A TOPOLOGICAL REPRESENTATION THEOREM FOR ORIENTED MATROIDS

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ABSTRACT. We present a new direct proof of a topological representation theorem for oriented matroids in the general rank case. Our proof is based on an earlier rank 3 version. It uses hyperline sequences and the generalized Schönflies theorem. As an application, we show that one can read off oriented matroids from arrangements of embedded spheres, even if wild spheres are involved.

Keywords: oriented matroid, pseudosphere arrangement, chirotope, Lawrence representation, hyperline sequences, generalized Schönflies theorem.

1. Introduction

When studying vector configurations or central hyperplane configurations, point sets on a sphere or great hypersphere arrangements, vector spaces or their duals, points on grassmannians, polytopes and their corresponding cellular decompositions in projective space, etc., an abstraction of an important equivalence class of matrices often plays a central role: an oriented matroid. The theory of oriented matroids (see [1]) provides us with a multitude of definitions for an oriented matroid that can be viewed as reflecting the variety of objects that a matrix can represent.

These definitions via circuit or cocircuit axioms ([1], p. 103), sphere systems ([1], p. 227), Graßmann Plücker relations (chirotope axioms) ([1], p. 126, p. 138, [13]), hull systems ([16]), to mention just a few of them, differ a lot with respect to their motivational aspects, their algorithmical efficiency or their relation to the actual application. Each definition provides in general an additional insight for the motivating problem. In the research monograph on oriented matroids, [1], three chapters are devoted to axiomatics concerning oriented matroids and to the topological representation theorem for oriented matroids.

This central theorem in the theory of oriented matroids due to Lawrence shows the equivalence of oriented matroids defined via sphere system axioms with oriented matroids defined via, say, the circuit axioms. This remarkable result asserts that each oriented matroid has a topological representation as an oriented pseudosphere arrangement, even a piecewise-linear one, cf. Edmonds and Mandel [6]. Other authors ([1], [14]) have later simplified or complemented the original proof, but all use fundamentally the same approach: the face lattice (tope) formalism for oriented matroids and a shelling order to carry through the construction.

Finding a reasonably direct proof in rank 3, one that would rely on the structural simplicity of the planar case, has been posed as an open problem in the research monograph [1] (Exercise 6.3). In [3] such a proof in the rank 3 case was given. Unlike the previous ones, this was based on *hyperline sequences*, an equivalent axiomatization for oriented matroids which is particularly natural in rank 3. In this article we generalize this proof to the arbitrary rank case. The proof is inductive, direct and uses only one advanced result from topology, the generalized Schönflies theorem.

The motivation for introducing hyperline sequences in rank 3 was presented in detail in our previous paper [3]. Here we use the generalization to arbitrary rank, formally defined in Section 2. The motivation is similar. Here is a brief account of it. We use an index

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set $E_n = \{1, \dots, n\}$ with its standard total order. We consider a labeled finite point set $X := \{x_1, x_2, \dots, x_n\}$, $x_i \in \mathbb{R}^{r-1}$, ordered according to their labels. We assume that the point set X affinely spans $\mathbb{R}^{r-1}$. We endow the affine Euclidean space $\mathbb{R}^{r-1}$ with an orientation, i.e., for any r–element subset of affinely independent points, we can tell whether the ordering of their indices defines a positive, or a negative, oriented simplex, respectively. Every $(r-1)$–element subset of affine independent points of X defines via the ordering of their indices an oriented hyperplane, i.e., an affine hull of codimension 1 together with an orientation. Furthermore, every $(r-2)$–element subset of affine independent points of X defines via the ordering of their indices an oriented hyperline, i.e., an affine hull of codimension 2 together with an orientation of it. Although this affine model can be described further and the concept we are going to describe is useful for affine applications as well, in what follows we prefer to use the notational advantage that occurs when embedding this concept in the usual way in projective space. First consider a bijective map between our former oriented $(r-1)$–dimensional Euclidean space $\mathbb{R}^{r-1}$ and a non-central oriented hyperplane of the oriented space $\mathbb{R}^r$. The former points x_i then become non-zero vectors v_i in $\mathbb{R}^r$, $i \in E_n$, the former oriented hyperplanes become oriented central hyperplanes in $\mathbb{R}^r$, and the former oriented hyperlines, defined via an index set $B \subset E_n$ with $|B| = r-2$, become oriented central hyperlines (subspaces of codimension 2 in $\mathbb{R}^r$ together with an orientation). For what follows we can also assume that we have started with an ordered set of vectors $V = \{v_1, v_2, \dots, v_n\}$ in our oriented space $\mathbb{R}^r$ that has not necessarily an affine pre-image.

The oriented central hyperline with index set B is an $\mathbb{R}^{r-2}$, and the restriction of V to all elements in that hyperline yields an arrangement Y of non-zero vectors in $\mathbb{R}^{r-2}$. We rotate an additional central oriented hyperplane in $\mathbb{R}^r$ around one of our oriented central hyperlines with index set B in the well defined positive sense, i.e., the ordered set of vectors with index set B together with two adjacent consecutive outer normal vectors of the hyperplane form a positive basis of our oriented space $\mathbb{R}^r$. An orthogonal projection π_B along the oriented hyperline onto its corresponding two-dimensional orthogonal oriented space $\pi_B(\mathbb{R}^r)$ (that forms together with the oriented hyperline a positive orientation of the whole space) shows the image of the additional central oriented hyperplane as a rotating (mathematically positive) central oriented line in the plane. Precisely all vectors v_i that are not in the hyperline defined by B appear after the projection as non-zero vectors $\pi_B(v_i)$. Their indices form a periodic circular sequence around the origin when we pursue the incidences of the vectors with the rotating line. We store the cyclic order of incidences in a cyclic order of sets, where we write an index i when the vector v_i is consistent with the orientation of the oriented line, and $\bar{\imath}$ otherwise. We refer to this notation as **signed indices**.

When an incidence position, say position m, of the rotating oriented line with a vector occurs, we write the signed indices of all vectors that are incident with the oriented line in a set Z^m. Note that the period of the cyclic sequence is even, since there are two positions of the rotating oriented line in which it is incident with the vector $\pi_B(v_i)$, corresponding to the signed indices i and $\bar{\imath}$. We obtain a well defined oriented circular sequence $Z^0, \dots, Z^{2k-1}$ (**oriented cycle**) of sets of signed indices, where after half the period there is a sign reversal for each element. The combinatorics of the hyperline is encoded in the arrangement Y with index set B together with the oriented circular sequence of incidences with the rotating line, $(Y|Z^0, \dots, Z^{2k-1})$, to which we also refer as a **hyperline**. We obtain a **hyperline sequence** when we write down all hyperlines arising in the vector arrangement.

This model can be generalized in two ways. The combinatorial abstraction leads on the one hand to oriented matroids characterized as hyperline sequences and on the other hand it leads to topological equivalence classes of arrangements of pseudospheres. We give a new proof of a one-to-one correspondence of oriented matroids and classes of arrangements of pseudospheres. We further generalize arrangements of pseudospheres, and we show that one can still read off a hyperline sequence.

The paper is organised as follows. We define hyperline sequences as a combinatorial abstraction of vector configurations in Section 2. We recall the chirotope concept in Section 3 and we show the equivalence of these two concepts in Theorem 1. Furthermore, we discuss the concepts of deletion and contraction for both settings. We introduce the topological representation of oriented matroids via arrangements of oriented pseudospheres in Section 4 with the corresponding concepts of deletion and contraction. In the proof of Theorem 2 we use the generalized Schönflies theorem to show the cell structure of an arrangement of oriented pseudospheres. In Theorem 3 we replace our two axioms for an arrangement of oriented pseudospheres with a single one. Section 5 deals with the easier part of the main representation theorem: we obtain chirotopes and hyperline sequences from the topological representation. We complete the proof of the one-to-one correspondence of hyperline sequences with their topological representation by induction. Section 6 is devoted to the base case and Section 7 contains the essential part. We finally discuss in Section 8 the wild arrangement case as an easy supplement of our approach.

The whole proof is based heavily on the ideas that have been worked out already in [3] by the first and the last two authors in the rank 3 case. But the experience with respect to topological arguments of the second author was decisive to arrive at our final version of the proof in the general rank case. The last section and many improvements of the proof compared with the rank 3 version are due to him. For instance, the uniform and non-uniform cases form no longer separate parts within the proof.

2. Hyperline sequences

Our aim in this section is to introduce the notion of hyperline sequences. The geometric motivation of our definition comes from vector arrangements in Euclidian space as explained in the introduction.

Let $(E,<)$ be a finite totally ordered set. Let $\overline{E}=\{\overline{e}|e\in E\}$ be a copy of E. The set $\mathbf{E}$ of **signed indices** is defined as the disjoint union of E and $\overline{E}$. By extending the map $e\mapsto\overline{e}$ to $\overline{e}\mapsto\overline{\overline{e}}=e$ for $e\in E$, we get an involution on $\mathbf{E}$. We define $e^*=\overline{e}^*=e$. For $X\subset\mathbf{E}$, define $\overline{X}=\{\overline{x}|\, x\in X\}$ and $X^*=\{x^*|\, x\in X\}$.

An **oriented** d**–simplex** in E is a $(d+1)$–tuple $\sigma=[x_1,\dots,x_{d+1}]$ of elements of $\mathbf{E}$, so that $x_1^*,\dots,x_{d+1}^*$ are pairwise distinct. Let an equivalence relation $\sim$ on oriented d–simplices in E be generated by $[x_1,\dots,x_{d+1}]\sim[x_1,\dots,x_{i-1},\overline{x_{i+1}},x_i,x_{i+2},\dots,x_{d+1}]$, for $i=1,\dots,d$. As usual, any oriented d–simplex is equivalent to one of the form $[e_1,\dots,e_{d+1}]$ or $[e_1,\dots,e_d,\overline{e_{d+1}}]$, with elements $e_1<e_2<\cdots<e_{d+1}$ of E. Define $-[x_1,\dots,x_{d+1}]=[x_1,\dots,x_d,\overline{x_{d+1}}]$. If $\phi\colon\ \mathbf{E}\to\mathbb{R}^d$ is a map with $\phi(\overline{e})=-\phi(e)$ for all $e\in E$, then $\phi(x_1),\dots,\phi(x_{d+1})$ are the vertices of a simplex in $\mathbb{R}^d$. Note that this simplex might be degenerate.

In the following iterative definition of hyperline sequences we denote with

$$C_m=\big(\{0,1,\dots,m-1\},+\big)$$

the cyclic group of order m.

Definition 1 (Rank 1)**.** A **hyperline sequence** X over $E(X)\subset E$ of rank 1 is a non-empty subset $X\subset E(X)\cup\overline{E(X)}$ so that $|X|=|X^*|$ and $X^*=E(X)$.

The oriented simplex $[x]$ is by definition a **positively oriented base** of X for $x\in X$. We define $-X=\overline{X}$.

Definition 2 (Rank 2)**.** Let $k\in\mathbb{N}$, $k\geq 2$. A **hyperline sequence** X of rank 2 over $E(X)\subset E$ is a map from C_{2k} to hyperline sequences of rank one, $a\mapsto X^a$, so that $X^{a+k}=-X^a$ for all $a\in C_{2k}$, and $E(X)\cup\overline{E(X)}$ is a disjoint union of $X^0,\dots,X^{2k-1}$.

We refer to $X^0,\dots,X^{2k-1}$ as the **atoms** of X and to $2k$ as the **period length** of X. We say that $e\in E(X)$ is **incident** to an atom X^a of X if $e\in(X^a)^*$. Let $x_1,x_2\in E(X)\cup\overline{E(X)}$ so that x_1^* and x_2^* are not incident to a single atom of X, and X induces the cyclic order $(x_1,x_2,\overline{x_1},\overline{x_2})$. Then, the oriented simplex $[x_1,x_2]$ is by definition a **positively oriented base** of X. We define the hyperline sequence $-X$ over $E(-X)=E(X)$ of rank 2 as the map $a\mapsto(-X)^a=X^{-a}$ for $a\in C_{2k}$.

A hyperline sequence X of rank 2 is determined by the sequence $(X^0, \dots, X^{2k-1})$ of atoms. We define that two hyperline sequences X_1 and X_2 of rank 2 are equal, $X_1 = X_2$, if $E(X_1) = E(X_2)$, the number $2k$ of atoms coincides, and X_1 is obtained from X_2 by a shift, i.e., there is an $s \in C_{2k}$ with $X_1^{a+s} = X_2^a$ for all $a \in C_{2k}$.

We prepare the axioms for hyperline sequences of rank $r > 2$ with the following definitions. Let X be a set of pairs $(Y|Z)$, where Y is a hyperline sequence of rank $r-2$ and Z is a hyperline sequence of rank 2. A **positively oriented base** of X in $(Y|Z) \in X$ is an oriented simplex $[x_1, \dots, x_r]$ in $E(X)$, where $[x_1, \dots x_{r-2}]$ is a positively oriented base of Y and $[x_{r-1}, x_r]$ is a positively oriented base of Z. We define $-X = \{(Y| - Z) \mid (Y|Z) \in X\}$. The elements of X are called **hyperlines**. An **atom** of X in a hyperline $(Y|Z) \in X$ is the pair $(Y|Z^a)$, where Z^a is an atom of Z.

