On subdirectly irreducible OMAs

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Abstract

In this paper some properties of epi-representations and Schmidt-congruence relations of orthomodular partial algebras are investigated and an infinite list of $\underline{\underline{\mathrm{OMA}}}$-epi subdirectly irreducible orthomodular partial algebras will be constructed.

Introducion

For every (existence) variety $\mathfrak{K}$ of partial algebras each element $\underline{A} \in \mathfrak{K}$ is a subdirect product of subdirectly irreducible algebras of $\mathfrak{K}$. If $\mathfrak{K}$ is only a quasivariety this theorem may not hold because $\mathfrak{K}$ is not closed with respect to homomorphic images, so for a subdirect representation $f : \underline{A} \to \prod_{i \in I} \underline{A}_i$ the partial algebras $\underline{A}_i$ need not to be in the quasivariety. Therefore we need a different concept of subdirect representation. We use the subdirect epi-representation of [B98]. In [B98] it is shown that for a quasivariety $\mathfrak{K}$ each element $\underline{A} \in \mathfrak{K}$ is a subdirect epi-product of $\mathfrak{K}$-epi subdirectly irreducible algebras of $\mathfrak{K}$. In this paper we consider the ECE-variety $\underline{\underline{\mathrm{OMA}}}$ of all orthomodular partial algebras. The sub-ECE-varieties of $\underline{\underline{\mathrm{OMA}}}$ are exactly the subquasivarieties because every closed homomorphism between OMAs is injective. It will be shown that there exist infinitly many finite $\underline{\underline{\mathrm{OMA}}}$-epi subdirectly irreducible orthomodular partial algebras.

1 Subdirect epi-representation and Schmidt-congruences

The partial and total algebras of a signature Σ are denoted by $\mathsf{PAlg}(\Sigma)$ and $\mathsf{TAlg}(\Sigma)$. A mapping $h : \underline{A} \to \underline{B}$ between partial algebras is a homomorphism if for each n-ary operation symbol f and for each tuple $\underline{a} = (a_1, a_2, \ldots, a_n)$ the existence of $f^{\underline{A}}(\underline{a})$ implies that $f^{\underline{B}}(h(a_1), \ldots, h(a_n))$ exists and $h(f^{\underline{A}}(\underline{a})) = f^{\underline{B}}(h(a_1), \ldots, h(a_n))$ holds. A homomorphism h is closed if for each n-ary operation symbol f and for each tuple $\underline{a} = (a_1, a_2, \ldots, a_n)$ the existence of $f^{\underline{B}}(h(a_1), \ldots, h(a_n))$ implies the existence of $f^{\underline{A}}(\underline{a})$. A homomorphism h is an isomorphism if h is closed and bijective. The free completion[1] of a partial algebra $\underline{A}$ is denoted by $\underline{\hat{A}} := \underline{F}(\underline{A}, \mathsf{TAlg}(\Sigma))$. A formula in the signature Σ is a pair (X, α) where X is a set of variables and α is constructed in the usual way from existence equations of the form $s \stackrel{e}{=} t$ (where s and t are terms over X) with the junctors $\vee, \wedge, \rightarrow, \neg$. The atomic formula $(X, s \stackrel{e}{=} t)$ is valid for a valuation $v : X \to A$ into a partial algebra $\underline{A}$ if the interpretations of s and t exist and are equal. A formula (X, α) is valid in $\underline{A}$ if it is valid for every valuation $v : X \to A$. The equation $t \stackrel{e}{=} t$ is denoted by $\mathbb{D}(t)$. For a set G of formulas let $Mod(G) := \{\underline{A} \in \mathsf{PAlg}(\Sigma) \mid \text{each formula } (X, \alpha) \in G \text{ is valid in } \underline{A}\}$ be the class of all models of G. Let $\mathfrak{K} \subseteq \mathsf{PAlg}(\Sigma)$. The class of all products in $\mathfrak{K}$ is denoted by $\mathcal{P}(\mathfrak{K})$. The class of all reduced products in $\mathfrak{K}$ is denoted by $\mathcal{P}_r(\mathfrak{K})$. The class of all subalgebras of elements of $\mathfrak{K}$ is denoted by $\mathcal{S}(\mathfrak{K})$. The class of all homomophic images of elements of $\mathfrak{K}$ is denoted by $\mathcal{H}(\mathfrak{K})$. The class of all closed homomophic images of elements of $\mathfrak{K}$ is denoted by $\mathcal{H}_c(\mathfrak{K})$. The class of all isomorphic images of elements of $\mathfrak{K}$ is denoted by $\mathcal{I}(\mathfrak{K})$. With [B86] we have that $\mathcal{HSP}(\mathfrak{K})$ is the E-variety generated by $\mathfrak{K}$, $\mathcal{H}_c\mathcal{SP}_r(\mathfrak{K})$ is the ECE-variety generated by $\mathfrak{K}$ and $\mathcal{I}SP_r(\mathfrak{K})$ is the quasivariety[2] generated by $\mathfrak{K}$.

Definition 1 *For a partial algebra $\underline{A} \in \mathsf{PAlg}(\Sigma)$ define the following order on $\underline{\hat{A}} = \underline{F}(\underline{A}, \mathsf{TAlg}(\Sigma))$: $x \leq y$ iff every relative subalgebra of $\underline{\hat{A}}$ which contains y and is generated by a subset of A also contains x. This order is called* natural order.

Note that the natural order is well founded: Every nonempty set contains a minimal element. The minimal elements of $\underline{\hat{A}}$ are the elements of A. The natural order differs from the algebraic quasiorder defined in [B86] because in the algebraic quasiorder of $\underline{\hat{A}}$ some elements of A may be comparable. For the elements of $\underline{\hat{A}}$ which are not in A the natural order coincides with the algebraic quasiorder.

Definition 2 *Let Σ be a signature and $\mathfrak{K} \subseteq \mathsf{PAlg}(\Sigma)$.*
A family $(f_i : \underline{A} \to \underline{A}_i)_{i \in I}$ of epimorphisms with $\underline{A} \in \mathsf{PAlg}(\Sigma)$ and $\underline{A}_i \in \mathfrak{K}$ for $i \in I$ is called a subdirect epi-representation[3] of $\underline{A}$ in $\mathfrak{K}$ if the induced homomorphism $f = < f_i >_{i \in I}: \underline{A} \to \prod_{i \in I} \underline{A}_i$ is injective and closed. In this case the partial algebra $\underline{A}$ is called a subdirect epi-product of $(\underline{A}_i)_{i \in I}$. The class of all subdirect epi-products in $\mathfrak{K}$ is denoted by $\mathcal{P}_s(\mathfrak{K})$.
A partial algebra $\underline{A} \in \mathsf{PAlg}(\Sigma)$ is called $\mathfrak{K}$-epi subdirectly irreducible if for every subdirect epi-representation $(f_i : \underline{A} \to \underline{A}_i)_{i \in I}$ in $\mathfrak{K}$ there exists an element $i \in I$ such that f_i is an isomorphism.
For $\mathfrak{L} \subseteq \mathsf{PAlg}(\Sigma)$ let $Ir_{\mathfrak{K}}(\mathfrak{L}) = \{\underline{A} \in \mathfrak{L} \mid \underline{A}$ is $\mathfrak{K}$-epi subdirectly irreducible$\}$.
the elements of $Ir_{\mathfrak{K}}(\mathfrak{L})$ are called $\mathfrak{K}$-irreducible $\mathfrak{L}$-algebras. For $\underline{A} \in \mathsf{PAlg}(\Sigma)$ let $Con_{\mathfrak{K}}(\underline{A}) = \{\theta \in Con(\underline{A}) \mid \underline{A}/\theta \in \mathfrak{K}\}$ be the set of all $\mathfrak{K}$-congruence relations on $\underline{A}$

[1] see [B86]
[2] or QE-variety (see [B86] and [B98])
[3] see [B98]

and let

$$S\text{-}Con_{\mathfrak{K}}(\underline{A}) = \{\theta \subseteq \hat{A}^2 \mid dom(\theta) \text{ is an initial segment of } \underline{\hat{A}} \\ \text{with } A \subseteq dom(\theta) \text{ and } \theta \text{ is a closed congruence relation on} \\ \text{the relative subalgebra } \underline{dom(\theta)} \text{ and } \underline{A}/\theta \in \mathfrak{K}\}$$

be the set of all $\mathfrak{K}$-Schmidt-congruence relations[4] *on $\underline{A}$.*
For $\mathfrak{K} = \mathsf{PAlg}(\Sigma)$ let $S\text{-}Con(\underline{A}) = S\text{-}Con_{\mathfrak{K}}(\underline{A})$.

