Justification of homogenized models for viscoplastic bodies with microstructure

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Abstract. We justify the formal homogenization of the quasistatic initial boundary value problem with internal variables, called the microscopic problem, which models the deformation behavior of viscoplastic bodies. To this end it is first shown that the formally derived homogenized initial-boundary value problem has a solution. From this solution an asymptotic solution of the microscopic problem is constructed, and it is shown that the difference of the exact solution and the asymptotic solution tends to zero if the lengthscale of the microstructure converges to zero. Our results are proved for viscoplastic material behavior that can be modeled by constitutive equations of monotone type with linear hardening terms. For technical reasons we are only able to prove the convergence result locally in time and for smooth data.

1 Introduction and statement of results

The numerical simulation of viscoplastic material behavior is expensive, since the dependence of the stress field on the deformation history must be taken into account. The difficulties increase for viscoplastic bodies with a microstructure caused by phase changes or by other spatial variations of the material properties, because of the fine discretization required by the microstructure. If the lengthscale of the microstructure is small, effective numerical simulations can thus not be based on a mathematical model which faithfully describes this microstructure. For viscoplastic bodies it is therefore of particular interest to derive from this faithful model, which we call the microscopic model, a homogenized or macroscopic model, which describes a body without microstructure, but which shows the same overall behavior as the body with microstructure.

In this article we study the justification of the formally derived homogenized model for a viscoplastic body. To this end we show that from the solution of the homogenized model an asymptotic solution of the microscopic model can be derived. We prove that the difference of the exact solution and the asymptotic solution tends to zero if the lengthscale of the microstructure converges to zero. For technical reasons we are only able to prove this result locally in time and for smooth data.

The microscopic model consists of a quasistatic initial-boundary value problem. The formulation of this problem is based on the assumption that only small strains occur: Let $\Omega \subseteq \mathbb{R}^3$ denote the set of material points of the body, let $\mathcal{S}^3$ denote the set of symmetric 3×3–matrices, and let $u(x,t) \in \mathbb{R}^3$ be the unknown displacement of the material point x at time t. Furthermore, $T(x,t) \in \mathcal{S}^3$ is the unknown Cauchy stress tensor and $z(x,t) \in \mathbb{R}^N$ denotes the unknown vector of internal variables. The model equations of the microscopic problem are

$$-\operatorname{div}_x T(x,t) = b(x,t)\,, \tag{1}$$

$$T(x,t) = \mathcal{D}[\frac{x}{\eta}]\Big(\varepsilon\big(\nabla_x u(x,t)\big) - Bz(x,t)\Big)\,, \tag{2}$$

$$\frac{\partial}{\partial t} z(x,t) \in f\Big(\frac{x}{\eta}\,,\, \varepsilon\big(\nabla_x u(x,t)\big)\,,\, z(x,t)\Big)\,, \tag{3}$$

$$z(x,0) = z^{(0)}(x)\,, \tag{4}$$

which must hold for $x \in \Omega$ and $t \in [0,\infty)$. For simplicity we only consider the Dirichlet boundary condition

$$u(x,t) = \gamma_D(x,t)\,, \tag{5}$$

which must be satisfied for $(x,t) \in \partial\Omega \times [0,\infty)$. Here $\nabla_x u(x,t)$ denotes the 3×3–matrix of first order derivatives of u, the deformation gradient, $\big(\nabla_x u(x,t)\big)^T$ denotes the transposed matrix,

$$\varepsilon\big(\nabla_x u(x,t)\big) = \frac{1}{2}\Big(\nabla_x u(x,t) + \big(\nabla_x u(x,t)\big)^T\Big) \in \mathcal{S}^3$$

is the strain tensor, and $B : \mathbb{R}^N \to \mathcal{S}^3$ is a linear mapping, which assigns to the vector $z(x,t)$ the plastic strain tensor $\varepsilon_p(x,t) = Bz(x,t)$. For every $y \in \mathbb{R}^3$ we denote by $\mathcal{D}[y] : \mathcal{S}^3 \to \mathcal{S}^3$ a linear, symmetric, positive definite mapping, the elasticity tensor. It is assumed that the mapping $y \mapsto \mathcal{D}[y]$ is periodic with a rectangular periodicity cell $Y \subseteq \mathbb{R}^3$. The number $\eta > 0$ is the scaling parameter of the microstructure.

The given data are the volume force $b : \Omega \times [0,\infty) \to \mathbb{R}^3$, the boundary displacement $\gamma_D : \partial\Omega \times [0,\infty) \to \mathbb{R}^3$ and the initial values $z^{(0)} : \Omega \to \mathbb{R}^N$ of the vector of internal variables. $f : \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ in (3) is a given function. The equation (2) and the differential inclusion (3) together determine the dependence of the stress $T(x,t)$ on the strain history $s \mapsto \varepsilon\big(\nabla_x u(x,s)\big)$. They are the constitutive relations which model the inelastic behavior of the body. The choice of f is restricted by thermodynamical and mathematical requirements. In this article we assume that (3) belongs to the class of constitutive relations of monotone type with positive definite free energy. For this class the function f can be written in the form

$$f(y,\varepsilon,z) = g\big(y, -\rho\nabla_z\psi(y,\varepsilon,z)\big)\,, \quad (y,\varepsilon,z) \in \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}^N\,, \tag{6}$$

with the constant mass density $\rho > 0$, with a suitable free energy ψ, which is a positive definite quadratic form

$$\rho\psi(y,\varepsilon,z) = \frac{1}{2}[\mathcal{D}[y](\varepsilon - Bz)]\cdot(\varepsilon - Bz) + \frac{1}{2}(Lz)\cdot z\,, \tag{7}$$

with respect to the variables (ε, z), and with a suitable function $g : \mathbb{R}^3\times\mathbb{R}^N \to 2^{\mathbb{R}^N}$, which satisfies $0 \in g(y,0)$ and for which the function $z \mapsto g(y,z) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is monotone for all $y \in \mathbb{R}^3$. In equation (7) we denote the scalar product of two matrices $\sigma, \tau \in \mathcal{S}^3$ by

$$\sigma\cdot\tau = \sum_{i,j=1}^{3} \sigma_{ij}\tau_{ij}\,,$$

and L denotes a symmetric $N\times N$–matrix. It is easily seen that this matrix is positive definite if and only if the quadratic form ψ is positive definite. We assume also that the function $y \mapsto g(y,z)$ is periodic with periodicity cell Y for all $z \in \mathbb{R}^N$.

We employ (2) and obtain by a simple computation $-\rho\nabla_z\psi(y,\varepsilon,z) = B^T\mathcal{D}[y](\varepsilon - Bz) - Lz = B^T T - Lz$, where $B^T : \mathcal{S}^3 \to \mathbb{R}^N$ is the mapping adjoint to B. Using this equation we can write the microscopic problem in the form

$$-\mathrm{div}_x\, T(x,t) = b(x,t)\,, \tag{8}$$

$$T(x,t) = \mathcal{D}[\frac{x}{\eta}]\Big(\varepsilon\big(\nabla_x u(x,t)\big) - Bz(x,t)\Big)\,, \tag{9}$$

$$z_t(x,t) \in g\big(\frac{x}{\eta}\,,\, B^T T(x,t) - Lz(x,t)\big)\,, \tag{10}$$

$$z(x,0) = z^{(0)}(x)\,, \tag{11}$$

$$u(x,t) = \gamma_D(x,t), \quad (x,t) \in \partial\Omega\times[0,\infty)\,. \tag{12}$$

The class of constitutive equations of monotone type, which was introduced in the book [1], extends the class of generalized standard materials introduced by Halphen and Nguyen Quoc Son [23]. The class of generalized standard materials includes the classical constitutive equations like the Prandtl-Reuss and Norton-Hoff laws, but it does not contain most constitutive equations developed in engineering in the last decades. To treat these constitutive equations it is therefore necessary to seek larger classes, for which existence theorems can be proved. One such class is the class of constitutive equations of monotone type. It has been shown in [1] that most constitutive equations lie outside even this larger class, and a further enlargement of this class by transformation methods has been discussed. Yet, a general mathematical existence theory for most of the constitutive equations used in practice is not available up to now. It is nevertheless an important mathematical goal to understand initial-boundary value problems to constitutive equations of monotone type as a basis for the investigation of still more general equations. For a discussion of these questions, for the existence theory and for an introduction to the mathematical literature in viscoplasticity we refer to [1,3,4,15–18,21] and to [19].

The class of constitutive equations of monotone type requires the free energy to be positive semi-definite. In this article we only consider the subclass of constitutive equations of monotone type with positive definite free energy because of the strong existence theorems available for this subclass, cf. [4,21], which allow to derive regularity and stability estimates. Constitutive equations with linear hardening are of this type.

The investigations in this article thus only form the beginning of the study of homogenization in viscoplasticity; in particular, homogenization of models to constitutive equations with positive semi-definite free energy remains to be considered in the future.