Definition 3 (Rank $r > 2$)**.** Let $X \neq \emptyset$ be a set whose elements are pairs $(Y|Z)$, where Y is a hyperline sequence of rank $r-2$ and Z is a hyperline sequence of rank 2. The set X is a **hyperline sequence** of rank $r > 2$ over $E(X) \subset E$ if it satisfies the following axioms.

(H1) $E(X)$ is a disjoint union of $E(Y)$ and $E(Z)$, for all $(Y|Z) \in X$.
(H2) Let $(Y_1|Z_1), (Y_2|Z_2) \in X$ and let $[x_1, \dots x_{r-2}]$ be a positively oriented base of Y_1. If $\{x_1^*, \dots, x_{r-2}^*\} \subset E(Y_2)$ then $(Y_1|Z_1) = (Y_2|Z_2)$ or $(Y_1|Z_1) = (-Y_2| - Z_2)$.
(H3) For all positively oriented bases $[x_1, \dots, x_r]$ and $[y_1, \dots, y_r]$ of X, there is some $j \in \{1, \dots, r\}$ so that $[x_1, \dots, x_{r-1}, y_j]$ or $[x_1, \dots, x_{r-1}, \overline{y_j}]$ is a positively oriented base of X.
(H4) For any positively oriented base $[x_1, \dots, x_r]$ of X, $[x_1, \dots, x_{r-3}, \overline{x_{r-1}}, x_{r-2}, x_r]$ is a positively oriented base of X.

We call an oriented simplex σ a negatively oriented base of a hyperline sequence X if $-\sigma$ is a positively oriented base of X.

We connect these axioms to the geometric motivation exposed in the introduction. Let $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^{r-1}$ be non-zero vectors spanning $\mathbb{R}^r$. Let $H \subset \mathbb{R}^r$ be a hyperline, i.e. a subspace of dimension $r-2$ spanned by elements of V. Let $H^\perp$ be the orthogonal complement of H in $\mathbb{R}^r$. Then $V \cap H$ is a set of non-zero vectors in $\mathbb{R}^{r-2}$, corresponding to the hyperline sequence Y in a hyperline $(Y|Z)$. The image of $V \setminus H$ under the orthogonal projection onto $H^\perp$ yields a set of non-zero vectors in $\mathbb{R}^2$ corresponding to the term Z in $(Y|Z)$. Axiom (H1) simply means that V is a disjoint union of $V \cap H$ and $V \setminus H$. Axiom (H2) corresponds to the fact that H is uniquely determined by any $(r-2)$–tuple of elements of $V \cap H$ in general position. Axiom (H3) is the Steinitz–McLane exchange lemma, stating that one can replace any vector in a base by some vector of any other base. Axiom (H4) ensures that if $[x_1, \dots, x_r]$ is a positively oriented base of X than so is any oriented simplex that is equivalent to $[x_1, \dots, x_r]$; this is a part of Theorem 1 below. Axiom (H4) is related to the "consistent abstract sign of determinant" in [3]. It means that if r points span an $(r-1)$–simplex, then any subset of $r-2$ points spans a hyperline, and the orientation of the $(r-1)$–simplex does not depend on the hyperline on which we consider the r points.

3. Chirotopes

We recall in this section the chirotope axioms for oriented matroids (see [1], p. 126,p. 138, [13]). Let $(E, <)$ be as in the preceding section. We denote by $\Delta_d(E)$ the set of all oriented d–simplices in E.

Definition 4. A **chirotope** χ of rank r over E is a map $\Delta_{r-1}(E) \to \{-1, 0, +1\}$, so that the following holds.

(C1) For any $e_1 \in E$, there are $e_2, e_3, \dots, e_r \in E$ with $\chi([e_1, \dots, e_r]) \neq 0$.
(C2) For any $\sigma \in \Delta_{r-1}(E)$ holds $\chi(-\sigma) = -\chi(\sigma)$, and if $\sigma \sim \tau$ then $\chi(\sigma) = \chi(\tau)$.
(C3) If $\chi([x_1, \dots, x_r]) \neq 0$ and $\chi([y_1, \dots, y_r]) \neq 0$ then there is some $i \in \{1, \dots, r\}$ with $\chi([x_1, \dots, x_{r-1}, y_i]) \neq 0$.

(C4) If $x_1, x_2, \ldots, x_r, y_1, y_2 \in \mathbf{E}$ so that

$$\begin{aligned}\chi([x_1, \ldots, x_{r-2}, y_1, x_r]) \cdot \chi([x_1, \ldots, x_{r-1}, y_2]) &\geq 0 \text{ and}\\ \chi([x_1, \ldots, x_{r-2}, y_2, x_r]) \cdot \chi([x_1, \ldots, x_{r-1}, \overline{y_1}]) &\geq 0,\end{aligned}$$

then $\chi([x_1, \ldots, x_r]) \cdot \chi([x_1, \ldots, x_{r-2}, y_1, y_2]) \geq 0$.

A simplex $\sigma \in \Delta_{r-1}(E)$ is a **positively oriented base** of χ if $\chi(\sigma) = +1$. By Axiom (C2), χ is completely described by the set of its positively oriented bases.

In order to give Axioms (C1)–(C4) a geometric meaning, we show how to construct a chirotope of rank r from a vector configuration in $\mathbb{R}^r$. Let $V = \{v_1, \ldots v_n\} \subset \mathbb{R}^r$ be a set of non-zero vectors spanning $\mathbb{R}^r$. Let $\phi\colon \mathbf{E} \to \mathbb{R}^r$ be defined by $\phi(e) = v_e$ and $\phi(\overline{e}) = -v_e$ for all $e \in E$. We get a map $\chi\colon \Delta_{r-1}(E) \to \{-1, 0, +1\}$ by setting

- $\chi([x_1, \ldots, x_r]) = +1$ if the determinant $[\phi(x_1), \ldots, \phi(x_r)]$ is positive (i.e. if $\phi(x_1), \ldots, \phi(x_r)$ is a positive base of $\mathbb{R}^r$),
- $\chi([x_1, \ldots, x_r]) = -1$ if $[\phi(x_1), \ldots, \phi(x_r)]$ is negative, and
- $\chi([x_1, \ldots, x_r]) = 0$ if $\phi(x_1), \ldots, \phi(x_r)$ are linearly dependent.

We show that χ is a chirotope. Axiom (C1) holds since V spans $\mathbb{R}^r$. Axiom (C2) follows from the symmetry of determinants. Axiom (C3) is the Steinitz–McLane exchange lemma, as Axiom (H3) above. Axiom (C4) sais that the signs of oriented bases as defined by χ do not contradict the three summand Graßmann–Plücker relation.

Note that we do distinguish between χ and $-\chi$, although this is not usual in the literature. Our reason comes from geometry. We believe if one deals with oriented objects (vectors, oriented hyperspheres, etc.) in $\mathbb{R}^d$ or $\mathbb{S}^d$, then it is consequent to distinguish not only the two different orientations of any object, but also the two different orientations of $\mathbb{R}^d$ or $\mathbb{S}^d$.

By the following theorem, the notions of chirotopes and hyperline sequences are equivalent. This connects our concept of hyperline sequences with other ways to look at oriented matroids. The cyclic structure of a hyperline captures many instances of the 3–term Graßmann–Plücker relations at once. Therefore the proof of the theorem becomes rather long and tedious. But we believe that this price is worth to pay. Namely, it is easier to deal with a few cyclic structures than with a multitude of Graßmann–Plücker relations, and therefore it is algorithmically more convenient and efficient to encode the structure of oriented matroids in hyperline sequences rather than in chirotopes.

Theorem 1. *The set of positively oriented bases of a hyperline sequence of rank r over E is the set of positively oriented bases of a chirotope of rank r over E, and vice versa.*

Proof. 1. Let X be a hyperline sequence of rank r over E, and let $[x_1, \ldots, x_r]$ be a positively oriented base of X. Since the cyclic orders $(x_{r-1}, x_r, \overline{x_{r-1}}, \overline{x_r})$ and $(\overline{x_r}, x_{r-1}, x_r, \overline{x_{r-1}})$ are equal up to a shift, $[x_1, \ldots, x_{r-2}, \overline{x_r}, x_{r-1}]$ is a positively oriented base of X as well. This together with Axiom (H4) and an induction on the rank implies that all oriented simplices equivalent to $[x_1, \ldots, x_r]$ are positively oriented bases of X, hence, yields Axiom (C2). Axiom (C1) follows from Axiom (H1), Axiom (C2) and induction on the rank. The Axioms (H3) and (C3) are equivalent.

In the next paragraphs we deduce Axiom (C4) from the cyclic order of hyperlines. We use Axiom (C2), that is already proven, but we do not mention any application explicitly. Let $x_1, x_2, \ldots, x_r, y_1, y_2 \in \mathbf{E}$ satisfy the first two inequalities in Axiom (C4). The third inequality is to prove. We can assume that $x_1, \ldots, x_{r-2}$ defines some hyperline $(Y|Z) \in X$ with period length $2k$, since otherwise Axiom (C4) is trivial. For simplicity, we write $[a, b]$ in place of $\chi([x_1, \ldots, x_{r-2}, a, b])$.

If $[y_1, x_r] = [x_{r-1}, y_2] = -1$ then we replace y_1 with x_r, x_r with y_1, x_{r-1} with $\overline{y_2}$ and y_2 with $\overline{x_{r-1}}$. This changes the signs of the factors in the first inequality, whereas the other inequalities remain unchanged. Similarly, we can assume without loss of generality that both factors of the second inequality and $[y_1, y_2]$ are non-negative. Let $a, b, c, d \in C_{2k}$ so that

$$x_{r-1} \in Z^a, x_r \in Z^b, y_1 \in Z^c, y_2 \in Z^d.$$

By shifting the hyperline, we assume that $c = 0$. Then we have

$$b, d-a, b-d, a, d \in \{0, 1, \ldots, k\} \subset C_{2k}.$$

If $[y_1, y_2] = 0$ then the third inequality is satisfied, and Axiom (C4) is proven. If $[y_1, y_2] = 1$ then $d \in \{1, \ldots, k-1\}$. Assume that $[x_{r-1}, x_r] = -1$, thus $a - b \in \{1, \ldots, k-1\}$. With the natural order on $0, 1, \ldots, k$ we find $0 \leq b < a \leq d < k$. This is a contradiction to $b - d \in \{0, 1, \ldots, k\}$. Hence $[x_{r-1}, x_r] \geq 0$, which finishes the proof of Axiom (C4).
2. Conversely, let χ be a chirotope of rank r over E. We wish to construct a hyperline sequence X with the same positively oriented bases. Note that in our proof we do not mention all applications of Axiom (C2) explicitly. If χ is of rank 1, then we define X as the set of all $x \in \mathbf{E}$ with $\chi([x]) = +1$. Since $\chi([x]) = -\chi([\overline{x}])$ and since $X \neq \emptyset$ by Axiom (C1), X is a hyperline sequence of rank 1, and it has the desired positively oriented bases by construction.

Let χ be of rank 2. Fix some element $e \in E$. We iteratively define a sequence $X^0, X^1, \ldots$ of subsets of $\mathbf{E}$ as follows. We start with

$$X^0 = \big\{x \in \mathbf{E} \mid \ \chi([e,x]) = 1, \text{ and for all } y \in \mathbf{E} \text{ with } \chi([e,y]) = 1 \text{ holds } \chi([x,y]) \geq 0\big\}.$$

For $a \geq 0$, pick some $x^{(a)} \in X^a$. Define

$$X^{a+1} = \big\{x \in \mathbf{E} \mid \ \chi([x^{(a)},x]) = 1, \text{ and for all } y \in \mathbf{E} \text{ with } \chi([x^{(a)},y]) = 1 \text{ holds } \chi([x,y]) \geq 0\big\}.$$

We prove that X^{a+1} (and similarly X^0) is not empty. Under the assumption $X^{a+1} = \emptyset$, we inductively define elements $x_0, x_1, \ldots \in \mathbf{E}$ as follows. By Axiom (C1), there is some $x_0 \in \mathbf{E}$ with $\chi([x^{(a)}, x_0]) = 1$. For $i \geq 0$, since $x_i \notin X^{a+1}$ there is some $x_{i+1} \in \mathbf{E}$ with $\chi([x^{(a)}, x_{i+1}]) = 1$ and $\chi([x_i, x_{i+1}]) = -1$. Since $\mathbf{E}$ is finite, we find an index $i \geq 2$ so that $x_{i+1} = x_k$ with $k < i-1$ (note that $\chi([x_i, x_{i-1}]) = 1$). We choose i minimal, which implies $\chi([x_{i-1}, x_k]) \geq 0$. It follows

$$\begin{array}{rcl}
\chi([x^{(a)}, x_k]) \cdot \chi([x_i, x_{i-1}]) & = & 1 \\
\chi([x^{(a)}, x_i]) \cdot \chi([x_{i-1}, x_k]) & \geq & 0 \\
\chi([x^{(a)}, x_{i-1}]) \cdot \chi([x_i, x_k]) & = & -1,
\end{array}$$

which contradicts Axiom (C4). Hence $X^{a+1} \neq \emptyset$. Since $\chi([x^{(a)}, x]) = 1$ for $x \in X^{a+1}$, we have $\chi([x^{(a)}, \overline{x}]) = -1$ by Axiom (C2), thus $\overline{x} \notin X^{a+1}$. In conclusion, X^{a+1} is a hyperline sequence of rank 1.

