# NONUNIFORM WEB-SPLINES

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ABSTRACT. The construction of weighted extended B-splines (web-splines), as recently introduced by the authors and J. Wipper for uniform knot sequences, is generalized to the non-uniform case. We show that web-splines form a stable basis for splines on arbitrary domains in $\mathbb{R}^m$ which provides optimal approximation power. Moreover, homogeneous boundary conditions, as encountered frequently in finite element applications, can be satisfied exactly by using an appropriate weight function. To illustrate the performance of the method, it is applied to a scattered data fitting problem and a finite element approximation of an elliptic boundary value problem.

## 1. INTRODUCTION

Tensor product B-splines have become a standard for approximation of functions and discrete data [2], computer-aided-design [5, 11], geometric modelling and computer graphics [4]. Among their many favorable properties, the stability of the B-spline basis is crucial for approximation purposes. However, stability is in general lost if the domain is trimmed to a bounded domain $D \subset \mathbb{R}^m$, whose boundaries are not aligned with the coordinate axes. This fact causes severe problems for instance in reverse engineering applications, where data are typically available only a bounded domain. Equally, it is a major obstacle to using B-splines as finite elements. As a generalization of the approach introduced in [8, 9, 10, 7] for the uniform case, we present a solution to this problem for nonuniform spline spaces.

The basic idea is simple. As is illustrated in Figure 1, we can approximate a function on a bounded domain $D \subset \mathbb{R}^m$ by forming a *spline*, i.e., a linear combination of all *relevant B-splines*

$$b_k, \quad k \in K,$$

which have some support in $D$. Depending on the degree, this yields approximations of arbitrary order and smoothness. However, numerical instabilities may arise due to the *outer B-splines*

$$b_j, \quad j \in J,$$

*Key words and phrases.* finite element, B-spline, web-spline, meshless method, nonuniform.

for which no complete grid cell of their support lies in $D$. Here and in the sequel, a *grid cell* is an interval which in every coordinate direction is bounded by two consecutive, but different knots, and an *inner grid cell* is a grid cell whose interior is completely contained in $D$. A further difficulty is that, in general, splines do not conform to homogeneous boundary conditions, which is essential for standard finite element schemes [16] or for matching boundaries in data fitting problems.

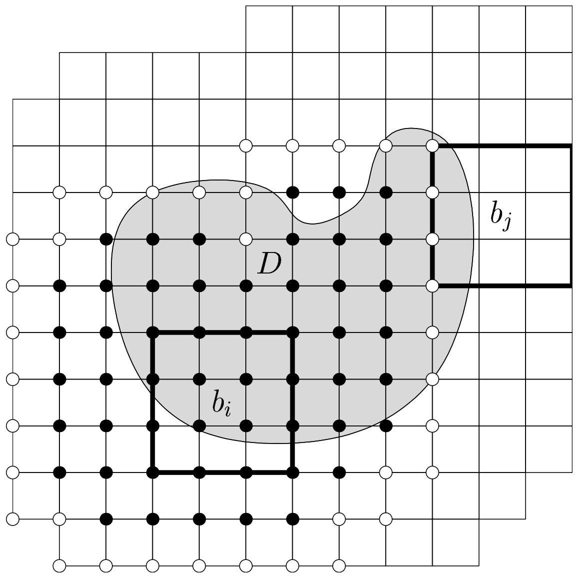


FIGURE 1. Relevant biquadratic B-splines for a domain $D$, marked at the lower left corners $(k_1h, k_2h)$ of their supports. Inner and outer B-splines are indicated with dots and circles, respectively.

Fortunately, both problems can be resolved. A stable basis is obtained by forming appropriate extensions of the *inner B-splines*

$$b_i, \quad i \in I := K \backslash J,$$

which have at least one inner grid cell in their support. If zero boundary data are required, we multiply by a positive *weight function* $w$ which vanishes on the boundary $\partial D$ of $D$. Otherwise, $w$ can be omitted or, to unify notation, be set to $w \equiv 1$. Combining both ideas led to the definition of *weighted extended B-splines* (web-splines) [8]. These new basis functions combine the computational advantages of B-splines and standard finite elements:

- The web-spline basis is stable.
- Homogeneous boundary conditions can be matched exactly.
- No mesh generation is required.
- Accurate numerical approximations are possible with relatively low-dimensional subspaces.

- Smoothness and approximation order can be chosen arbitrarily.
- Hierarchical bases permit adaptive refinement und multigrid algorithms.

Given the difficulty of meshing 2d and even more 3d domains, (cf., e.g., [13]), the third property is of great importance for finite element applications. Utilizing a regular grid not only eliminates a complicated and time-consuming preprocessing step, but also permits a very efficient implementation of algorithms.

In [8], web-splines have been constructed with uniform B-splines. This is adequate for smooth problems and also gives acceptable results for moderate singularities. To gain more flexibility, nonuniform knot spacing can be used to adapt the spline space to the requested resolution, or to suit the grid structure to the geometry of the domain. On the left hand side, Figure 2 shows an example, where nonuniform knots are used to resolve the potential fine structure of the function to be approximated in a vicinity of the small cut out circle. The right hand side depicts a typical situation, where the grid lines are aligned in a natural way to horizontal and vertical boundaries. We shall come back to these two examples in Section 5. In such situations, the use of nonuniform knot sequences, as described in this paper, is particularly competitive since it combines relatively low-dimensional spaces with the computational advantages of a regular grid.

Of course, in the multivariate case, nonuniform knot sequences are not always useful. The point is that knot insertion is not local in the sense that the complete domain is subdivided. This leads to an unjust increase of the dimension of the spline space if several, unaligned features are to be resolved. In such cases, hierarchical bases, as described in [6, 7], are the method of choice. Here, the finite elements are defined on a nested sequence of grids with scaled uniform knot sequences.

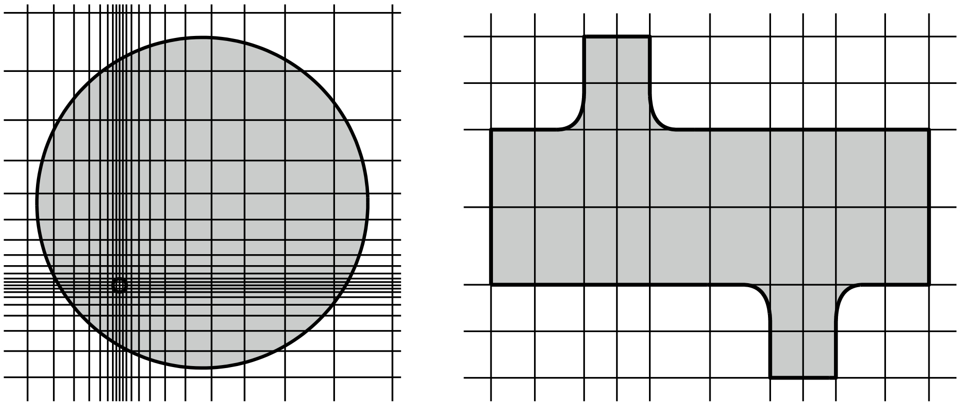

FIGURE 2. Domains with nonuniform grids.