We are interested in the solution of (8) – (12) to quasiperiodic initial data of the form

$$z^{(0)}(x) = z_\eta^{(0)}(x) = z_0^{(0)}(x, \frac{x}{\eta}) \,, \tag{13}$$

with a given function $z_0^{(0)} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^N$, such that for all $x \in \Omega$ the function $y \mapsto z_0^{(0)}(x,y)$ is periodic with periodicity cell Y. We denote a solution of the microscopic problem to such initial data by (u_η, T_η, z_η). Since for small values of η the function $x \mapsto z_0^{(0)}(x, \frac{x}{\eta})$ is close to a periodic function with periodicity cell ηY, and since $x \mapsto \mathcal{D}[\frac{x}{\eta}]$ and $x \mapsto g(\frac{x}{\eta}, z)$ are periodic with this periodicity cell, one expects that also (u_η, T_η, z_η) will be close to a quasiperiodic function $(\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)$ of the form

$$\hat{u}_\eta(x,t) = u_0(x,t) + \eta u_1(x, \frac{x}{\eta}, t) \,, \tag{14}$$

$$\hat{T}_\eta(x,t) = T_0(x, \frac{x}{\eta}, t) \,, \tag{15}$$

$$\hat{z}_\eta(x,t) = z_0(x, \frac{x}{\eta}, t) \,, \tag{16}$$

where the function $(x,y,t) \mapsto (u_1, T_0, z_0)(x,y,t) : \Omega \times \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}^N$ is required to be periodic with respect to y and to have periodicity cell Y. In [2] it has been shown that if $(\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)$ is asymptotically equal to the solution (u_η, T_η, z_η) for $\eta \to 0$, then (u_0, u_1, T_0, z_0) and the overall stress T_∞ must satisfy the *homogenized initial-boundary value problem* formed by the equations

$$-\operatorname{div}_x T_\infty(x,t) = b(x,t)\,, \tag{17}$$

$$T_\infty(x,t) = \frac{1}{|Y|}\int_Y T_0(x,y,t)dy\,, \tag{18}$$

$$-\operatorname{div}_y T_0(x,y,t) = 0\,, \tag{19}$$

$$T_0(x,y,t) = \mathcal{D}[y]\Big(\varepsilon\big(\nabla_y u_1(x,y,t)\big) - Bz_0(x,y,t) + \varepsilon\big(\nabla_x u_0(x,t)\big)\Big), \tag{20}$$

$$\frac{\partial}{\partial t} z_0(x,y,t) \in g\big(y, B^T T_0(x,y,t) - Lz_0(x,y,t)\big)\,, \tag{21}$$

$$z_0(x,y,0) = z_0^{(0)}(x,y)\,, \tag{22}$$

which must hold for $(x,y,t) \in \Omega \times Y \times [0,\infty)$, and by the boundary condition

$$u_0(x,t) = \gamma_D(x,t)\,, \tag{23}$$

which must hold for $(x,t) \in \partial\Omega \times [0,\infty)$. The symbol $|Y|$ in (18) denotes the measure of Y.

Note that for x fixed the equations (19) – (22) together with the requirement that $y \mapsto (u_1, T_0)(x,y,t)$ must be periodic, which can be considered to be a boundary condition, define an initial-boundary value problem, the cell problem, in the domain $Y \times [0,\infty)$, the representative volume element. The cell problem is of the same form as the microscopic problem. u_1 is the microdisplacement, T_0 the microstress; the overall stress T_∞ is obtained via (18) by averaging of T_0 over the representative volume element, u_0 is the macrodisplacement. The term $\varepsilon(\nabla_x u_0(x,t))$ in (20) can be considered to be a homogeneous strain imposed on the representative volume element by the macrodisplacement. If the history $t \mapsto \varepsilon(\nabla_x u_0(x,t))$ of the macrostrain is known, then the function $(y,t) \mapsto (u_1, T_0, z_0)(x,y,t)$ and therefore also the function $t \mapsto T_\infty(x,t)$ can be determined from the cell problem. This dependence of T_∞ on $\varepsilon(\nabla_x u)$ defines a history functional

$$\big[t \mapsto \varepsilon\big(\nabla_x u_0(x,t)\big)\big] \mapsto \big[t \mapsto T_\infty(x,t) = \mathop{\mathcal{F}}_{s\le t}\big(\varepsilon\big(\nabla_x u_0(x,s)\big)\big)\big]\,,$$

the constitutive relation for the homogenized material modeled by the balance law (17), by this constitutive relation and by the boundary condition (23).

The goal of this article is to prove that indeed the solution (u_η, T_η, z_η) of (8) – (12) with initial data given by (13) is asymptotically equal to $(\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)$ for $\eta \to 0$, with u_0, u_1, T_0, z_0 determined from (17) – (23). However, since our estimates are not sharp enough to decide whether the term ηu_1 is present in (14), we actually prove that the solution is asymptotically equal to $(u_0, \hat{T}_\eta, \hat{z}_\eta)$. Moreover, because of technical reasons we are only able to prove that this result holds in a finite interval of time, and for smooth data and smooth functions $\mathcal{D}$ and g.

Rigorous mathematical investigation of homogenization has been carried out for many problems. Of particular importance for our problem are the investigations to the linear theory of elasticity. Other examples are the investigations to the nonlinear theory of elasticity, to the transport of neutrons, to problems of hydrodynamics and porous media, to linear viscoelasticity and electrodynamics, cf. for example [5–11,13,22,24–30,34,36]. However, the only rigorous mathematical investigations of homogenization in the theory of plastic or viscoplastic solids known to the author are [2,21]; this is in contrast to the importance of homogenization in solid mechanics, which is demonstrated by the many engineering publications devoted to the study of different aspects of homogenization in plasticity and viscoplasticity; we only mention [31–33,35,37–41].

In this article we study periodic microstructures. In [21] a homogenization result has been proved for a material with a random microstructure in one space dimension. This result is strictly one-dimensional and cannot be transfered to higher space dimensions, since in one space dimension the amplitude of the fast oscillations of the stress tends to zero when the lengthscale of the microstructure decreases to zero. This is not true in the higher dimensional case studied here.

Statement of the main results. To state the main results we need some notations and preparations.

If not stated otherwise we assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded open set with C^1-boundary $\partial\Omega$. The periodicity cell $Y \subseteq \mathbb{R}^3$ is a parallelepiped. Let ∂Y_i and ∂Y_{-i} be parallel faces for $i = 1, 2, 3$, and let $y_i \in \mathbb{R}^3$ be the vector such that $\partial Y_i = y_i + \partial Y_{-i}$. We make Y into a manifold $Y_{\rm per}$ without boundary by identifying the points x and $x + y_i$ for all $x \in \partial Y_{-i}$, and by choosing the appropriate topology and parametrization. It is clear that every function on $Y_{\rm per}$ can be identified with a function on $\mathbb{R}^3$, which is periodic and has periodicity cell Y. If a function belongs to $C^m(Y_{\rm per})$, then the corresponding periodic function belongs to $C^m(\mathbb{R}^3)$.

By T_e we denote a positive number (time of existence), and for $0 \le t \le T_e$ we set

$$\Omega_t = \Omega \times [0,t], \quad Y_{{\rm per},t} = Y_{\rm per} \times [0,t], \quad (\Omega \times Y_{\rm per})_t = \Omega \times Y_{\rm per} \times [0,t]\,.$$

If w is a function defined on $\Omega_t, Y_{{\rm per},t}$ or $(\Omega \times Y_{\rm per})_t$ and if $0 \le s \le t$, we denote the function $x \mapsto w(x,s)$ by $w(s)$. For a suitable subset Γ of $\mathbb{R}^n$ the scalar products on $L^2(\Gamma, \mathbb{R}^m)$ and on $L^2(\Gamma, \mathcal{S}^3)$ are denoted by

$$(\sigma, \tau)_\Gamma = \int_\Gamma \sigma(x) \cdot \tau(x)\, dx\,.$$

For $1 \le p \le \infty$ and for a Banach space V the Sobolev space of all functions which together with their weak derivatives up to order m belong to $L^p(\Gamma, V)$ is denoted by $H^p_m(\Gamma, V)$. The norm of $L^p(\Gamma, V)$ is $\|u\|_{\Gamma,p}$, and the norm of

$H^p_m(\Gamma, V)$ is $\|u\|_{\Gamma,p;m}$; for $p = 2$ we set $\|u\|_\Gamma = \|u\|_{\Gamma,2}$, $\|u\|_{\Gamma;m} = \|u\|_{\Gamma,2;m}$. Also, $\overset{\circ}{H}_m(\Gamma, V)$ denotes the closure of $C^\infty_0(\Gamma, V)$ in $H_m(\Gamma, V) = H^2_m(\Gamma, V)$.

We assume that the symmetric linear mapping $\mathcal{D}[y] : \mathcal{S}^3 \to \mathcal{S}^3$ is positive definite uniformly with respect to y, and that the mapping $y \mapsto \mathcal{D}[y]$ is bounded, periodic with periodicity cell Y and measurable. Measurability means that the coefficients in the tensorial representation of $\mathcal{D}[y]$ are measurable functions of y. These assumptions imply that a bounded, selfajoint, positive definite linear mapping $\sigma \mapsto \mathcal{D}\sigma : L^2(Y_{\text{per}}, \mathcal{S}^3) \to L^2(Y_{\text{per}}, \mathcal{S}^3)$ is defined by

$$(\mathcal{D}\sigma)(y) = D[y]\sigma(y)\,, \quad y \in Y_{\text{per}}\,.$$

With this mapping

$$[\sigma, \tau]_{\Omega\times Y} = (D\sigma, \tau)_{\Omega\times Y}$$

is a scalar product on $L^2(\Omega \times Y_{\text{per}}, \mathcal{S}^3)$. The norm associated to this scalar product is equivalent to the norm $\|\cdot\|_{\Omega\times Y}$.