We show that for $a > 0$ and for all $x_1, y_2 \in X^a$ and all $x_2 \in \mathbf{E}$ we have $\chi([x_1, x_2]) = \chi([y_2, x_2])$. Let $y_1 = x^{(a-1)} \in X^{a-1}$ be the element that appears in the definition of X^a. By definition $\chi([y_1, x_1]) = \chi([y_1, y_2]) = 1$. Hence we have $\chi([x_1, y_2]) \geq 0$ and $\chi([y_2, x_1]) \geq 0$, thus $\chi([x_1, y_2]) = 0$. Assume that $\chi([x_1, x_2]) \neq \chi([y_2, x_2])$. Without loss of generality, we assume $\chi([x_1, x_2]) = -1$ and $\chi([y_2, x_2]) \geq 0$. We obtain

$$\begin{array}{rcl}
\chi([y_1, x_2]) \cdot \chi([x_1, y_2]) & = & 0 \\
\chi([y_1, x_1]) \cdot \chi([y_2, x_2]) & \geq & 0 \\
\chi([y_1, y_2]) \cdot \chi([x_1, x_2]) & = & -1.
\end{array}$$

This is a contradiction to Axiom (C4), hence $\chi([x_1, x_2]) = \chi([y_2, x_2])$. This implies that X^{a+1} does not depend on the choice of $x^{(a)} \in X^a$.

Let $p \in \mathbb{N}$ be the least number so that $e \in X^{p-1}$. Since X^{a+1} is independent of the choice of $x^{(a)} \in X^a$, it follows $X^a = X^{a+p}$ for all $a \in \mathbb{N}$. Moreover, if $q \in \mathbb{N}$ is the least number with $\overline{e} \in X^{q-1}$, we have $X^{a+q} = \overline{X^a}$ for all $a \in \mathbb{N}$.

In order to prove that $(X^0, X^1, \dots, X^{p-1})$ yields a hyperline sequence of rank 2 over E with period length $2q = p$, it remains to show that $\mathbf{E}$ is a disjoint union of $X^0, \dots, X^{p-1}$. By choosing p minimal and by the independence of X^{a+1} from the choice of $x^{(a)}$, it follows that $X^0, \dots, X^p$ are disjoint. Let $x_1 \in \mathbf{E}$. There is some $x_2 \in \mathbf{E}$ with $\chi([x_1, x_2]) = 1$, by Axiom (C1). Since $\chi([x^{(0)}, x^{(1)}]) = 1$ by definition, Axiom (C3) implies that $\chi([x_1, x^{(0)}])$ or $\chi([x_1, x^{(1)}])$ does not vanish. Therefore and by Axiom (C2), there is a least index i so that $\chi([x^{(i)}, x_1]) = 1$ and $\chi([x^{(i+1)}, x_1]) \leq 0$. Let $y \in \mathbf{E}$ with $\chi([x^{(i)}, y]) = 1$. By definition of $x^{(i+1)}$, we have

$$\begin{aligned}\chi([y, x^{(i)}]) \cdot \chi([x^{(i+1)}, x_1]) &\geq 0,\\ \chi([y, x^{(i+1)}]) \cdot \chi([x_1, x^{(i)}]) &\geq 0\end{aligned}$$

and $\chi([x^{(i)}, x^{(i+1)}]) = 1$, and therefore $\chi([x_1, y]) \geq 0$ by Axiom (C4). But this means $x_1 \in X^{(i+1)}$. In conclusion, $\mathbf{E} = X^0 \cup \dots \cup X^p$. Hence $(X^0, X^1, \dots, X^{p-1})$ yields a hyperline sequence of rank 2 over E, and by construction it has the same positively oriented bases than χ.

Finally, we come to the case of rank $r \geq 3$. Let $x_1, \dots, x_{r-2} \in \mathbf{E}$ be so that there are two elements $x_{r-1}, x_r \in \mathbf{E}$ with $\chi([x_1, \dots, x_r]) = 1$. Such an $(r-2)$–tuple exists by Axiom (C1). It is easy to verify that

$$C(x_1, \dots, x_{r-2}) = \{[y_{r-1}, y_r] \in \Delta_1(E) |\ \chi([x_1, \dots, x_{r-2}, y_{r-1}, y_r]) = 1\}$$

is the set of positively oriented bases of a chirotope of rank 2 over some subset of E, thus, of a hyperline sequence $Z(x_1, \dots, x_{r-2})$ of rank 2. By application of Axiom (C2), one can also show that

$$\begin{aligned}B(x_1, \dots, x_{r-2}) = \{[y_1, \dots, y_{r-2}] \in \Delta_{r-3}(E) \quad | \quad & \chi([y_1, \dots, y_r]) = 1\\ & \text{for all } [y_{r-1}, y_r] \in C(x_1, \dots, x_{r-2})\}\end{aligned}$$

is the set of positively oriented bases of a chirotope of rank $r-2$ over some subset of E, with $[x_1, \dots, x_{r-2}] \in B(x_1, \dots, x_{r-2})$. By induction, it is equivalent to a hyperline sequence $Y(x_1, \dots, x_{r-2})$ of rank $r-2$.

We collect all pairs $\big(Y(x_1, \dots, x_{r-2}) | Z(x_1, \dots, x_{r-2})\big)$ to form a set (not a multi-set) X, where $x_1, \dots, x_{r-2} \in \mathbf{E}$. It has the same positively oriented bases as χ, by construction. It remains to show that X is a hyperline sequence of rank r over E. The most difficult to prove is Axiom (H1). Let $Y = Y(x_1, \dots, x_{r-2})$ and $Z = Z(x_1, \dots, x_{r-2})$. By definition, in an oriented simplex over E any element of $\mathbf{E}$ occurs at most once. This implies that $E(Y) \cap E(Z) = \emptyset$. Let $e \in E \setminus E(Z)$. It remains to show that $e \in E(Y)$, hence $E = E(Y) \cup E(Z)$. Let $[x_1, \dots, x_{r-2}]$ and $[x_{r-1}, x_r]$ be positively oriented bases of Y and Z, thus $\chi([x_1, \dots, x_r]) = 1$. By multiple application of Axioms (C2) and (C3), there is an index i so that $\chi([x_1, \dots, x_{i-1}, e, x_{i+1}, \dots, x_r]) \neq 0$. We have $i < r-1$ since $e \notin E(Z)$.

We claim that for all $[y_1, y_2] \in C(x_1, \dots, x_{r-2})$ holds

$$\chi([x_1, \dots, x_{i-1}, e, x_{i+1}, \dots, x_{r-2}, y_1, y_2]) = \chi([x_1, \dots, x_{i-1}, e, x_{i+1}, \dots, x_r]).$$

By replacing e with $\overline{e}$ if necessary, we can assume that $\chi([x_1, \dots, x_{i-1}, e, x_{i+1}, \dots, x_r]) = 1$. By Axiom (C2), it suffices to consider the case $y_1 = x_{r-1}$. For simplicity, we denote $[a, b]$ for $\chi([x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_{r-1}, b])$. We have $[x_i, x_r] = [x_i, y_2] = [e, x_r] = 1$, and $[x_i, e] = 0$ since $e \notin E(Z)$. If $[e, y_2] = 0$, then

$$\begin{aligned}[x_r, y_2] \cdot [e, x_i] &= 0\\ [x_r, x_i] \cdot [e, y_2] &= 0\\ [x_r, e] \cdot [x_i, y_2] &= -1,\end{aligned}$$

in contradiction to Axiom (C4). Hence $[e, y_2] \neq 0$. Since

$$\begin{aligned} [x_i, y_2] \cdot [e, x_r] &= 1 \\ [x_i, e] \cdot [x_r, y_2] &= 0 \end{aligned}$$

and $[x_i, x_r] = 1$, it follows $[e, y_2] = 1$ from Axiom (C4). This proves our claim. The claim implies that $e \in E(Y)$, which finishes the proof of Axiom (H1).

We check the remaining axioms. Since $\chi([x_1, \ldots, x_r]) = \chi([x_1, \ldots, x_{r-3}, \overline{x_{r-2}}, x_r, x_{r-1}])$ by Axiom (C2), we have

$$\begin{aligned} Y(x_1, \ldots, x_{r-3}, \overline{x_{r-2}}) &= -Y(x_1, \ldots, x_{r-2}), \\ Z(x_1, \ldots, x_{r-3}, \overline{x_{r-2}}) &= -Z(x_1, \ldots, x_{r-2}), \end{aligned}$$

and this implies Axiom (H2). Axiom (H3) is Axiom (C3). Finally, Axiom (H4) is a special case of Axiom (C2). Thus, X is a hyperline sequence. $\square$

Our proof of the topological representation theorem in Sections 6 and 7 is by induction on the rank and the number of elements of a chirotope or hyperline sequence. In the rest of this section, we expose the basic techniques for this induction. Let χ be a chirotope of rank r over E, and let $R \subset E$. We define the map $\chi \setminus R\colon \Delta_{r-1}(E \setminus R) \to \{-1, 0, +1\}$ as the restriction of χ to $\Delta_{r-1}(E \setminus R)$. We call $\chi \setminus R$ the **deletion** of R in χ. In general, $\chi \setminus R$ is not a chirotope. However if the rank of χ is smaller than $|E|$, then we find by the following lemma an element that can be deleted such that $\chi \setminus \{i\}$ is a chirotope of the same rank.

Lemma 1. *Let χ be a chirotope of rank r over E. If $|E| > r$ then there is an $i \in E$ so that $\chi \setminus \{i\}$ is a chirotope of rank r over $E \setminus \{i\}$.*

Proof. By Axioms (C1) and (C2), there are $f_1, \ldots, f_r \in E$ with $\chi([f_1, \ldots, f_r]) \neq 0$. Since $|E| > r$, we can pick some $i \in E \setminus \{f_1, \ldots, f_r\}$. It is obvious that $\chi \setminus \{i\}$ satisfies Axioms (C2), (C3) and (C4).

It remains to show that $\chi \setminus \{i\}$ satisfies Axiom (C1). Let $e_1 \in E \setminus \{i\}$. By Axiom (C1), there are $e_2, \ldots, e_r \in E$ with $\chi([e_1, \ldots, e_r]) \neq 0$. If $i \in \{e_2, \ldots, e_r\}$ then by Axiom (C2) we can assume that $i = e_r$. Since $\chi([f_1, \ldots, f_r]) \neq 0$ from the preceding paragraph, we find some f_j so that $\chi([e_1, \ldots, e_{r-1}, f_j]) \neq 0$, by Axiom (C3). Since $[e_1, \ldots, e_{r-1}, f_j] \in \Delta_{r-1}(E \setminus \{i\})$, the map $\chi \setminus \{i\}$ satisfies Axiom (C1) and is therefore a chirotope. $\square$

Let $R = \{e_1, \ldots, e_k\} \subset E$, with $e_1 < \cdots < e_k$ and $k < r$. We define E/R as the set of all $e \in E$ for which there exist $e_{k+1}, \ldots e_r \in E$ so that $\chi([e_1, \ldots, e_{r-1}, e]) \neq 0$. The map $\chi/R\colon \Delta_{r-1-k}(E/R) \to \{-1, 0, +1\}$ is then defined by $\chi/R([e_{k+1}, \ldots, e_r]) = \chi([e_1, \ldots, e_r])$, for all $[e_{k+1}, \ldots, e_r] \in \Delta_{r-1}(E/R)$. It is obvious that χ/R satisfies Axioms (C2), (C3) and (C4). It satisfies Axiom (C1) by definition of E/R. Hence, χ/R is a chirotope of rank $r - k$ over E/R, that is called the **contraction** of χ on R.

Finally, let X be a hyperline sequence of rank r over E. Let χ be the chirotope that corresponds to X. For any $R \subset E$ so that $\chi \setminus R$ (resp. χ/R) is a chirotope, we define the **deletion** $X \setminus R$ of R in X (resp. the **contraction** X/R of X on R) as the hyperline sequence associated to $\chi \setminus R$ (resp. χ/R).

4. Arrangements of oriented pseudospheres

In the preceding sections, we extracted combinatorial data from vector arrangements and turned properties of these data into axioms, yielding hyperline sequences and chirotopes. In this section, we will generalize vector arrangements in a geometric way. The main aim of this paper is to prove the equivalence of these geometric structures with hyperline sequences.

The idea is as follows. Let $\mathbb{S}^{r-1} \subset \mathbb{R}^r$ be the unit sphere, and let $V \subset R^r$ be a finite multiset of non-zero vectors. Any $v \in V$ yields a pair $\pm\frac{v}{\|v\|}$ of points in $\mathbb{S}^{r-1}$. This is dual to a hypersphere $S_v \in \mathbb{S}^{r-1}$, namely the intersection of $\mathbb{S}^{r-1}$ with the hyperplane $H_v \subset \mathbb{R}^r$

that is perpendicular to v. We put an orientation on H_v, so that a positive base of H_v together with v is a positive base of $\mathbb{R}^r$. This induces an orientation on S_v. In conclusion, V is dual to an arrangement of oriented hyperspheres in $\mathbb{S}^{r-1}$. The geometric idea is now to consider arrangements of oriented embedded spheres of codimension 1 in $\mathbb{S}^{r-1}$, intersecting each other similarly to hyperspheres, though not being hyperspheres in general.