In this paper, we show how web-splines can be generalized to nonuniform knot sequences, where emphasis is put on the extension procedure.

The key tool are dual functionals, which we review in Section 2 along with some definitions and basic facts about B-splines. In Section 3, we illustrate the main idea of our basis construction for a simple univariate model. The definition of multivariate nonuniform web-splines is then given in Section 4. Finally, we consider in Section 5 two applications: We show how web-splines avoid boundary artifacts in scattered data approximation on trimmed domains, and we demonstrate their excellent performance as finite elements at hand of a simple model problem.

Throughout, we use the following notational conventions. For an interval $Q \subset \mathbb{R}^m$, we denote by $|Q|$ and $\mu(Q)$ its diameter and measure, respectively. The linear space of all polynomial of degree $\le n$ is denoted by $\mathcal{P}_n$, where in the multivariate case, $n = [n_1, \ldots, n_m]$ is understood as the coordinate degree. In estimates, constants $\mathrm{const}(p_1, p_2, \ldots)$ depending on parameters $p_\nu$ are always positive. If the constants are clear form the context, we drop them and use the symbols $\preceq, \succeq$, and $\asymp$, instead. The $p$-norm of a vector or sequence $C = \{c_k\}_{k \in K}$ is denoted by $\|C\|_{p,K}$, and the $L_p$-norm of a function $u$ on a domain $D$ by $\|u\|_{p,D}$. Finally, $\|\cdot\|_{p,\ell,D}$ is the norm of the Sobolev space $W_p^\ell(D)$, see [1].

## 2. Some Facts about B-Splines

The material presented in this section is well known, but briefly compiled here for later reference and convenience of the reader. For given degree $n$ and a bi-infinite nondecreasing knot sequence

$$t: \ \cdots t_k \le t_{k+1} \le \cdots$$

we denote by $b_k, k \in \mathbb{Z}$, the B-splines of degree $n$ with support

$$\operatorname{supp} b_k = [t_k, t_{k+n+1}].$$

We choose the standard normalization $\sum_k b_k = 1$ and assume $t_k < t_{k+n+1}$ to avoid degenerate cases, i.e., the multiplicity $m_k$ of the knot $t_k$ is at most $n+1$. As is well-known [2], the B-splines $\{b_k, k \in \mathbb{Z}\}$ form a basis for the piecewise polynomials of degree $\le n$ which are $(n-m_k)$-times continuously differentiable at $t_k$. If higher order derivatives are required at a knot, they are understood as right-sided limits of adjacent polynomials. Moreover, the B-spline basis is uniformly stable,

$$\mathrm{const}(n)^{-1}\|C\|_{\infty,\mathbb{Z}} \le \left\| \sum_{k \in \mathbb{Z}} c_k b_k \right\|_{\infty,\mathbb{R}} \le \|C\|_{\infty,\mathbb{Z}}. \tag{1}$$

Hence, in the bi-infinite case, the condition number of the basis does not depend on the knot sequence $t$.

The estimate (1) and many other results concerning stability and approximation power of B-splines can be proven using *dual functionals*. From the plethora of possible constructions, we consider here the classical definition of de Boor and Fix [3], which is explicit and particularly elegant. For $k \in \mathbb{Z}$ and $\tau_k \in [t_k, t_{k+n+1})$, we define

$$(2)\quad \lambda_k f := \sum_{\ell=0}^{n} (-1)^{n-\ell} \psi_k^{(n-\ell)}(\tau_k) f^{(\ell)}(\tau_k), \quad \psi_k(x) := \frac{1}{n!} \prod_{\ell=1}^{n} (t_{k+\ell} - x).$$

The essential properties of this family of functionals are *bi-orthogonality* and uniform *boundedness* on the space of polynomials of degree $\leq n$.

**Theorem 1** (Dual Functionals for B-Splines). *i) B-splines and de Boor-Fix functionals are bi-orthogonal,*

$$\lambda_k b_{k'} = \delta_{k,k'}, \quad k, k' \in \mathbb{Z}.$$

*ii) If $Q$ is an interval in the support of $b_k$ with length bounded by $|Q| \geq \alpha |\operatorname{supp} b_k|$ for some constant $\alpha \in (0, 1]$, then*

$$|\lambda_k p| \leq \operatorname{const}(n, \alpha) \|p\|_{\infty, Q}, \quad p \in \mathcal{P}_n.$$

*Proof.* The proof of bi-orthogonality is based on Taylor's theorem and Marsden's identity, see [2] for details. To verify boundedness, we note that definition (2) is compatible with translation and scaling. More precisely, if $\tilde{\lambda}_k$ is the dual functional corresponding to the knot sequence $\tilde{t} := ht + s$ and the parameter $\tilde{\tau}_k := h\tau_k + s$, then

$$\lambda_k f = \tilde{\lambda}_k f\big((\cdot - s)/h\big).$$

This implies that we may assume $Q_k = [0, 1]$ without loss of generality. It is easily checked that $\|\psi_k\|_{\infty,n,[0,1]} \leq \operatorname{const}(n) |t_{k+n+1} - t_k|^n \leq \operatorname{const}(n) \alpha^{-n}$. Thus, by equivalence of norms on $\mathcal{P}_n$,

$$|\lambda_k p| \leq \operatorname{const}(n) \|\psi_k\|_{\infty,n,[0,1]} \|p\|_{\infty,n,[0,1]} \leq \operatorname{const}(n, \alpha) \|p\|_{\infty,[0,1]}.$$

□

Clearly, if we choose $Q$ as the largest grid interval in $\operatorname{supp} b_k$, we can take $\alpha = 1/(n+1)$, and the constant in the estimate depends only on the degree. With this choice and $\tau_k \in Q$ we obtain a proof for the nontrivial left inequality of the stability result (1). We simply observe that, by bi-orthogonality,

$$c_k = \lambda_k q, \quad q = \sum_{k'} c_{k'} b_{k'}$$

for any spline $q$, and that Theorem 1 applies since $q$ is a polynomial on $Q$.

However, it should be noted that this argument can fail for finite knot sequences. For a B-spline with exterior knots, the largest grid interval $Q$ may lie outside the natural domain of definition $D$ of the spline space. Hence, as is easily overlooked, for finite dimensional spline spaces, (1) does not hold. This problem can be eliminated by requiring that the *mesh ratio*, i.e., the maximal quotient of the lengths of adjacent grid cells, is bounded by a constant $\varrho$. In this case, every grid cell $Q_k$ in the support of $b_k$ has length

$$|Q_k| \geq \operatorname{const}(n, \varrho)|\operatorname{supp} b_k|,$$

and the constant in (1) has to be replaced by $\operatorname{const}(n, \varrho)$.