Lemma 3 *Let Σ be a signature, $\mathfrak{K}_1' \subseteq \mathfrak{K}_1 \subseteq \mathsf{PAlg}(\Sigma)$ and let $\mathfrak{K}_2 \subseteq \mathfrak{K}_2' \subseteq \mathsf{PAlg}(\Sigma)$. Then $Ir_{\mathfrak{K}_1}(\mathfrak{K}_2) \subseteq Ir_{\mathfrak{K}_1'}(\mathfrak{K}_2')$ holds.*

Proof. Let $\underline{A} \in Ir_{\mathfrak{K}_1}(\mathfrak{K}_2)$. Then $\underline{A} \in \mathfrak{K}_2 \subseteq \mathfrak{K}_2'$ holds. Let $(f_i : \underline{A} \to \underline{A_i})_{i\in I}$ be a subdirect epi-representation in $\mathfrak{K}_1'$. Then $(f_i : \underline{A} \to \underline{A_i})_{i\in I}$ is also a subdirect epi-representation in $\mathfrak{K}_1$ because of $\mathfrak{K}_1' \subseteq \mathfrak{K}_1$. Therefore f_i is an isomorphism for some $i \in I$ because of $\underline{A} \in Ir_{\mathfrak{K}_1}(\mathfrak{K}_2)$. So we get $\underline{A} \in Ir_{\mathfrak{K}_1'}(\mathfrak{K}_2')$ and $Ir_{\mathfrak{K}_1}(\mathfrak{K}_2) \subseteq Ir_{\mathfrak{K}_1'}(\mathfrak{K}_2')$ holds. ∎

Lemma 4 *Let $\mathfrak{K} \subseteq \mathsf{PAlg}(\Sigma)$ and $\underline{A} \in \mathsf{PAlg}(\Sigma)$ for a signature Σ. Then $id_{\underline{A}} \in S\text{-}Con_{\mathfrak{K}}(\underline{A})$ holds iff $\underline{A} \in \mathfrak{K}$.*

Proof. $\underline{A}/id_A \cong \underline{A}$ ∎

With [B98] a partial algebra $\underline{A} \in \mathfrak{K} \subseteq \mathsf{PAlg}(\Sigma)$ is $\mathfrak{K}$-irreducible iff id_A is completely meet-irreducible in $S\text{-}Con_{\mathfrak{K}}(\underline{A})$. Note that for every quasivariety $\mathfrak{K} \subseteq \mathsf{PAlg}(\Sigma)$ of a signature Σ and for all $\underline{A} \in \mathfrak{K}$ the set $S\text{-}Con_{\mathfrak{K}}(\underline{A})$ is an algebraic lattice and $\underline{A}$ has a subdirect epi-representation into $\mathfrak{K}$-irreducible $\mathfrak{K}$-algebras.[5]

Remark:
Let a and b be constants and $\mathfrak{K} = Mod(\{(\emptyset, \mathbb{D}(a) \to \mathbb{D}(b)), (\emptyset, \mathbb{D}(b) \to \mathbb{D}(a))\})$. Let $\underline{A}$ be the empty partial algebra with two discrete operations $a^{\underline{A}}$ and $b^{\underline{A}}$. Then $\underline{A} \in \mathfrak{K}$ holds, but $\underline{A}$ does not have a subdirect epi-representation into $\mathsf{PAlg}(\Sigma)$-irreducible $\mathfrak{K}$-algebras, because $id_{\underline{A}}$ is completely meet-irreducible in $S\text{-}Con_{\mathfrak{K}}(\underline{A})$ but not in $S\text{-}Con(\underline{A})$.

The following theorem shows that if the set of all $\mathfrak{K}$-Schmidt-congruence relations for a class $\mathfrak{K}$ is an algebraic lattice, then $\mathfrak{K}$ is generated by its $\mathfrak{K}$-irreducible $\mathfrak{K}$-algebras.

Theorem 5 [6] *Let $\mathfrak{K} \subseteq \mathsf{PAlg}(\Sigma)$ for a signature Σ such that $S\text{-}Con_{\mathfrak{K}}(\underline{A})$ is an algebraic lattice for all $\underline{A} \in \mathfrak{K}$. Then $\mathfrak{K} \subseteq \mathcal{ISP}(Ir_{\mathfrak{K}}(\mathfrak{K}))$ holds.*

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[4] see [B98]
[5] see [B98]
[6] see [B98]

Proof. Let $A \in \mathfrak{K}$. In the algebraic lattice $S\text{-}Con_{\mathfrak{K}}(\underline{A})$ the relation $id_{\underline{A}}$ is the intersection of completely meetirreducible elements: $id_{\underline{A}} = \bigcap_{i \in I} \theta_i$. With [B98] there exists an injective closed homomorphism $h : \underline{A} \to \prod_{i \in I} \underline{dom}(\theta_i)/\theta_i$. Each $\underline{dom}(\theta_i)/\theta_i$ is $\mathfrak{K}$-irreducible because θ_i is completely meetirreducible. Therefore $A \in \mathcal{ISP}(Ir_{\mathfrak{K}}(\mathfrak{K}))$ holds. ∎

Therefore every quasivariety $\mathfrak{K}$ is generated by its $\mathfrak{K}$-irreducible $\mathfrak{K}$-algebras:

Corollary 6 *$\mathcal{HSP}(Ir_{\mathfrak{K}}(\mathfrak{K})) = \mathfrak{K}$ holds for every E-variety $\mathfrak{K}$.*
$\mathcal{H}_c\mathcal{SP}_r(Ir_{\mathfrak{K}}(\mathfrak{K})) = \mathfrak{K}$ holds for every ECE-variety $\mathfrak{K}$.
$\mathcal{ISP}_r(Ir_{\mathfrak{K}}(\mathfrak{K})) = \mathfrak{K}$ holds for every quasivariety $\mathfrak{K}$. ∎

Corollary 7 *Every quasivariety $\mathfrak{K}$ is uniquely determined by its $\mathfrak{K}$-irreducible $\mathfrak{K}$-algebras.* ∎

In the following theorem it will be proved that the restriction of a $\mathfrak{K}$-Schmidt-congruence θ of a partial algebra $\underline{A}$ to the free completion of a subalgebra is again a Schmidt-congruence, and that the induced factor algebra is isomorphic to a subalgebra of $\underline{dom}(\theta)/\theta$.