We formalize this idea. Let $\mathbb{S}^d$ denote the d–dimensional oriented sphere

$$\mathbb{S}^d = \left\{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid x_1^2 + \dots + x_{d+1}^2 = 1\right\},$$

and let

$$B^d = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d | x_1^2 + \dots + x_d^2 \leq 1\right\}$$

denote the closed d–dimensional ball. A submanifold N of codimension m in a d–dimensional manifold M is **tame** if any $x \in N$ has an open neighbourhood $U(x) \subset M$ such that there is a homeomorphism $\overline{U(x)} \to B^d$ sending $U(x) \cap N$ to $B^{d-m} \subset B^d$.

An **oriented pseudosphere** $S \subset \mathbb{S}^d$ is a tame embedded $(d-1)$–dimensional sphere with a choice of an orientation. Obviously any oriented hypersphere is an oriented pseudosphere. Let $\psi\colon \mathbb{S}^{d-1} \to \mathbb{S}^d$ be an embedding with image S, inducing the desired orientation of S. By a result of M. Brown [5], the image of ψ is tame if and only if ψ can be extended to an orientation preserving embedding

$$\tilde{\psi}\colon \mathbb{S}^{d-1} \times [-1, 1] \to \mathbb{S}^d \text{ with } \psi(\cdot) = \tilde{\psi}(\cdot, 0).$$

The image of an oriented pseudosphere S under a homeomorphism $\phi\colon \mathbb{S}^d \to \mathbb{S}^d$ is again an oriented pseudosphere, since the defining embedding $\phi \circ \psi\colon \mathbb{S}^{d-1} \to \mathbb{S}^d$ can be extended to $\phi \circ \tilde{\psi}\colon \mathbb{S}^{d-1} \times [-1, 1] \to \mathbb{S}^d$. By the generalized Schönflies theorem, that was also proven by M. Brown [4], $\mathbb{S}^d \setminus S$ is a disjoint union of two open balls whose closures are closed balls. We call the connected component of $\mathbb{S}^d \setminus S$ containing $\tilde{\psi}(\mathbb{S}^{d-1} \times \{1\})$ (resp. $\tilde{\psi}(\mathbb{S}^{d-1} \times \{-1\})$) the **positive side** S^+(resp. **negative side** S^-) of S. The following definition of arrangements of oriented pseudospheres is similar to [1], p. 227. Recall $E_n = \{1, \dots, n\}$.

Definition 5. Let $S_1, \dots, S_n \subset \mathbb{S}^d$ be not necessarily distinct oriented pseudospheres, ordered according to their indices. Assume that the following conditions hold:

(A1) $S_R = \mathbb{S}^d \cap \bigcap_{i \in R} S_i$ is empty or homeomorphic to a sphere, for all $R \subset E_n$.

(A2) Let $R \subset E_n$ and $i \in E_n$ with $S_R \not\subset S_i$. Then $S_R \cap S_i$ is a pseudosphere in S_R, and $S_R \cap S_i^+$ and $S_R \cap S_i^-$ are both non-empty.

Then the ordered multiset $\{S_1, \dots, S_n\}$ is an **arrangement of oriented pseudospheres** over E_n. The arrangement is called **of full rank** if the intersection of its members is empty.

We omitted Axiom (A3) from [1], p. 227, since it follows from the other two axioms. We remark that in [1], arrangements of full rank are called *essential*. Obviously if $S_1, \dots, S_n$ are oriented hyperspheres then $\{S_1, \dots, S_n\}$ is an arrangement of oriented pseudospheres. At the end of this section, we will characterize arrangements of oriented pseudospheres by a single axiom.

Definition 6. Two ordered multisets $\{S_1, \dots, S_n\}$ and $\{S_1', \dots, S_n'\}$ of oriented pseudospheres in $\mathbb{S}^d$ are **equivalent** if there is an orientation preserving homeomorphism $\mathbb{S}^d \to \mathbb{S}^d$ sending S_i^+ to $(S_i')^+$ and S_i^- to $(S_i')^-$, simultaneously for all $i \in E_n$. We do not allow renumbering of the pseudospheres.

Since the image of a pseudosphere under a homeomorphism $\phi\colon \mathbb{S}^d \to \mathbb{S}^d$ is a pseudosphere, it is easy to observe that if $\{S_1, \dots, S_n\}$ is an arrangement of oriented pseudospheres, then so is $\{\phi(S_1), \dots, \phi(S_n)\}$.

Example 1. We construct non-equivalent arrangements $\mathcal{A}(d, +), \mathcal{A}(d, -)$ of $d + 1$ oriented pseudospheres $S_1, \dots, S_{d+1} \subset \mathbb{S}^d$ of full rank as follows. For $i = 1, \dots, d + 1$, set

$$S_i = \left\{(x_1, \dots, x_{d+1}) \in \mathbb{S}^d \mid x_i = 0\right\}.$$

In $\mathcal{A}(d,+)$, define $S_1^+ = \{(x_1,\dots,x_{d+1}) \in \mathbb{S}^d \mid x_1 > 0\}$. In $\mathcal{A}(d,-)$, define $S_1^+ = \{(x_1,\dots,x_{d+1}) \in \mathbb{S}^d \mid x_1 < 0\}$. In both $\mathcal{A}(d,+)$ and$\mathcal{A}(d,-)$, define $S_i^+ = \{(x_1,\dots,x_{d+1}) \in \mathbb{S}^d \mid x_i > 0\}$, for $i = 2,\dots,d+1$.

$\mathcal{A}(d,+)$ and $\mathcal{A}(d,-)$ are of full rank. If $\phi\colon \mathbb{S}^d \to \mathbb{S}^d$ is any homeomorphism that fixes $S_2^+,\dots,S_{d+1}^+$ setwise and maps $\{(x_1,\dots,x_{d+1}) \in \mathbb{S}^d \mid x_1 > 0\}$ to $\{(x_1,\dots,x_{d+1}) \in \mathbb{S}^d \mid x_1 < 0\}$, then ϕ is orientation reversing. Thus, $\mathcal{A}(d,+)$ is not equivalent to $\mathcal{A}(d,-)$.

In the preceding section, we have defined deletion and contraction of chirotopes and hyperline sequences. There is a similar notion for arrangements of oriented pseudospheres, as follows. Fix an arrangement $\mathcal{A} = \{S_1,\dots S_n\}$ of oriented pseudospheres in $\mathbb{S}^d$. For any $R \subset E_n$, we obtain an arrangement

$$\mathcal{A} \setminus \{S_r \mid r \in R\}$$

of oriented pseudospheres over $E_n \setminus R$ in $\mathbb{S}^d$ (Axioms (A1) and (A2) are easy to verify). We denote this arrangement by $\mathcal{A} \setminus R$ and call it the **deletion** of R in $\mathcal{A}$. In general, $\mathcal{A} \setminus R$ is not of full rank, even if $\mathcal{A}$ is.

It is intuitively clear from Axiom (A2), that for $R \subset E_n$ one gets an arrangement of oriented pseudospheres on S_R, induced by $\mathcal{A}$. In the following iterative definition of this induced arrangement, the orientation of S_R requires some care. Let $r \in E_n$, and let $\psi_r\colon \mathbb{S}^{d-1} \to \mathbb{S}^d$ be a tame embedding defining S_r with the correct orientation. Denote $S_i' = \psi_r^{-1}(S_i \cap S_r)$, for $i \in E_n$. We obtain an ordered multiset

$$\mathcal{A}/\{r\} = \{S_i' \mid i \in E_n, S_r \not\subset S_i\}$$

of oriented pseudospheres in $\mathbb{S}^{d-1}$, where $(S_i')^+ = \psi_r^{-1}(S_i^+ \cap S_r)$. Axioms (A1) and (A2) are easy to verify, thus, $\mathcal{A}/\{r\}$ is an arrangement of oriented pseudospheres over a subset of $E_n \setminus \{r\}$.

This construction can be iterated. Let $R \subset E_n$ so that $\dim S_R = d - |R|$. List the elements of R in ascending order, $r_1 < r_2 < \cdots < r_{|R|}$. The **contraction** of $\mathcal{A}$ in R is then the arrangement of oriented pseudospheres

$$\mathcal{A}/R = \Big(\dots((\mathcal{A}/\{r_1\})/\{r_2\})\dots\Big/\{r_{|R|}\}\Big).$$

Note that $(\mathcal{A}/\{r_1\})/\{r_2\}$ and $(\mathcal{A}/\{r_2\})/\{r_1\}$ are related by an orientation *reversing* homeomorphism $\mathbb{S}^{d-2} \to \mathbb{S}^{d-2}$. It is easy to see that $\mathcal{A}/R$ is of full rank if and only if $\mathcal{A}$ is of full rank.

An arrangement $\mathcal{A} = \{S_1,\dots,S_n\}$ of oriented pseudospheres in $\mathbb{S}^d$ yields a cellular decomposition of $\mathbb{S}^d$, as follows. For any subset $I \subset E_n$, we consider the parts of the intersection of the pseudospheres with label in I that are not contained in pseudospheres with other labels,

$$\begin{aligned}\mathcal{C}(I,\mathcal{A}) &= \{x \in \mathbb{S}^d \mid x \in S_i \iff i \in I, \text{ for all } i \in E_n\}\\ &= (\bigcap_{i\in I} S_i) \setminus (\bigcup_{j \in E_n\setminus I} S_j).\end{aligned}$$

By the next theorem, the connected components of $\mathcal{C}(I,\mathcal{A})$ are topological cells if $\mathcal{A}$ is of full rank. Hence, it makes sense to refer to a connected component of $\mathcal{C}(I,\mathcal{A})$ as a **cell of $\mathcal{A}$ with label** I. The d–**skeleton** $\mathcal{A}^{(d)}$ of $\mathcal{A}$ is the union of its cells of dimension $\le d$. By Axiom (A1), $\bigcap_{i\in R} S_i$ is empty or a sphere, for $R \subset E_n$. Thus, for any 0-dimensional cell of $\mathcal{A}$ exists exactly one other cell of $\mathcal{A}$ with the same label, corresponding to the two points in $\mathbb{S}^0$. The statement that the cells of an arrangement of oriented pseudospheres of full rank are in fact topological cells is essential for the proof of the Topological Representation Theorem in Sections 6 and 7.

Theorem 2. *Let $\mathcal{A} \neq \emptyset$ be an arrangement of oriented pseudospheres over E_n. Any connected component of $\mathcal{C}(\emptyset,\mathcal{A})$ is a d–dimensional cell whose closure is a closed ball. If $\mathcal{A}$ is of full rank, then for any $I \subset E_n$, any connected component of $\mathcal{C}(I,\mathcal{A})$ is an open cell whose closure is a closed ball.*

Proof. First, we prove that the theorem holds for $\mathcal{C}(\emptyset, \mathcal{A})$, by induction on the number n of elements of $\mathcal{A}$. Since $\mathcal{A} \neq \emptyset$, we have $n > 0$. The base case $n = 1$ is the generalized Schönflies theorem [4], stating that an embedded tame $(d-1)$–sphere in $\mathbb{S}^d$ is the image of a hypersphere under a homeomorphism $\mathbb{S}^d \to \mathbb{S}^d$. In particular, the complement of a pseudosphere is a disjoint union of two d–dimensional cells whose closures are balls.

Let $n > 1$, and let c be the closure of a connected component of $\mathcal{C}(\emptyset, \mathcal{A} \setminus \{n\})$. By induction on n, it is a cell of dimension d. The connected components of $S_n \cap c$ correspond to the closures of connected components of $\mathcal{C}(\emptyset, \mathcal{A}/\{n\})$. By induction hypothesis, applied to the arrangement $\mathcal{A}/\{n\}$ of $n-1$ pseudospheres in $\mathbb{S}^{d-1}$, these are closed balls of dimension $d-1$ whose boundary lies in ∂c. Since S_n is tame, $S_n \cap c$ is tame in c. Hence, it follows from the generalized Schönflies theorem [4] that the closure of any connected component of $c \setminus S_n$ is a ball of dimension d. In conclusion, the connected components of $\mathcal{C}(\emptyset, \mathcal{A})$ are cells of dimension d whose closures are balls.

It remains to prove the theorem in the full rank case with $I \neq \emptyset$. We can assume $\mathcal{C}(I, \mathcal{A}) \neq \emptyset$. Since $\mathcal{A}$ is of full rank, $\mathcal{C}(E_n, \mathcal{A}) = \emptyset$, hence $I \neq E_n$. By Axiom (A1), S_I is homeomorphic to some sphere $\mathbb{S}^e$. There is an $R \subset I$ with $|R| \leq d - e$ and $S_R = S_I$. The set $\mathcal{C}(I, \mathcal{A})$ is mapped to $\mathcal{C}(\emptyset, \mathcal{A}/R)$ by the restriction of a homeomorphism $S_I \to \mathbb{S}^e$. It has already been proven that the closure of any connected component of $\mathcal{C}(\emptyset, \mathcal{A}/R)$ is a ball. Thus, the closure of any connected component of $\mathcal{C}(I, \mathcal{A})$ is a ball, as well. □

We remark that the preceding theorem becomes wrong by dropping the hypothesis that pseudospheres are tame. In fact, there are wild 2-spheres in $\mathbb{S}^3$ (e.g. the famous Horned Sphere of Alexander), whose complement is not a union of cells. We will consider arrangements of oriented embedded spheres (not necessarily tame) in Section 8 and prove that these wild arrangements have the same combinatorics than tame arrangements.

