Asymptotic Expansions for Solutions of Linear Differential-Algebraic Equations

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ABSTRACT

Asymptotic expansions for solutions of linear differential-algebraic ordinary differential equation with variable matrix coefficients are considered. The solution is being sought in the form of a formal power series. The coefficients of this series satisfies linear infinite-dimensional system of the algebraic equations with triangular matrix of coefficients. Existence and uniqueness theorem is proved for such equations and initial manifolds are described. The Drazin inverse matrices are used to demonstrate the existence of asymptotic expansion.

1. Introduction

Let's consider a linear differential-algebraic equation with variable matrix coefficients in a C^m space

$$A(t) \cdot \dot{x}(t) + x(t) = f(t) \tag{1}$$

with an initial condition

$$\lim_{t \to 0} x(t) = x_0, \qquad t \in S, \tag{2}$$

where $S = \{t \in C : 0 < |t| \leq t_0 \wedge \alpha < \arg t < \beta, -\pi/2 \leq \alpha < \beta \leq \pi/2\}$ - sector of complex plane with a corner in zero (t_0-some positive constant), $A(t)$ is $(m \times m)-$matrix function of variable t, $f(t)$ is $(m \times 1)-$vector function of variable t. Vector function $x(t)$ is called the solution of initial problem (1)-(2), if it is holomorphic function in S and satisfied the equation (1) and the condition (2) for all $t \in S$. Let $A(t)$ and $f(t)$ have the following asymptotic expansions on S [1]

$$\begin{aligned} f(t) &\sim \sum_{r=0}^{\infty} f_r t^r, \qquad t \to 0, \ t \in S; \\ A(t) &\sim \sum_{r=0}^{\infty} A_r t^r, \qquad t \to 0, \ t \in S, \end{aligned}$$

and holomorphic in sector S. Let's find the solution in the such form

$$x(t) \sim \sum_{r=0}^{\infty} x_r t^r, \qquad t \to 0, \ t \in S,$$

where power series $\sum_{r=0}^{\infty} x_r t^r$ satisfies the equation formally. It means that this power series after inserting instead of $x(t)$ into the equation (1) leads to the linear infinite-dimensional system of the algebraic equations with triangular matrix of coefficients:

$$\begin{array}{rcl}
A_0x_1 + x_0 & = & f_0 \\
2\cdot A_0x_2 + (E + A_1)\,x_1 & = & f_1 \\
3\cdot A_0x_3 + (E + 2\cdot A_1)\,x_2 & = & f_2 - A_2x_1 \\
. & . & . \\
(n+1)A_0x_{n+1} + (E + n\cdot A_1)\,x_n & = & f_n - (n-1)A_2x_{n-1} - \ldots - \\
 & & -2\cdot A_{n-1}x_2 - A_nx_1, \quad n = 0,1,2,\ldots
\end{array} \tag{3}$$

with a unique solution. Solving of this infinite-dimensional system of the algebraic equations has some difficulties so as $\det A_0 = 0$ [1]. However this system could be solved and has a unique solution under some limitation on coefficients in asymptotic expansion of matrix $A(t)$. This task will be proved below (*Lemma*). Finally we will prove *Theorem* that power series $\sum_{r=0}^{\infty} x_r t^r$ is the asymptotic expansion for the unique solution $x(t)$ of differential-algebraic equation (1).

2. Solving the implicit system of algebraic equations

In this section we will prove that system (3) has unique solution under some limitation on coefficients in asymptotic expansion of matrix $A(t)$.

Definition. The *index* of an $(m \times m)-$matrix A of complex numbers, denoted by $ind\,A$ is the smallest integer $k \geq 0$ such that $rank\,(A^k) = rank\,(A^{k+1})$.