Theorem 8 *Let Σ be a signature, $\mathfrak{K} \subseteq \mathsf{PAlg}(\Sigma)$ and $\underline{A} \in \mathfrak{K}$. Let $\theta \in S\text{-}Con_{\mathfrak{K}}(\underline{A})$ and $\underline{B} \leq \underline{A}$ and $\rho = \theta \cap \hat{B}^2$. Then $\rho \in S\text{-}Con(\underline{B})$ holds and $\underline{dom}(\rho)/\rho$ is isomorphic to a subalgebra of $\underline{dom}(\theta)/\theta$. If $\mathfrak{K}$ is a quasivariety, then $\rho \in S\text{-}Con_{\mathfrak{K}}(\underline{B})$ holds.*

Proof. Obviously ρ is a closed congruence relation on $\underline{dom}(\rho)$ because θ is a closed congruence relation on $\underline{dom}(\theta)$. The set $dom(\rho)$ is an initial segment of $\underline{\hat{B}}$. Therefore we have $\rho \in S\text{-}Con(\underline{B})$. Let $h : \underline{dom}(\rho)/\rho \to \underline{dom}(\theta)/\theta$ with $h(x/\rho) = x/\theta$. Then h is a welldefined homorphism because of $\rho \subseteq \theta$. The function h is injective because $h(x/\rho) = h(y/\rho)$ implies $(x, y) \in \theta \cap \hat{B}^2 = \rho$. Now we show that h is closed. Let f be an n-ary operation symbol and $\underline{x} = (x_1, x_2, \dots, x_n) \in dom(\rho)^n$ such that $f^{\underline{dom}(\theta)/\theta}(\underline{x}/\theta)$ exists. Then $f^{\underline{dom}(\theta)}(\underline{x})$ exists because θ is closed. We get $(f^{\underline{dom}(\theta)}(\underline{x}), f^{\underline{dom}(\theta)}(\underline{x})) \in \theta \cap \hat{B}^2 = \rho$, so $f^{\underline{dom}(\rho)/\rho}(\underline{x}/\rho)$ exists and h is closed. Therefore $\underline{dom}(\rho)/\rho$ is isomorphic to a subalgebra of $\underline{dom}(\theta)/\theta$. If $\mathfrak{K}$ is a quasivariety then this subalgebra is an element of $\mathfrak{K}$, so $\rho \in S\text{-}Con_{\mathfrak{K}}(\underline{B})$ holds. ∎

For every quasivariety $\mathfrak{K}$ the $\mathfrak{K}$-Schmidt-congruence relation generated by a $\mathfrak{K}$-congruence relation θ is the kernel of the closed homomorphic extension $nat_\theta^\sim$ of the natural factor homomorphism nat_θ:

Theorem 9 *Let $\mathfrak{K} \subseteq \mathsf{PAlg}(\Sigma)$ be a quasivariety and let $\underline{A} \in \mathsf{PAlg}(\Sigma)$. Let $\theta \in Con_{\mathfrak{K}}(\underline{A})$ and let $<\theta>_{S\text{-}Con_{\mathfrak{K}}(\underline{A})}$ be the smallest $\mathfrak{K}$-Schmidt-congruence of $\underline{A}$ which contains θ. Then*
$<\theta>_{S\text{-}Con_{\mathfrak{K}}(\underline{A})} = <\theta>_{S\text{-}Con(\underline{A})} = ker(nat_{\theta}^{\sim})$,
$<\theta>_{S\text{-}Con_{\mathfrak{K}}(\underline{A})} \cap A^2 = \theta$
hold.

Proof. Let $\rho = ker(nat_{\theta}^{\sim})$. We have $\rho \in S\text{-}Con_{\mathfrak{K}}(\underline{A})$ because of $dom(\rho)/\rho \cong \underline{A}/\theta \in \mathfrak{K}$. Because of $\theta \subseteq \rho$ we get $<\theta>_{S\text{-}Con_{\mathfrak{K}}(\underline{A})} \subseteq \rho$.
The inclusion $<\theta>_{S\text{-}Con(\underline{A})} \subseteq <\theta>_{S\text{-}Con_{\mathfrak{K}}(\underline{A})}$ is trivial and $<\theta>_{S\text{-}Con(\underline{A})} = \rho$ was proved in [Sch70]. Therefore $<\theta>_{S\text{-}Con_{\mathfrak{K}}(\underline{A})} = <\theta>_{S\text{-}Con(\underline{A})} = ker(nat_{\theta}^{\sim})$ holds. We have $<\theta>_{S\text{-}Con_{\mathfrak{K}}(\underline{A})} \cap A^2 = ker(nat_{\theta}^{\sim}) \cap A^2 = \theta$. ∎

2 The quasivariety of orthomodular partial algebras

In this section we use the definitions of chapter 1 to investigate the ECE-variety of all orthomodular partial algebras.

Definition 10 *Let $\underline{\underline{\mathrm{OMA}}}$ be the class of all orthomodular partial algebras.*[7]
Let $\underline{2}$ be the OMA with two elements. An OMA $\underline{A}$ is called Boolean if the corresponding ordered set[8] *$(A, \leq)$ is a Boolean lattice. A pair $(a, b) \in A^2$ in an OMA $\underline{A}$ is called boolean if there exist elements $x, y, z \in A$ with $x \oplus y = a$, $y \oplus z = b$ such that $x \oplus z$ exists.*

Note that an OMA $\underline{A}$ is Boolean iff every pair $(a, b) \in A^2$ is a Boolean pair.[9]

Theorem 11 *Let $\mathfrak{K} \subseteq \underline{\underline{\mathrm{OMA}}}$. Then $\mathcal{H}_c\mathcal{S}\mathcal{P}_r(\mathfrak{K}) = \mathcal{I}\mathcal{S}\mathcal{P}_r(\mathfrak{K})$ holds.*

Proof. Let $A \in \mathcal{S}\mathcal{P}_r(\mathfrak{K})$ and $h : \underline{A} \to \underline{B}$ be a closed surjective homomorphism. Then $\underline{A}, \underline{B} \in \underline{\underline{\mathrm{OMA}}}$ holds because $\underline{\underline{\mathrm{OMA}}}$ is an ECE-variety. $ker(h)$ is a closed congruence relation on $\underline{A}$ and with [BM94] we get $ker(h) = id_A$, therefore $B \in \mathcal{I}\mathcal{S}\mathcal{P}_r(\mathfrak{K})$. ∎

Corollary 12 *The sub-ECE-varieties of $\underline{\underline{\mathrm{OMA}}}$ are exactly the subquasivarieties.* ∎

[7] see [BM94]
[8] see [BM94]
[9] see [BM98] and [BM94]

If E is a set of QE-equations then the corresponding set E' of ECE-equations can easily be computed: Each equation $s \stackrel{e}{=} t$ occuring in a formula of E can be replaced by $\mathbb{D}(s \oplus t') \wedge \mathbb{D}(t \oplus s')$. Together with the axioms of the class $\underline{\underline{\mathrm{OMA}}}$ these two sets E and E' are equivalent: $Mod(E \cup \{A0, A1, \ldots, A9\}) = Mod(E' \cup \{A0, A1, \ldots, A9\})$.

Lemma 13 *Let $\underline{A}$ be a Boolean OMA, then $\underline{A} \in ISP(\underline{2})$ holds.*

Proof. In the signature of Boolean algebras the induced Boolean lattice $(A, \wedge, \vee, ', 0, 1)$ is isomorphic to a subalgebra of a power of the lattice $\underline{C}_2$ with two elements, so there exists an injective homomorhpism $f : (A, \wedge, \vee, ', 0, 1) \to \underline{C}_2^I$ for a set I. With [BM98] f is also a homomorphism in the signature of OMAs. Now we show that f is closed. Let $b, c \in A$ such that $f(b) \oplus f(c)$ exists. Then we get $f(b) \leq f(c)' = f(c')$. Every injective lattice homomorphism reflects the order, so $b \leq c'$ and $b \oplus c$ exists. So f is closed and $\underline{A} = (A, \oplus, ', 0)$ is isomorphic to a subalgebra of $\underline{2}^I$. ∎

In the following theorem it will be shown that every reduced product of Boolean OMAs is Boolean.