Let $\mathcal{A} = \{S_1, \dots, S_n\}$ be an ordered multiset of oriented pseudospheres in $\mathbb{S}^d$. For $R \subset E_n$, denote $\mathcal{A}_R = \{S_j | j \in R\}$. In the remainder of this section, we show that one can replace Axioms (A1) and (A2) by the following single **Axiom (A')**.

(A') Let $R \subset E_n$ so that $S_{R'} \neq S_R$ for any proper subset R' of R. Then, $\mathcal{A}_R$ is equivalent to an arrangement of $|R|$ oriented hyperspheres in $\mathbb{S}^d$.

Theorem 3. *An ordered multiset $\mathcal{A} = \{S_1, \dots, S_n\}$ of oriented pseudospheres in $\mathbb{S}^d$ satisfies Axiom (A') if and only if it is an arrangement of oriented pseudospheres.*

Proof. First, we assume that $\mathcal{A}$ satisfies Axiom (A') and prove that $\mathcal{A}$ is an arrangement of oriented pseudospheres. Let $\emptyset \neq R \subset E_n$, and assume that $S_R \neq \emptyset$. Replacing R by a subset if necessary, we can assume that $S_{R'} \neq S_R$ for any proper subset R' of R. By Axiom (A'), $\mathcal{A}_R$ is equivalent to an arrangement of oriented hyperspheres. Thus, S_R is homeomorphic to a sphere, and Axiom (A1) holds. Let $i \in E_n$ with $S_R \not\subset S_i$; we wish to prove (A2). There is some $\tilde{R} \subset R \cup \{i\}$ with $S_{\tilde{R}} = S_{R \cup \{i\}}$, so that $S_{R'} \neq S_{\tilde{R}}$ for any proper subset R' of $\tilde{R}$. Since by Axiom (A'), $\mathcal{A}_{\tilde{R}}$ is equivalent to an arrangement of oriented hyperspheres, it follows

$$\dim S_{\tilde{R} \setminus \{i\}} = \dim S_{R \cup \{i\}} + 1.$$

Since both S_R and $S_{R \cup \{i\}}$ are spheres by Axiom (A1) (that has already been proven) and since $S_R \not\subset S_i$ by hypothesis, it follows

$$\dim S_{R \cup \{i\}} + 1 = \dim S_{\tilde{R} \setminus \{i\}} \geq \dim S_R > \dim S_{R \cup \{i\}}.$$

Thus, $S_{\tilde{R} \setminus \{i\}} \subset S_R$ is a pair of speres of the same dimension $\dim S_{R \cup \{i\}} + 1$, hence, $S_{\tilde{R} \setminus \{i\}} = S_R$. Since $\mathcal{A}_{\tilde{R}}$ is equivalent to an arrangement of oriented hyperspheres, Axiom (A2) holds for $S_{\tilde{R} \setminus \{i\}} = S_R$ and S_i. In conclusion, $\mathcal{A}$ is an arrangement of oriented pseudospheres.

Secondly, we assume that $\mathcal{A}$ is an arrangement of oriented pseudospheres and prove by induction on $|R|$ that (A') holds. If $|R| = 1$, then (A') is nothing but the generalized Schönflies theorem [4]. In the general case, let $\emptyset \neq R \subset E_n$ so that $S_{R'} \neq S_R$ for any proper subset R' of R, and let $i \in R$. It follows easily that $S_{R'} \neq S_{R \setminus \{i\}}$ for any proper subset R'

of $R \setminus \{i\}$. Thus, by Axiom (A') we can assume that $\mathcal{A}_{R\setminus\{i\}}$ is an arrangement of oriented hyperspheres. Note that the connected components of $\mathcal{C}(I, \mathcal{A}_{R\setminus\{i\}})$ are topological cells, for any proper subset I of $R \setminus \{i\}$.

Let $H \subset \mathbb{S}^d$ be a hypersphere that does not contain $S_{R\setminus\{i\}}$. Our aim is to transform S_i into H, fixing $\mathcal{A}_{R\setminus\{i\}}$ cellwise, which implies Axiom (A') for $\mathcal{A}_R$. By Axiom (A2) and by the generalized Schönflies theorem, $S_{R\setminus\{i\}} \cap S_i$ can be mapped to $S_{R\setminus\{i\}} \cap H$ by some orientation preserving homeomorphism $S_{R\setminus\{i\}} \to S_{R\setminus\{i\}}$, The homeomorphism can be extended to a homeomorphism $\mathbb{S}^d \to \mathbb{S}^d$ fixing all cells of $\mathcal{A}_{R\setminus\{i\}}$, by the cone construction [22].

We now proceed with transforming S_i in cells of $\mathcal{A}_{R\setminus\{i\}}$ of higher dimension. Let R' be a proper subset of $R \setminus \{i\}$. By induction on $|R \setminus R'|$, we can assume that $S_{R'\cup\{j\}} \cap S_i = S_{R'\cup\{j\}} \cap H$, for all $j \in R \setminus (R' \cup \{i\})$. Let $C \subset S_{R'}$ be the closure of a cell of $\mathcal{A}_{R\setminus\{i\}}$ with $\dim C = \dim S_{R'}$. It follows from Axiom (A2) that $B = S_i \cap C$ is a tame ball in C. As a consequence of the generalized Schönflies theorem and since $\partial B \subset H$, we can map $C \cap S_i$ to $C \cap H$ by an orientation preserving homeomorphism $C \to C$ that is the identity on ∂C. It can be extended to a homeomorphism $\mathbb{S}^d \to \mathbb{S}^d$ by the cone construction, fixing all cells of $\mathcal{A}_{R\setminus\{i\}}$. Thus, we can transform $S_{R'} \cap S_i$ into $S_{R'} \cap H$. Finally, when we achieve $R' = \emptyset$, we have $S_{R'} \cap S_i = S_i = H$. Therefore, $\mathcal{A}_R$ is equivalent to an arrangement of oriented hyperspheres. This proves (A'), as claimed. $\square$

Corollary 1. *If an arrangement $\mathcal{A}$ of oriented pseudospheres in $\mathbb{S}^d$ is of full rank and $n = d + 1$, then $\mathcal{A}$ is equivalent to $\mathcal{A}(d, +)$ or $\mathcal{A}(d, -)$.*

Proof. Let $R \subset E_{d+1}$ be minimal so that $\mathcal{A}_R$ is of full rank. By the preceding theorem, $\mathcal{A}_R$ is equivalent to an arrangement of oriented hyperspheres of full rank. Such an arrangement consists of at least $d + 1$ hyperspheres, hence $R = E_{d+1}$ and $\mathcal{A}_R = \mathcal{A}$.

Any arrangement of $d + 1$ oriented hyperspheres of full rank is dual to an arrangement of $d + 1$ unit vectors in $\mathbb{R}^{d+1}$ that span $\mathbb{R}^{d+1}$. Since $GL_{d+1}(\mathbb{R})$ acts transitively on those vector arrangements, it follows that $\mathcal{A}$ is equivalent to $\mathcal{A}(d, +)$ or $\mathcal{A}(d, -)$, depending on its orientation. $\square$

5. Chirotopes and hyperline sequences associated to arrangements of oriented pseudospheres

The aim of our paper is to prove a topological representation theorem for hyperline sequences, i.e., to establish a one-to-one correspondence between hyperline sequences and equivalence classes of arrangements of oriented pseudospheres. In this section, we settle one direction of this correspondence. We associate to any arrangement of oriented pseudospheres a hyperline sequence, compatible with deletions and contractions.

We first expose the geometric idea. By a **cycle** of an arrangement $\mathcal{A}$ of oriented pseudospheres in $\mathbb{S}^d$, we mean an embedded circle $\mathbb{S}^1 \subset \mathbb{S}^d$ that is the intersection of some elements of $\mathcal{A}$. Let L be a cycle of $\mathcal{A}$ with a choice of an orientation. It corresponds to a hyperline $(Y|Z)$, as follows. The positively oriented bases of Y correspond to $(d-1)$–tuples of pseudospheres containing L so that the system of positive normal vectors of the pseudospheres together with a positive tangential vector of L forms a direct base of the oriented vector space $\mathbb{R}^d$. The set of all elements of $\mathcal{A}$ containing L corresponds to $E(Y)$. The 0-dimensional cells of $\mathcal{A}$ on L occur in a cyclic order, corresponding to the cyclic order of Z. Let $S_e \in \mathcal{A}$. If in a point of $L \cap S_e$ the cycle L passes from the negative side of S_e to the positive side, then we have an element e in the corresponding atom of $(Y|Z)$ (see Figure 1). Further, in the second point of $L \cap S_e$, the cycle L passes from the positive to the negative side, yielding an element $\overline{e}$ in the atom that is opposite to the first atom.

The meaning of Axiom (H3) in the setting of arrangements of oriented pseudospheres is that any two cycles of $\mathcal{A}$ have non-empty intersection. Figure 2 provides a visualization of Axiom (H4) in an arrangement of three oriented pseudospheres in $\mathbb{S}^2$: If we get

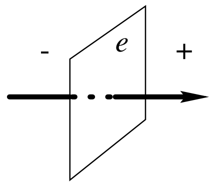

FIGURE 1. The atom $\{e\}$ on a cycle

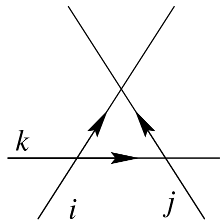

FIGURE 2. Axiom (H4) in rank 3

from the pseudosphere k the cyclic order $\big(\{i\},\{j\},\{\overline{\imath}\},\{\overline{\jmath}\}\big)$, then we get the cyclic order $\big(\{\overline{k}\},\{j\},\{k\},\{\overline{\jmath}\}\big)$ from i and the cyclic order $\big(\{\overline{k}\},\{\overline{\imath}\},\{k\},\{i\}\big)$ from j.

We formalize this idea in the rest of this section. Let $\mathcal{A} = \{S_1, \ldots, S_n\}$ be an arrangement of n oriented pseudospheres of full rank in $\mathbb{S}^d$. Our aim is to associate to $\mathcal{A}$ a hyperline sequence $X(\mathcal{A})$ of rank $d+1$ over E_n. By the equivalence of hyperline sequences and chirotopes, established in Theorem 1, this also allows to associate a chirotope $\chi(\mathcal{A})$ to $\mathcal{A}$, with the same positively oriented bases. If $d = 0$ then define

$$X(\mathcal{A}) = \big\{e \in E_n |\ S_e^+ = \{+1\}\big\} \cup \big\{\overline{e} \in \overline{E_n} |\ S_e^+ = \{-1\}\big\},$$

which is obviously a hyperline sequence over E_n of rank 1.

In the case $d = 1$, the orientation of $\mathbb{S}^1$ yields a cyclic order $p_0, p_1, \ldots, p_{2k-1}$ on the points of $S_1 \cup \cdots \cup S_n$. For $a \in \{0, \ldots, 2k-1\}$, we define $X^a \subset \mathbf{E_n}$ by

1. $e \in X^a$ if $p_a \in S_e$ and, along the cyclic orientation of $\mathbb{S}^1$, one passes in p_a from S_e^- to S_e^+, and
2. $\overline{e} \in X^a$ if $p_a \in S_e$ and one passes in p_a from S_e^+ to S_e^-.

It is easy to check that $(X^0, \ldots, X^{2k-1})$ yields a hyperline sequence of rank 2 over E_n.

In the case $d = 2$, let $\gamma \subset \mathbb{S}^2$ be an oriented cycle of $\mathcal{A}$. Let $R \subset E_n$ be the indices of oriented pseudospheres containing γ. There is a hyperline sequence $Y(\gamma)$ of rank 1 over R, with $e \in Y(\gamma)$ (resp. $\overline{e} \in Y(\gamma)$) if the orientation of S_e coincides (resp. does not coincide) with the orientation of γ. As in the preceding paragraph, we obtain a hyperline sequence $Z(\gamma)$ of rank 2 over $E_n \setminus R$. We collect the pairs $(Y(\gamma), Z(\gamma))$ to form a set $X(\mathcal{A})$, where γ runs over all oriented cycles of $\mathcal{A}$. Axioms (H1) and (H2) are obvious for $X(\mathcal{A})$. To prove Axiom (H3), let $[x_1, x_2, x_3]$ and $[y_1, y_2, y_3]$ be two positively oriented bases of $X(\mathcal{A})$. By definition, the pseudospheres $S_{y_1^*}$, $S_{y_2^*}$, $S_{y_3^*}$ have no point in common. In particular, one of them, say, $S_{y_1^*}$, does not contain $S_{x_1^*} \cap S_{x_2^*}$. Thus $S_{y_1^*}$ intersects $S_{x_1^*}$ transversely in $S_{x_1^*} \setminus S_{x_2^*}$, hence $[x_1, x_2, y_1]$ or $[x_1, x_2, \overline{y_1}]$ is a positively oriented base of $X(\mathcal{A})$. It remains to prove Axiom (H4). Here we use the Jordan–Schönflies theorem [22], stating that the complement of an embedded 1–sphere in $\mathbb{S}^2$ is a disjoint union of two discs. This holds even without the assumption of tameness. With this in mind, Axiom (H4) can be read off from Figure 2.