Lemma. Let the system (3) of linear equation is given in a C^m space. Here x_0-initial vector, vectors f_n are given, matrices A_n are also given and at least $\det A_0 = 0$. Let $ind\,A = 1$, and suppose that

$$rank\,P\cdot(E + n\cdot A_1) = rank\,P, \quad \forall n = 0,1,2,..., \tag{4}$$

where $P : C^m \to \ker A_0$ is a projector onto null-space of A_0. Then the implicit system has unique solution $\{x_n\}_0^\infty$ for each vector x_0 which satisfies the condition

$$(x_0 - f_0) \in Im\,A_0. \tag{5}$$

That system doesn't have solutions for other vectors x_0.

Proof: As $ind\,A = 1$ then $C^m = Im\,A_0 \dot{+} \ker A_0$ and projector P exists such that

$$P : C^m \to \ker A_0, \quad G : C^m \to Im\,A_0, \quad G\cdot P = 0, \quad G + P = E.$$

From supposing of lemma $rank\,A_0 = p$ we get that $rank\,G = p,\ rank\,P = m - p$. From Grassmann's formula and the condition (4) we obtain that

$$rank\begin{pmatrix} G \\ P\cdot(E + n\cdot A_1) \end{pmatrix} = m, \qquad \forall n = 0,1,2,\ldots \tag{6}$$

Matrix $\theta = A_0 + P$ has the inverse one. So as from equation $\theta\cdot x = 0$ follow $A_0\cdot x = 0$ and $P\cdot x = 0$, i.e. $x = 0$. It's easy to verify such correlations

$$\theta^{-1}\cdot A_0 = G, \quad P\cdot\theta^{-1} = P, \quad \theta^{-1} = P + \theta^{-1}\cdot G. \tag{7}$$

Let's denote

$$\varphi_n := f_n - (n-1)A_2 x_{n-1} - ... - 2\cdot A_{n-1}x_2 - A_n x_1, \quad n = 0,1,2,...$$

Let's multiply each equality (3) by matrix θ^{-1} and using the first equality (7), we obtain equivalent equation

$$(n+1)\cdot G\cdot x_{n+1} + \theta^{-1}\cdot (E + n\cdot A_1)\cdot x_n = \theta^{-1}\cdot \varphi_n.$$

After applying the projectors G, P we have

$$(n+1)\cdot G\cdot x_{n+1} + G\cdot\theta^{-1}\cdot (E + n\cdot A_1)\cdot x_n = G\cdot\theta^{-1}\cdot \varphi_n, \tag{8}$$

$$P\cdot\theta^{-1}\cdot (E + n\cdot A_1)\cdot x_n = P\cdot\theta^{-1}\cdot \varphi_n. \tag{9}$$

Equation (9) is simplified by using the second correlation (7):

$$P\cdot (E + n\cdot A_1)\cdot x_n = P\cdot \varphi_n. \tag{10}$$

At $n = 0$ equality (10) gives us the limitation of the initial vector $P\cdot x_0 = P\cdot f_0$, i.e. $(x_0 - f_0) \in Im\, A_0$. From (8),(9) follows that

$$\begin{array}{c} (n+1)\cdot G\cdot x_{n+1} = -\,G\cdot\theta^{-1}\cdot (E + n\cdot A_1)\cdot x_n + G\cdot\theta^{-1}\cdot\varphi_n \\ P\cdot (E + (n+1)\cdot A_1)\cdot x_{n+1} = P\cdot \varphi_{n+1} \end{array} \tag{11}$$

For fixed n we assume that vectors $x_n, \varphi_n, \varphi_{n+1}$ are known and consider (11) as system of $2m$ linear algebraic equations regarding m components of vector x_{n+1}. The matrix

$$\begin{pmatrix} (n+1)\cdot G & G\cdot\xi_n \\ P\cdot (E + (n+1)\cdot A_1) & P\cdot\varphi_{n+1} \end{pmatrix}, \qquad \xi_n = \theta^{-1}\cdot(\varphi_n - (E + n\cdot A_1)\cdot x_n),$$

has the rank m, so as matrices $((n+1)\cdot G \quad G\cdot\xi_n)$ and $(P\cdot (E + (n+1)\cdot A_1) \quad P\cdot\varphi_{n+1})$ have p and $(m-p)$ linear independent lines respectively. System (11) has a unique solution by the theorem of Kronecker-Capelli. Solving the system (11) sequentially at $n = 0, 1, ...$, we obtain the claim of lemma.