Theorem 14 *Let $\mathfrak{K}$ be a class of Boolean OMAs. Every $\underline{A} \in P_r(\mathfrak{K})$ is a Boolean OMA.*

Proof. For $\underline{A} \in P_r(\mathfrak{K})$ there exists a family $(\underline{A}_i)_{i \in I}$ of OMAs in $\mathfrak{K}$ and a filter $\mathcal{F}$ on I with $\underline{A} = \prod\limits_{i \in I} \underline{A}_i / \mathcal{F}$. The partial algebra $\underline{A}$ is an OMA because $\underline{\underline{\mathrm{OMA}}}$ is an ECE-variety. With [BM98] and [BM94] we only need to show that every pair $(a, b) \in A^2$ is a Boolean pair. Let $a = (a_i)_{i \in I}/\mathcal{F} \in A$ and $b = (b_i)_{i \in I}/\mathcal{F} \in A$. Let $c = (c_i)_{i \in I}/\mathcal{F}$ with $c_i = a_i \wedge b_i'$ for $i \in I$. Let $d = (d_i)_{i \in I}/\mathcal{F}$ with $d_i = a_i \wedge b_i$ for $i \in I$. Let $e = (e_i)_{i \in I}/\mathcal{F}$ with $e_i = a_i' \wedge b_i$ for $i \in I$. Then $a = c \oplus d$, $b = d \oplus e$ and $c \oplus e$ exists because each $\underline{A}_i$ is Boolean, so (a, b) is a Boolean pair and $\underline{A}$ is Boolean. ∎

With this theorem we get a characterisation of the ECE-variety generated by the OMA with two elements: An OMA $\underline{A}$ is generated by $\underline{2}$ iff it is isomorphic to a subalgebra of a power of $\underline{2}$:

Corollary 15 $P_r(\underline{2}) \subseteq \mathcal{ISP}(\underline{2}) = \mathcal{ISP}_r(\underline{2}) = \mathcal{H}_c\mathcal{SP}_r(\underline{2}) = \mathcal{P}_s(\underline{2}) =$
$\{\underline{A} \in \underline{\underline{\mathrm{OMA}}} \mid \underline{A}$ *is isomorphic to a subalgebra of* $\underline{2}^I$ *for a set* $I\}$.

Proof. For $\underline{A} \in \mathcal{ISP}(\underline{2})$ there exists a set I and an injective closed homomorphism $f : \underline{A} \to \underline{2}^I$, so $\pi_i \circ f : \underline{A} \to \underline{2}$ is epi for all $i \in I$ and we get $\mathcal{ISP}(\underline{2}) = \mathcal{P}_s(\underline{2})$. The other properties follow from Theorem 14, Lemma 13 and Theorem 11. ∎

The class of all Boolean OMAs is not closed with respect to arbitrary homomorphic images, but if the image is again an OMA then it is Boolean:

Theorem 16 *Let $\underline{B}$ be an OMA and let $\underline{A}$ be a Boolean OMA. Let $h : \underline{A} \to \underline{B}$ be a surjective homomorphism. Then $\underline{B}$ is Boolean.*

Proof. Let $a, b \in \underline{B}$. There exist $c, d \in \underline{A}$ with $h(c) = a$ and $h(d) = b$. (c, d) is a Boolean pair in $\underline{A}$, so there exist $x, y, z \in \underline{A}$ with $x \oplus^{\underline{A}} y = c$, $y \oplus^{\underline{A}} z = d$ and $x \oplus^{\underline{A}} z$ exists. Therefore $h(x) \oplus^{\underline{B}} h(y) = a$, $h(y) \oplus^{\underline{B}} h(z) = b$ and $h(x) \oplus^{\underline{B}} h(z)$ exists, so (a, b) is a Boolean pair in $\underline{B}$ and $\underline{B}$ is Boolean. ■

Theorem 17 $\underline{F}_{\underline{\underline{\mathrm{OMA}}}}(X) \in \mathcal{ISP}(\underline{2})$ *holds for every set X.*

Proof. We can assume that $|X| > 2$ holds, because for $|X| \leq 2$ we have $\underline{F}_{\underline{\underline{\mathrm{OMA}}}}(X) \in \mathcal{ISP}(\underline{F}_{\underline{\underline{\mathrm{OMA}}}}(\{1, 2, 3\}))$. Let $Y = \{1, 2\} \times X$ and $f : X \to \underline{2}^Y$ with

$$
\begin{aligned}
f(x)(1, z) &= 1 = f(x)(2, x) \\
f(x)(1, x) &= 0 = f(x)(2, z)
\end{aligned}
$$

for $x, z \in X$ with $x \neq z$. Let $h : \underline{F}_{\underline{\underline{\mathrm{OMA}}}}(X) \to \underline{2}^Y$ be the unique homomorphic extension of f. Now we show that h is injective. With [BM94] we have $F_{\underline{\underline{\mathrm{OMA}}}}(X) = X \cup \{x' \mid x \in X\} \cup \{0, 1\}$. Of course $h(x) = 0$ iff $x = 0$ holds, and $h(x) = 1$ iff $x = 1$ holds. Let $a, b \in X$ with $h(a) = h(b)$. Then $0 = f(a)(1, a) = h(a)(1, a) = h(b)(1, a) = f(b)(1, a)$, so $a = b$ holds. Let $a, b \in X$ with $h(a) = h(b')$. Because of $|X| > 2$ there exist elements $c, d \in X$ with $a \neq c \neq d \neq a$. Then $1 = f(a)(1, c) = h(a)(1, c) = h(b')(1, c) = (h(b))'(1, c)$, so $0 = h(b)(1, c) = f(b)(1, c)$ and $b = c$ holds. Analoguosly we have $b = d$ which is a contradiction to $c \neq d$. The cases $h(a') = h(b)$ and $h(a') = h(b')$ work analogously, so h is injective.
Now we show that h is closed. Let $x, y \in h(F_{\underline{\underline{\mathrm{OMA}}}}(X))$ such that $x \oplus y$ exists. Let $a, b \in F_{\underline{\underline{\mathrm{OMA}}}}(X)$ with $h(a) = x$ and $h(b) = y$. We only have to consider the following four cases:
Case 1: $a, b \in X$
Then there exists an element $c \in X$ with $a \neq c \neq b$, so $h(a)(1, c) = 1 = h(b)(1, c)$ and $h(a) \oplus h(b)$ does not exist which is a contradiction.
Case 2: $a', b' \in X$:
Analogously to case 1 $h(a) \oplus h(b)$ can not exist because of $h(a')(2, c) = 0 = h(b')(2, c)$ and therefore $h(a)(2, c) = 1 = h(b)(2, c)$.
Case 3: $a, b' \in X$:

We have $h(a)(2,a) = 1$, so we get $h(b)(2,a) = 0$ because of the existence of $h(a) \oplus h(b)$ and therefore $h(b')(2,a) = 1$, so $a = b'$ and $a \oplus b$ exists.
Case 4: $a', b \in X$:
Analogously case 3.
Therefore h is closed and we get $\underline{F}_{\underline{\underline{\mathrm{OMA}}}}(X) \cong h(\underline{F}_{\underline{\underline{\mathrm{OMA}}}}(X)) \in \mathcal{ISP}(\underline{2})$. $\blacksquare$

As a corollary we get that all orthomodular partial algebras are contained in the E-variety generated by the OMA with two elements:

Corollary 18 $\underline{\underline{\mathrm{OMA}}}$ *is a proper subclass of* $\mathcal{HSP}(\underline{2})$.

Proof. For all $\underline{A} \in \underline{\underline{\mathrm{OMA}}}$ there exists a surjective homomorphism $h : F_{\underline{\underline{\mathrm{OMA}}}}(A) \to \underline{A}$ extending id_A, so we get $\underline{\underline{\mathrm{OMA}}} \subseteq \mathcal{HSP}(\underline{2})$ because of Theorem 17. The sup-semilattice with two elements is a homomorphic image of $\underline{2}$, but it is not an OMA, so $\underline{\underline{\mathrm{OMA}}}$ is a proper subclass of $\mathcal{HSP}(\underline{2})$. $\blacksquare$

In the following theorem it will be proved that the factor algebra $\underline{dom}(\theta)/\theta$ for an $\underline{\underline{\mathrm{OMA}}}$-Schmidt-congruence θ of $\underline{2}^n$ for a natural number n is isomorphic to a Boolean powerset algebra.

Theorem 19 *Let* $n \in \mathbb{N}$ *and* $T = \{1, 2, ..., n\}$ *and* $\theta \in S\text{-}Con_{\underline{\underline{\mathrm{OMA}}}}(\underline{2}^n)$.
Let $S = \{i \in T \mid (e^i, 0) \notin \theta\}$, *where* $e^i \in \underline{2}^n$ *with* $e^i_i = 1$ *and* $e^i_j = 0$ *for* $i \neq j$. *Then the mapping* $f : 2^S \to dom(\theta)/\theta$ *with* $f(x) = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in S \text{ with } x_i = 1\}$ *is an isomorphism:* $f : \underline{2}^S \to \underline{dom}(\theta)/\theta$.

