lie group $C^\infty(M,S;K)$.*

Proof. First we reduce the problem to the assertion that for the Lie algebra $\mathfrak{k}$ of K the subspace $C^\infty_c(M\setminus S;\mathfrak{k})$ is dense in $C^\infty(M,S;\mathfrak{k})$.

Let $U\subseteq K$ be an open identity neighborhhod and $\varphi\colon V\to U$ a chart of K with $V\subseteq\mathfrak{k}$ an open convex subset and $\varphi(0)=e$. Then $\{f\in C^\infty(M,S;K)\colon f(M)\subseteq U\}$ is an open subset of $C^\infty(M,S;K)$ because it is already open in the compact open topology. We choose an open convex 0-neighborhood $V_1\subseteq V$ with $\varphi(V_1)^{-1}\varphi(V_1)\subseteq\varphi(V)$.

Let $f\in C^\infty(M,S;K)$. As f vanishes on S, the set $f^{-1}(\varphi(V_1))$ is an open subset of M containing S. Therefore its complement X is a compact subset of $M\setminus S$. Arguing as in the proof of Lemma A.8, we find a function $\widetilde f\in C^\infty_c(M\setminus S;K)$ with $\widetilde f|_X=f|_X$ and $\widetilde f(M\setminus X)\subseteq\varphi(V_1)$. Now it suffices to show that $h:=f^{-1}\widetilde f$, whose values are contained in $\varphi(V_1)^{-1}\varphi(V_1)\subseteq\varphi(V)$, is contained in the closure of $C^\infty_c(M\setminus S;K)$. As $\varphi^{-1}\circ h\colon M\to\mathfrak{k}$ is a well-defined smooth map, we see that it suffices to prove the theorem for $\mathfrak{k}$ instead of K. In this setting we have to show that if $V\subseteq\mathfrak{k}$ is an open convex 0-neighborhood with $f(M)\subseteq V$, then f can be approximated by functions in $C^\infty_c(M;\mathfrak{k})$ whose values lie in V.

Let $f\in C^\infty(M,S;\mathfrak{k})$. Using Lemma A.17 and a smooth partition of unity on M, we may assume that the support of f lies in a coordinate neighborhood which we may identify with $\mathbb{R}^n$. We are therefore led to the following situation. We consider a smooth function $f\in C^\infty_c(\mathbb{R}^n;\mathfrak{k})$ all of whose derivatives vanish on the closed subset $S\subseteq\mathbb{R}^n$, and we are looking for a sequence of functions with compact support in $\mathbb{R}^n\setminus S$ converging to f in $C^\infty(\mathbb{R}^n;\mathfrak{k})$ whose supports are uniformly contained in a compact set. The existence of such a sequence is proved in Proposition A.22 below. ∎

An Approximation Lemma

Let $\emptyset\neq S\subseteq\mathbb{R}^d$ be a closed subset, Y a Banach space, and $f\in C^\infty_X(\mathbb{R}^d;Y)$ for a compact subset $X\subseteq\mathbb{R}^d$ such that f and all its partial derivatives vanish on $S\cap X$. We want to see that f is contained in the closure of the subspace $C^\infty_c(\mathbb{R}^d\setminus S;Y)\cap C^\infty_X(\mathbb{R}^d;Y)$. In the following $d(S,x)$ denotes the euclidean distance of the set S and x. We write $\|\cdot\|$ for the euclidean norm on $\mathbb{R}^d$.

Lemma A.19. *For each $k\in\mathbb{N}$ and each $f\in C^\infty_c(\mathbb{R}^d,S;Y)$ there exists a constant $C_k>0$ with*

$$\|f(x)\|\leq C_k d(S,x)^k.$$

Proof. We prove the assertion by induction over k. For $k=0$ it follows from the compactness of the support of f.

Now we assume that the assertion holds for $k\in\mathbb{N}_0$. Let $h\in C^\infty_c(\mathbb{R}^d,S;Y)$. Then the induction hypothesis applies to $dh\in C^\infty_c(\mathbb{R}^d,S;\mathrm{Lin}(\mathbb{R}^d;Y))$, and we obtain a constant D_k with $\|dh(x)\|\leq D_k d(S,x)^k$ for all $x\in\mathbb{R}^d$. For $x\in\mathbb{R}^d$ we find an $x_0\in S$ with $\|x-x_0\|\leq 2d(S,x)$. Then

$$h(x)=h(x_0)+\int_0^1 dh(x_0+t(x-x_0))(x-x_0)\,dt=\int_0^1 dh(x_0+t(x-x_0))(x-x_0)\,dt$$

leads to

$$\begin{aligned}\|h(x)\|&\leq\|x-x_0\|\sup\nolimits_{0\leq t\leq 1}\|dh(x_0+t(x-x_0))\|\\&\leq 2d(S,x)D_k\sup\nolimits_{0\leq t\leq 1}d(S,x_0+t(x-x_0))^k\\&\leq 2D_k d(S,x)2^k d(S,x)^k=2^{k+1}D_k d(S,x)^{k+1}.\end{aligned}$$