In the case $d \geq 3$, let γ be an oriented cycle of $\mathcal{A}$. Let $R_\gamma = \{r \in E_n | \, \gamma \subset S_r\}$. As in the preceding paragraph, the cyclic orientation of γ induces a cyclic order of the oriented points

$$\gamma \cap \bigcup_{e \in E_n \setminus R_\gamma} S_e,$$

yielding a hyperline sequence $Z(\gamma)$ of rank 2 over $E_n \setminus R_\gamma$. Since $\mathcal{A}$ is of full rank, there are $i, j \in E_n$ so that $S_i \cap S_j$ is a sphere of dimension $d-2$ disjoint to γ. We may assume that $Z(\gamma)$ yields the cyclic order $(\{i\}, \{j\}, \{\bar{\imath}\}, \{\bar{\jmath}\})$, by changing the roles of i and j if necessary. Then

$$\mathcal{A}(i,j) = \big((\mathcal{A}/\{i\})/\{j\}\big)_{R_\gamma}$$

is an arrangement of oriented pseudospheres in $\mathbb{S}^{d-2}$ of full rank over R_γ, and by induction it corresponds to a hyperline sequence $Y(i,j)$ of rank $d-1$ over R_γ.

We show that $Y(i,j)$ does not depend on the choice of i, j. By symmetry of i and j, it suffices to pick $k \in E_n$ so that $S_i \cap S_k$ is a sphere of dimension $d-2$ disjoint to γ and $Z(\gamma)$ yields the cyclic order $(\{i\}, \{k\}, \{\bar{\imath}\}, \{\bar{k}\})$, and to show that $Y(i,j) = Y(i,k)$. We will prove that the cyclic order of signed points on oriented cycles of $\mathcal{A}(i,j)$ coincides with those of $\mathcal{A}(i,k)$. It is clear that this implies $Y(i,j) = Y(i,k)$.

Let $R \subset R_\gamma$ so that $S_R \cap S_i \cap S_j$ is a cycle of $\mathcal{A}(i,j)$. We consider the 2–sphere $S = S_R \cap S_i$. Both $s_j = S \cap S_j$ and $s_k = S \cap S_k$ are embedded 1–spheres in S that are either equal or intersect in two points. Both s_j and s_k have the positive (resp. negative) point of $\gamma \cap S$ on their positive (resp. negative) side. Thus, again using the Jordan–Schönflies theorem, the situation is as in Figure 3. Let C be a connected component of $S \setminus (s_j \cup s_k)$ disjoint from γ.

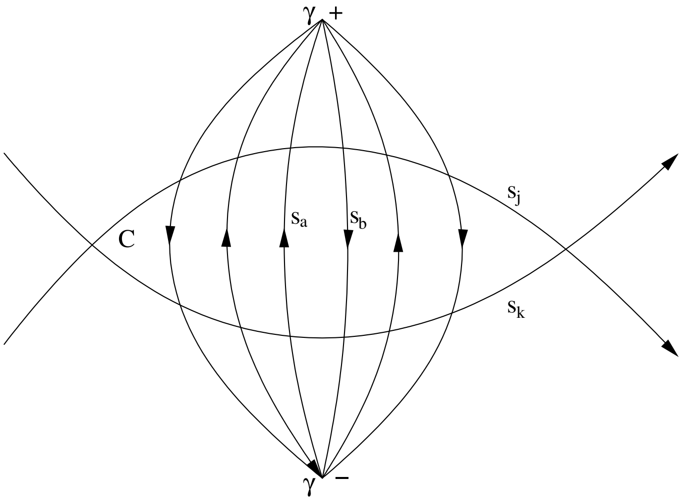

FIGURE 3. Parallel cycles

Let $a, b \in R_\gamma \setminus R$. Since $s_a = S \cap S_a$ is equal to $s_b = S \cap S_b$ or s_a intersects s_b transversaly in $S \cap \gamma$, it follows that $C \cap s_a$ and $C \cap s_b$ are parallel arcs. Hence, the cyclic order of the four signed points of $s_j \cap (s_a \cup s_b)$ coincides with the order on $s_k \cap (s_a \cup s_b)$. This proves our claim $Y(i,j) = Y(i,k)$. Since $Y(i,j)$ does not depend on the choice of i and j, we denote $Y(\gamma) = Y(i,j)$.

Now, we define $X(\mathcal{A})$ as the set of all pairs $\big(Y(\gamma) | Z(\gamma)\big)$, where γ runs over all oriented cycles of $\mathcal{A}$, each cycle occuring in both orientations. It remains to show that $X(\mathcal{A})$ is a hyperline sequence of rank $d+1$ over E_n. First of all, $X(\mathcal{A})$ is not empty. It remains to check the four axioms. Axiom (H1) is trivial. Any sub-arrangement of $d-1$ pseudospheres defines a cycle γ of $\mathcal{A}$, up to orientation. Hence Axiom (H2) follows from $Y(\gamma) = -Y(-\gamma)$

and $Z(\gamma) = -Z(-\gamma)$. Axioms (H3) and (H4) follow from the corresponding axioms in the case $d = 2$, since we can argue by contraction onto a 2–sphere containing the two cycles of $\mathcal{A}$ involved in Axioms (H3) and (H4).

In conclusion, $X(\mathcal{A})$ is a hyperline sequence. By Theorem 1, we can define $\chi(\mathcal{A})$ as the chirotope that is associated to the hyperline sequence $X(\mathcal{A})$ with the same positively oriented bases. Let $R \subset E_n$ so that $\mathcal{A} \setminus R$ (resp. $\mathcal{A}/R$) is an arrangement of oriented pseudospheres of full rank. It is easy to see that $\chi(\mathcal{A} \setminus R) = \chi(\mathcal{A}) \setminus R$ (resp. $\chi(\mathcal{A}/R) = \chi(\mathcal{A})/R$).

6. The topological representation theorem — statement and base case

Theorem 4 (Topological Representation Theorem)**.** *To any hyperline sequence X of rank r over E_n, there is an arrangement $\mathcal{A}(X)$ of n oriented pseudo hyperspheres in $\mathbb{S}^{r-1}$ of full rank with $X = X(\mathcal{A}(X))$. The equivalence class of $\mathcal{A}(X)$ is unique.*

We prove Theorem 4 by induction on the number of elements and the rank of X. In this section, we prove the base cases $r \leq 2$ and $n = r$. The next section is devoted to the inductive step.

Lemma 2. *Theorem 4 holds for $r = 1$.*

Proof. Let X be a hyperline sequence of rank 1 over E_n. We will construct an arrangement $\mathcal{A}(X)$ of oriented pseudospheres in $\mathbb{S}^0$, with $X(\mathcal{A}(X)) = X$.

An oriented pseudosphere S in $\mathbb{S}^0 = \{+1, -1\}$ is the empty set, together with the information whether $S^+ = \{+1\}$ or $S^+ = \{-1\}$. Let $\mathcal{A}(X) = \{S_1, \dots, S_n\}$ be defined as follows. For $i \in E_n$, set $S_i^+ = \{+1\}$ if $i \in X$ and $S_i^+ = \{-1\}$ otherwise. It is obvious that $X(\mathcal{A}(X)) = X$ and that $\mathcal{A}(X)$ is unique with this property. □

Lemma 3. *Theorem 4 holds for $r = 2$.*

Proof. Let X be a hyperline sequence of rank 2 over E_n. We will construct an arrangement $\mathcal{A}(X)$ of oriented pseudospheres in $\mathbb{S}^1$ with $X(\mathcal{A}(X)) = X$, and show that it is unique up to equivalence.

The hyperline sequence X is a map from some cyclic group C_{2k} to non-empty subsets of $\mathbf{E}_n$. We consider C_{2k} as a subgroup of $\mathbb{S}^1$. An oriented pseudosphere corresponds to an embedding of $\mathbb{S}^0 = \{+1, -1\}$ into $\mathbb{S}^1$. For $i \in E_n$, let $a_i \in C_{2k}$ so that $i \in X^a$. Define $\psi_i(+1) = a_i \in \mathbb{S}^1$ and $\psi_i(-1) = -a_i \in \mathbb{S}^1$. It follows easily that $X(\mathcal{A}(X)) = X$ and that $\mathcal{A}(X)$ is essentially unique with this property. □

Lemma 4. *There are exactly two chirotopes of rank $|E|$ over E, namely one with $[1, \dots, r]$ as positively oriented base, and the other with $[1, \dots, r-1, \overline{r}]$ as positively oriented base.*

Proof. Set $r = |E|$. Without loss of generality, let χ be a chirotope of rank r over $E_r = E$. There are exactly two equivalence classes of oriented $(r-1)$–simplices in E_r, namely those equivalent to $[1, \dots, r]$ and those equivalent to $[1, \dots, r-1, \overline{r}]$. Thus, by Axiom (C2), χ is completely determined by $\chi([1, \dots, r])$. Axiom (C1) implies $\chi([1, \dots, r]) = \pm 1$. Hence there are at most two chirotopes of rank $|E|$ over E. The conditions in Axioms (C3) and (C4) are empty for $|E| = r$. Thus, there are two chirotopes of rank $|E|$ over E. □

Lemma 5. *Theorem 4 holds for $n = r$.*

Proof. We prove the lemma for chirotopes rather than hyperline sequences. Let χ be a chirotope of rank r over E_r. By Lemma 4, χ is determined by whether $[1, \dots, r]$ is a positively oriented base of χ or not. In the former case, define $\mathcal{A}(\chi) = \mathcal{A}(r-1, +)$, in the latter case define $\mathcal{A}(\chi) = \mathcal{A}(r-1, -)$. We have $\chi(\mathcal{A}(\chi)) = \chi$ by construction, and the uniqueness of $\mathcal{A}(\chi)$ follows from Corollary 1. □

7. The topological representation theorem — general case

This section is devoted to the inductive step in the proof of Theorem 4. Let $n > r > 2$. Suppose that Theorem 4 holds for all hyperline sequences of rank r with less than n elements and for all hyperline sequences of rank less than r. Thus, for any non-empty $R \subset E_n$ if the contraction X/R (resp. the deletion $X \setminus R$) is defined, then there is an essentially unique arrangement $\mathcal{A}(X/R)$ (resp. $\mathcal{A}(X \setminus R)$) of oriented pseudospheres in $\mathbb{S}^{r-1-|R|}$ (resp. in $\mathbb{S}^{r-1}$) of full rank with $X/R = X(\mathcal{A}(X/R))$ (resp. with $X \setminus R = X(\mathcal{A}(X \setminus R))$).

By Lemma 1, there is an element of X, say, n for simplicity, so that $X \setminus \{n\}$ is a hyperline sequence of rank r. Denote $\{S_1, \dots, S_{n-1}\} = \mathcal{A}(X \setminus \{n\})$. Our aim is to construct an oriented pseudosphere $S_n \subset \mathbb{S}^{r-1}$ as the image of a tame embedding $\psi\colon \mathbb{S}^{r-2} \to \mathbb{S}^{r-1}$, so that $\{S_1, \dots, S_n\}$ is an arrangement of oriented pseudospheres with $X(\{S_1, \dots, S_n\}) = X$.

We outline informally the idea of the construction of ψ. We start with the arrangement $\mathcal{A}(X/\{n\})$ in $\mathbb{S}^{r-2}$. We require that ψ maps this arrangement "consistently" to the arrangement $\mathcal{A}(X \setminus \{n\})$, in the sense that any cell in $\mathcal{C}\big(I, \mathcal{A}(X/\{n\})\big)$ is mapped to a cell in $\mathcal{C}\big(I, \mathcal{A}(X \setminus \{n\})\big)$ in the correct orientation. It turns out that this forces $\{S_1, \dots, S_n\}$ to be an arrangement of oriented pseudospheres. Moreover, we show that if S_n intersects the cycles of $\mathcal{A}(X \setminus \{n\})$ in a way consistent with the rank 2 contractions of X (i.e., the cyclic order on its hyperlines), then $X(\{S_1, \dots, S_n\}) = X$. Our construction of ψ is iterative. We start with defining ψ on 0–dimensional cells of $\mathcal{A}(X/\{n\})$ and show that if it is defined on d–dimensional cells then it can be consistently extended to $(d+1)$–dimensional cells. It turns out that this is possible in an essentially unique way.

Let us formalize this idea. By induction hypothesis, the arrangement $\mathcal{A}(X/\{n\})$ exists and is unique up to equivalence. For any element $i \in E(X/\{n\})$ of $X/\{n\}$, we denote by s_i the oriented pseudosphere of $\mathcal{A}(X/\{n\})$ that corresponds to i. For any $R \subset E(X/\{n\})$, set $s_R = \mathbb{S}^{r-2} \cap \bigcap_{j \in R} s_j$, and similarly $S_R = \mathbb{S}^{r-1} \cap \bigcap_{j \in R} S_j$ for $R \subset E_{n-1}$. Recall that the d–dimensional skeleton $\mathcal{A}^{(d)}$ of an arrangement $\mathcal{A}$ of oriented pseudospheres is the union of its cells of dimension $\leq d$. For any $R \subset E_n$, define $R/n = R \cap E(X/\{n\})$.