Proof. Let $B := \{\bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in I\} \mid I \subseteq T\}$. First we show that $\underline{B}$ is a subalgebra of $\underline{dom}(\theta)/\theta$. We have $0 = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in \emptyset\} \in B$ and for $x = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in I\} \in B$ we get $x' = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in T \setminus I\} \in B$. Let $x, y \in B$ sucht that $x \oplus^{\underline{dom}(\theta)/\theta} y$ exists. Then there exist $I, J \subseteq T$ with $x = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in I\}$ and $y = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in J\}$. With the axioms[10] (A4) and (A5) we get the existence of $e^i/\theta \oplus^{\underline{dom}(\theta)/\theta} e^i/\theta$ for all $i \in I \cap J$, so $(e^i, 0) \in \theta$ for $i \in I \cap J$. Therefore $x \oplus^{\underline{dom}(\theta)/\theta} y = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in (I \setminus J) \cup (J \setminus I)\} \in B$ holds, and B is a subalgebra of $\underline{dom}(\theta)/\theta$. The set $dom(\theta)$ is generated by 2^n and $\underline{2}^n$ is generated by $\{e^i \mid i \in T\}$, so $dom(\theta)/\theta$ is generated by $\{e^i/\theta \mid i \in T\}$ and we get $dom(\theta)/\theta = B$.
For $x \in \underline{2}^S$ we have $f(x') = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in S \text{ with } x_i = 0\} = \bigoplus^{\underline{dom}(\theta)/\theta}(\{e^i/\theta \mid i \in S \text{ with } x_i = 0\} \cup (T \setminus S)) = f(x)'$, so f is compatible with $'$. The function f is compatible with 0 and $\oplus$, so it is a homomorphism.
Surjectiviy of f:

[10] see [BM94]

Let $y = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in I\} \in dom(\theta)/\theta$. Then $y = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in I \cap S\}$ holds, and we get $f(x) = y$ for $x \in \underline{2}^S$ with $x_i = 1$ for $i \in I \cap S$ and $x_i = 0$ for $i \in S \setminus I$.

Injectivity of f:

Let $x, y \in \underline{2}^S$ with $f(x) = f(y)$. Assume that there exists an $i \in S$ with $x_i \neq y_i$. We can assume $x_i = 1$ and $y_i = 0$. Then $e^i/\theta \oplus^{\underline{dom}(\theta)/\theta} f(y) = e^i/\theta \oplus^{\underline{dom}(\theta)/\theta} f(x)$ exists, so $e^i/\theta \oplus^{\underline{dom}(\theta)/\theta} e^i/\theta$ exists and $(e^i, 0) \in \theta$ which is a contradiction to $i \in S$.

Closedness of f:

Let $x, y \in \underline{2}^S$ such that $f(x) \oplus f(y)$ exists. Let $I = x^{-1}(1)$ and $J = y^{-1}(1)$ be the carriers of x and y. If there exists an element $i \in I \cap J$ then we would get the existence of $e^i/\theta \oplus e^i/\theta$ because of the existence of $f(x) \oplus f(y)$ which is a contradiction to $i \in S$. Therefore $I \cap J = \emptyset$ holds, $x \oplus y$ exists and f is closed and an isomorphism. ∎

For every natural number n the lattices $Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n)$ and $S\text{-}Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n)$ are isomorphic to the Boolean powerset algebra $\mathcal{P}(\{1, 2, \dots, n\})$:

Theorem 20 *Let $n \in \mathbb{N}$ and $T = \{1, 2, ..., n\}$.*
Then $Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n) \cong S\text{-}Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n) \cong \mathcal{P}(T)$ holds where $\mathcal{P}(T)$ is the powerset of T.

Proof. Let $f : \mathcal{P}(T) \to Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n)$ with $f(A) := \{(x, y) \mid x, y \in 2^n \text{ and } x_i = y_i \text{ for all } i \in T \setminus A\}$.

Let $g : Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n) \to S\text{-}Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n)$ whith $g(\theta) =< \theta >_{S\text{-}Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n)}$.

Let $h : S\text{-}Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n) \to \mathcal{P}(T)$ whith $h(\theta) = \{i \in T \mid (e^i, 0) \in \theta\}$.

Obviously f, g, h are monotone and f and g are injective (see Theorem 9). Now we show that h is injective. Let $\theta, \rho \in S\text{-}Con_{\underline{\underline{\text{OMA}}}}(\underline{2}^n)$ with $h(\theta) = h(\rho)$. First we show that $\rho \cap (2^n)^2 = \theta \cap (2^n)^2$ holds. Assume that there exist $x, a \in \underline{2}^n$ with $(x, a) \in \rho$ but $(x, a) \notin \theta$. Let $I = x^{-1}(1) = \{i \in T \mid x_i = 1\}$ and $J = a^{-1}(1)$. For $i \in J \setminus I$ we get the existence of $x/\rho \oplus^{\underline{dom}(\rho)/\rho} e^i/\rho = a/\rho \oplus^{\underline{dom}(\rho)/\rho} e^i/\rho$, therefore $e^i/\rho \oplus^{\underline{dom}(\rho)/\rho} e^i/\rho$ exists and $(e^i, 0) \in \rho$ holds. Because of $h(\theta) = h(\rho)$ we get $(e^i, 0) \in \theta$ for all $i \in J \setminus I$ and analogously $(e^i, 0) \in \theta$ for all $i \in I \setminus J$, so $x/\theta = \bigoplus^{\underline{dom}(\theta)/\theta}\{e^i/\theta \mid i \in I \cap J\} = a/\theta$ and $(x, a) \in \theta$ which is a contradiction.

Therefore we get $\rho \cap (2^n)^2 \subseteq \theta \cap (2^n)^2$ and analogously $\theta \cap (2^n)^2 \subseteq \rho \cap (2^n)^2$. Now assume that there exist $x \in dom(\rho)$ and $a \in \underline{2}^n$ with $(x, a) \in \rho$ and $(x, a) \notin \theta$. Here we choose x minimal (with respect to the natural order) such that there exists an $a \in \underline{2}^n$ with $(x, a) \in \rho$ and $(x, a) \notin \theta$. Then we get $x \notin \underline{2}^n$. We have to consider the following two cases:

Case 1: There exists an element $y \in dom(\rho)$ with $x = y'^{\underline{dom}(\rho)}$

We have $(x', a') \in \rho$ because ρ is a closed congruence relation. Because of $\underline{dom}(\rho)/\rho \in \underline{\underline{\text{OMA}}}$ we have $(y, x') = (y, y'') \in \rho$, so $(y, a') \in \rho$ and by the minimality of x we get $(y, a') \in \theta$ and $(x, a) = (y', a'') \in \theta$ which is a contradiction.

Case 2: There exist $y, z \in dom(\rho)$ with $x = y \oplus^{\underline{dom}(\rho)} z$.