This completes the induction, and hence the proof of the lemma. ■

Now let δ be a smooth function supported in the closed unit ball $B_1(0)$ in $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \delta(x)\,dx = 1$ and $\mathrm{im}(\delta) \subseteq [0,1]$. We define

$$\delta_n(x) := n^d \delta(nx)$$

and observe that these functions form a smooth Dirac sequence. For each multiindex $J = (j_1, \ldots, j_d) \in \mathbb{N}_0^d$ we have

$$\|\partial^J \delta_n\|_\infty = n^{d+|J|} \|\partial^J \delta\|_\infty.$$

Let $S_n := \{x \in \mathbb{R}^d : d(S,x) \leq \frac{2}{n}\}$ and

$$\chi_{S_n}(x) := \begin{cases} 1 & \text{for } x \in S_n \\ 0 & \text{for } x \notin S_n \end{cases}$$

the characteristic function of S_n. Then we define

$$\varphi_n(x) := 1 - (\delta_n * \chi_{S_n})(x) = 1 - \int_{x - S_n} \delta_n(y)\,dy \in [0,1].$$

Then each function φ_n is smooth with $\varphi_n(x) = 1$ for $d(S,x) \geq \frac{3}{n}$ and $\varphi_n(x) = 0$ for $d(S,x) \leq \frac{1}{n}$.

Lemma A.20. *For each multiindex J there exists a constant D_J such that*

$$\|\partial^J \varphi_n(x)\| \leq D_J d(S,x)^{-|J|}, \quad x \in \mathbb{R}^d, n \in \mathbb{N}.$$

Proof. For $|J| = 0$ the assertion follows from $\mathrm{im}(\varphi_n) \subseteq [0,1]$.

Suppose that $|J| > 0$ and that $d(S,x) \in [\frac{1}{n}, \frac{3}{n}]$. Otherwise $\partial^J \varphi_n(x)$ vanishes anyway. Then we have

$$\begin{aligned} \|\partial^J \varphi_n(x)\| &= \|\big((\partial^J \delta_n) * \chi_{S_n}\big)(x)\| \leq \mathrm{vol}(B_{\frac{1}{n}}(0)) \|\partial^J \delta_n\|_\infty \\ &\leq C n^{-d} n^{d+|J|} \|\partial^J \delta\|_\infty = C n^{|J|} \|\partial^J \delta\|_\infty \leq C 3^{|J|} d(S,x)^{-|J|} \|\partial^J \delta\|_\infty. \end{aligned}$$

■

Lemma A.21. *For all multiindices J with $|J| > 0$ we have uniformly $\partial^J \varphi_n \cdot f \to 0$.*

Proof. Combining Lemma A.19 and A.20, we get for each $k \in \mathbb{N}$ a constant C_k with

$$\|(\partial^J \varphi_n(x)) f(x)\| \leq C_k d(S,x)^{-|J|} d(S,x)^{|J|+k} = C_k d(S,x)^k.$$

As $\partial^J \varphi_n(x) = 0$ for $d(S,x) \geq \frac{3}{n}$ (here we need $|J| > 0$), this leads to

$$\|(\partial^J \varphi_n(x)) f(x)\| \leq C_k 3^k n^{-k}$$

for all $x \in \mathbb{R}^d$, and this implies the assertion. ■

Proposition A.22. *For each locally convex space Y and $f \in C_c^\infty(\mathbb{R}^d, S; Y)$ we have $\varphi_n f \to f$ in $C^\infty(\mathbb{R}^d; Y)$.*

Proof. As every locally convex space can be embedded into a product of Banach spaces, it suffices to assume that Y is a Banach space. Since the supports of the functions $\varphi_n f$ and f are contained in one compact subset of $\mathbb{R}^d$, we have to show $\|\partial^J(\varphi_n f - f)\|_\infty \to 0$ for all multiindices J.

For $|J| = 0$ this follows easily from the support properties of φ_n and $\|f(x)\| \leq C d(S,x)$.