3. The main asymptotic theorem

In this section we will prove the main asymptotic theorem for solutions of linear differential-algebraic equation (1).

Definition: If A is an $n\times n$ matrix of complex numbers, then the Drazin inverse of A, denoted by A^D, is the unique solution of three equations

$$\begin{aligned} AX &= XA, \\ XAX &= X, \\ XA^{k+1} &= A^k, \qquad k = Ind(A) \end{aligned}$$

Theorem. Let matrix $A(t)$ in differential-algebraic equation (1) satisfies the conditions $rank\, A(t) = const$ and $ind\, A(t) = 1$ for $\forall\, t \in \bar{S}$. Suppose that the coefficients A_r in asymptotic expansion of $A(t)$ satisfy the conditions of lemma. If series $\sum_0^\infty x_r t^r$ satisfies the equation (1) formally and x_0 satisfies the condition (5) of lemma then the solution $x = x(t)$ of the initial problem (1)-(2) exists and unique in the sector S, and

$$x(t) \sim \sum_{r=0}^{\infty} x_r t^r, \qquad t \to 0,\ t \in S.$$

Proof:

1). *Existence of asymptotic expansion.* Let's consider a closed sector S_1 such that $S_1 \subset S$. Using the constant rank of matrix $A(t)$, the form of matrix $A^D(t)$ [2] and *theorem* 1 [3] we get that matrix $A^D(t)$ is holomorphic in S_1. From integral form of matrix $A^D(t)$ [5] we can prove that $A^D(t)$ has asymptotic expansion on S_1. It follows from opportunity to choose the constant way of integration so as matrix $A(t)$ has constant rank and is holomorphic in S. Using the theorem 9.3 [1] we obtain that $\exists \hat{x} \in S_1$ such that

$$\hat{x}(t) \sim \sum_{r=0}^{\infty} x_r t^r, \quad t \to 0,\ t \in S_1.$$

Let's suppose that $x(t) = z(t) \cdot e^t + \hat{x}(t)$,where function $z(t)$ must be defined. After transformation the equation (1) we obtain

$$A(t) \cdot \dot{z}(t) + z(t) = \varphi(t, z) \tag{12}$$

Here $\varphi(t, z) = -\, A(t) \cdot z(t) - e^{-t} \cdot a(t)$ and $a(t) = A(t) \cdot \dot{\hat{x}}(t) + \hat{x} - f(t) \sim 0, t \to 0,\ t \in S_1$. Now we must prove that the solution of equation (12) has asymptotic expansion which equals zero. Differential equation (1) has been written in the form (12). It's necessary for preparing the next step consisted of transformation the differential equation to the integral equation. Let us multiply (12) by matrix $\big(E - A^D(t) \cdot A(t)\big)$. According to the definition of $A^D(t)$ and the conditions of theorem the correlation $\big(E - A^D(t) \cdot A(t)\big) \cdot A(t) = 0$ is true and then we obtain

$$z(t) = A^D(t) \cdot A(t) \cdot z(t) + \xi(t), \tag{13}$$

where $\xi(t) = -\,(E - A^D(t) \cdot A(t)) \cdot e^{-t} \cdot a(t)$. After taking the derivative (13) and multiplying the result equation by $A^D(t) \cdot A(t)$ we obtain

$$A^D(t) \cdot A(t) \cdot \frac{d\xi}{dt} = -\, A^D(t) \cdot A(t) \frac{dA^D A}{dt} \cdot z(t), \tag{14}$$

and thereto using the definition of matrix $A^D(t)$:

$$A^D(t) \cdot A(t) \cdot \xi(t) = 0, \tag{15}$$

$$A^D(t) \cdot \xi(t) = 0, \qquad A(t) \cdot \xi(t) = 0 \tag{16}$$

$$A(t) \cdot \frac{d\xi}{dt} = -\, A(t) \frac{dA^D A}{dt} \cdot \xi(t), \tag{17}$$