Let $S = T \setminus h(\rho) = T \setminus h(\theta)$. Because of Theorem 19 there exist sets $I, J \subseteq S$ with

$y/\rho = \bigoplus^{\underline{dom}(\rho)/\rho}\{e^i/\rho \mid i \in I\}$ and $z/\rho = \bigoplus^{\underline{dom}(\rho)/\rho}\{e^i/\rho \mid i \in J\}$. For $i \in I \cap J$ we get the existence of $e^i/\rho \oplus^{\underline{dom}(\rho)/\rho} e^i/\rho$, so $(e^i, 0) \in \rho$ which is a contradiction to $I \subseteq S$, so $I \cap J = \emptyset$ holds. Let $b := \bigoplus^{\underline{dom}(\rho)}\{e^i \mid i \in I\} \in 2^n$ and $c := \bigoplus^{\underline{dom}(\rho)}\{e^i \mid i \in J\} \in 2^n$. We have $(y, b) \in \rho$ and $(z, c) \in \rho$ and because of the minimality of x we get $(y, b) \in \theta$ and $(z, c) \in \theta$. The sum $b \oplus^{\underline{dom}(\theta)} c$ exists, so $x = y \oplus^{\underline{dom}(\theta)} z$ exists because θ is a closed congruence relation. Let $d = b \oplus^{\underline{dom}(\rho)} c = b \oplus^{\underline{dom}(\theta)} c \in \underline{2}^n$. We get $(x, d) \in \theta$. We have $(y \oplus^{\underline{dom}(\rho)} z, a) \in \rho$, so $(b \oplus^{\underline{dom}(\rho)} c, a) \in \rho \cap (2^n)^2 = \theta \cap (2^n)^2$ holds, and we get $(d, a) \in \theta$, which is a contradiction to $(x, a) \notin \theta$.
Therefore $(x, a) \in \rho$ implies $(x, a) \in \theta$ for all $x \in dom(\rho)$ and $a \in 2^n$. Now let $(x, y) \in \rho$ be an arbitrary pair, then with Theorem 19 there exists an element $a \in 2^n$ with $(x, a) \in \rho$ and $(y, a) \in \rho$, so $(x, a) \in \theta$ and $(y, a) \in \theta$, and we get $(x, y) \in \theta$. Therefore $\rho \subseteq \theta$ holds and analogously we get $\theta \subseteq \rho$. Therefore h is injective, and f, g, h are bijective because $\mathcal{P}(T)$ is finite. So they are lattice isomorphism because of the finiteness of the lattices. ∎

Now we construct some Greechie diagrams of $\underline{\overline{\overline{\text{OMA}}}}$-epi subdirectly irreducible orthomodular partial algebras. For $n \geq 3$ let $D_n = (P, R)$ be the hypergraph[11] with
$P = \bigcup\{\{a_i, b_i, c_i\} \mid 1 \leq i \leq n\} \cup \{x\}$ and
$R = \{r_1, r_2, r_3\} \cup \{s_i \mid 1 \leq i \leq n\}$ with
$r_1 = \{a_1, a_2, \ldots, a_n\}$,
$r_2 = \{b_1, b_2, \ldots, b_n\}$,
$r_3 = \{c_1, c_2, \ldots, c_n, x\}$,
$s_i = \{a_i, b_i, c_i\}$ for $i \leq n$.
Let $D_n^* = (P^*, R^*)$ be the hypergraph with
$P^* = \bigcup\{\{a_i, b_i, c_i\} \mid 1 \leq i \leq n\} \cup \{x, y\}$ and
$R^* = \{r_1^*, r_2^*, r_3^*\} \cup \{s_i^* \mid 1 \leq i \leq n\}$ with
$r_1^* = \{a_1, a_2, \ldots, a_n\}$,
$r_2^* = \{b_1, b_2, \ldots, b_n, x\}$,
$r_3^* = \{c_1, c_2, \ldots, c_n, y\}$,
$s_i^* = \{a_i, b_i, c_i\}$ for $i \leq n$.
Then D_n and D_n^* are complete OMA-diagrams:[12]

[11] see [Ho01]
[12] see [Ho01]

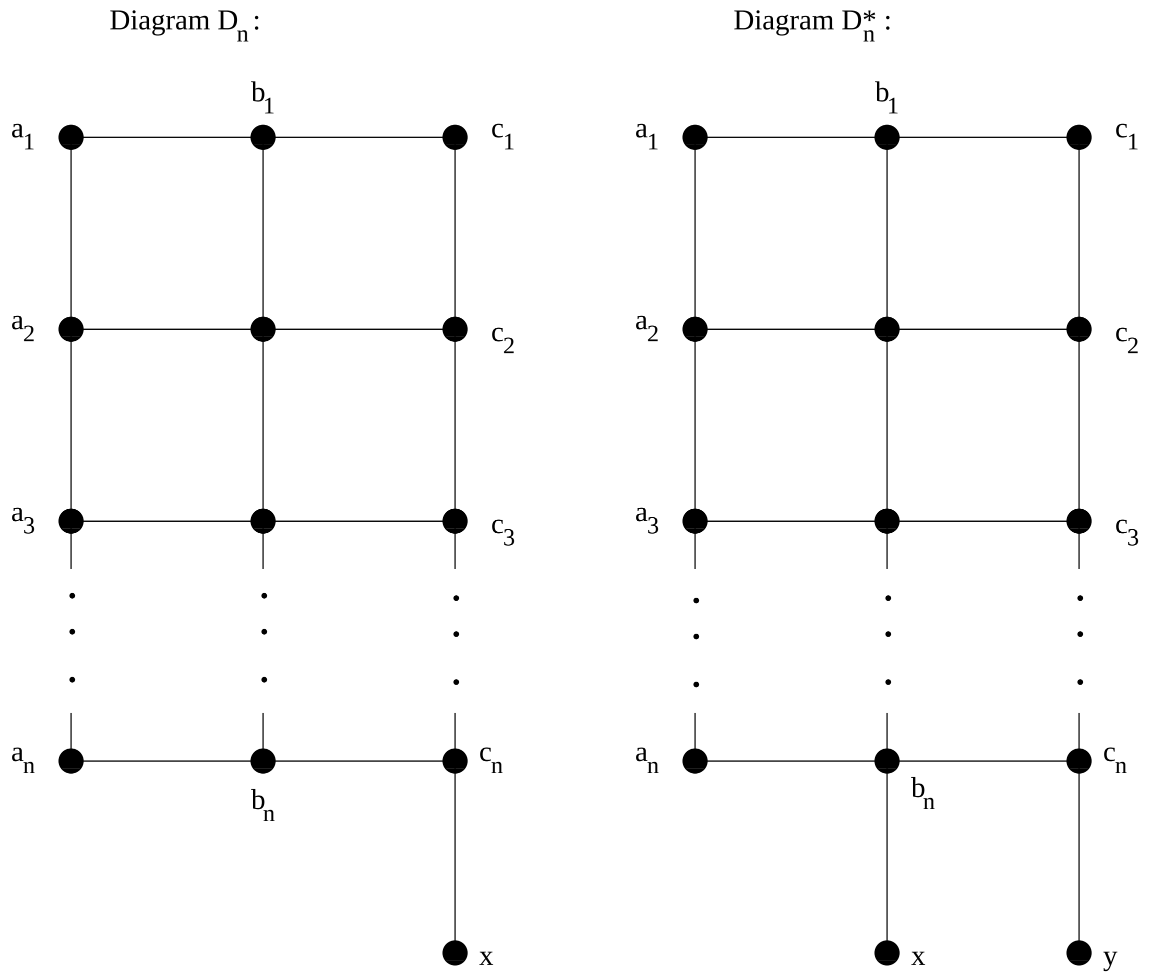

Figure 1: Diagrams D_n and D_n^*

Theorem 21 *$\underline{2} \in Ir_{\underline{\underline{\mathrm{OMA}}}}(\underline{\underline{\mathrm{OMA}}})$ and $[\![D_n]\!] \in Ir_{\underline{\underline{\mathrm{OMA}}}}(\underline{\underline{\mathrm{OMA}}})$ and $[\![D_n^*]\!] \in Ir_{\underline{\underline{\mathrm{OMA}}}}(\underline{\underline{\mathrm{OMA}}})$ hold for all $n \geq 3$.*[13]