Next we note that for each multiindex J the function $\partial^J f$ also has the property that all its partial derivatives vanish on S. Therefore Lemma A.21 implies that $\partial^J \varphi_n \cdot \partial^J f \to 0$ uniformly whenever $|J| > 0$. In view of the Leibniz rule, the problem reduces to showing that $\varphi_n \partial^J f$ converges uniformly to $\partial^J f$, but this follows from the case $|J| = 0$, applied to $\partial^J f$ instead of f. ■

Appendix B. Locally convex direct limit spaces

In this section we discuss the discreteness of certain subgroups of direct limits of locally convex spaces. In this paper we only use Lemma B.4. Nevertheless Proposition B.3 provides a much more direct way to prove the discreteness of the groups $H^k(M;Y,\Gamma)$ if Y is finite-dimensional and $\Gamma \subseteq Y$ is a discrete subgroup (cf. Corollary IV.21).

Lemma B.1. *If X is a locally convex space, $Y \subseteq X$ a closed subspace and $F \subseteq X$ a finite-dimensional subspace complementing Y, then $X \cong Y \oplus F$ as topological vector spaces.*

Proof. The quotient map $q\colon X \to X/Y$ induces an isomorphism $q\,|_F\colon F \to X/Y$. Hence q has a continuous linear section $\sigma\colon X/Y \to X$ whose range is F, and therefore the addition map $a\colon Y \times F \to X$ is a topological isomorphism because $a^{-1}(x) = \big(x - \sigma(q(x)), \sigma(q(x))\big)$ is continuous. ∎

Lemma B.2. *Let X be a locally convex space which is the locally convex direct limit of the subspaces X_n, $n \in \mathbb{N}$, where each X_n is a closed subspace of X_{n+1}. Further let $F \subseteq X$ be a subspace such that for each $n \in \mathbb{N}$ the intersection $F_n := F \cap X_n$ is finite-dimensional. Then the following assertions hold:*

(i) *There exists a continuous linear projection $p\colon X \to F$ with $p(X_n) = F_n$ for each $n \in \mathbb{N}$. In particular we have $X \cong \ker p \oplus F$.*

(ii) *F is closed.*

(iii) *F is the topological direct limit of the subspaces F_n, $n \in \mathbb{N}$, which means that F carries the finest locally convex topology.*

Proof. (i) We argue by induction. As F_1 is finite-dimensional, the Hahn–Banach Theorem yields a continuous extension $p_1\colon X_1 \to F_1$ of the identity map id_{F_1}. Then p_1 can be viewed as a continuous projection of X_1 to F_1.

Now let $n \in \mathbb{N}$ and assume that $p_n\colon X_n \to F_n$ is a continuous projection. Then we choose a complement E_{n+1} of F_n in F_{n+1}. As X_n is a closed subspace of the locally convex space $X_n + F_{n+1} = X_n \oplus E_{n+1}$, it follows from Lemma B.1 that $X_n + F_{n+1} \cong X_n \oplus E_{n+1}$ as topological vector spaces. The linear map $q_n := p_n \oplus \mathrm{id}_{E_{n+1}}$ is a continuous projection of $X_n + F_{n+1}$ onto F_{n+1}. We use the Hahn–Banach Theorem again to extend q_n to a continuous linear map $p_{n+1}\colon X_{n+1} \to F_{n+1}$ which then also is a continuous projection. We thus obtain a sequence $(p_n)_{n\in\mathbb{N}}$ of continuous linear maps $p_n\colon X_n \to F$ with $p_{n+1}|_{X_n} = p_n$. Now the universal property of X yields the existence of a continuous linear map $p\colon X \to F$ with $p\,|_{X_n} = p_n$ for each $n \in \mathbb{N}$. As $p|_F = \mathrm{id}_F$, we are done.

(ii) follows from (i).

(iii) Let Z be a locally convex space and $f\colon F \to Z$ be a linear map. We claim that f is continuous. To this end, we consider the map $h := f \circ p\colon X \to Z$. Then $h|_{X_n} = (f\,|_{F_n}) \circ p_n$, and p_n is continuous, as well as the map $f\,|_{F_n}$ on the finite-dimensional vector space F_n. Therefore all the restrictions $h\,|_{X_n}$ are continuous, and we conclude that h is continuous, which in turn implies that f is continuous. The fact that all linear maps from F to locally convex spaces are continuous shows that F carries the finest locally convex topology. Furthermore, F is countably dimensional because all the spaces F_n are finite-dimensional. Using [KK63], we now conclude that the topology on F coincides with the finite open topology, i.e., the direct limit topology with respect to the directed system of all finite-dimensional subspaces. As the sequence $(F_n)_{n\in\mathbb{N}}$ is cofinal, this topology coincides with the direct limit topology with respect to the sequence $(F_n)_{n\in\mathbb{N}}$. ∎