If we fix t in the equations (17) – (20), (23) we obtain a linear boundary value problem, which slightly extends the classical *homogenized problem of linear elasticity theory*. Since we need solutions of this problem in the formulation of our results and in the proofs, we introduce and shortly discuss this problem here: To given functions $\hat{b} : \Omega \to \mathbb{R}^3$, $\hat{\gamma}_D : \partial\Omega \to \mathbb{R}^3$ and $\hat{\varepsilon}_p : \Omega \times Y \to \mathcal{S}^3$ we seek a solution $(u_0, u_1, T_\infty, T_0)$ with $(u_0, T_\infty) : \Omega \to \mathbb{R}^3 \times \mathcal{S}^3$ and $(u_1, T_0) : \Omega \times Y_{\text{per}} \to \mathbb{R}^3 \times \mathcal{S}^3$ of the equations

$$-\operatorname{div}_x T_\infty(x) = \hat{b}(x)\,, \tag{24}$$

$$T_\infty(x) = \frac{1}{|Y|}\int_Y T_0(x, y)\, dy\,, \tag{25}$$

$$-\operatorname{div}_y T_0(x, y) = 0\,, \tag{26}$$

$$T_0(x, y) = \mathcal{D}[y]\Big(\varepsilon\big(\nabla_x u_0(x)\big) + \varepsilon\big(\nabla_y u_1(x, y)\big) - \hat{\varepsilon}_p(x, y)\Big)\,, \tag{27}$$

$$u_0(x) = \hat{\gamma}_D(x)\,, \quad x \in \partial\Omega\,. \tag{28}$$

To define a weak solution of this problem assume that $\hat{b} \in L^2(\Omega, \mathbb{R}^3)$, $\hat{\varepsilon}_p \in L^2(\Omega \times Y, \mathcal{S}^3)$ and $\hat{\gamma}_D \in H_1(\Omega, \mathbb{R}^3)$. We combine (24) and (25), multiply the resulting equation by $v_0 \in \overset{\circ}{H}_1(\Omega, \mathbb{R}^3)$ and integrate by parts. If we identify v_0 with the function $(x, y) \mapsto v_0(x)$, the resulting equation can be written as

$$\big(T_0, \varepsilon(\nabla_x v_0)\big)_{\Omega\times Y} = (\hat{b}, v_0)_{\Omega\times Y}\,, \tag{29}$$

where we also used that $T_0(x, y)$ is a symmetric matrix, hence $T_0 \cdot \nabla_x v_0 = T_0 \cdot \varepsilon(\nabla_x v_0)$. Furthermore, we multiply (26) by $v_1 \in L^2\big(\Omega, H_1(Y_{\text{per}}, \mathbb{R}^3)\big)$, integrate by parts and add the resulting equation to (29) to obtain

$$\big(T_0, \varepsilon(\nabla_x v_0) + \varepsilon(\nabla_y v_1)\big)_{\Omega\times Y} = (\hat{b}, v_0)_{\Omega\times Y}\,.$$

Insertion of (27) yields

$$[\varepsilon(\nabla_x u_0 + \nabla_y u_1) - \hat{\varepsilon}_p\,,\, \varepsilon(\nabla_x v_0 + \nabla_y v_1)]_{\Omega\times Y} = (\hat{b}, v_0)_{\Omega\times Y}\,. \tag{30}$$

A function

$$(u_0, u_1, T_\infty, T_0) \\ \in H_1(\Omega,\mathbb{R}^3) \times L^2\big(\Omega, H_1(Y_{\text{per}},\mathbb{R}^3)\big) \times L^2(\Omega,\mathcal{S}^3) \times L^2(\Omega\times Y_{\text{per}},\mathcal{S}^3)$$

is called weak solution of (24) – (28), if (25), (27) hold, if u_0 can be represented in the form $u_0 = \hat{\gamma}_D + w_0$ with $w_0 \in \overset{\circ}{H}_1(\Omega,\mathbb{R}^3)$, and if (30) is satisfied for all $(v_0, v_1) \in \overset{\circ}{H}_1(\Omega,\mathbb{R}^3) \times L^2\big(\Omega, H_1(Y_{\text{per}},\mathbb{R}^3)\big)$.

The following existence result is well known:

Lemma 1 *Let $\hat{b} \in L^2(\Omega,\mathbb{R}^3)$, $\hat{\gamma}_D \in H_1(\Omega,\mathbb{R}^3)$ and $\hat{\varepsilon}_p \in L^2(\Omega\times Y_{\text{per}},\mathcal{S}^3)$. Then there is a unique weak solution $(u_0, u_1, T_\infty, T_0)$ of the Dirichlet problem* (24) – (28) *satisfying*

$$\int_Y u_1(x,y)\,dy = 0$$

for all $x \in \Omega$. Moreover, there is a constant C such that for $\hat{b} = \hat{\gamma}_D = 0$ the solution satisfies

$$\|u_0\|_{1,\Omega} + \Big(\int_\Omega \|u_1(x,\cdot)\|^2_{1,Y}\,dx\Big)^{1/2} \le C\|\hat{\varepsilon}_p\|_{\Omega\times Y}\,. \tag{31}$$

Proofs can be found for example in [7,30]. [6, pp. 1494] contains an existence proof for the corresponding scalar boundary value problem; the formulation of the homogenized boundary value problem given there is similar in spirit to (24) – (28), and the proof can be generalized to (24) – (28).

Before we can state our existence theorem for the homogenized problem of viscoplasticity, we finally need some assumptions and definitions for the function g: We assume that the mapping $y \mapsto g(y,z) : \mathbb{R}^3 \to 2^{\mathbb{R}^N}$ is periodic with periodicity cell Y for all $z \in \mathbb{R}^N$. As usual, $z \mapsto g(y,z)$ is said to be monotone if

$$(\zeta - \hat{\zeta}) \cdot (z - \hat{z}) \ge 0$$

for all $z, \hat{z} \in \mathbb{R}^N$ and all $\zeta \in g(y,z)$, $\hat{\zeta} \in g(y,\hat{z})$. This function is said to be maximal monotone if it does not have a proper monotone extension. It is well known that if $z \mapsto g(y,z)$ is maximal monotone, then the mapping $z \mapsto z + \lambda g(y,z)$ has a single valued inverse $j_\lambda[y] : \mathbb{R}^N \to \mathbb{R}^N$ for all $\lambda > 0$.

Theorem 2 (Existence and uniqueness of solutions for the homogenized problem of viscoplasticity) *Assume that the $N\times N$–matrix L in* (21) *is positive definite and that the mapping $g : \mathbb{R}^3 \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfies the following three conditions:*

(i) $0 \in g(0)$.

(ii) $z \mapsto g(y,z)$ *is maximal monotone for all* $y \in \mathbb{R}^3$.

(iii) $y \mapsto j_\lambda[y](z) : \mathbb{R}^3 \to \mathbb{R}^N$ *is measurable for all* $\lambda > 0$ *and all* $z \in \mathbb{R}^N$.

Suppose that $b \in H_2^1\big(0,T_e; L^2(\Omega,\mathbb{R}^3)\big)$ *and* $\gamma_D \in H_2^1\big(0,T_e; H_1(\Omega,\mathbb{R}^3)\big)$. *Finally, assume that* $z_0^{(0)} \in L^2(\Omega \times Y_{\rm per}, \mathbb{R}^N)$ *and that there is* $\zeta \in L^2(\Omega \times Y_{\rm per}, \mathbb{R}^N)$ *such that*

$$\zeta(x,y) \in g\big(y, B^T T^{(0)}(x,y) - L z_0^{(0)}(x,y)\big)\,, \quad a.e.\ in\ \Omega \times Y_{\rm per}\,, \tag{32}$$

where $(u_0^{(0)}, u_1^{(0)}, T_\infty^{(0)}, T_0^{(0)})$ *is a weak solution of the linear problem* (24) – (28) *to the data* $\hat{b} = b(0)$, $\hat{\varepsilon}_p = B z_0^{(0)}$, $\hat{\gamma}_D = \gamma_D(0)$.

Then to every $T_e > 0$ *there are solutions*

$$\begin{aligned}&(u_0,u_1,T_\infty,T_0,z_0) \in L^2\big(0,T_e; H_1(\Omega,\mathbb{R}^3)\big) \times L^2\big(\Omega_{T_e}, H_1(Y_{\rm per},\mathbb{R}^3)\big)\\ &\quad \times L^2(\Omega_{T_e},\mathcal{S}^3) \times L^2\big((\Omega \times Y_{\rm per})_{T_e},\mathcal{S}^3\big) \times C\big(0,T_e; L^2(\Omega \times Y_{\rm per},\mathbb{R}^N)\big)\end{aligned}$$

of the homogenized initial-boundary value problem (17) – (23). *If a solution is given by* $(u_0,u_1,T_\infty,T_0,z_0)$, *then all solutions are obtained in the form* $(u_0, u_1 + a, T_\infty, T_0, z_0)$ *with* $a \in L^2(\Omega_{T_e},\mathbb{R}^3)$.

We are able to show that the function $(u_0, \hat{T}_\eta, \hat{z}_\eta)$ is asymptotic to the solution of the microscopic problem and thus justify the homogenized problem only when the solution of this homogenized problem is of higher regularity as given in this theorem. When g is a single valued function, i.e. $g : Y_{\rm per} \times \mathbb{R}^N \to \mathbb{R}^N$, and when g, $\mathcal{D}$, the domain Ω and the data are regular, one can show that up to a certain time the solution is regular. This result is formulated in the following

Theorem 3 (Higher regularity locally) *Let* $n \geq 1$ *be an integer, suppose that the assumptions of the preceding theorem are satisfied and that* $\Omega \in C^n$, $g \in C^n(Y_{\rm per} \times \mathbb{R}^N, \mathbb{R}^N)$, $b \in C^n(\overline{\Omega} \times [0,\infty), \mathbb{R}^3)$, $\gamma_D \in C^n(\partial\Omega \times [0,\infty), \mathbb{R}^3)$, $z_0^{(0)} \in C^n(\overline{\Omega} \times Y_{\rm per}, \mathbb{R}^N)$. *Furthermore, suppose that* $y \mapsto \mathcal{D}[y]$ *is* n*–times continuously differentiable on* $Y_{\rm per}$. *Then there exists a time* $T_r > 0$ *such that the solution of the homogenized problem satisfies* $(u_0, T_\infty) \in C^n(\overline{\Omega}_{T_r}, \mathbb{R}^3 \times \mathcal{S}^3)$ *and* $(u_1, T_0, z_0) \in C^n(\overline{\Omega} \times Y_{\rm per})_{T_r}, \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}^N)$.

Remark. $y \mapsto \mathcal{D}[y]$ is n–times continuously differentiable on $Y_{\rm per}$ if the coefficients of the tensorial representation of $\mathcal{D}[y]$ belong to $C^n(Y_{\rm per})$.

Theorem 4 (Justification of the homogenized problem)
(i) *Suppose that* Ω, L, $\mathcal{D}$, g, b, γ_D *and* $z_0^{(0)}$ *satisfy the assumptions of Theorem 2. Let* $T_e > 0$ *and* $\eta > 0$. *Then there is a unique solution*

$$(u_\eta, T_\eta, z_\eta) \in L^2\big(0,T_e; H_1(\Omega,\mathbb{R}^3)\big) \times L^2(\Omega_{T_e},\mathcal{S}^3) \times C\big(0,T_e; L^2(\Omega,\mathbb{R}^N)\big)$$

of the microscopic problem (8) – (12) *to the inital data* $z^{(0)}(x) = z_0^{(0)}(x, \frac{x}{\eta})$.