Proof. $\underline{2} \in Ir_{\underline{\underline{\mathrm{OMA}}}}(\underline{\underline{\mathrm{OMA}}})$ follows from Theorem 20 (or see [BM98]). Let $\underline{A} = [\![D_n]\!]$ and $\theta \in S\text{-}Con_{\underline{\underline{\mathrm{OMA}}}}(\underline{A})$ with $id_A \neq \theta$. Let $\underline{B} = \underline{dom}(\theta)/\theta$. There is no nontrivial closed congruence relation on $\underline{A}$ (see [BM98]), so A is a proper subset of $dom(\theta)$. The set $dom(\theta)$ is generated by A so there exist $d_1, e_1 \in \underline{A}$ such that $d_1 \oplus^{\underline{A}} e_1$ does not exist, but $d_1 \oplus^{\underline{B}} e_1$ exists. Each element of $\underline{A}$ is a sum of atoms because $\underline{A}$ is finite. Because of the axioms[14] (A4) and (A8) there exist atoms $d \leq d_1$ and $e \leq e_1$ of $\underline{A}$ such that $d \oplus^{\underline{A}} e$ does not exist. Because of the axioms (A4) and (A5) $d \oplus^{\underline{B}} e$ exists. Now we show $([\![x]\!], 0) \in \theta$. We only consider the following six cases because all other cases

[13] for the definition of $[\![D]\!]$ for a diagram see [Ho01]

[14] see [BM94]

can be transformed into these cases by a diagram isomorphism.
Case 1: $d = [\![a_1]\!]$ and $e = [\![x]\!]$
Because of axiom (A8) and the existence of $[\![a_1]\!]/\theta \oplus^{\underline{B}} [\![c_1]\!]/\theta$ and $[\![c_1]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta$ we get the existence of $p := ([\![a_1]\!]/\theta \oplus^{\underline{B}} [\![c_1]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta)'$. We have

$$\begin{aligned}1 &= [\![a_1]\!]/\theta \oplus^{\underline{B}} [\![c_1]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta \oplus^{\underline{B}} p\\ &= ([\![b_1]\!]/\theta)' \oplus^{\underline{B}} [\![x]\!]/\theta \oplus^{\underline{B}} p\\ &= \bigoplus^{\underline{B}}\{[\![b_i]\!]/\theta \mid 2 \le i \le n\} \oplus^{\underline{B}} [\![x]\!]/\theta \oplus^{\underline{B}} p\end{aligned}$$

Therefore $[\![b_2]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta$ exists and with axiom (A8) we get the existence of $q := ([\![b_2]\!]/\theta \oplus^{\underline{B}} [\![c_2]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta)'$. We have

$$\begin{aligned}1 &= [\![b_2]\!]/\theta \oplus^{\underline{B}} [\![c_2]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta \oplus^{\underline{B}} q\\ &= ([\![a_2]\!]/\theta)' \oplus^{\underline{B}} [\![x]\!]/\theta \oplus^{\underline{B}} q\\ &= \bigoplus^{\underline{B}}\{[\![a_i]\!]/\theta \mid 2 \ne i \le n\} \oplus^{\underline{B}} [\![x]\!]/\theta \oplus^{\underline{B}} q\end{aligned}$$

Therefore $[\![a_i]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta$ exists for all $i \le n$ with $i \ne 2$. Analogously $[\![a_i]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta$ exists for all $i \le n$ with $i \ne 3$, so with axiom (A8) we get the existence of $\bigoplus^{\underline{B}}\{[\![a_i]\!]/\theta \mid 1 \le i \le n\} \oplus^{\underline{B}} [\![x]\!]/\theta = 1 \oplus^{\underline{B}} [\![x]\!]/\theta$ and $[\![x]\!]/\theta = 0$.
Case 2: $d = [\![a_1]\!]$ and $e = [\![b_2]\!]$
With axiom (A8) we get the existence of $p := ([\![a_1]\!]/\theta \oplus^{\underline{B}} [\![a_2]\!]/\theta \oplus^{\underline{B}} [\![b_2]\!]/\theta)'$. We have

$$\begin{aligned}1 &= [\![a_1]\!]/\theta \oplus^{\underline{B}} [\![a_2]\!]/\theta \oplus^{\underline{B}} [\![b_2]\!]/\theta \oplus^{\underline{B}} p\\ &= [\![a_1]\!]/\theta \oplus^{\underline{B}} ([\![c_2]\!]/\theta)' \oplus^{\underline{B}} p\\ &= [\![a_1]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta \oplus^{\underline{B}} \bigoplus^{\underline{B}}\{[\![c_i]\!]/\theta \mid 2 \ne i \le n\} \oplus^{\underline{B}} p\end{aligned}$$

Therefore $[\![a_1]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta$ exists, and like in case 1 we get $[\![x]\!]/\theta = 0$.
Case 3: $d = [\![a_1]\!]$ and $e = [\![c_2]\!]$
With axiom (A8) we get the existence of $p := ([\![a_1]\!]/\theta \oplus^{\underline{B}} [\![c_1]\!]/\theta \oplus^{\underline{B}} [\![c_2]\!]/\theta)'$. We have

$$\begin{aligned}1 &= [\![a_1]\!]/\theta \oplus^{\underline{B}} [\![c_1]\!]/\theta \oplus^{\underline{B}} [\![c_2]\!]/\theta \oplus^{\underline{B}} p\\ &= ([\![b_1]\!]/\theta)' \oplus^{\underline{B}} [\![c_2]\!]/\theta \oplus^{\underline{B}} p\\ &= [\![b_2]\!]/\theta \oplus^{\underline{B}} [\![c_2]\!]/\theta \oplus^{\underline{B}} \bigoplus^{\underline{B}}\{[\![b_i]\!]/\theta \mid 3 \le i \le n\} \oplus^{\underline{B}} p\\ &= ([\![a_2]\!]/\theta)' \oplus^{\underline{B}} \bigoplus^{\underline{B}}\{[\![b_i]\!]/\theta \mid 3 \le i \le n\} \oplus^{\underline{B}} p\\ &= [\![a_1]\!]/\theta \oplus^{\underline{B}} [\![a_3]\!]/\theta \oplus^{\underline{B}} [\![b_3]\!]/\theta \oplus^{\underline{B}} \bigoplus^{\underline{B}}\{[\![a_i]\!]/\theta \mid 4 \le i \le n\} \oplus^{\underline{B}}\\ &\quad \bigoplus^{\underline{B}}\{[\![b_i]\!]/\theta \mid 4 \le i \le n\} \oplus^{\underline{B}} p\end{aligned}$$

Therefore $[\![a_1]\!]/\theta \oplus^{\underline{B}} [\![b_3]\!]/\theta$ exists, and like in case 2 we get $[\![x]\!]/\theta = 0$.
Case 4: $d = [\![a_1]\!] = e$
Then $[\![a_1]\!]/\theta = 0$ holds. Therefore $[\![a_1]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta$ exists, and like in case 1 we get $[\![x]\!]/\theta = 0$.
Case 5: $d = [\![x]\!] = e$
Then $[\![x]\!]/\theta = 0$ holds.
Case 6: $d = [\![c_2]\!] = e$
Then $[\![c_2]\!]/\theta = 0$ holds. Therefore $[\![a_1]\!]/\theta \oplus^{\underline{B}} [\![c_2]\!]/\theta$ exists, and like in case 3 we get $[\![x]\!]/\theta = 0$.
In all cases we get $([\![x]\!], 0) \in \theta$, so id_A is completely meetirreducible and $\underline{A}$ is $\underline{\underline{\text{OMA}}}$-epi subdirectly irreducible.
Now let $\underline{A} = [\![D_n^*]\!]$ and $\theta \in S\text{-}Con_{\underline{\underline{\text{OMA}}}}(\underline{A})$ with $id_A \neq \theta$. Let $\underline{B} = \underline{dom}(\theta)/\theta$. Analogously to the proof above we get atoms $d, e \in \underline{A}$ such that $d \oplus^{\underline{A}} e$ does not exist but $d \oplus^{\underline{B}} e$ exists. Now we show $([\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!], [\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!]) \in \theta$.
Case 1: $d = [\![a_1]\!]$ and $e = [\![x]\!]$
With axiom (A8) we get the existence of

$$[\![x]\!]/\theta \oplus^{\underline{B}} [\![a_1]\!]/\theta \oplus^{\underline{B}} [\![b_1]\!]/\theta = [\![x]\!]/\theta \oplus^{\underline{B}} ([\![c_1]\!]/\theta)'$$
$$= [\![x]\!]/\theta \oplus^{\underline{B}} [\![y]\!]/\theta \oplus^{\underline{B}} \bigoplus\nolimits^{\underline{B}} \{[\![c_i]\!]/\theta \mid 2 \leq i \leq n\}$$