Proposition B.3. *Let X be a locally convex space which is the locally convex direct limit of the subspaces X_n, $n \in \mathbb{N}$, with $X_n \subseteq X_{n+1}$, where X_n is closed in X_{n+1}. Let further $\Gamma \subseteq X$ be a subgroup such that for each $n \in \mathbb{N}$ the group $\Gamma \cap X_n$ is discrete and finitely generated. Then Γ is a discrete subgroup of X.*

Proof. For each $n \in \mathbb{N}$ we consider the finite-dimensional subspace $F_n := \operatorname{span}\Gamma_n$ for the discrete finitely generated subgroup $\Gamma_n := \Gamma \cap X_n$ of X_n. Let $F := \bigcup_n F_n = \operatorname{span}\Gamma$. We claim that $F_n = F \cap X_n$ holds for each $n \in \mathbb{N}$. Fix $n, m \in \mathbb{N}$ with $n < m$. As Γ_n is discrete in the finite-dimensional space F_n, there exists a basis B_n of F_n with $\Gamma_n = \operatorname{span}_{\mathbb{Z}} B_n$. Further $\Gamma_n = \Gamma \cap X_n = \Gamma_m \cap X_n$ is a pure subgroup of Γ_m, so that Γ_m/Γ_n is a free abelian group. Hence we find a subset $C_m \subseteq \Gamma_m$ such that the image of C_m is a basis in $(F_m + X_n)/X_n \cong F_m/F_m \cap X_n$ generating the subgroup $(\Gamma_m + X_n)/X_n \cong \Gamma_m/\Gamma_n$. Now $B_m := B_n \cup C_m$ is a basis of F_m with $\Gamma_m = \operatorname{span}_{\mathbb{Z}} B_m$. In particular, it follows that $F_m \cap X_n = \operatorname{span}_{\mathbb{R}} B_n = F_n$. As m was arbitrary, we conclude that $F \cap X_n = F_n$.

Next Lemma B.2 applies to the subspace $F \subseteq X$ and shows that F is closed and carries the finite open topology. Let $O := (F \backslash \Gamma) \cup \{0\}$. For each $n \in \mathbb{N}$ we then have $O \cap F_n = (F_n \backslash \Gamma_n) \cup \{0\}$, which is an open set because Γ_n is discrete in F_n. Therefore O is an open subset of F (Lemma B.2(iii)), and since F carries the subspace topology of X, there exists an open subset $O_X \subseteq X$ with $O_X \cap F = O$. Now O_X is an open 0-neighborhood in X with $O_X \cap \Gamma = \{0\}$. This shows that Γ is discrete. ■

Lemma B.4. *Let $X = \varinjlim X_j$ be a locally convex direct limit of the spaces X_j.*

(i) *If $F \subseteq X$ is a closed subspace, then $X/F \cong \varinjlim X_j/(F \cap X_j)$.*

(ii) *A subspace $F \subseteq X$ is closed if and only if all intersections $F \cap X_j$ are closed.*

Proof. (i) (cf. [Kö79, p.42]) Since F is closed, all the spaces $F_j := F \cap X_j$ are closed. Let $Z := \varinjlim X_j/F_j$ denote the locally convex direct limit of the spaces X_j/F_j. Then we have natural continuous maps $\varphi_j : X_j/F_j \to X/F$ which define a continuous linear map $\varphi : Z \to X/F$. On the other hand the continuous linear maps $X_j \to Z$ combine to a continuous linear map $X \to Z$ which then factors through a continuous linear map $\psi : X/F \to Z$. Now $\varphi \circ \psi = \mathrm{id}_{X/F}$ and $\psi \circ \varphi = \mathrm{id}_Z$ imply (i).
(ii) If F is closed, then the subspaces $F \cap X_j$ are trivially closed in X_j. If, conversely, this condition is satisfied, then we can form the locally convex direct limit space $Z := \varinjlim X_j/(F \cap X_j)$. The natural maps $X_j \to Z$ are continuous, hence combine to a continuous map $X \to Z$ whose kernel F is a closed subspace. ■

Problem B.1. Does Proposition B.3 also hold without the assumption that the groups $\Gamma \cap X_n$ are finitely generated? If this is true, then the proof of the discreteness of the groups $H^1_{\mathrm{dR,c}}(M;\Gamma)$ in Section IV would be much easier because we would not need the complicated approximation procedure from Section III. ■

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