For getting equality (18) we use the definition of $\xi(t)$ and properties of matrix $A^D(t)$ [5]. The solution z(t) of the equation (12) satisfy the following correlation

$$A^DA^2\frac{dz}{dt}+A^DA^2(A^D+\frac{dA^DA}{dt})z-A^DA^2\frac{dA^DA}{dt}\xi-A^DA^2A^D\varphi(t,z)=0. \qquad (18)$$

It follows from (14), (15), (17) and that matrix $A^D(t)\cdot A(t)$ is a projector. From the equation (18) we get such vector $\delta(t)\in\ker A(t)$ exists and the following correlation is true for it:

$$\frac{dz}{dt}=-\left(A^D+\frac{dA^DA}{dt}\right)z+\frac{dA^DA}{dt}\xi+A^D\varphi(t,z)+\delta(t). \qquad (19)$$

Let $Z(t)$ is fundamental matrix of system (19). It's known [4] that $Z(t)$ can be the solution of the following matrix Cauchy's task:

$$\frac{dZ\,(t)}{dt}=-\left(A^D+\frac{dA^DA}{dt}\right)Z(t), \qquad Z(a)=E, \quad a\in S_1, \quad t\in S_1.$$

Let us multiply (19) by matrix $Z^{-1}(t)$ and integrate the result equation from a to t:

$$\int_a^t Z^{-1}\frac{dz}{d\tau}d\tau=-\int_a^t Z^{-1}\left(A^D+\frac{dA^DA}{d\tau}\right)zd\tau+\int_a^t Z^{-1}\left(\frac{dA^DA}{d\tau}\xi+A^D\varphi+\delta\right)d\tau.$$

After using the integration by parts for the left side of the last equation, a correlation $dZ^{-1}/dt=Z^{-1}\cdot\left(A^D+d(A^DA)/dt\right)$, $Z^{-1}(a)=E$ and multiplying by the matrix $Z(t)$ we get the result

$$z(t)=Z(t)\cdot z(a)+Z(t)\int_a^t Z^{-1}(\tau)\cdot\left(\frac{dA^DA}{d\tau}\cdot\xi(\tau)+A^D(\tau)\cdot\varphi(\tau,z)+\delta(\tau)\right)d\tau. \quad (20)$$

Note that the next correlation is true:

$$\eta(t)\equiv A^D(t)\cdot A(t)\cdot Z(t)\cdot\int_a^t Z^{-1}(\tau)\cdot\delta(\tau)d\tau\equiv 0, \qquad (21)$$

so as, using the definition of $Z(t)$, $\delta(t)\in\ker A(t)$ and properties of matrix $A^D(t)$ we obtain that $\eta(t)$ is the solution of the system:

$$\frac{d\eta(t)}{dt}=\left(-A^D+\frac{dA^DA}{dt}\right)\cdot\eta(t), \qquad \eta(a)=0, \quad a\in S_1, \quad t\in S_1.$$

This system has only zero solutions because this is the Cauchy's task with zero initial data. It verifies directly. Substituting (20) in (13) and using (21) we obtain that $z(t)$ satisfies the integral equation:

$$z(t)=A^D\cdot A\cdot Z\cdot z(a)+A^D\cdot A\cdot Z\int_a^t Z^{-1}\cdot\left(\frac{dA^DA}{d\tau}\cdot\xi+A^D\cdot\varphi\right)d\tau+\xi(t). \qquad (22)$$