Finally, with the aid of dual functionals, we can define a canonical projector onto splines via

$$Pf := \sum_k (\lambda_k f) b_k.$$

Because of bi-orthogonality, $P$ reproduces B-splines, which implies in particular *polynomial precision*, i.e.,

$$Pp = \sum_k (\lambda_k p) b_k = p \tag{3}$$

for all polynomials $p \in \mathcal{P}_n$.

## 3. Stability via Extension

As we have seen, stability problems are caused by B-splines with small support in $D$. While the mesh ratio can be controlled quite easily, the support of some B-splines in $D$ can still become small if $D$ is not a union of grid cells. This phenomenon is persistent only in the multivariate case, but shall at first be studied in one variable for the sake of simplicity.

Let

$$\sum_{k \in K} c_k b_k(x), \quad x \in D,$$

be a spline on a bounded interval $D \subset \mathbb{R}$, where the index set $K$ comprises all relevant B-splines with some support in $D$. The example of a quadratic spline space on $D = (0, 1)$, depicted in Figure 3, captures the essential difficulty. If the interval endpoints do not coincide with knots, there exist outer B-splines

$$b_j, \quad j \in J,$$

for which $\operatorname{supp} b_j$ does not contain an inner grid cell. These outer B-splines cause stability problems even if the mesh ratio is small. In the example, we have $J = \{-1, 0\}$ and

$$b_{-1}(x) = O(\varepsilon^2), \quad b_0(x) = O(\varepsilon)$$

for $x \in D$. Hence, the first two coefficients of a spline $q$ with $\|q\|_{\infty,D} \leq 1$ can become arbitrarily large as $\varepsilon \to 0$. For the inner B-splines

$$b_i, \quad i \in I,$$

$\operatorname{supp} b_i$ contains at least one inner grid cell. In the example, $I = \{1, 2, 3, 4\}$. This part of the basis is stable regardless of the size of $\varepsilon$.

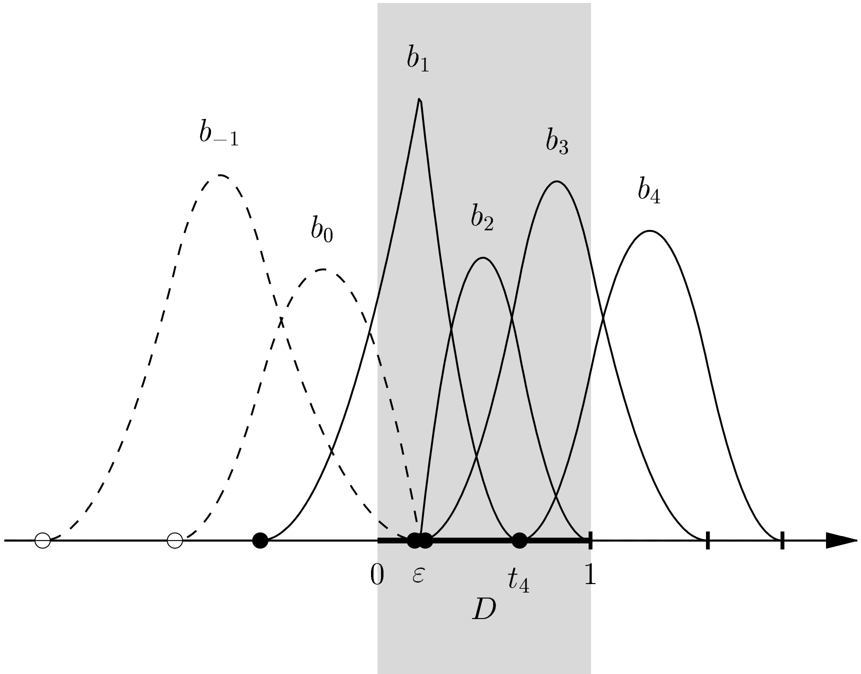


FIGURE 3. Inner B-splines $b_i$, $i \in I$, and outer B-splines $b_j$, $j \in J$, on a bounded interval $D$.

We would like to select a subspace with a stable, local basis while maintaining polynomial precision. This is accomplished by adjoining outer to inner B-splines via appropriate linear combinations. To this end, for an outer index $j \in J$, we denote the inner grid cell closest to $\operatorname{supp} b_j$ by $Q_j$ and define the set of related inner indices by

$$I(j) := \{i \in I : \ Q_j \subset \operatorname{supp} b_i\}.$$

It is easy to see that $I(j)$ consists of $n+1$ consecutive inner indices, $I(j) = \{\ell, \dots, \ell+n\}$. Conversely, for an inner index $i$, we define the set of related outer indices by

$$J(i) := \{j \in J : \ i \in I(j)\}.$$

In the example, $Q_{-1} = Q_0 = [\varepsilon, t_4]$ and

$$I(-1) = I(0) = \{1, 2, 3\}, \quad J(1) = J(2) = J(3) = \{-1, 0\}, \quad J(4) = \emptyset.$$

With these notions, we define extended B-splines as follows:

**Definition 1.** *For $i \in I, j \in J(i)$, and $Q_j$ as defined above, we denote by $p_{i,j}$ the polynomial which agrees with $b_i$ on $Q_j$ and define the* extension coefficients

$$e_{i,j} := \lambda_j p_{i,j}. \tag{4}$$

*Then, the* extended B-splines (eb-splines) *are*

$$B_i := b_i + \sum_{j \in J(i)} e_{i,j} b_j, \quad i \in I.$$

*The linear span of eb-splines is denoted by $\mathcal{B}$.*

The computation of the extension coefficients is straightforward: We generate the polynomials $p_{i,j}$ in Taylor form using the recurrence relation for B-splines. Expanding at an arbitrary point $\tau_j$, which appears in the definition of the dual functional $\lambda_j$, the coefficients yield the relevant data for applying formula (2). This procedure is slightly more involved than for uniform knots ($t_k = kh$), where we have the simple expression

$$e_{i,j} = \prod_{\substack{\nu=0 \\ \ell+\nu \neq i}}^{n} \frac{j - \ell - \nu}{i - \ell - \nu},$$

derived via Lagrange interpolation, see [8] for details. In any case, the overhead is small since only few B-splines near the interval endpoints are extended.

We show now that extended B-splines inherit all properties of standard B-splines which are crucial for approximation purposes, namely locality, boundedness, existence of dual functionals, and polynomial precision.

**Theorem 2** (Locality and Boundedness)**.** *i) The distance between $i \in I$ and $j \in J(i)$ is bounded by $|i - j| \leq 2n + 1$. In particular,*

$$|\operatorname{supp} B_i| \leq \operatorname{const}(n, \varrho) |\operatorname{supp} b_i|. \tag{5}$$

*ii) eb-splines are uniformly bounded by*

$$\|B_i^{(\ell)}\|_{\infty,D} \leq \operatorname{const}(n, \varrho) |\operatorname{supp} b_i|^{-\ell}. \tag{6}$$

*Proof.* To prove locality, we consider, e.g., the left boundary of $D$. If $i_0$ is the smallest inner index, then $i_0 - n - 2$ is an index which certainly corresponds to a non-relevant B-spline. Hence, $i_0 - n - 1 \leq j < i_0$, and the corresponding set of inner indices is $I(j) = i_0 + \{0, \ldots, n\}$. The bound on the number of outer B-splines $b_j$ attached to $b_i$ combined with the bound $\varrho$ on the mesh ratio yields (5).