(ii) *Suppose that additionally the assumptions of Theorem 3 are satisfied with* $n = 3$. *Let* $(u_0, u_1, T_\infty, T_0, z_0)$ *be a solution of the homogenized initial-boundary value problem* (17) – (23), *and let* T_r *be the positive time given in Theorem 3 such that this solution is* 3*–times continuously differentiable for* $t \in [0, T_r]$. *Let the functions* $\hat{T}_\eta$ *and* $\hat{z}_\eta$ *be defined by the equations* (15) *and* (16). *Then* $(u_0, \hat{T}_\eta, \hat{z}_\eta)$ *is asymptotic to the solution* (u_η, T_η, z_η) *of the microscopic problem in the time interval* $[0, T_r]$, *i.e. for all* $0 \le t \le T_r$

$$\lim_{\eta \to 0} \left[\|u_\eta(t) - u_0(t)\|_\Omega + \|T_\eta(t) - \hat{T}_\eta(t)\|_\Omega + \|z_\eta(t) - \hat{z}_\eta(t)\|_\Omega \right] = 0 \,. \quad (33)$$

The proof of Theorems 2 is given in Section 2, whereas the proof of Theorem 3 is only sketched there. Theorem 4 is proved in Section 3. In the proof of Theorem 4 we apply a well known homogenization result for the linear boundary value problem of elasticity theory derived by the energy method of Tartar, cf. the proof of Lemma 14 in Section 3. At one place in this proof we need that $\partial_t \mathrm{div}_x\, T_0(x, y, t)\Big|_{y = \frac{x}{\eta}}$ and $\partial_t \mathrm{rot}_x \nabla_y\, u_1(x, y, t)\Big|_{y = \frac{x}{\eta}}$ belong to compact subsets of H_{-1}^{loc}, where T_0 and u_1 are the functions in the solution of the homogenized problem (17) – (23). The regularity of the global solution obtained from Theorem 2 is slightly too small to prove this. There are other subtleties in the proof of Theorem 4, but this is the main point why we need higher regularity and why we can only prove the convergence result (33) locally in time. We surmise, however, that a similar inequality is valid for the global solution obtaind from Theorem 2 without the regularity assumptions of Theorem 3.

2 The homogenized initial boundary value problem

In this section we prove Theorem 2 and sketch the proof of Theorem 3. The proof of Theorem 2 is based on the reduction of the homogenized initial-boundary value problem (17) – (23) to an evolution equation in the Hilbert space L^2 with a maximal monotone evolution operator. Existence of solutions of the initial-boundary value problem follows from the standard existence theorems for such evolution equations. This proof follows in the essential details the proof of existence for the microscopic initial-boundary value problem in [4], and we thus refer several times to that proof.

We start with a definition based on Lemma 1.

Definition 5 *Let the linear operator* $P : L^2(\Omega \times Y_{\mathrm{per}}, \mathcal{S}^3) \to L^2(\Omega \times Y_{\mathrm{per}}, \mathcal{S}^3)$ *be defined by*

$$P\hat{\varepsilon}_p = \varepsilon(\nabla_x u_0 + \nabla_y u_1)$$

for every $\hat{\varepsilon}_p \in L^2(\Omega \times Y_{\mathrm{per}}, \mathcal{S}^3)$, *where* $(u_0, u_1, T_\infty, T_0)$ *is the unique weak solution of the Dirichlet boundary value problem* (24) – (28) *to* $\hat{b} = \hat{\gamma}_D = 0$

given by Lemma 1. Furthermore, with the identity I we define the linear operator $Q = I - P$.

Lemma 6 (i) *The operators P and Q are bounded projection operators orthogonal with respect to the scalar product $[\sigma,\tau]_{\Omega\times Y}$ on $L^2(\Omega\times Y_{\mathrm{per}},\mathcal{S}^3)$.*
(ii) *The operator $B^T\mathcal{D}QB : L^2(\Omega\times Y_{\mathrm{per}},\mathbb{R}^N) \to L^2(\Omega\times Y_{\mathrm{per}},\mathbb{R}^N)$ is selfajoint and non-negative with respect to the scalar product $(z,\hat{z})_{\Omega\times Y}$.*

Proof: (i) The boundedness of P follows from (31). To see that P is a projection, assume that $\hat{\varepsilon}_p$ belongs to the range of P, hence $\hat{\varepsilon}_p = \varepsilon(\nabla_x w_0 + \nabla_y w_1)$ for a suitable pair (w_0, w_1). Thus, by definition of P we have $P\varepsilon(\nabla_x w_0 + \nabla_y w_1) = P\hat{\varepsilon}_p = \varepsilon(\nabla_x u_0 + \nabla_y u_1)$, where $(u_0,u_1) \in \overset{\circ}{H}_1(\Omega)\times L^2\big(\Omega, H_1(Y_{\mathrm{per}},\mathbb{R}^3)\big)$ is the unique function with $\int_Y u_1(x,y)dy = 0$ satisfying (30) for $\hat{b} = 0$. Clearly, if we insert (w_0,w_1) for (u_0,u_1) then (30) is satisfied, hence $(u_0,u_1) = (w_0,w_1)$ and $P\hat{\varepsilon}_p = \varepsilon(\nabla_x w_0 + \nabla_y w_1) = \hat{\varepsilon}_p$, which shows that P is a projection. To prove that P is orthogonal, note that for all $\tau \in L^2(\Omega\times Y_{\mathrm{per}},\mathcal{S}^3)$ the function $P\tau$ is of the form $\varepsilon(\nabla_x v_0 + \nabla_y v_1)$. Therefore we can plug $P\tau$ into the second argument of the scalar product in (30) and obtain by definition of P for all $\sigma \in L^2(\Omega\times Y_{\mathrm{per}},\mathcal{S}^3)$

$$[P\sigma - \sigma, P\tau]_{\Omega\times Y} = 0\,.$$

Interchanging the roles of σ and τ yields

$$[P\tau - \tau, P\sigma]_{\Omega\times Y} = 0\,.$$

From these two equations we conclude for all $\sigma,\tau \in L^2(\Omega\times Y_{\mathrm{per}},\mathcal{S}^3)$

$$[\tau, P\sigma]_{\Omega\times Y} = [P\tau, P\sigma]_{\Omega\times Y} = [\sigma, P\tau]_{\Omega\times Y} = [P\tau,\sigma]_{\Omega\times Y}\,.$$

This yields $P^* = P$, whence P is selfajoint. Therefore P is an orthogonal projection, which clearly implies that also $Q = I - P$ is an orthogonal projection.

(ii) For $z,\hat{z} \in L^2(\Omega\times Y_{\mathrm{per}},\mathbb{R}^n)$ we have

$$\begin{aligned}
&(B^T\mathcal{D}QBz,\hat{z})_{\Omega\times Y} = (\mathcal{D}QBz, B\hat{z})_{\Omega\times Y} = [QBz, B\hat{z}]_{\Omega\times Y}\\
&= [QBz, QB\hat{z}]_{\Omega\times Y} = [Bz, QB\hat{z}]_{\Omega\times Y} = (\mathcal{D}Bz, QB\hat{z})_{\Omega\times Y}\\
&= (Bz, \mathcal{D}QB\hat{z})_{\Omega\times Y} = (z, B^T\mathcal{D}QB\hat{z})_{\Omega\times Y}\,,
\end{aligned}$$

which implies that $B^T\mathcal{D}QB$ is selfajoint and non-negative. This completes the proof.

Since by assumption the symmetric $N\times N$–matrix L is positive definite, it follows from this lemma that the operator $L + B^T\mathcal{D}QB : L^2(\Omega\times Y_{\mathrm{per}},\mathbb{R}^N) \to L^2(\Omega\times Y_{\mathrm{per}},\mathbb{R}^N)$ is bounded, selfadjoint and positive definite. Therefore

$$\langle z,\hat{z}\rangle_{\Omega\times Y} = \big((L + B^T\mathcal{D}QB)^{-1}z,\hat{z}\big)_{\Omega\times Y}$$

defines a scalar product on $L^2(\Omega\times Y_{\rm per},\mathbb{R}^N)$. The associated norm $[\![z]\!]_{\Omega\times Y} = \langle z,z\rangle_{\Omega\times Y}^{1/2}$ is equivalent to the norm $\|z\|_{\Omega\times Y}$.

After these preparations we can reduce the initial-boundary value problem (17) – (23) to an evolution equation. To this end we note that (20) yields

$$B^T T_0 - Lz_0 = B^T\mathcal{D}\big(\varepsilon(\nabla_x u_0 + \nabla_y u_1) - Bz_0\big) - Lz_0 \,. \tag{34}$$

Assume that $(u_0,u_1,T_\infty,T_0,z_0)$ is a solution of the initial-boundary value problem (17) – (23). We fix t. If $z_0(t)$ is known, then (17) – (20), (23) is a boundary value problem for the components $u_0(t),u_1(t),T_\infty(t),T_0(t)$ of the solution, the homogenized problem from linear elasticity theory. Consequently, these functions are obtained in the form

$$\begin{aligned}\big(u_0(t),u_1(t),T_\infty(t),T_0(t)\big) = \big(\tilde u_0(t),\tilde u_1(t),\tilde T_\infty(t),\tilde T_0(t)\big)\\ + \big(v_0(t),v_1(t),\sigma_\infty(t),\sigma_0(t)\big)\,,\end{aligned}$$

with a solution $\big(v_0(t),v_1(t),\sigma_\infty(t),\sigma_0(t)\big)$ of the Dirichlet boundary value problem (24) – (28) to the data $\hat b = b(t)$, $\hat\gamma_D = \gamma_D(t)$, $\hat\varepsilon_p = 0$, and with a solution $\big(\tilde u_0(t),\tilde u_1(t),\tilde T_\infty(t),\tilde T_0(t)\big)$ of the boundary value problem (24) – (28) to the data $\hat b = \hat\gamma_D = 0$, $\hat\varepsilon_p = Bz_0(t)$. With the projector P from Definition 5 we thus obtain

$$\varepsilon((\nabla_x u_0 + \nabla_y u_1)(t)) - Bz_0(t) = (P-I)Bz_0(t) + \varepsilon((\nabla_x v_0 + \nabla_y v_1)(t))\,.$$