Now let's prove that the integral equation (22) is equivalent to the differential equation (12). Let $z(t)$ is the solution of the integral equation (22). Then it looks like

$$z(t)=A^D(t)\cdot A(t)\cdot z(t)+\xi(t), \qquad (23)$$

and $z(t)$ satisfies the correlation (19) at $\delta(t) \equiv 0$. Therefore using (17), (23) we obtain

$$\begin{aligned} A \cdot \frac{dz}{dt} &= A \cdot \left(\frac{dA^D A}{dt} \cdot z + A^D \cdot A \cdot \frac{dz}{dt} + \frac{d\xi}{dt} \right) = A \cdot \frac{dA^D A}{dt} \cdot z + A \cdot \frac{dA^D A}{dt} \cdot \xi - \\ &- A^D \cdot A \cdot z - A \cdot \frac{dA^D A}{dt} \cdot z + A^D \cdot A \cdot \varphi + A \cdot \frac{d\xi}{dt} = -z + \xi + A^D \cdot A \cdot \varphi. \end{aligned}$$

Hence the correlation

$$\theta = A \cdot \frac{dz}{dt} + z - \varphi = -z + \xi + A^D \cdot A \cdot \varphi + z - \varphi = \xi - (E - A^D \cdot A) \cdot \varphi = 0.$$

is true. And thus the equivalence of equations (12) and (22) is proved. So as the integrand function in the right part of (22) is continuous at the zero, then we can make the limit transformation at a approaching to 0 and the integral from zero to t exists and does not depend on the way of integration if the way of integration belongs to S_1. Let's choose the straight line way of integration.

$$z(t) = A^D(t) \cdot A(t) \cdot Z(t) \int_0^t Z^{-1}(\tau) \cdot \left(\frac{dA^D A}{d\tau} \cdot \xi(\tau) + A^D(\tau) \cdot \varphi(\tau, z) \right) d\tau + \xi(t). \quad (24)$$

We will seek the solution of the equation (24) using iterative method. Let $z_0(t) \equiv 0$ and define $z_{r+1}(t)$ by the recurrent way, i.e.

$$\begin{aligned} z_{r+1}(t) = -A^D \cdot A \cdot Z \textstyle\int_0^t Z^{-1} \cdot A^D \cdot \left(A \cdot z_r(\tau) + (E + A \cdot \frac{dA^D A}{d\tau}) \cdot e^{-\tau} \cdot a(\tau) \right) d\tau - \\ -(E - A^D(t) \cdot A(t)) \cdot e^{-t} \cdot a(t). \end{aligned} \quad (25)$$

Note that $\|a(t)\| \le c \cdot |t|^l$, $t \in S_1$ so as $a(t) \sim 0$, where l is any positive integer number and c is some constant which depends on l. Let's denote

$$k_1 = max \left\{ \sup_{t \in S_1} \|E - A^D A\|, \sup_{t \in S_1} \|Z^{-1} A^D (t \cdot E + A \frac{dA^D A}{dt})\| \right\}$$

$$\sup_{t \in S_1} \|A^D A Z\| \le k_2$$

$$\sup_{t \in S_1} \|Z^{-1} A^D A\| \le k_3, \qquad and \quad k := \frac{k_2 \cdot k_3}{l+1}.$$

Then one can find that

$$\|z_{r+1} - z_r\| \le c \cdot k_1 \cdot k^r \cdot |t|^{l+r} \cdot (\frac{k_2}{l+1} + 1) \quad (26)$$

and the norm of $z_{r+1}(t)$

$$\|z_{r+1}(t)\| = \| \sum_{i=o}^{r} (z_{i+1} - z_i) \| \le \sum_{i=o}^{r} \|(z_{i+1} - z_i)\| \le c \cdot k_1 \cdot (\frac{k_2}{l+1} + 1) \cdot \sum_{i=o}^{r} k^i \cdot |t|^{l+i}.$$

At $|t| < 1/k$ the last numerical series converges as geometrical progression then

$$\|z_{r+1}(t)\| \le c \cdot k_1 \cdot (\frac{k_2}{l+1} + 1) \cdot \frac{|t|^l}{1 - k \cdot |t|} \quad (27)$$