Therefore $[\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!]$ exists.
Case 2: $d = [\![c_2]\!]$ and $e = [\![x]\!]$
With axiom (A8) we get the existence of

$$[\![x]\!]/\theta \oplus^{\underline{B}} [\![b_2]\!]/\theta \oplus^{\underline{B}} [\![c_2]\!]/\theta = [\![x]\!]/\theta \oplus^{\underline{B}} ([\![a_2]\!]/\theta)'$$
$$= [\![x]\!]/\theta \oplus^{\underline{B}} \bigoplus\nolimits^{\underline{B}} \{[\![a_i]\!]/\theta \mid 2 \neq i \leq n\}$$

Therefore $[\![a_1]\!] \oplus^{\underline{B}} [\![x]\!]$ exists and like in case 1 we get the existence of $[\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!]$.
Case 3: $d = [\![a_1]\!]$ and $e = [\![c_2]\!]$
With axiom (A8) we get the existence of

$$[\![c_2]\!]/\theta \oplus^{\underline{B}} [\![a_1]\!]/\theta \oplus^{\underline{B}} [\![c_1]\!]/\theta = [\![c_2]\!]/\theta \oplus^{\underline{B}} ([\![b_1]\!]/\theta)'$$
$$= [\![c_2]\!]/\theta \oplus^{\underline{B}} [\![x]\!]/\theta \oplus^{\underline{B}} \bigoplus\nolimits^{\underline{B}} \{[\![b_i]\!]/\theta \mid 2 \leq i \leq n\}$$

Therefore $[\![c_2]\!] \oplus^{\underline{B}} [\![x]\!]$ exists and like in case 2 we get the existence of $[\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!]$.
Case 4: $d = [\![b_3]\!]$ and $e = [\![c_2]\!]$
With axiom (A8) we get the existence of

$$[\![c_2]\!]/\theta \oplus^{\underline{B}} [\![b_3]\!]/\theta \oplus^{\underline{B}} [\![c_3]\!]/\theta = [\![c_2]\!]/\theta \oplus^{\underline{B}} ([\![a_3]\!]/\theta)'$$
$$= [\![c_2]\!]/\theta \oplus^{\underline{B}} \bigoplus\nolimits^{\underline{B}} \{[\![a_i]\!]/\theta \mid 3 \neq i \leq n\}$$

Therefore $[\![a_1]\!] \oplus^{\underline{B}} [\![c_2]\!]$ exists and like in case 3 we get the existence of $[\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!]$.
Case 5: $d = [\![x]\!]$ and $e = [\![y]\!]$
Then $[\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!]$ exists.
Case 6: $d = e$
Then $d/\theta = 0 = e/\theta$ and we get the same situation like in the other cases.
In all cases we get $([\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!], [\![x]\!] \oplus^{\underline{dom}(\theta)} [\![y]\!]) \in \theta$, so id_A is completely meetirreducible and $\underline{A}$ is $\underline{\underline{\text{OMA}}}$-irreducible. ∎

So there exist an infinite number of finite $\underline{\underline{\text{OMA}}}$-epi subdirectly irreducible OMAs.
The following theorem shows that the quasivariety generated by the OMA with two elements contains (up to isomorphy) only one $\underline{\underline{\text{OMA}}}$-irreducible OMA.

Theorem 22 *Let $\underline{A} \in \mathcal{H}_c\mathcal{S}\mathcal{P}_r(\underline{2})$ with $\underline{A} \not\cong \underline{2}$, then $\underline{A}$ is not $\underline{\underline{\text{OMA}}}$-irreducible.*

Proof. Because of Corollary 15 there exists a set I and a closed injective homomorphism $h : \underline{A} \to \underline{2}^I$. For $i \in I$ the mapping $\pi_i \circ h : \underline{A} \to \underline{2}$ is surjective, so it is an epimorphism. $\pi_i \circ h$ is not an isomorphism because of $\underline{A} \not\cong \underline{2}$, so $\underline{A}$ is not $\underline{\underline{\text{OMA}}}$-irreducible. ∎

Corollary 23 *$\mathcal{H}_c\mathcal{S}\mathcal{P}_r(\underline{2})$ is a proper subclass of $\mathcal{H}_c\mathcal{S}\mathcal{P}_r([\![D_n]\!])$ for $n \geq 3$.*
$\mathcal{H}_c\mathcal{S}\mathcal{P}_r(\underline{2})$ is a proper subclass of $\mathcal{H}_c\mathcal{S}\mathcal{P}_r([\![D_n^*]\!])$ for $n \geq 3$.

Proof. $\underline{2}$ is a subalgebra of $[\![D_n]\!]$ and $[\![D_n^*]\!]$, so we get $\mathcal{H}_c\mathcal{S}\mathcal{P}_r(\underline{2}) \subseteq \mathcal{H}_c\mathcal{S}\mathcal{P}_r([\![D_n]\!])$ and $\mathcal{H}_c\mathcal{S}\mathcal{P}_r(\underline{2}) \subseteq \mathcal{H}_c\mathcal{S}\mathcal{P}_r([\![D_n^*]\!])$. With Theorem 22 we get $[\![D_n]\!] \notin \mathcal{H}_c\mathcal{S}\mathcal{P}_r(\underline{2})$ and $[\![D_n^*]\!] \notin \mathcal{H}_c\mathcal{S}\mathcal{P}_r(\underline{2})$. ∎

So there exists an ECE-equation which holds in $\underline{2}$ but not in $[\![D_n]\!]$ and not in $[\![D_n^*]\!]$ for $n \geq 3$. This ECE-equation is specified in the following theorem.

Theorem 24 *Let*
$\alpha = (X, \mathbb{D}(a_1 \oplus a_2) \wedge \mathbb{D}(c_1 \oplus c_2) \wedge \mathbb{D}((a_1 \oplus c_1)' \oplus (a_2 \oplus c_2)') \to \mathbb{D}((a_1 \oplus a_2)' \oplus (c_1 \oplus c_2)'))$ *with $X = \{a_1, a_2, c_1, c_2\}$. Then α is valid in $\underline{2}$ but not in $[\![D_n]\!]$ and not in $[\![D_n^*]\!]$ for $n \geq 3$.*

Proof. Let $v : X \to \underline{2}$ be a mapping such that the premise is satisfied for v. Then we get $1 \in \{v(a_1), v(a_2), v(c_1), v(c_2)\}$ because otherwise $(v(a_1) \oplus v(c_1))' \oplus (v(a_2) \oplus v(c_2))'$ would not exist. Therefore the conclusion of α is satisfied for v, so α is valid in $\underline{2}$. Now let $v : X \to [\![D_n]\!]$ be the mapping with $v(z) = [\![z]\!]$ for $z \in X$. Then the premise of α is satisfied for v because $([\![a_1]\!] \oplus [\![c_1]\!])' = [\![b_1]\!]$ and $([\![a_2]\!] \oplus [\![c_2]\!])' = [\![b_2]\!]$ holds. We have

$([\![a_1]\!] \oplus [\![a_2]\!])' = \bigoplus\{[\![a_i]\!] \mid 3 \le i \le n\}$ and $([\![c_1]\!] \oplus [\![c_2]\!])' = [\![x]\!] \oplus \bigoplus\{[\![c_i]\!] \mid 3 \le i \le n\}$, so the existence of $([\![a_1]\!] \oplus [\![a_2]\!])' \oplus ([\![c_1]\!] \oplus [\![c_2]\!])'$ would imply the existence of $[\![a_3]\!] \oplus [\![x]\!]$ which is a contradiction, because there does not exist a block containing $[\![a_3]\!]$ and $[\![x]\!]$. Therefore the conclusion of α is not satisfied for v, and α is not valid in $[\![D_n]\!]$. Analogously α is not valid in $[\![D_n^*]\!]$. ∎

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