We insert this equation into (34) and obtain that (21) can be written in the form

$$\frac{\partial}{\partial t} z_0(t) \in G\Big(\big(B^T\mathcal{D}(P-I)B - L\big)z_0(t) + B^T\sigma_0(t)\Big), \tag{35}$$

with the mapping $G : L^2(\Omega\times Y_{\rm per},\mathbb{R}^N)\to 2^{L^2(\Omega\times Y_{\rm per},\mathbb{R}^N)}$ defined by

$$G(z) = \{\zeta\in L^2(\Omega\times Y_{\rm per},\mathbb{R}^n) \mid \zeta(x,y)\in g\big(y,z(x,y)\big) \text{ a.e.}\}\,.$$

Since σ_0 is computed from the data b and γ, and thus is known, (35) is an evolution equation for z_0. If we define the evolution operator $A(t)$ by

$$A(t)z_0 = -G\big(-(B^T\mathcal{D}QB + L)z_0(t) + B^T\sigma_0(t)\big)$$

and note that $Q = I - P$, this evolution equation can be written as

$$\frac{\partial}{\partial t} z_0(t) + A(t)\, z_0(t) \ni 0\,.$$

To transform this equation to an autonomous equation, insert

$$h = -(B^T\mathcal{D}QB + L)z_0 + B^T\sigma_0$$

into (35). This yields the evolution equation

$$\frac{\partial}{\partial t} h(t) \in -\big(B^T\mathcal{D}QB + L\big)G(h(t)) + B^T\frac{\partial}{\partial t}\sigma_0(t)\,. \tag{36}$$

Definition 7 *We define the operator* $\mathcal{C} : L^2(\Omega \times Y_{\text{per}}, \mathbb{R}^N) \to 2^{L^2(\Omega \times Y_{\text{per}}, \mathbb{R}^N)}$ *and the domain* $\Delta(\mathcal{C})$ *of* $\mathcal{C}$ *by*

$$\mathcal{C} = (L + B^T \mathcal{D} Q B) G, \qquad \Delta(\mathcal{C}) = \{h \in L^2(\Omega \times Y_{\text{per}}, \mathbb{R}^N) \mid \mathcal{C}h \neq \emptyset\}.$$

With this operator we finally write the evolution equation (36) on $L^2(\Omega \times Y_{\text{per}}, \mathbb{R}^N)$ in the form

$$h_t(t) + \mathcal{C}h(t) \ni B^T \sigma_{0t}.$$

The existence proof is now based on the following fundamental

Theorem 8 (i) *Let the mapping* $z \mapsto g(y,z) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ *be monotone for all* $y \in Y_{\text{per}}$. *Then the operator* $\mathcal{C}$ *is monotone with respect to the scalar product* $\langle z, \hat{z} \rangle_{\Omega \times Y}$.
(ii) *If* g *and* j_λ *satisfy the conditions of Theorem 2, then* $\mathcal{C}$ *is maximal monotone with respect to this scalar product.*

We omit the proof of this theorem, since it coincides essentially with the proof of Theorem 3.3 in [4].

Corollary 9 *Suppose that* g *and* j_λ *satisfy the conditions of Theorem 2. Also, let* $\sigma_0 \in H_2^1\big(0, T_e; L^2(\Omega \times Y_{\text{per}}, \mathcal{S}^3)\big)$ *and let* $h^{(0)} \in \Delta(\mathcal{C})$.
Then the evolution equation

$$h_t + \mathcal{C}h \ni B^T \sigma_{0t} \tag{37}$$

has a unique solution $h \in H_1^\infty\big(0, T_e; L^2(\Omega \times Y_{\text{per}}, \mathbb{R}^N)\big)$ *with*

$$h(0) = h^{(0)}. \tag{38}$$

This solution satisfies

$$\|h_t(t)\|_{\Omega \times Y} \leq |||\mathcal{C}h^{(0)} + B^T \sigma_{0t}(0)||| + \int_0^t \|B^T \sigma_{0tt}(s)\|_{\Omega \times Y}\, ds \qquad a.e.,$$

where

$$|||\mathcal{C}h^{(0)} + B^T \sigma_{0t}(t)||| = \inf\{\|\zeta\|_{\Omega \times Y} \mid \zeta \in \mathcal{C}h^{(0)} + B^T \sigma_{0t}(0)\}.$$

Proof: Since $\mathcal{C}$ is maximal monotone and since for $\sigma_0 \in H_2^1(0, T_e; L^2\big(\Omega \times Y_{\text{per}}, \mathcal{S}^3)\big)$ the function $B^T \sigma_{0t}$ belongs to $H_1^1\big(0, T_e; L^2(\Omega \times Y_{\text{per}}, \mathcal{S}^3)\big)$, this theorem is an immediate consequence of [12, Theorem 2.2, p. 131].

The **proof of Theorem 2** follows from this corollary, since it can be easily shown that the reduction of the initial-boundary value problem (17) – (22) to the evolution equation (37) can be reversed and that a solution of the initial value problem (37), (38) yields a solution of the initial-boundary value problem (17) – (22). The assumptions for $z_0^{(0)}$ in Theorem 2 guarantee that the inital data $h^{(0)} = -\big(B^T\mathcal{D}QB + L\big)z_0^{(0)} + B^T\sigma_0(0)$ belong to the domain $\Delta(\mathcal{C})$. We omit the proof, since it is essentially the same as the proof of Theorem 1.3 in [4].

The **proof of Theorem 3** is based on the standard construction of local solutions to the evolution equation (35) in the Banach space $C^n(\overline{\Omega}\times Y_{\rm per}, \mathbb{R}^N)$ using contraction estimates. Since by assumption Ω belongs to the class C^n and $y \mapsto \mathcal{D}[y]$ is n–times continuously differentiable, it can be shown by the usual regularity theory for the boundary value problem (24) – (28) that the operator P from Definition 5 maps $C^n(\overline{\Omega} \times Y_{\rm per}, \mathcal{S}^3)$ boundedly into itself. Therefore $B^T\mathcal{D}(P-I)B - L$ maps $C^n(\overline{\Omega} \times Y_{\rm per}, \mathbb{R}^N)$ boundedly into itself, which together with the assumed regularity of g allows to prove local contraction estimates for the operator

$$z_0 \mapsto G\big((B^T\mathcal{D}(P-I)B - L)z_0 + B^T\sigma_0\big).$$

We omit the details of the proof.

3 Justification of the homogenized problem

Here we prove Theorem 4. In the proof we need a stability estimate for the microscopic problem (8) – (12), which is obtained using the framework of the proof of existence of solutions for this problem. This existence proof is given in [4,21]; it is similar to the proof of Theorem 2, as mentioned in the preceding section. To set up this framework, we first give the definitions and state the results from [4] needed in the proof of Theorem 4, which follows afterwards:

To begin with, consider the boundary value problem formed by the equations (8), (9), (12): To given functions $\hat{b} : \Omega \to \mathbb{R}^3$, $\hat{\gamma}_D : \partial\Omega \to \mathbb{R}^3$, $\hat{\varepsilon}_p : \Omega \to \mathcal{S}^3$ and to a given number $\eta > 0$ we seek solutions $(u, T) : \Omega \to \mathbb{R}^3 \times \mathcal{S}^3$ of the equations

$$-{\rm div}_x\, T(x) = \hat{b}(x)\,, \tag{39}$$

$$T(x) = \mathcal{D}[\frac{x}{\eta}]\big(\varepsilon\big(\nabla_x u(x)\big) - \hat{\varepsilon}_p(x)\big)\,, \tag{40}$$

$$u(x) = \hat{\gamma}_D(x), \quad x \in \partial\Omega\,. \tag{41}$$

This is the linear problem of elasticity theory. To define weak solutions let $\hat{b} \in H_{-1}(\Omega, \mathbb{R}^3)$, $\hat{\varepsilon}_p \in L^2(\Omega, \mathcal{S}^3)$ and $\hat{\gamma}_D \in H_1(\Omega, \mathbb{R}^3)$. A function $(u, T) \in H_1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)$ is called weak solution of (39) – (41), if (40) is satisfied, if the equation

$$\big(D(\frac{\cdot}{\eta})\big(\varepsilon(\nabla_x u) - \hat{\varepsilon}_p\big),\, \varepsilon(\nabla_x v)\big)_\Omega = (\hat{b}, v)_\Omega \tag{42}$$

holds for all $v \in \overset{\circ}{H}_1(\Omega, \mathbb{R}^3)$, and if u can be represented as $u = \hat{\gamma}_D + w$ with $w \in \overset{\circ}{H}_1(\Omega, \mathbb{R}^3)$.

Since $y \mapsto \mathcal{D}[y]$ is bounded and uniformly positive definite, it follows that

$$\big(\mathcal{D}(\frac{\cdot}{\eta})\sigma, \tau\big)_\Omega$$

is a scalar product on $L^2(\Omega, \mathcal{S}^3)$, for which constants c_1, c_2 exist such that

$$c_1\|\sigma\|_\Omega \le \big(\mathcal{D}(\frac{\cdot}{\eta})\sigma, \sigma\big)_\Omega^{1/2} \le c_2\|\sigma\|_\Omega$$

holds for all $\sigma \in L^2(\Omega, \mathcal{S}^3)$ and all $\eta > 0$. Using this fact, we obtain by the well known theory for the boundary value problem (39) – (41) that to $\hat{b} \in H_{-1}(\Omega, \mathbb{R}^3)$, $\hat{\varepsilon}_p \in L^2(\Omega, \mathcal{S}^3)$, $\hat{\gamma}_D \in H_1(\Omega, \mathbb{R}^3)$ and $\eta > 0$ there is a unique weak solution (u, T) satisfying

$$\|u\|_{1,\Omega} + \|T\|_\Omega \le C(\|\hat{b}\|_{\Omega;-1} + \|\hat{\varepsilon}_p\|_\Omega + \|\hat{\gamma}_D\|_{1,\Omega}), \tag{43}$$

with a constant C independent of η.