Inequality (27) does not break the condition of the arbitrariness of t_0, because we can choose number l, such that $t_0 < 1/k$. Let's prove that such limit exists

$$z(t) = \lim_{r\to\infty} z_r(t). \tag{28}$$

From (26) we obtain that series

$$\sum_{r=0}^{\infty} \|z_{r+1} - z_r\|$$

may be majority by convergent series on the set S_1. Therefore the series

$$\sum_{r=0}^{\infty} (z_{r+1} - z_r)$$

converges uniformly on the set S_1 according to K.Weierstrass's theorem. It means that following limit exists

$$\lim_{r\to\infty} z_r(t) = \lim_{r\to\infty} \sum_{i=0}^{r} (z_{i+1} - z_i)$$

and the limit function $z(t)$ is holomorphic in S_1 . So $z_r(t)$ converges to $z(t)$ uniformly on the set S_1 at $r \to \infty$. Let's prove that $z(t)$ satisfies integral equation (24). Let's consider

$$\|z_r(t) + A^D \cdot A \cdot Z \int_0^t Z^{-1} \cdot A^D \cdot \left(A \cdot z(\tau) + (E + A \cdot \frac{dA^D A}{d\tau}) \cdot e^{-\tau} \cdot a(\tau) \right) d\tau -$$

$$+(E - A^D \cdot A) \cdot e^{-t} \cdot a(t)\| = \|A^D \cdot A \cdot Z \int_0^t Z^{-1} \cdot A^D \cdot A \cdot (z(\tau) - z_{r-1}(\tau))\| \leq$$

$$\leq k_2 \cdot k_3 \cdot t_0 \cdot \max_{t\in S_1} \|z(t) - z_{r-1}(t)\| \to 0$$

at $r \to \infty$. Hence $z(t)$ satisfies integral equation (24). Let's prove uniqueness of $z(t)$. Suppose that $y(t)$ is the other solution of the integral equation (24). The solutions $y(t)$ and $z(t)$ are different. The function $y(t)$ is holomorphic and satisfies the initial condition $\lim_{t\to 0} y(t) = 0$ for $\forall t \in S_1$and

$$\sup_{t\in S_1} \|y(t)\| \leq M,$$

where M - some constant.

$$y(t) = A^D(t) \cdot A(t) \cdot Z(t) \int_0^t Z^{-1}(\tau) \cdot \left(\frac{dA^D A}{d\tau} \cdot \xi(\tau) + A^D(\tau) \cdot \varphi(\tau, y) \right) d\tau + \xi(t).$$

Using the method of induction, the last correlation and the definition of $z_r(t)$ we obtain the inequality

$$\|y(t) - z_r(t)\| \leq M \cdot \frac{(k_2 \cdot k_3)^r \cdot t^r}{r!}, \qquad t \in S_1,$$

hence

$$y(t) = \lim_{t\to\infty} = z(t), \qquad t \in S_1.$$

So $z(t)$ is the unique solution of the integral equation (24), hence it's also the solution of the differential equation (12). Using (27), (28) and formulas for the calculation of coefficients of asymptotic expansion [1] we obtain

$$z(t) \sim 0, \qquad t \to 0, \qquad t \in S_1.$$

Thus we may conclude that power series $\sum_{r=0}^{\infty} x_r t^r$ is asymptotic expansion for the solution of equation (1) on set S, i.e.

$$x(t) \sim \sum_{r=0}^{\infty} x_r t^r, \qquad t \to 0, \ t \in S.$$

2). *Uniqueness of solution.* Let two solutions $x_1(t)$ and $x_2(t)$ exist and satisfy the equation (1). Then let's consider the following initial problem

$$A(t) \cdot \dot{u}(t) + u(t) = 0, \qquad \lim_{t \to 0} u(t) = 0,$$

where $u(t) = x_1(t) - x_2(t)$.