To prove the second statement, we first show that the extension coefficients are uniformly bounded. The construction of eb-splines is invariant under affine transformations of the abscissa. Hence, for $j \in J$ and $Q_j$ the nearest inner grid cell, we may assume $Q_j = [0, 1]$. Being part of a standard B-spline, the polynomial $p_{i,j}$ is bounded by $\|p_{i,j}\|_{\infty,[0,1]} \leq 1$. This implies that $p_{i,j}(x)$ is bounded by a constant depending only on $n$ and $x$. By (5), $|x| \leq \operatorname{const}(n, \varrho)$ for $x \in \operatorname{supp} b_j$. So, we obtain using Theorem 1 with $Q = \operatorname{supp} b_j$

$$|e_{i,j}| = |\lambda_j p_{i,j}| \leq \operatorname{const}(n) \|p_{i,j}\|_{\infty,\operatorname{supp} b_j} \leq \operatorname{const}(n, \varrho).$$

Boundedness of extension coefficients combined with the known estimate $\|b_k^{(\ell)}\|_{\infty,\mathbb{R}} \leq \mathrm{const}(n,\varrho)|\operatorname{supp} b_k|^{-\ell}$, which holds for standard B-splines, proves the claim. $\square$

Now, we show that $\{\lambda_i, i \in I\}$ is a family of bounded dual functionals for the eb-splines $\{B_i, i \in I\}$:

**Theorem 3** (Dual Functionals for eb-Splines). *i) eb-splines and de Boor-Fix functionals are bi-orthogonal,*

$$\lambda_i b_{i'} = \delta_{i,i'}, \quad i, i' \in I.$$

*ii) If $Q$ is an inner grid cell in the support of $b_i$, then*

$$|\lambda_i p| \leq \mathrm{const}(n,\varrho)\|p\|_{\infty,Q}, \quad p \in \mathcal{P}_n.$$

*Proof.* Bi-orthogonality follows from $\lambda_k b_{k'} = \delta_{k,k'}$ and the definition of $B_{i'}$ since $\lambda_i b_j = 0$ for $j \in J$, while boundedness just recalls Theorem 1. $\square$

The existence of dual functionals implies linear independence, i.e., eb-splines form a basis for the spline space $\mathcal{B}$. Moreover, like standard B-splines, eb-splines are a *local basis* in the sense that for any grid cell $Q$ intersecting $D$ the eb-splines which do not vanish on $Q$ are linearly independent. This can easily be shown by selecting $\tau_i \in Q \cap D$ for all dual functionals $\lambda_i$ corresponding to eb-splines with $Q$ in their support. Since all polynomials $p \in \mathcal{P}_n$ are contained in $\mathcal{B}$, as will follow from the next theorem, there exist exactly $n+1$ eb-splines which do not vanish on $Q$, and they span the space of all polynomials of degree $\leq n$ on $Q$.

Defining the canonical projector $P$ onto $\mathcal{B}$ by

$$Pf := \sum_{i \in I} (\lambda_i f) B_i,$$

we can establish polynomial precision.

**Theorem 4** (Polynomial Precision). *For all polynomials $p \in \mathcal{P}_n$,*

$$Pp = p.$$

*In particular, the spline space $\mathcal{B}$ contains all polynomials of degree $\leq n$ on $D$.*

*Proof.* Substituting the definition of $B_i$ and interchanging sums, we have for $x \in D$

$$Pp = \sum_{i \in I} (\lambda_i p) B_i(x) = \sum_{i \in I} (\lambda_i p) b_i + \sum_{j \in J} \left[ \sum_{i \in I(j)} e_{i,j} (\lambda_i p) \right] b_j.$$

Now, because of (3), $Pp = p$ is equivalent to

$$\sum_{i \in I(j)} e_{i,j} (\lambda_i p) = \lambda_j p. \tag{7}$$

Since both sides are linear in $p$, it suffices to check this identity for a basis. Taking $p = p_{i',j}$ with $i' \in I(j)$ and $\tau_i \in Q_j$, we have $\lambda_i p_{i',j} = \lambda_i b_{i'} = \delta_{i,i'}$, and (7) reduces to the definition (4) of the extension coefficients $e_{i,j}$. □

After establishing locality and boundedness, dual functionals, and polynomial precision, we have all essential ingredients at our disposal to derive standard results on stability and approximation power. Exemplarily, we establish optimal convergence rates when approximating smooth functions.

**Theorem 5** (Approximation power)**.** *For $x \in D$, we denote by $Q$ the union of supports of eb-splines containing $x$, and by $h$ the length of the grid cell containing $x$. Then, for a smooth function $f$, the approximation error $d := Pf - f$ is pointwise bounded by*

$$\left|d^{(\ell)}(x)\right| \leq \operatorname{const}(n, \varrho) \left\|f^{(n+1)}\right\|_{\infty,Q} h^{n+1-\ell}.$$

*Proof.* The proof is routine. We denote by $\tilde{I}$ the set of inner indices which are relevant for $x$. Since $\tilde{I}$ contains $n+1$ elements, $|Q| \asymp h$. Let $p \in \mathcal{P}_n$ be the Taylor polynomial of $f$ at $x$. Then, with $\Delta := f - p$,

$$\left\|\Delta^{(\ell)}\right\|_{\infty,Q} \preceq \left\|f^{(n+1)}\right\|_{\infty,Q} h^{n+1-\ell}.$$

Further, by polynomial precision and boundedness of eb-splines,

$$\left|d^{(\ell)}(x)\right| = \left|(P\Delta)^{(\ell)}(x)\right| = \left|\sum_{i \in \tilde{I}} (\lambda_i \Delta) B_i^{(\ell)}(x)\right| \preceq \max_{i \in \tilde{I}} |\lambda_i \Delta|\, h^{-\ell}.$$

It remains to consider $\lambda_i \Delta$. The point $\tau_i$ in the definition of $\lambda_i$ lies in $Q$. Hence, $\left|\psi_i^{(n-\ell')}(\tau_i)\right| \preceq h^{\ell'}$, and

$$|\lambda_i \Delta| \leq \sum_{\ell'=0}^{n} \left|\psi_i^{(n-\ell')}(\tau_i)\right| \; \left|\Delta^{(\ell')}(\tau_i)\right| \preceq \left\|f^{(n+1)}\right\|_{\infty,Q} h^{n+1}$$

□

We note that similar results for the approximation of less regular functions can be obtained exactly in the same way using dual functionals which are bounded, e.g., with respect to the sup-norm. The special choice that we made here is merely due to the explicit character of the de Boor-Fix functionals, which is favorable for the definition of extension coefficients.