Definition 10 *To $\eta > 0$ let the linear operator $P_\eta : L^2(\Omega, \mathcal{S}^3) \to L^2(\Omega, \mathcal{S}^3)$ be defined by*

$$P_\eta \hat{\varepsilon}_p = \varepsilon(\nabla_x u)$$

for every $\hat{\varepsilon}_p \in L^2(\Omega, \mathcal{S}^3)$, where (u, T) is the unique weak solution of the Dirichlet problem (39) – (41) *to $\hat{\varepsilon}_p$ and to $\hat{b} = \hat{\gamma}_D = 0$. Also, we define $Q_\eta = I - P_\eta$.*

Lemma 11 *Let $\eta > 0$.*
(i) *The operators P_η and Q_η are projection operators bounded uniformly with respect to η and orthogonal with respect to the scalar product $\big(\mathcal{D}(\frac{\cdot}{\eta})\sigma, \tau\big)_\Omega$ on $L^2(\Omega, \mathcal{S}^3)$.*
(ii) *The operator $B^T\mathcal{D}(\frac{\cdot}{\eta})Q_\eta B : L^2(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$ is selfadjoint and non-negative with respect to the scalar product $(z, \hat{z})_\Omega$. Moreover, there is a constant $C > 0$ such that*

$$\|B^T\mathcal{D}(\frac{\cdot}{\eta})Q_\eta B z\|_\Omega \le C\|z\|_\Omega, \tag{44}$$

for all $\eta > 0$. Hence $B^T\mathcal{D}(\frac{\cdot}{\eta})Q_\eta B$ is bounded uniformly with respect to η.

The **proof** is essentially equal to the proof of Lemma 6 if we use the estimate (43) instead of (31).

Since L is positive definite, it follows from this lemma that the operator $L + B^T\mathcal{D}(\frac{\cdot}{\eta})Q_\eta B$ is uniformly positive definite. This implies that

$$\langle z, \hat{z}\rangle_{\Omega,\eta} = \Big(\big(L + B^T\mathcal{D}[\frac{\cdot}{\eta}]Q_\eta B\big)^{-1} z, \hat{z}\Big)_\Omega$$

defines a scalar product on $L^2(\Omega,\mathbb{R}^N)$. Furthermore, together with (44) we obtain that to the associated norm

$$|\!|\!| z |\!|\!|_{\Omega,\eta} = \langle z, z\rangle_{\Omega,\eta}^{1/2}$$

there are constants $C_1, C_2 > 0$ such that for all $\eta > 0$ and all $z \in L^2(\Omega,\mathbb{R}^N)$

$$C_1\|z\|_\Omega \le |\!|\!| z |\!|\!|_{\Omega,\eta} \le C_2\|z\|_\Omega\,. \tag{45}$$

Using the projection Q_η we define an evolution operator $\mathcal{C}_\eta : \Delta(\mathcal{C}_\eta) \subseteq L^2(\Omega,\mathbb{R}^N) \to 2^{L^2(\Omega,\mathbb{R}^N)}$ by

$$\begin{aligned}&\mathcal{C}_\eta h \\ &= \{(L + B^T\mathcal{D}(\frac{\cdot}{\eta})Q_\eta B)\zeta \mid \zeta \in L^2(\Omega,\mathbb{R}^N),\, \zeta(x) \in g(\frac{x}{\eta}, h(x)) \text{ a.e. in } \Omega\}\,.\end{aligned} \tag{46}$$

Theorem 12 *If g satisfies the conditions of Theorem 2, then the operator $\mathcal{C}_\eta$ is maximal monotone with respect to the scalar product $\langle z,\hat{z}\rangle_{\Omega,\eta}$.*

The **proof** is obtained by a slight modification of the proof of Theorem 3.3 in [4].

Corollary 13 *For all $F_i \in H_1^1(0,T_e;L^2(\Omega,\mathbb{R}^N))$ and $h_i^{(0)} \in \Delta(\mathcal{C}_\eta)$, $i = 1, 2$, the initial value problem*

$$\frac{\partial}{\partial t}h_i + \mathcal{C}_\eta h_i \ni F_i\,, \tag{47}$$

$$h_i(0) = h_i^{(0)}\,, \tag{48}$$

has unique weak solutions $h_i \in H_1^\infty(0,T_e;L^2(\Omega,\mathbb{R}^N))$. These solutions satisfy

$$|\!|\!| h_1(t) - h_2(t) |\!|\!|_{\Omega,\eta} \le |\!|\!| h_1^{(0)} - h_2^{(0)} |\!|\!|_{\Omega,\eta} + \int_0^t |\!|\!| F_1(s) - F_2(s) |\!|\!|_{\Omega,\eta}\, ds. \tag{49}$$

Proof: Cf. [14, Lemma 3.1 and Theorem 3.4, pp. 64, 65].

After these preparations we come to the
Proof of Theorem 4: We assume that the data b, γ_D and $z^{(0)}(x) = z_0^{(0)}(x,\frac{x}{\eta})$ have the properties required in Theorem 4.

Let $\big(v_\eta(t),\, \sigma_\eta(t)\big)$ be a solution of the boundary value problem (39) – (41) to the data

$$\hat{b} = b(t)\,, \quad \hat{\varepsilon}_p = 0\,, \quad \hat{\gamma}_D = \gamma_D(t)\,. \tag{50}$$

By the same procedure as in in the preceding section it is shown in [4] that if (u_η, T_η, z_η) is the solution of the microscopic problem (8) – (12), then the function

$$h_\eta = -\big(B^T\mathcal{D}(\frac{\cdot}{\eta})Q_\eta B + L\big)z_\eta + B^T\sigma_\eta \tag{51}$$

satisfies the initial value problem

$$h_{\eta t}(t) + \mathcal{C}_\eta h_\eta(t) = B^T \sigma_{\eta t}(t)\,, \tag{52}$$

$$h_\eta(0) = -\big(B^T \mathcal{D}(\frac{\cdot}{\eta}) Q_\eta B + L\big) z_\eta(0) + B^T \sigma_\eta(0). \tag{53}$$

The approximate solution $(u_0, \hat{T}_\eta, \hat{z}_\eta)$ constructed from $(u_0, u_1, T_\infty, T_0, z_0)$, the solution of the homogenized problem (17) – (23), can be reduced to the solution of the same initial value problem, however with different data. For, observing (17) – (23), we obtain by a simple computation that $(u_0, \hat{T}_\eta, \hat{z}_\eta)$ satisfies the equations

$$-\mathrm{div}_x\, \hat{T}_\eta(x,t) = -\mathrm{div}_x\, T_0(x,y,t)\Big|_{y=\frac{x}{\eta}}\,, \tag{54}$$

$$\hat{T}_\eta(x,t) = \mathcal{D}[\frac{x}{\eta}]\Big(\varepsilon(\nabla_x u_0(x,t)) - B\hat{z}_\eta(x,t) + \varepsilon(\nabla_y u_1(x,y,t))\Big|_{y=\frac{x}{\eta}}\Big)\,, \tag{55}$$

$$\frac{\partial}{\partial t}\hat{z}_\eta(x,t) = g\big(\frac{x}{\eta},\, B^T \hat{T}_\eta(x,t) - L\hat{z}_\eta(x,t)\big)\,, \tag{56}$$

$$\hat{z}_\eta(x,0) = z_0^{(0)}(x, \frac{x}{\eta})\,, \tag{57}$$

$$u_0(x,t) = \gamma_D(x,t)\,, \quad x \in \partial\Omega\,. \tag{58}$$

Since these equations have the same form as the equations of the microscopic problem, we can again employ the procedure from the last section and obtain that if $\big(\hat{v}_\eta(t),\, \hat{\sigma}_\eta(t)\big)$ is the solution of the linear boundary value problem (39) – (41) to the data

$$\hat{b}(x) = -\mathrm{div}_x\, T_0(x,y,t)\Big|_{y=\frac{x}{\eta}}\,, \tag{59}$$

$$\hat{\varepsilon}_p(x) = -\varepsilon\big(\nabla_y u_1(x,y,t)\big)\Big|_{y=\frac{x}{\eta}}\,, \tag{60}$$

$$\hat{\gamma}_D(x) = \gamma_D(x,t)\,, \tag{61}$$

then the function

$$\hat{h}_\eta = -\big(B^T \mathcal{D}(\frac{\cdot}{\eta}) Q_\eta B + L\big)\hat{z}_\eta + B^T \hat{\sigma}_\eta \tag{62}$$

satisfies the initial value problem

$$\hat{h}_{\eta t}(t) + \mathcal{C}_\eta \hat{h}_\eta(t) = B^T \hat{\sigma}_{\eta t}(t)\,, \tag{63}$$

$$\hat{h}_\eta(0) = -\big(B^T \mathcal{D}(\frac{\cdot}{\eta}) Q_\eta B + L\big)\hat{z}_\eta(0) + B^T \hat{\sigma}_\eta(0). \tag{64}$$

Thus, if we note that $\hat{z}_\eta(x,0) = z_\eta(x,0) = z_0^{(0)}(x,\frac{x}{\eta})$ and apply Corollary 13, it follows from (52), (53) together with (63), (64) that for all $0 \le t \le T_r$

$$
|\!|\!| h_\eta(t) - \hat{h}_\eta(t) |\!|\!|_{\Omega,\eta} \le |\!|\!| B^T(\sigma_\eta(0) - \hat{\sigma}_\eta(0)) |\!|\!|_{\Omega,\eta} + \int_0^t |\!|\!| B^T\big(\sigma_{\eta t}(s) - \hat{\sigma}_{\eta t}(s)\big) |\!|\!|_{\Omega,\eta}\, ds\,. \tag{65}
$$