Definition 7. Let $t < r$. A t–**admissible** embedding is an embedding

$$\psi^{(t)}\colon \big(\mathcal{A}(X/\{n\})\big)^{(t-1)} \to \mathbb{S}^{r-1}$$

so that for any $R \subset E_{n-1}$ with $\dim s_{R/n} \leq t-1$ holds

1. $\psi^{(t)}(s_{R/n}) = S_R$ or $\psi^{(t)}(s_{R/n})$ is a pseudosphere in S_R,
2. if $\psi^{(t)}(s_{R/n}) \neq S_R$, then any cycle of $\mathcal{A}(X \setminus \{n\})$ in S_R is either contained in $\psi^{(t)}(s_{R/n})$ or meets both connected components of $S_R \setminus \psi(s_{R/n})$, and
3. for any $i \in E(X/\{n\}) \setminus R$ holds $\psi(s_{R/n} \cap s_i^+) \subset S_R \cap S_i^+$ and $\psi(s_{R/n} \cap s_i^-) \subset S_R \cap S_i^-$.

By the following two lemmas, in our request for the pseudosphere S_n it suffices to study $(r-1)$–admissible embeddings.

Lemma 6. *Let $\psi\colon \mathbb{S}^{r-2} \to \mathbb{S}^{r-1}$ be a tame embedding that defines an oriented pseudosphere S_n. If ψ is $(r-1)$–admissible then $\mathcal{A} = \{S_1, \dots, S_n\}$ is an arrangement of oriented pseudospheres.*

Proof. We prove that $\mathcal{A} = \{S_1, \dots, S_n\}$ satisfies Axioms (A1) and (A2).

(A1) Let $R \subset E_n$. It is to show that S_R is empty or homeomorphic to a sphere. If $n \notin R$ then we are done since $\{S_1, \dots, S_{n-1}\}$ is an arrangement. If $n \in R$ then we have $S_R = \psi(s_{R/n})$, and $s_{R/n}$ is empty or homeomorphic to a sphere since $\mathcal{A}(X/\{n\})$ is an arrangement.

(A2) Let $R \subset E_n$ and $i \in E_n$ with $S_R \not\supset S_i$. Since ψ is $(r-1)$–admissible, $S_R \cap S_i$ is a pseudosphere in S_R. It remains to show that $S_R \cap S_i^+$ and $S_R \cap S_i^-$ are both non-empty. If $R \cup \{i\} \subset E_{n-1}$ then we are done, since $\{S_1, \dots, S_{n-1}\}$ is an arrangement of oriented pseudospheres.

If $n \in R$ then $S_R = \psi(s_{R/n})$ and $S_R \cap S_i = \psi(s_{(R\cup\{i\})/n})$. Since $S_R \not\supset S_i$, we have $s_{R/n} \not\supset s_i$. Thus in this case, Axiom (A2) for $\mathcal{A}$ follows from Axiom (A2) for $\mathcal{A}(X/\{n\})$ applied to $s_{R/n}$ and s_i, since $\psi(s_{R/n} \cap s_i^+) \subset S_R \cap S_i^+$ and $\psi(s_{R/n} \cap s_i^-) \subset S_R \cap S_i^-$.

If $n = i$, then $R = R/n$ since otherwise $S_n = S_i \subset S_R$ by definition of S_n. Hence $S_R \cap S_n = \psi(s_R)$ is a pseudosphere in S_R. Since $\mathcal{A} \setminus \{n\}$ is of full rank, S_R contains a cycle of $\mathcal{A} \setminus \{n\}$. By the second property in the definition of $(r-1)$–admissible embeddings, applied to the empty set, this circle meets both connected components of $\mathbb{S}^{r-1} \setminus S_n$, which implies Axiom (A2).

Thus, $\mathcal{A}$ is an arrangement of oriented pseudospheres. ☐

Lemma 7. *Let $\tilde{\psi}\colon \mathbb{S}^{r-2} \to \mathbb{S}^{r-1}$ be a tame embedding that defines an oriented pseudosphere $\tilde{S}_n$. Assume that $\tilde{\mathcal{A}} = \{S_1, \ldots, S_{n-1}, \tilde{S}_n\}$ is an arrangement of oriented pseudospheres. If $X(\tilde{\mathcal{A}}) = X$ then there is an orientation preserving homeomorphism $\phi\colon \mathbb{S}^{r-2} \to \mathbb{S}^{r-2}$ so that $\tilde{\psi} \circ \phi$ is $(r-1)$-admissible.*

Proof. The arrangement $\tilde{\mathcal{A}}/\{n\}$ of oriented pseudospheres is given by the pre-images of $S_1, \ldots, S_{n-1}$ under $\tilde{\psi}$. By induction hypothesis in the proof of Theorem 4, the equivalence class of $\mathcal{A}(X/\{n\})$ is uniquely determined by the property $X(\mathcal{A}(X/\{n\})) = X/\{n\}$. If $X(\tilde{\mathcal{A}}) = X$, then $X(\tilde{\mathcal{A}}/\{n\}) = X/\{n\}$. Hence, the arrangement $\mathcal{A}(X/\{n\})$ is equivalent to $\tilde{\mathcal{A}}/\{n\}$. Let $\phi\colon \mathbb{S}^{r-2} \to \mathbb{S}^{r-2}$ be the orientation preserving homeomorphism realizing this equivalence. It easily follows that $\psi \circ \phi$ is $(r-1)$–admissible. ☐

According to the preceding two lemmas, we shall construct S_n via a tame $(r-1)$–admissible embedding. Moreover, we need to take into account the rank 2 contractions of X, as follows. Let $\psi^{(1)}$ be an 1–admissible embedding. Assume that for any contraction X/R of rank 2 with $n \in E(X/R)$, the oriented 0–dimensional sphere $\psi(s_R)$ extends the arrangement of oriented pseudospheres on the oriented cycle S_R that is induced by $\mathcal{A}(X \setminus \{n\})$ to an arrangement equivalent to $\mathcal{A}(X/R)$. Then we call $\psi^{(1)}$ **compatible with** X. In the next three lemmas, we prove that there is an essentially unique $(r-1)$–admissible embedding whose restriction to the 0–skeleton of $\mathcal{A}(X/\{n\})$ is compatible with X.

Lemma 8. *There is a tame 1–admissible embedding $\psi^{(1)}$ that is compatible with X. It is unique up to composition with a homeomorphism $\big(\mathcal{A}(X \setminus \{n\})\big)^{(1)} \to \big(\mathcal{A}(X \setminus \{n\})\big)^{(1)}$ that fixes $\big(\mathcal{A}(X \setminus \{n\})\big)^{(0)}$.*

Proof. Let $R \subset E_{n-1}$ so that X/R is of rank 2. We first prove the uniqueness of $\psi^{(1)}$.

If $s_{R/n} \approx \mathbb{S}^0$, then the cyclic order of the signed elements of X/R uniquely determines in which cells of $\mathcal{A}(X \setminus \{n\})$ on S_R the two points of s_R must be mapped to, provided $\psi^{(1)}$ is compatible with X. If they are mapped to 0–dimensional cells of $\mathcal{A}(X \setminus \{n\})$ then their image is unique. If they are mapped to 1–dimensional cells, then their image is unique up to a homeomorphism of these cells fixing the boundary.

If $\dim s_{R/n} > 0$, then $R \neq R/n$, and $s_{R/n}$ is a cycle of $\mathcal{A}(X/\{n\})$. Any 0–cell on $s_{R/n}$ is contained in some pseudosphere s_i of $\mathcal{A}(X/\{n\})$ that intersects $s_{R/n}$ transversely. Then $s_{R/n} \cap s_i \approx \mathbb{S}^0$, and if $\psi^{(1)}$ is 1–admissible then $\psi(s_{R/n} \cap s_i) \subset S_R \cap S_i \approx \mathbb{S}^0$. Moreover, if $\psi^{(1)}$ is 1–admissible then the intersection of $s_{R/n} \cap s_i$ with one side of a pseudosphere s_j is mapped to the corresponding side of S_j. This uniquely determines the image of the two points of $s_{R/n} \cap s_i$ under $\psi^{(1)}$.

We prove the existence of $\psi^{(1)}$. According to the preceding two paragraphs, for any 0–dimensional cell p of $\mathcal{A}(X/\{n\})$, a candidate for $\psi^{(1)}(p)$ is given by the cyclic order of the rank 2 contractions of X. We must ensure that the candidate does not depend on the choice of the contraction. Since any two cycles of $\mathcal{A}(X \setminus \{n\})$ are contained in some 2–sphere that is the intersection of pseudospheres in $\mathcal{A}(X \setminus \{n\})$, we can assume by a contraction that X is of rank 3. Then, the pseudospheres $S_1, \ldots, S_{n-1}$ are cycles.

Let $i,j \in E_{n-1}$. If $i \notin E(X/\{n\})$ then $X/\{i\} = \pm X/\{n\}$, hence, the cyclic order of signed points on S_i is a copy of $\mathcal{A}(X/\{n\})$, possibly with opposite orientation. In this case (and similarly if $j \notin E(X/\{n\})$) it is easy to show that the candidates for $\psi^{(1)}$ imposed by i and j coincide. It remains the case $i,j \in E(X/\{n\})$ with $s_i \cap s_j \neq \emptyset$. The positive point p of the oriented pseudosphere s_i shall be mapped into the cell C_i of $\mathcal{A}(X \setminus \{n\})$ on S_i that corresponds to the atom of $X/\{i\}$ containing $\overline{n}$. If $(X/\{n\})/\{i\} = (X/\{n\})/\{j\}$ (resp. $(X/\{n\})/\{i\} = -(X/\{n\})/\{j\}$), then p is the positive (resp. negative) point of s_j, thus shall be mapped into the cell C_j on S_j that corresponds to the atom of $X/\{j\}$ containing $\overline{n}$ (resp. containing n).

If $X/\{i\} = \pm X/\{j\}$ then obviously $C_i = C_j$. Otherwise, $S_i \cap S_j$ is a 0–sphere containing both C_i and C_j. Up to symmetry, we can assume that $(X/\{i\})/\{n\} = (X/\{i\})/\{j\}$, which means that the atom of $X/\{i\}$ containing $\overline{n}$ does also contain $\overline{j}$. Hence, C_i corresponds to the atom of $X/\{j\}$ containing i. This atom also contains $\overline{n}$ (resp. n) if and only if

$$(X/\{j\})/\{n\} = -(X/\{j\})/\{i\} = (X/\{i\})/\{j\} = X/\{i\})/\{n\}$$

(resp. (X/{j})/{n} = - (X/{i})/{n}). Thus $C_i = C_j$ by construction of C_i and C_j. In conclusion, the candidates for $\psi^{(1)}(p)$ imposed by i and j coincide, which is enough to prove the existence of $\psi^{(1)}$ □

Lemma 9. *Let $t < r-1$, and let $\psi^{(t)}$ be a t–admissible embedding. For any t–dimensional cell c of $\mathcal{A}(X/\{n\})$ there is a cell c' of $\mathcal{A}(X \setminus \{n\})$ of dimension t or $t+1$ so that $\psi^{(t)}(\partial c) \subset \partial c'$.*

Proof. Let $R \subset E_{n-1}$ be maximal so that $c \subset s_{R/n}$. In particular, $\dim s_{R/n} = \dim c = t$ and $\dim S_R \leq \dim s_{R/n} + 1 = t+1$. For any cell b of dimension $\dim b = \dim c - 1$ in ∂c and any $j_b \in E(X/\{n\})$ with $b \subset s_{R/n} \cap s_{j_b}$, we have $\psi^{(t)}(b) \subset S_R \cap S_{j_b}$ since $\psi^{(t)}$ is t–admissible. By consequence, $\psi^{(t)}(\partial c) \subset S_R$.