Summarizing, the material presented in this section admits to derive standard approximation and stability properties for spline spaces with small parameter intervals at the endpoints of $D$. The modifications are crucial for splines in several variables, where we can in general not align the grid lines to the domain boundaries.

## 4. Multivariate Web-Splines

Generalizing the univariate definitions and results of the last section to $m \geq 2$ variables is straightforward. The arguments are completely analogous. Merely the notation needs to be adapted to the multivariate setting.

We consider a tensor product grid in $\mathbb{R}^m$ with knot sequences $t = [t^1, \dots, t^m]$,

$$t^\nu : \ \cdots \leq t_k^\nu \leq t_{k+1}^\nu \leq \cdots , \quad \nu = 1, \dots , m,$$

and denote by

$$b_k = b_{k,t}^n(x) := b_{k_1,t^1}^{n_1}(x_1) \cdots b_{k_m,t^m}^{n_m}(x_m), \quad k \in \mathbb{Z}^m,$$

the corresponding tensor product B-splines of degree $n = [n_1, \dots , n_m]$. For a grid cell $Q$ with side lengths $l_1, \dots , l_m$ we define its *distortion* by

$$\max_{\nu,\nu'} l_\nu / l_{\nu'} .$$

The distortion of the knot sequence $t$ is the maximal distortion of its grid cells, and $\delta$ will denote an upper bound on it. Like the mesh ratio in the univariate case, the distortion quantifies the deviation from a uniform setting in the multivariate case. It is easy to see that if the distortion of $t$ is bounded by $\delta$, then the mesh ratios of all knot sequences $t^1, \dots , t^m$ are bounded by $\delta^2$.

For a bounded domain $D \subset \mathbb{R}^m$ we define the sets $K, I, J$ of relevant, inner, and outer indices as in the univariate case (cf. also Figure 1):

$$\begin{aligned} K &:= \{k \in \mathbb{Z}^m : D \cap \operatorname{supp} b_k \neq \emptyset\} \\ I &:= \{i \in \mathbb{Z}^m : \operatorname{supp} b_i \text{ contains an inner grid cell}\} \\ J &:= K \backslash I. \end{aligned}$$

For $j \in J$, the inner grid cell whose midpoint is closest to the midpoint of $\operatorname{supp} b_j$ is denoted by $Q_j$. The B-splines which do not vanish on $Q_j$ have indices in

$$I(j) = I^1(j) \times \cdots \times I^m(j) = \ell + \{0, \dots , n\}^m$$

with $\ell = \ell(j) \in \mathbb{Z}^m$, see Figure 4. The complementary sets $J(i)$ are defined as before.

The multivariate de Boor-Fix functionals are constructed from the univariate ones as follows: For $k \in \mathbb{Z}^m$ and $\tau_k = [\tau_{k_1}, \dots , \tau_{k_m}]$,

$$\lambda_k := \lambda_{k_1}^1 \circ \cdots \circ \lambda_{k_m}^m , \tag{8}$$

where $\lambda_{k_\nu}^\nu$ is acting on the $\nu$th variable. It is easily checked that biorthogonality and uniform boundedness are kept.

Except for the incorporation of an additional weight function, the definition of multivariate extended B-splines is completely analogous to the univariate case:

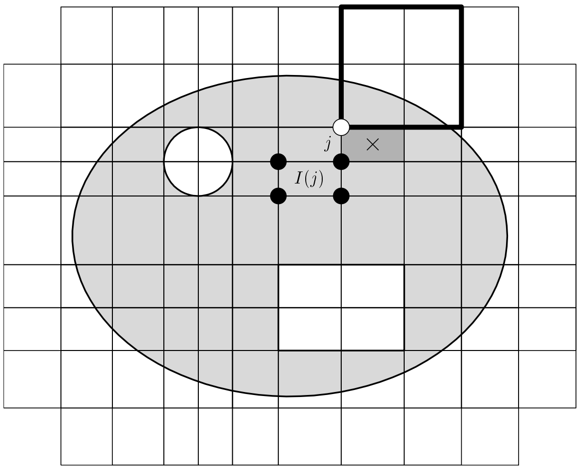


FIGURE 4. Grid points $(t^1_{i_1}, t^2_{i_2})$, $i \in I(j)$, for a bilinear outer B-spline $b_j$. The nearest inner grid cell $Q_j$ is highlighted, and the point $x_i$ marked by a cross.

**Definition 2.** *For $i \in I, j \in J(i)$, and $Q_j$ defined as above, we denote by $p_{i,j}$ the polynomial which agrees with $b_i$ on $Q_j$ and define the* extension coefficients

$$e_{i,j} := \lambda_j p_{i,j}.$$

*Further, let $w$ be a positive* weight function *which is smooth on $D$ and equivalent to some power $r \geq 0$ of the boundary distance function,*

$$w(x) \asymp \operatorname{dist}(x, \partial D)^r, \tag{9}$$

*and denote by $x_i$ the center of an inner grid cell in* $\operatorname{supp} b_i$. *Then, the* weighted extended B-splines (web-splines) *are defined by*

$$B_i := \frac{w}{w(x_i)} \left( b_i + \sum_{j \in J(i)} e_{i,j} b_j \right), \quad i \in I.$$

*The linear span of web-splines is the* web-space $\mathcal{B}$.

In particular, the weight function is essential for finite element applications. It allows us to satisfy homogeneous Dirichlet boundary conditions simply by requiring that $w$ vanishes on the appropriate component of the boundary $\partial D$. Using such weighted finite element bases was already suggested by Kantorowitsch and Krylow [12] and has been extensively studied by Rvachev et al. (cf., e.g., the survey [15] and the literature cited there). Rvachev developed the so-called R-function method, which is particularly suited for domains constructed from simple primitives with Boolean operations. For planar domains bounded piecewise by NURBS-curves, weight functions are constructed in [14].

With dual functionals according to (8) and $p_{i,j}(x) = \prod_\nu p_{i_\nu,j_\nu}(x_\nu)$, we obtain

$$e_{i,j} = \left(\lambda^1_{j_1} \circ \cdots \circ \lambda^m_{j_m}\right) p_{i,j} = \prod_{\nu=1}^m \lambda^\nu_{j_\nu} p_{i_\nu,j_\nu} = \prod_{\nu=1}^m e_{i_\nu,j_\nu}.$$

That is, multivariate extension coefficients can be conveniently computed as products of univariate ones.

Again, the web-splines $B_i$ inherit all basic properties of standard nonuniform B-splines, except positivity. However, constants typically depend now on a bound $\delta$ on the distortion instead of the mesh ratio.