We next use that $\big(B^T \mathcal{D}[\frac{\cdot}{\eta}]Q_\eta B + L\big)^{-1}$ is uniformly bounded with respect to η. Consequently (62) and (51) yield that there is a constant C_3 such that

$$
\|z_\eta(t) - \hat{z}_\eta(t)\|_\Omega \le C_3\big(\|h_\eta(t) - \hat{h}_\eta(t)\|_\Omega + \|\sigma_\eta(t) - \hat{\sigma}_\eta(t)\|_\Omega\big)\,, \tag{66}
$$

for all $0 \le t \le T_r$ and all $\eta > 0$. This estimate, (45) and (65) imply for $0 \le t \le T_r$

$$
\|z_\eta(t) - \hat{z}_\eta(t)\|_\Omega \le C_4\Big(\|\sigma_\eta(0) - \hat{\sigma}_\eta(0)\|_\Omega + \int_0^{T_r} \|\sigma_{\eta t}(s) - \hat{\sigma}_{\eta t}(s)\|_\Omega\, ds\Big)\,, \tag{67}
$$

with a constant C_4 independent of t and of η. Thus, we can estimate the difference $z_\eta - \hat{z}_\eta$ if estimates for the differences $\sigma_\eta(0) - \hat{\sigma}_\eta(0)$ and $\sigma_{\eta t} - \hat{\sigma}_{\eta t}$ can be obtained. Since the functions σ_η and $\hat{\sigma}_\eta$ both are solutions of the same elliptic boundary value problem, the problem of linear elasticity theory, we obtain such estimates from the well known homogenization theory for this boundary value problem. The estimates are stated in the following lemma, whose proof is postponed:

Lemma 14 *For all* $0 \le t \le T_r$

$$
\lim_{\eta\to 0} \|\partial_t^i\big(v_\eta(t) - \hat{v}_\eta(t)\big)\|_\Omega = 0\,, \quad i = 0,1\,, \tag{68}
$$

$$
\lim_{\eta\to 0} \|\partial_t^i\big(\sigma_\eta(t) - \hat{\sigma}_\eta(t)\big)\|_\Omega = 0\,, \quad i = 0,1\,. \tag{69}
$$

Moreover, there is a constant K *such that for all* $0 \le t \le T_r$ *and all* $\eta > 0$

$$
\|\sigma_{\eta t}(t) - \hat{\sigma}_{\eta t}(t)\|_\Omega \le K. \tag{70}
$$

From (69) we conclude that the term $\|\sigma_\eta(0) - \hat{\sigma}_\eta(0)\|_\Omega$ tends to zero for $\eta \to 0$, and we conclude that the integrand in (67) tends to zero for $\eta \to 0$, pointwise for every s. Since this integrand is uniformly bounded, by (70), Lebesgue's convergence theorem implies that the right hand side of (67) tends to zero for $\eta \to 0$, whence

$$
\lim_{\eta\to 0} \|z_\eta(t) - \hat{z}_\eta(t)\|_\Omega = 0\,, \tag{71}
$$

for all $0 \le t \le T_r$.

To obtain the estimate (33) we observe that the equations (8), (9), (12) form a boundary value problem for (u_η, T_η), and that the equations (54), (55), (58) form a boundary value problem for $(u_0, \hat{T}_\eta)$. The Definition 10 of P_η and the definitions of (v_η, σ_η) and $(\hat{v}_\eta, \hat{\sigma}_\eta)$ thus yield the decomposition

$$u_\eta = w_\eta + v_\eta\,, \;\; T_\eta = \mathcal{D}[\frac{\cdot}{\eta}](P_\eta - I)Bz_\eta + \sigma_\eta\,,$$

$$u_0 = \hat{w}_\eta + \hat{v}_\eta\,, \;\; \hat{T}_\eta = \mathcal{D}[\frac{\cdot}{\eta}](P_\eta - I)B\hat{z}_\eta + \hat{\sigma}_\eta\,,$$

where $w_\eta(t), \hat{w}_\eta(t) \in \overset{\circ}{H}_1(\Omega, \mathbb{R}^3)$ are the unique functions from Definition 10 which satisfy $\varepsilon(\nabla_x w_\eta(t)) = P_\eta B z_\eta(t)$ and $\varepsilon(\nabla_x \hat{w}_\eta(t)) = P_\eta B \hat{z}_\eta(t)$. We thus have

$$\varepsilon\big(\nabla_x(w_\eta - \hat{w}_\eta)\big) = P_\eta B(z - \hat{z}_\eta)\,, \tag{72}$$

$$T_\eta - \hat{T}_\eta = -\mathcal{D}[\frac{\cdot}{\eta}]Q_\eta B(z_\eta - \hat{z}_\eta) + (\sigma_\eta - \hat{\sigma}_\eta)\,, \tag{73}$$

$$u_\eta - u_0 = (w_\eta - \hat{w}_\eta) + (v_\eta - \hat{v}_\eta)\,. \tag{74}$$

From (69), (71), (73) and the uniform boundedness of $\mathcal{D}[\frac{\cdot}{\eta}]Q_\eta B$ we infer that

$$\lim_{\eta\to 0} \|T_\eta(t) - \hat{T}_\eta(t)\|_\Omega = 0\,.$$

Since $(w_\eta - \hat{w}_\eta)(t) \in \overset{\circ}{H}_1(\Omega, \mathbb{R}^3)$, we infer from the first Korn's inequality $\|(w_\eta - \hat{w}_\eta)(t)\|_{\Omega;1} \le c\|\varepsilon(\nabla_x(w_\eta - \hat{w}_\eta)(t))\|_\Omega$ and from (71), (72) that $\|w_\eta(t) - \hat{w}_\eta(t)\|_{\Omega;1} \to 0$ for $\eta \to 0$; from (74) and (68) we thus conclude

$$\lim_{\eta\to 0} \|u_\eta(t) - u_0(t)\|_\Omega = 0$$

for all $0 \le t \le T_r$. These two relations and (71) together yield (33).

To finish the proof of Theorem 4 it thus remains to verify Lemma 14. This lemma is a consequence of the following well known result from homogenization theory; similar results can be found in many places, cf. for example [7,30]:

Lemma 15 *Let the functions* $\tau \in L^2(\Omega, \mathcal{S}^3)$, $b \in L^2(\Omega, \mathbb{R}^3)$ *satisfy*

$$b - \operatorname{div} \tau = 0,$$

and let the families $\{\tau_\eta\}_{\eta>0}$ *and* $\{\kappa_\eta\}_{\eta>0}$ *with* $\tau_\eta \in L^2(\Omega, \mathcal{S}^3)$ *and* $\kappa_\eta \in L^2(\Omega, \mathbb{R}^{3\times 3})$ *have the following properties:*

(i) $\tau_\eta \rightharpoonup \tau$ *for* $\eta \to 0$*, weakly in* $L^2(\Omega, \mathcal{S}^3)$*,*

(ii) *The set* $\{\operatorname{div} \tau_\eta\}_{\eta>0}$ *is a subset of a compact subset of* $H^{\mathrm{loc}}_{-1}(\Omega, \mathbb{R}^3)$*,*

(iii) $\kappa_\eta \rightharpoonup 0$ *for* $\eta \to 0$*, weakly in* $L^2(\Omega, \mathbb{R}^{3\times 3})$*,*

(iv) The set $\{\mathrm{rot}\,\kappa_\eta\}_{\eta>0}$ *is a subset of a compact subset of* $H_{-1}^{\mathrm{loc}}(\Omega,\mathbb{R}^{3\times 3})$.

Let $(\overline{v}_\eta,\overline{\sigma}_\eta) \in \overset{\circ}{H}_1(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathcal{S}^3)$ *be a weak solution of the boundary value problem formed by the equations*

$$-\mathrm{div}\,\overline{\sigma}_\eta = b - \mathrm{div}\,\tau_\eta\,, \tag{75}$$

$$\overline{\sigma}_\eta = \mathcal{D}[\frac{\cdot}{\eta}]\big(\varepsilon(\nabla\,\overline{v}_\eta)+\varepsilon(\kappa_\eta)\big), \tag{76}$$

which must hold in Ω*, and by the boundary condition*

$$\overline{v}_\eta(x) = 0\,,\quad x\in\partial\Omega\,. \tag{77}$$

Then

$$\lim_{\eta\to 0}(\|\overline{v}_\eta\|_\Omega + \|\overline{\sigma}_\eta\|_\Omega) = 0. \tag{78}$$

For completeness we present the short **proof**, which is based on the energy method of Tartar:

We first observe that the symmetry of the matrix $\overline{\sigma}_\eta(x)$ and the equations (75) – (77) yield

$$\begin{aligned}\int_\Omega (\mathcal{D}[\frac{x}{\eta}]^{-1}\,\overline{\sigma}_\eta)\cdot\overline{\sigma}_\eta\,dx &= \int_\Omega \big(\varepsilon(\nabla\,\overline{v}_\eta)+\varepsilon(\kappa_\eta)\big)\cdot\overline{\sigma}_\eta\,dx \\ = \int_\Omega (\nabla\,\overline{v}_\eta+\kappa_\eta)\cdot\overline{\sigma}_\eta\,dx &= \int_\Omega \overline{v}_\eta\cdot b + \nabla\,\overline{v}_\eta\cdot\tau_\eta\,dx + \int_\Omega \kappa_\eta\cdot\overline{\sigma}_\eta\,dx\,.\end{aligned} \tag{79}$$