Since c is a cell of $\mathcal{A}(X/\{n\})$, $\partial c \cap s_j^+ = \emptyset$ or $\partial c \cap s_j^- = \emptyset$, for all $j \in E(X/\{n\})$. Since $\psi^{(t)}$ is t–admissible, $\psi^{(t)}(\partial c) \cap S_j^+ = \emptyset$ or $\psi^{(t)}(\partial c) \cap S_j^- = \emptyset$, for all $j \in E_{n-1}$. Thus, $\psi^{(t)}(\partial c)$ is contained in the closure of a connected component c' of

$$S_R \setminus \bigcup_{j \in E_{n-1} \setminus R} S_j$$

which is a cell of $\mathcal{A}(X \setminus \{n\})$ of dimension $\dim S_R \leq t+1$. □

Theorem 5. *There is an $(r-1)$–admissible embedding ψ whose restriction to $\left(\mathcal{A}(X/\{n\})\right)^0$ is compatible with X. It is unique, up to a homeomorphism $\mathbb{S}^{r-1} \to \mathbb{S}^{r-1}$ that fixes $\mathcal{A}(X \setminus \{n\})$ cellwise.*

Proof. Let $\psi^{(t)}$ denote the restriction of ψ to $\left(\mathcal{A}(X/\{n\})\right)^{(t-1)}$, for $t = 1, \dots, r-1$. We start with inductively proving the uniqueness of ψ. An 1–admissible embedding $\psi^{(1)}$ that is compatible with X is essentially unique, by Lemma 8. For $t < r-1$, assume that $\psi^{(t)}$ is unique, up to a homeomorphism $\mathbb{S}^{r-1} \to \mathbb{S}^{r-1}$ that fixes $\mathcal{A}(X \setminus \{n\})$ cellwise. Let c be a t–dimensional cell of $\mathcal{A}(X/\{n\})$, and let $R \subset E_{n-1}$ be maximal so that $\partial c \subset s_{R/n}$. If ψ is $(r-1)$–admissible, then $\psi(c) \subset S_R$. Let $c' \subset S_R$ be the cell of $\mathcal{A}(X \setminus \{n\})$ containing $\psi(c)$. Since $\mathcal{A}(X \setminus \{n\})$ is of full rank, the cell c' is uniquely determined by $\psi(\partial c)$. If $\dim c' = \dim c$ then the $(r-1)$–admissibility of ψ imposes $c' = \psi(c)$. Otherwise, it is a consequence of the generalized Schönflies theorem [4] that $\psi(c) \subset c'$, being a tame ball of codimension 1, is unique up to a homeomorphism $\overline{c}' \to \overline{c}'$ that fixes $\partial c'$ pointwise. This can be extended to a homeomorphism $\mathbb{S}^{r-1} \to \mathbb{S}^{r-1}$ fixing all cells of $\mathcal{A}(X \setminus \{n\})$, by the so-called cone construction [22]. In conclusion, $\psi^{(t+1)}$ is essentially uniquely determined by $\psi^{(t)}$, which completes the proof of the uniqueness of ψ.

Secondly, we expose an iterative construction of ψ. An 1–admissible embedding $\psi^{(1)}$ that is compatible with X exists, by Lemma 8. For $t < r-1$, assume that $\psi^{(t)}$ is a t–admissible embedding, and let c be a t–dimensional cell of $\mathcal{A}(X/\{n\})$. If there is a t–dimensional cell c' of $\mathcal{A}(X \setminus \{n\})$ so that $\partial c' = \psi^{(t)}(\partial c)$ then we define $\psi^{(t+1)}(c) = c'$. Otherwise, by the preceding lemma there is a $(t+1)$–dimensional cell c' of $\mathcal{A}(X \setminus \{n\})$ so that $\psi^{(t)}(\partial c) \subset \partial c'$. Since $\psi^{(t)}(\partial c)$ is a tame union of tame cells of codimension one in $\partial c'$, it follows from [15] that $\psi^{(t)}(\partial c)$ is a pseudosphere in $\partial c'$. Therefore $\psi^{(t)}(\partial c)$ bounds a tame cell $c'' \subset c'$. We define $\psi^{(t+1)}(c) = c''$.

We show that we can do this construction so that $\psi^{(t+1)}$ is an embedding. Let $\Psi^{(t)}$ denote the image of $\psi^{(t)}$, and let c' be a $(t+1)$–dimensional cell of $\mathcal{A}(X \setminus \{n\})$. If $\psi^{(t)}$ is an embedding, then $\Psi^{(t)} \cap \partial c'$ is a disjoint union of t–dimensional spheres. If $t > 1$ then these spheres bound a system of disjoint $(t+1)$–dimensional cells in c'. If $t = 1$ then a priori the spheres might be linked, as depicted in Figure 4. The thick dots indicate the images under ψ of four 0–dimensional cells on a cycle of $\mathcal{A}(X/\{n\})$, whose cyclic order corresponds to the numbering. It is shown in [3] that this case does not occur. In conclusion, our construction

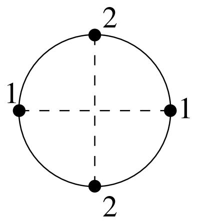

FIGURE 4. Two linked 0–spheres in $\mathbb{S}^1$

of $\psi^{(t+1)}$ produces an embedding.

It remains to show that $\psi^{(t+1)}$ is $(t+1)$–admissible. Let $R \subset E_{n-1}$ with $\dim s_{R/n} \leq t$.

1. Either $\psi^{(t+1)}(s_{R/n})$ is equal to S_R or it is composed by tame cells of codimension one in S_R. In the latter case, $\psi^{(t+1)}(s_{R/n})$ is a pseudosphere (i.e., tame) by [15].
2. Let $\psi^{(t+1)}(s_{R/n}) \neq S_R$, and let $\gamma \subset S_R$ be a cycle of $\mathcal{A}(X \setminus \{n\})$ that is not contained in $\psi^{(t+1)}(s_{R/n})$. Chose a subset $T \subset E_{n-1}$ with $\gamma = S_T$ and $|T| = r - 2$. The contraction X/T is a hyperline sequence of rank 2. Since $\psi^{(1)}$ is compatible with X, $\psi^{(t+1)}(s_{R/n}) \cap \gamma$ comprizes exactly two points x, y, corresponding to the elements $n, \overline{n}$ in X/T.

   There is some $x \in \mathbf{E}(X/T) \setminus \{n\}$, so that X/T induces the cyclic order $(n, x, \overline{n}, \overline{x})$. Let $j = x^*$. Since $\psi^{(t)}$ is t–admissible, it follows from Lemma 6 that $\mathcal{A}(X \setminus \{n\})$ and $\psi^{(t+1)}(s_{R/n})$ induce on $S_{R\cup\{j\}}$ an arrangement of oriented pseudospheres. If γ does not meet both connected components of $S_R \setminus \psi^{(t+1)}(s_{R/n})$, then the two points of $\gamma \cap S_j$ are contained in a single component of $S_{R\cup\{j\}} \setminus \psi^{(t+1)}(s_{R/n})$, which is impossible for arrangements of oriented pseudospheres. Hence, $\psi^{(t+1)}$ satisfies the second property in the definition of $(t+1)$–admissible embeddings.
3. We observe that an open cell is contained in the positive side of an oriented pseudosphere if and only if some point in its boundary is contained in the positive side of the oriented pseudosphere. Thus, if c is a t–dimensional cell of $\mathcal{A}(X/\{n\})$ and $c \subset s_i^+$, then $\partial c \cap s_i^+ \neq \emptyset$. Since $\psi^{(t)}$ maps s_i^+ into S_i^+, it follows
$$\psi^{(t)}(\partial c) \cap S_i^+ = \psi^{(t+1)}(\partial c) \cap S_i^+ \neq \emptyset,$$
   and therefore $\psi^{(t+1)}(c) \subset S_i^+$. Similarly, if $c \subset S_i^-$ then $\psi^{(t+1)}(c) \subset S_i^-$.

Therefore $\psi^{(t+1)}$ is $(t+1)$–admissible, which finishes the proof of Theorem 5. □

In the remainder of this section, we finish the proof of Theorem 4. Let $\psi\colon\ \mathbb{S}^{r-2} \to \mathbb{S}^{r-1}$ be a $(r-1)$–admissible embedding whose restriction to the 1–skeleton of $\mathcal{A}(X/\{n\})$ is compatible with X. By Theorem 5, ψ exists. Let $S_n = \psi(\mathbb{S}^{r-2})$ be the oriented pseudosphere defined by ψ, and set $\mathcal{A} = \{S_1, \dots, S_n\}$. By Lemma 6, $\mathcal{A}$ is an arrangement of oriented pseudospheres.

We show that $X(\mathcal{A}) = X$, hence, that the hyperlines $(Y|Z) \in X$ coincide with those of $X(\mathcal{A})$. If $n \notin E(Y)$ then $(Y|Z) \in X(\mathcal{A})$, since Z is a rank 2 contraction of X and $\psi^{(1)}$ is 1-admissible and compatible with X. If $n \in E(Y)$, then let $R = E(Y) \setminus \{n\}$. Since ψ is $(r-1)$–admissible, we have $S_R \subset S_n$, thus, S_R corresponds to the cycle $s_{R/n} \subset \mathbb{S}^{r-2}$ of $\mathcal{A}(X/\{n\})$. Since $X\big(\mathcal{A}(X/\{n\})\big) = X/\{n\}$, the cyclic order of points on this cycle coincides with Z. If $i, j \in E(Z)$ so that Z induces the cyclic order $(i, j, \bar{\imath}, \bar{\jmath})$, then $X/\{i, j\} = Y$. We have $X(\mathcal{A}/\{i, j\}) = X/\{i, j\}$, since the Topological Representation Theorem 4 holds in rank $r-2$ by induction hypothesis. Therefore, $(Y|Z) = (X(\mathcal{A}/\{i, j\})|Z) \in X(\mathcal{A})$.

It remains to prove the uniqueness of $\mathcal{A}$ stated in the Topological Representation Theorem 4. Let $\{\tilde{S}_1, \dots, \tilde{S}_n\}$ be an arrangement of oriented pseudospheres with $X(\{\tilde{S}_1, \dots, \tilde{S}_n\}) = X$. By induction hypothesis, we may use the uniqueness of $\mathcal{A}(X \setminus \{n\})$ and can assume that $\{\tilde{S}_1, \dots, \tilde{S}_{n-1}\} = \{S_1, \dots, S_{n-1}\}$. Let $\tilde{S}_n$ be the image of a tame embedding $\tilde{\psi}\colon \mathbb{S}^{r-2} \to \mathbb{S}^{r-1}$. Then, $\tilde{\psi}$ is $(r-1)$–admissible by Lemma 7, and it is obvious that its restriction to the 1-skeleton of $\mathcal{A}(X/\{n\})$ is compatible with X. Thus by Theorem 5, $\tilde{\psi}$ coincides with ψ up to a homeomorphism $\mathbb{S}^{r-1} \to \mathbb{S}^{r-1}$ that fixes $\{S_1, \dots, S_{n-1}\}$ cellwise. Hence, $\mathcal{A}$ is equivalent to $\{\tilde{S}_1, \dots, \tilde{S}_n\}$. This finishes the proof of the Topological Representation Theorem 4.

8. Wild arrangements

This section is an appendix to Section 5. We show here that one can get a hyperline sequence from an arrangement of oriented pseudospheres, even if one allows pseudospheres to be not tame.

Let $S_1, \dots, S_n \subset \mathbb{S}^d$ be embedded $(d-1)$–dimensional spheres with a choice of an orientation. We call the ordered multiset $\mathcal{A} = \{S_1, \dots, S_n\}$ an **arrangement of oriented topological spheres** over E_n if it satisfies Axioms (A1) and (A2), where the word "pseudosphere" is replaced by "embedded sphere of codimension one".

An embedded sphere $S \subset \mathbb{S}^d$ of codimension one is **wild**, if there is no homeomorphism $\mathbb{S}^d \to \mathbb{S}^d$ mapping S to $\mathbb{S}^{d-1} \subset \mathbb{S}^d$. Since there are infinitely many wild spheres in $\mathbb{S}^d$ for all $d \geq 3$ (see [21]), we are no longer allowed to use the generalized Schönflies theorem [4]. However, $\mathbb{S}^d \setminus S$ has exactly two connected components, even if S is wild.

Similarly to Section 4, we say that two arrangements $\{S_1, \dots, S_n\}$ and $\{\tilde{S}_1, \dots, \tilde{S}_n\}$ of oriented topological spheres in $\mathbb{S}^d$ are equivalent if there is an orientation preserving homeomorphism $\phi\colon \mathbb{S}^d \to \mathbb{S}^d$ with $\phi(S_i) = \tilde{S}_i$ in the correct orientation, for $i = 1, \dots, n$. For $R \subset E_n$, the definition of the contraction $\mathcal{A}/R$ and the deletion $\mathcal{A} \setminus R$ is identic to the corresponding definition for arrangements of oriented *pseudo*spheres.

We wish to define a hyperline sequence $X(\mathcal{A})$ associated to an arrangement $\mathcal{A}$ of oriented topological spheres. By Section 5, we know how to proceed if all spheres in $\mathcal{A}$ are tame. Both in the construction of $X(\mathcal{A})$ in Section 5 and in the proof that $X(\mathcal{A})$ is indeed a hyperline sequence, we were using induction on the contractions of $\mathcal{A}$, based on contractions of rank 1 and 2. The only topological argument in the induction step was the use of the Jordan–Schönflies theorem, that holds also without the assumption of tameness though, in the step from rank 2 to rank 3. It remains to remove the tameness condition in the base cases. Rank 1 is trivial. For rank 2, observe that any two different points $x, y \in \mathbb{S}^1$ can be separated by small intervalls around x and y. Thus, any embedded sphere $\mathbb{S}^0$ in $\mathbb{S}^1$ is tame. Therefore, even if $\mathcal{A}$ is not equivalent to an arrangement of pseudospheres, any rank 2 contraction $\mathcal{A}/R$ actually is an arrangement of pseudospheres, and we can read off a hyperline sequence $X(\mathcal{A}/R)$. As in Section 5, these contractions yield a hyperline sequence $X(\mathcal{A})$.

In conclusion, although arrangements of oriented topological spheres are very complicated from a topological point of view, their combinatorics is simple enough to read off ordinary hyperline sequences. Nevertheless, there are "more" arrangements of oriented topological spheres than hyperline sequences, in the sense that there are non-equivalent arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ of oriented topological spheres (for instance, a tame arrangement and a wild arrangement) with $X(\mathcal{A}_1) = X(\mathcal{A}_2)$. Hence, there is no analogue of the Topological Representation Theorem 4 in the setting of arrangements of oriented topological spheres.

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