**Theorem 6** (Locality and Boundedness). *i) If $D \subset \mathbb{R}^m$ is a Lipschitz-domain, then the distance between $i \in I$ and $j \in J(i)$ is bounded by $\|i - j\|_\infty \le \operatorname{const}(n, m, \delta, D)$. In particular,*

$$|\operatorname{supp} B_i| \le \operatorname{const}(n, m, \delta, D) |\operatorname{supp} b_i|$$
$$\mu(\operatorname{supp} B_i) \le \operatorname{const}(n, m, \delta, D) \mu(\operatorname{supp} b_i).$$

*ii) web-splines are uniformly bounded by*

$$\|B_i\|_{\infty,D} \le \operatorname{const}(n, m, \delta, D, w). \tag{10}$$

*Proof.* To prove the first statement, we observe that the ratio of diameters of any two grid cells $Q, Q'$ is bounded by $|Q|/|Q'| \le \delta^2$. In particular, if $\|t\|$ denotes the maximal diameter of grid cells, $\delta^{-2}\|t\| \le |Q| \le \delta^2\|t\|$. The diameter of $Q$ is bounded in terms of its side lengths $l_1, \dots, l_m$ by $|Q| \le \delta\sqrt{m}\, l_\nu$. Since the domain is assumed to be Lipschitz, there exist constants $\alpha, h_0$ depending on $D$ such that for all $h \in (0, h_0)$ and $x \in D$ there exists a point $y \in D$ with $\|x - y\|_2 < h < \alpha \operatorname{dist}(y, \partial D)$.

If $\|t\| < h_0/(\alpha\delta^2)$, we consider an outer index $j \in J$ and a point $x \in \operatorname{supp} b_j \cap D$. With $h := \|t\|\alpha\delta^2$ and $y$ as above, the grid cell $Q$ containing $y$ is inner since $|Q| \le \|t\|\delta^2 = h/\alpha < \operatorname{dist}(y, \partial D)$. Since $\|x - y\| \preceq \|t\|$, the distance between $\operatorname{supp} b_j$ and $Q$ is $\preceq \|t\|$. Consequently, the distance between $\operatorname{supp} b_j$ and the nearest inner grid cell $Q_j$ is $\preceq \|t\|$. All side lengths of all B-splines are $\succeq \|t\|$. So, the difference between $j$ and inner indices $i \in I(j)$ is $\preceq 1$.

If $\|t\| \ge h_0/(\alpha\delta^2)$, then the side lengths $l_1, \dots, l_m$ of any grid cell $Q$ are bounded by

$$l_\nu \ge \frac{|Q|}{\delta\sqrt{m}} \ge \frac{\|t\|}{\delta^3\sqrt{m}} \ge \frac{h_0}{\alpha\delta^5\sqrt{m}}.$$

Since $D$ is bounded, the lower bound on the side lengths yields an upper bound on the number $\#K$ of relevant indices, and $\|i - j\|_\infty \le \#K \le \operatorname{const}(n, m, \delta, D)$. The inequalities for the diameter and the measure of $\operatorname{supp} B_i$ follow immediately from the boundedness of the number outer B-splines attached to $b_i$ and the boundedness of distortion.

To prove the second statement, we conclude from (9) that the weight factor in the definition of web-splines is bounded by

$$\left\| \frac{w}{w(x_i)} \right\|_{\infty, \operatorname{supp} B_i} \preceq 1.$$

It remains to show that the extension coefficients are uniformly bounded by

$$|e_{i,j}| \le \operatorname{const}(n, m, \delta, D),$$

which can be done following exactly the arguments given in the univariate case. □

It can be shown by carefully constructed examples that the upper bound on $\|i-j\|_\infty$ in fact depends on the distortion. However, such cases are rarely encountered in applications. The examples in Section 5 show that $\|i-j\|_\infty$ is typically close to $n$ if the knot sequences are fine.

The dual functionals need to be adapted to the weight function. With $x_i$ as in Definition 2, we define the *weighted functionals*

$$\Lambda_i f := w(x_i)\lambda_i(f/w), \quad i \in I.$$

Uniform boundedness is now required on the space of weighted polynomials. On the inner grid cell $Q \subset \operatorname{supp} B_i$ containing $x_i$ the weight function can get arbitrarily small. The resulting problem can be circumvented by restriction to a sub-interval $\tilde{Q}$ of $Q$ which has the same center, but halved side lengths. From (9) we conclude that

$$\left\| \frac{w(x_i)}{w} \right\|_{\infty, \tilde{Q}} \preceq 1 \tag{11}$$

with constants depending on $\delta$ and $w$. Now, we are prepared to establish the analogue of theorems 1 and 3 for web-splines.

**Theorem 7** (Dual Functionals for web-Splines). *i) web-splines and weighted de Boor-Fix functionals are bi-orthogonal,*

$$\Lambda_i B_{i'} = \delta_{i,i'}, \quad i, i' \in I.$$

*ii) If $Q$ is the inner grid cell in the support of $B_i$ containing $x_i$, and $\tilde{Q}$ the half-size sub-interval as defined above, then*

$$|\Lambda_i(wp)| \le \operatorname{const}(n, \varrho, w)\|wp\|_{\infty, \tilde{Q}}, \quad p \in \mathcal{P}_n.$$

*Proof.* Bi-orthogonality is verified by inspection. To show boundedness on weighted polynomials, we note that the multivariate de Boor-Fix functionals are bounded by

$$|\lambda_i p| \le \operatorname{const}(n, \delta)\|p\|_{\infty, \tilde{Q}}.$$

Further, with (11),

$$\begin{aligned}|\Lambda_i(wp)| &= w(x_i)\,|\lambda_i p| \le \mathrm{const}(n,\varrho)w(x_i)\|p\|_{\infty,\tilde{Q}} \\ &\le \mathrm{const}(n,\varrho)\|w(x_i)/w\|_{\infty,\tilde{Q}}\|wp\|_{\infty,\tilde{Q}} \\ &\le \mathrm{const}(n,\varrho,w)\|wp\|_{\infty,\tilde{Q}}.\end{aligned}$$

□

The canonical projector $P$ onto the spline space $\mathcal{B}$ is defined as before by

$$Pf := \sum_{i\in I}(\Lambda_i f)B_i.$$

Now, polynomial precision is replaced by weighted polynomial precision.