Condition (i) of the lemma implies that the set $\{\tau_\eta \mid \eta>0\}$ is bounded in $L^2(\Omega)$, hence the set of functions $\{b-\mathrm{div}\,\tau_\eta \mid \eta>0\}$ on the right hand side of (75) is bounded in $H_{-1}(\Omega)$. Moreover, condition (iii) implies that the set $\{\varepsilon(\kappa_\eta)\mid\kappa>0\}$ is bounded in $L^2(\Omega)$. Since the problem (75) – (77) coincides with the boundary value problem (39) – (41), we thus obtain from (43) that there is C with

$$\|\overline{\sigma}_\eta\|_\Omega + \|\overline{v}_\eta\|_{\Omega;1} \le C$$

for all $\eta>0$. Consequently, we can choose a sequence $\{\eta_k\}_{k=1}^\infty$ with $\eta_k\to 0$ such that $\{\overline{v}_k\}_{k=1}^\infty = \{\overline{v}_{\eta_k}\}_{k=1}^\infty$ converges strongly in $L^2(\Omega,\mathbb{R}^3)$ to a function $v\in\overset{\circ}{H}_1(\Omega,\mathbb{R}^3)$, and such that

$$\nabla\,\overline{v}_k \rightharpoonup \nabla v\,,\quad \overline{\sigma}_k \rightharpoonup \tilde{\sigma}\,, \tag{80}$$

weakly in $L^2(\Omega,\mathbb{R}^3)$, with a suitable function $\tilde{\sigma}$. Equation (75) and condition (ii) of the lemma imply that $\{\mathrm{div}\,\overline{\sigma}_k\}_{k=1}^\infty$ belongs to a compact subset of H_{-1}^{loc}. Furthermore, $\mathrm{rot}(\nabla\,\overline{v}_k)=0$. These properties, the properties (i) - (iv) of τ_k

and κ_k, and the rot-div-Lemma imply that we can pass to the limit on the right hand side of (79) and obtain with a constant $c > 0$

$$c\|\overline{\sigma}_k\|_\Omega^2 \le \int_\Omega (\mathcal{D}[\frac{x}{\eta_k}]^{-1}\,\overline{\sigma}_k)\cdot\overline{\sigma}_k\,dx \tag{81}$$
$$\to (v,b)_\Omega + (\nabla v,\tau)_\Omega + (0,\tilde{\sigma})_\Omega = (v, b-\operatorname{div}\tau)_\Omega = 0\,.$$

Observe next that (76), (81) and the property (iii) of the lemma together yield

$$\varepsilon(\nabla\,\overline{v}_k) \rightharpoonup 0 \qquad \text{for } k\to\infty\,,$$

weakly in $L^2(\Omega,\mathcal{S}^3)$. Since $\nabla\,\overline{v}_k \rightharpoonup \nabla v$, by (80), it follows that $\varepsilon(\nabla v) = 0$. Using that $v \in \overset{\circ}{H}_1(\Omega,\mathbb{R}^3)$ we conclude from Korn's first inequality that $v = 0$. Relation (78) follows from this result, from the fact that $\overline{v}_k$ converges to v strongly in $L^2(\Omega,\mathbb{R}^3)$, and from (81). The proof of Lemma 15 is complete.

Proof of Lemma 14: We fix t and set

$$\tau_\eta(x) = -T_0(x,\frac{x}{\eta},t)\,,\quad \kappa_\eta(x) = -\nabla_y u_1(x,y,t)\Big|_{y=\frac{x}{\eta}}\,,\quad \tau = -T_\infty(t)\,,$$

$$\overline{v}_\eta = v_\eta(t) - \hat{v}_\eta(t)\,,\quad \overline{\sigma}_\eta = \sigma_\eta(t) - \hat{\sigma}_\eta(t)\,,\quad b = b(t)\,,$$

and verify that under the assumptions of Theorem 4 these functions satisfy the hypotheses of Lemma 15.

Note first that (19) yields

$$\operatorname{div}\tau_\eta(x) = -\operatorname{div}_x T_0(x,y,t)\Big|_{y=\frac{x}{\eta}}\,.$$

Since by definition $\big(\hat{v}_\eta(t),\hat{\sigma}_\eta(t)\big)$ is a solution of the boundary value problem (39) – (41) to the data

$$\hat{b}(x) = -\operatorname{div}_x T_0(x,y,t)\Big|_{y=\frac{x}{\eta}} = \operatorname{div}\tau_\eta(x), \tag{82}$$

$$\hat{\varepsilon}_p(x) = -\varepsilon(\nabla_y u_1(x,y,t)\Big|_{y=\frac{x}{\eta}}) = \varepsilon(\kappa_\eta(x)), \tag{83}$$

$$\hat{\gamma}_D = \gamma_D(t), \tag{84}$$

cf. (59) – (61), and since $\big(v_\eta(t),\sigma_\eta(t)\big)$ is a solution of the same boundary value problem to the data $\hat{b} = b(t)$, $\hat{\varepsilon}_p = 0$, $\hat{\gamma}_D = \gamma_D(t)$, cf. (50), it follows that $(\overline{v}_\eta,\overline{\sigma}_\eta)$ is a solution of the boundary value problem (75) – (77). Moreover, (17) implies

$$b - \operatorname{div}\tau = b(t) + \operatorname{div}_x T_\infty(t) = 0.$$

It thus remains to verify the conditions (i) – (iv) of Lemma 15. The condition (i) is satisfied, since by assumption $T_0 \in C^3\big(\overline{(\Omega \times Y_{\text{per}})_{T_r}}, \mathcal{S}^3\big)$, from which it can be shown by a modification of the proof given in [20, pp. 21] that for $w \in L^2(\Omega, \mathcal{S}^3)$

$$\lim_{\eta\to 0}(\tau_\eta, w)_\Omega = -\lim_{\eta\to 0}\int_\Omega T_0(x, \frac{x}{\eta}, t)w(x)dx$$
$$= -\frac{1}{|Y|}\int_\Omega\int_Y T_0(x,y,t)dy\, w(x)dx = -(T_\infty, w)_\Omega = (\tau, w)_\Omega\,.$$

Clearly, this relation implies (i). Also, $\operatorname{div}_x T_0 \in C^2\big(\overline{(\Omega \times Y_{\text{per}})_{T_r}}, \mathbb{R}^3\big)$ yields

$$\|\operatorname{div}\tau_\eta\|_\Omega = \|\operatorname{div}_x T_0(\cdot, y, t)\Big|_{y=\frac{\cdot}{\eta}}\|_\Omega \le \|\operatorname{div}_x T_0(t)\|_{\Omega\times Y,\infty}|\Omega|^{1/2}\,,$$

from which we conclude that condition (ii) holds. To prove (iii) we note that $\nabla_y u_1 \in C^2\big(\overline{(\Omega \times Y_{\text{per}})_{T_r}}, \mathbb{R}^{3\times 3}\big)$ implies for $w \in L^2(\Omega, \mathbb{R}^{3\times3})$ that

$$\lim_{\eta\to 0}(\kappa_\eta, w)_\Omega = -\lim_{\eta\to 0}\int_\Omega \nabla_y\, u_1(x, \frac{x}{\eta}, t)w(x)dx$$
$$= -\frac{1}{|Y|}\int_\Omega\int_Y \nabla_y\, u_1(x,y,t)dy\, w(x)dx = 0,$$

hence $\kappa_\eta \rightharpoonup 0$, weakly in $L^2(\Omega, \mathbb{R}^{3\times3})$. Again this is shown by a modification of the proof in [20, pp. 21]. Finally, to verify (iv) we note that

$$\operatorname{rot}_x \kappa_\eta(x) = [\operatorname{rot}_x \nabla_y u_1(x,y,t)]_{y=\frac{x}{\eta}}\,.$$

Thus, $\operatorname{rot}_x(\nabla_y u_1) \in C^1\big(\overline{(\Omega \times Y_{\text{per}})_{T_r}}, \mathbb{R}^3\big)$ implies

$$\|\operatorname{rot}\kappa_\eta\|_\Omega = \|[\operatorname{rot}_x \nabla_y u_1(x,y,t)]_{y=\frac{x}{\eta}}\|_\Omega \le \|\operatorname{rot}_x\big(\nabla_y u_1(t)\big)\|_{\Omega\times Y,\infty}|\Omega|^{1/2}\,.$$

Condition (iv) is a consequence of this estimate.

Thus, we can apply Lemma 15 and obtain from (78)

$$\lim_{\eta\to 0}(\|v_\eta(t) - \hat{v}_\eta(t)\|_\Omega + \|\sigma_\eta(t) - \hat{\sigma}_\eta(t)\|_\Omega) = 0,$$

which yields (68) and (69) for $i = 0$. To obtain these relations for $i = 1$, we replace the functions $v_\eta, \hat{v}_\eta, \sigma_\eta, \hat{\sigma}_\eta, T_0, u_1, T_\infty, b$ by their time derivatives and argue in exactly the same way.

Finally, to prove (70) we note that $\big(\hat{v}_{\eta t}(t), \hat{\sigma}_{\eta t}(t)\big)$ and $\big(v_{\eta t}(t), \sigma_{\eta t}(t)\big)$, respectively, are solutions of the boundary value problem (39) – (41) to the data

$$\hat{b} = -\operatorname{div}_x T_{0t}(t)\Big|_{y=\frac{x}{\eta}}\,, \quad \hat{\varepsilon}_p = -\varepsilon\big(\nabla_y u_{1t}(t)\big)\Big|_{y=\frac{x}{\eta}}\,, \quad \hat{\gamma}_D = \gamma_{Dt}(t)\,,$$

cf. (59) – (61), and

$$\hat{b} = b_t(t)\,, \quad \hat{\varepsilon}_p = 0\,, \quad \hat{\gamma}_D = \gamma_{Dt}(t)\,,$$

cf. (50), respectively. Since by the regularity assumptions of Theorem 4 we have $\operatorname{div}_x T_{0t} \in C^1\big((\overline{\Omega} \times Y_{\mathrm{per}})_{T_r}\big)$, $\nabla_y u_{1t} \in C^1\big((\overline{\Omega} \times Y_{\mathrm{per}})_{T_r}\big)$, $b_t \in C^2(\overline{\Omega}_{T_r})$, $\gamma_{Dt} \in C^2(\partial\Omega_{T_r})$, we conclude from (43) that there is a constant K with

$$\|\sigma_{\eta t}(t) - \hat{\sigma}_{\eta t}(t)\|_\Omega \le \|\sigma_{\eta t}(t)\|_\Omega + \|\hat{\sigma}_{\eta t}(t)\|_\Omega \le K$$

for all $0 \le t \le T_r$ and all $\eta > 0$. This completes the proof of Lemma 14.

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