**Theorem 8** (Weighted Polynomial Precision)**.** *For all polynomials $p \in \mathcal{P}_n$,*

$$P(wp) = wp.$$

*In particular, the spline space $\mathcal{B}$ contains all weighted polynomials of degree $\le n$ on $D$.*

*Proof.* We obtain

$$P(wp) = \sum_{i\in I}\Lambda_i(wp)B_i = w\sum_{i\in I}(\lambda_i p)\left(b_i + \sum_{j\in J(i)} e_{i,j}b_j\right) = wp,$$

where the last identity is verified exactly as in the proof of Theorem 4. □

Proving approximation results for weighted spline spaces is slightly more involved than in standard cases. The technical details are described in [7]. Here, we consider stability of the web-basis and show the following generalization of (1) and (10):

**Theorem 9** (Stability)**.** *Appropriately normalized, web-splines are uniformly stable with respect to p-norms, i.e.,*

$$\|C\|_{p,I} \asymp \left\|\sum_{i\in I} c_i\ (\gamma_i B_i)\right\|_{p,D},$$

*where the normalization factor is*

$$\gamma_i := \begin{cases}\mu(\operatorname{supp} b_i)^{-1/p} & \text{for } 1 \le p < \infty \\ 1 & \text{for } p = \infty,\end{cases}$$

*and the constants depend only on $n, m, \delta, D, w$.*

*Proof.* The line of arguments is well known: Since the support of each web-spline contains only $\preceq 1$ grid cells, it suffices to prove the local estimates

$$\left|\gamma_i^{-1}\Lambda_i q\right| \preceq \|q\|_{p,\tilde{Q}}, \quad \|\gamma_i B_i\|_{p,D} \preceq 1,$$

where $q = \sum_{i\in I} c_i(\gamma_i B_i)$ and $\tilde{Q}$ is the half-size sub-interval of the inner grid cell containing $x_i$. The first inequality is invariant under affine transformations of the arguments. Hence, we may assume $\tilde{Q} = [0,1]^m$. By Theorem 6, $\gamma_i^{-1} = \mu(\operatorname{supp} b_i)^{1/p} \preceq 1$. Further, since $q$ is a weighted polynomial on $[0,1]^m$, we can use Theorem 7 and equivalence of norms to obtain $|\Lambda_i q| \preceq \|q\|_{\infty,[0,1]^m} \preceq \|q\|_{p,[0,1]^m}$.

For $p = \infty$, the second inequality is just (10). For $p < \infty$, Theorem 6 yields

$$\|\gamma_i B_i\|_{p,D} \preceq \left(\frac{\mu(\operatorname{supp} B_i)}{\mu(\operatorname{supp} b_i)}\right)^{1/p} \preceq 1,$$

and the proof is complete. □

## 5. Applications

In this section, we discuss two typical applications of web-splines. First, we consider a scattered data approximation problem on a trimmed domain. Second, we illustrate their performance as finite elements at hand of a simple model problem.

Scattered data approximation problems on trimmed domains occur, for instance, in reverse engineering applications. Let $D \subset \mathbb{R}^2$ be a bounded domain. For given data points $(x_\nu, y_\nu, z_\nu) \in D \times \mathbb{R}$ we seek a bivariate spline $q : D \to \mathbb{R}$ which approximates in a least squares sense:

$$\sum_\nu (q(x_\nu, y_\nu) - z_\nu)^2 \to \min .$$

Figure 5 shows a domain and the location of data points together with knot lines, which are aligned with the boundary of $D$ in a natural way. In the example, height values are sampled from the smooth function $z = f(x,y) = 2\cos(x/3)\cos(y/2)$. No weighting is required, so we set $w \equiv 1$. On the left hand side, Figure 6 shows the best approximating cubic web-spline $q_{\mathrm{web}}$. In contrast, on the right hand side, standard B-splines are used to obtain the approximation $q_{\mathrm{std}}$. The artifacts at the rounded corners of the domain are clearly visible. The point is that outer B-spline coefficients may get very large in order to slightly reduce the approximation error at the data points near the boundary. The advantages of the web-method become obvious when comparing the Euclidean error at the data points, the maximal error on $D$, and the condition number of the Gramian matrix $G$:

$$\begin{aligned} \|q_{\text{web}}(X,Y) - Z\|_2 &\approx 8.6\text{e-}4, & \|q_{\text{std}}(X,Y) - Z\|_2 &\approx 8.2\text{e-}4 \\ \|q_{\text{web}} - f\|_{\infty,D} &\approx 2.2\text{e-}4, & \|q_{\text{std}} - f\|_{\infty,D} &\approx 2.8\text{e-}1 \\ \operatorname{cond} G_{\text{web}} &\approx 7.7\text{e}3, & \operatorname{cond} G_{\text{std}} &\approx 6.2\text{e}13. \end{aligned}$$

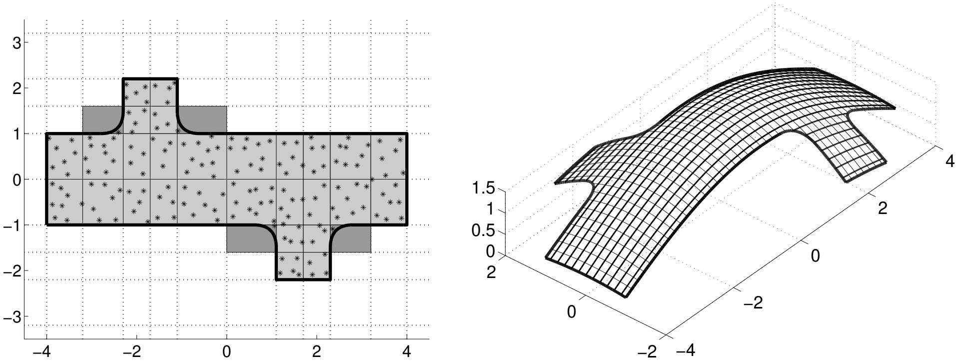


FIGURE 5. Domain with grid lines and scattered data points *(left)* and sampled function $f(x) = 2\cos(x/3)\cos(y/2)$ *(right.)*

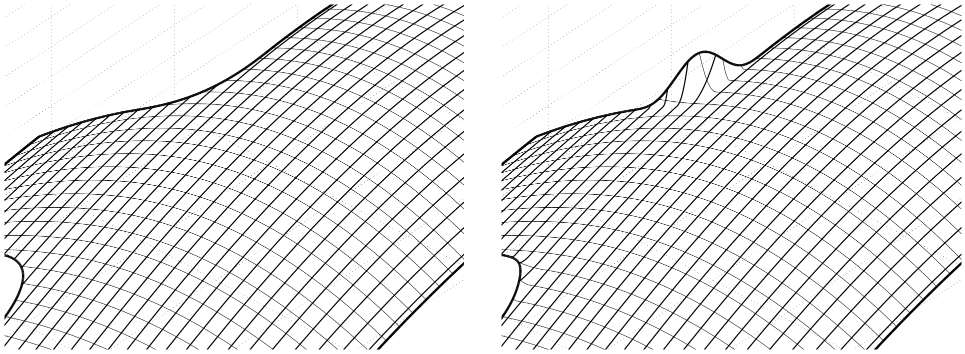

FIGURE 6. Approximation with extension *(left)* and without extension *(right)*.

As a second example, we consider Poisson's equation with Dirichlet boundary conditions,

$$-\Delta u(x,y) = f(x,y) = 25x^2 \text{ on } D, \quad u = 0 \text{ at } \partial D. \tag{12}$$

The domain $D$ is the unit disk with a small circular hole with radius $r = 0.04$ located at $(x_0, y_0) = (-1/2, -1/2)$, see Figure 7, left. Non-uniform knot spacing is used in order to resolve the expected high curvature of the solution near the small hole. In this case, an appropriate weight function is easily constructed,

$$w(x,y) = \big(1 - x^2 - y^2\big)\big((x - x_0)^2 + (y - y_0)^2 - r^2\big),$$

see Figure 8, left. Each grid cell that intersects the boundary has an adjacent inner grid cell. That is, despite the relatively high distortion $\delta \approx 18$, the difference between inner and outer indices is optimally small, $\|i-j\|_\infty \le n+1 = 5$. The coefficient vector $U$ of an approximate solution is obtained by solving the Galerkin system $GU = F$ resulting from the standard finite element discretization of (12). The moderate condition number $\operatorname{cond} G_{\text{web}} \approx 1700$, obtained after scaling the diagonal to 1, admits efficient solution with standard solvers. The approximation $u_{\text{web}}$ that we obtain using quartic web-splines is fairly accurate in view of the small number of coefficients,

$$\|u_{\text{web}} - u\|_{\infty,D} \approx 3.2\text{e-}4 \quad \text{with} \approx 450 \text{ coefficients},$$

see also Figure 9, left. Let us compare this result with uniform web-splines and standard hat functions.

- For uniform knot sequences and equal degree $n = 4$, a rather fine grid is requested to obtain an approximation $u_{\text{uni}}$ with similar accuracy,

  $$\|u_{\text{uni}} - u\|_{\infty,D} \approx 3.7\text{e-}4 \quad \text{with} \approx 5250 \text{ coefficients}.$$

  On the right hand side, Figure 9 shows that the error is highly concentrated near the hole, i.e., the global fine resolution is in fact not necessary.
- The MATLAB pde-toolbox, which uses standard algorithms based on a triangulation of the domain and piecewise linear basis functions, provides a comparable approximation $u_\Delta$ only for a very fine triangulation,

  $$\|u_\Delta - u\|_{\infty,D} \approx 4.8\text{e-}4 \quad \text{with} \approx 16.000 \text{ coefficients},$$

  see also Figure 7, right.

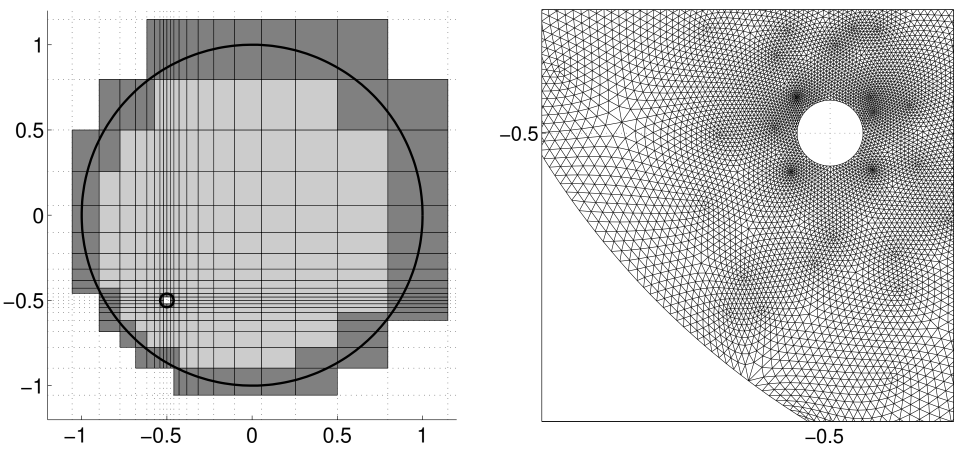


FIGURE 7. Domain with non-uniform grid *(left)* and part of the triangulation required to achieve similar accuracy *(right)*.

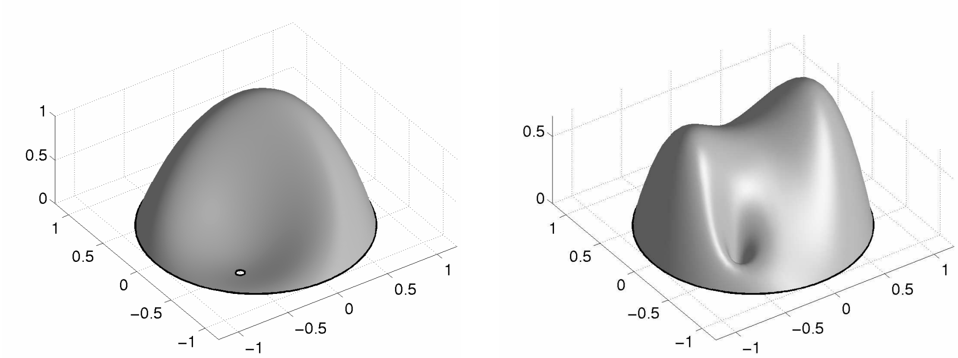


FIGURE 8. Weight function $w$ *(left)* and approximation $u_{\text{web}}$ *(right)*.

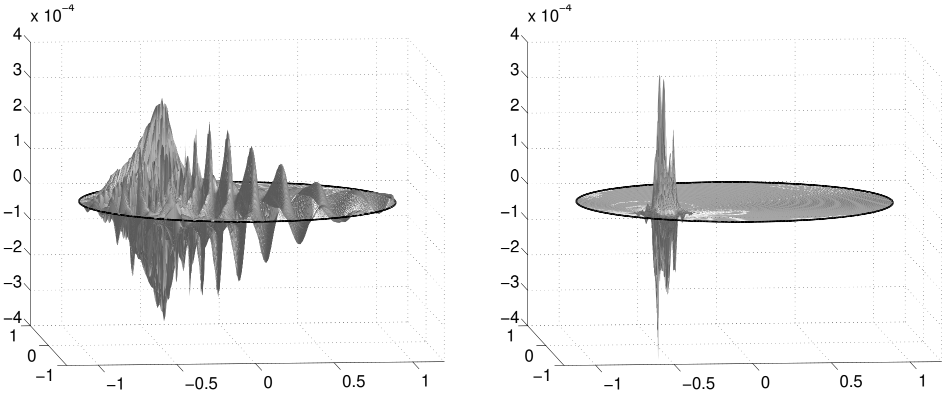


FIGURE 9. Error for non-uniform knots *(left)* and for uniform knots *(right)*.

The examples presented in this section illustrate that non-uniform web-splines are a competitive tool for approximating discrete data and solutions of pdes.

## 6. CONCLUSION

The web-method is a new meshless finite element technique combining the advantages of B-splines and standard mesh-based trial functions (cf. http://www.web.spline.de). In particular, highly accurate numerical solutions are possible with relatively few parameters and boundary conditions are matched exactly. Moreover, smoothness and approximation order can be chosen arbitrarily without significantly increasing the computational complexity.

Initially, web-splines were defined for uniform grids. As is shown in this paper, the concept naturally extends to arbitrary knot sequences. This provides additional flexibility for meeting design specifications and adapting the spline basis to the structure of the approximated data or functions. Perhaps more importantly, the nonuniform web-method conforms to the NURBS-standard, used in many industrial applications. We hope that our work will contribute to unifying methods in CAD/CAM and FEM, advertising B-splines as a convenient tool for all stages of the manufacturing process.

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