Let's make transformation $u(t) = N(t) \cdot w(t)$, $w(t) = (w_1(t), w_2(t))^{tr}$ and multiply the result correlation by matrix $N^{-1}(t)$. In result we obtain the equivalent system

$$\begin{array}{rcl} w_1(t) & = & 0 \\ \dot{w}_2(t) & = & I^{-1}(t) \cdot M(t) \cdot w_2(t) \end{array}$$

where $M(t)$ is some matrix of the blocks of matrix $N^{-1}(t) \cdot \dot{N}(t)$ and $I(t)$ is an invertible Jordan's block of matrix $A(t)$. Let's prove that the following initial problem has zero solution:

$$\dot{w}_2(t) = I^{-1}(t) \cdot M(t) \cdot w_2(t), \qquad \lim_{t \to 0} w_2(t) = 0. \tag{29}$$

For that let's consider simply connected region D such that $S_1 \subset D$ and matrix $I^{-1}(t) \cdot M(t)$ is holomorphic in it. The initial condition (29) has equivalent form

$$\lim_{t \to 0} w_2(t) = 0 \quad \Leftrightarrow \quad \forall \, \{t_n\}_{n=0}^{\infty} \subset D : \, t \to 0 \Rightarrow \lim_{n \to \infty} w_2(t_n) = 0.$$

Let's fix t_n. Let $W(t)$ is fundamental matrix of system (29). It's known [4] that $W(t)$ can be the solution of the following matrix Cauchy's task:

$$\frac{dW(t)}{dt} = I^{-1}(t) \cdot M(t) \cdot W(t), \quad W(t_n) = E, \quad t_n \in D, \quad t \in D.$$

After multiplying (29) by matrix $W^{-1}(t)$ and integrating from t_n till t we obtain

$$\int_{t_n}^{t} W^{-1}(\tau) \cdot \frac{dw_2(\tau)}{d\tau} d\tau = \int_{t_n}^{t} W^{-1}(\tau) \cdot I^{-1}(\tau) \cdot M(\tau) \cdot w_2(\tau) d\tau.$$

Applying integration by parts to the right part of the last equation and taking into account the correlation

$$\frac{dW^{-1}(t)}{dt} = -W^{-1}(t) \cdot I^{-1}(t) \cdot M(t), \quad W^{-1}(t_n) = E, \quad t_n \in D, \quad t \in D,$$

we obtain $W^{-1}(t)\cdot w_2(t) = W^{-1}(t_n)\cdot w_2(t_n)$, and after multiplying by matrix $W(t)$ we obtain the final result

$$w_2(t) = W(t)\cdot W^{-1}(t_n)\cdot w_2(t_n) = W(t)\cdot w_2(t_n).$$

After applying the limit transformation at $n\to\infty$, one can see $w_2(t) = 0$. So as for $\forall t\in S_1$ $\det N(t)\neq 0$ then $u(t) = 0$ and the theorem is proved completely.

4. Examples

Example 1: For illustration of the theorem let's consider the homogeneous system

$$A(t)\dot{x}(t) + x(t) = 0, \qquad t\in S, \tag{30}$$

where $A(t) = \begin{pmatrix} t+1 & 1 \\ (t+1)^2 & t+1 \end{pmatrix}$, S is arbitrary sector which satisfies the conditions of the theorem. Then $A(t)\sim A_0 + A_1 t + A_2 t^2$, where

$$A_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_3 = ... = 0.$$

It's evident that the lemma condition (4) is satisfied as $\det(E + (n+1)A_1) = (n+2)^2\neq 0$. All conditions of the theorem are satisfied and the obtained solution is:

$$x(t)\sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{r=1}^{\infty}\begin{pmatrix} (-1)^r \\ 0 \end{pmatrix} t^r, \qquad t\to 0, \quad t\in S.$$

Example 2: The conditions of the theorem can not be weakened. Let's consider the system (30) at $m = 3$ on the set $S = \{t\in C:\ |t| > 0\wedge |\arg t| < \alpha,\ 0 < \alpha\leq \pi/2\}$ with matrix

$$A(t) = \begin{pmatrix} \cos t - t & 0 & 0 \\ 0 & -\sin t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A(t)\sim \sum_{r=0}^{\infty} A_r t^r,$$

where $A_r = \frac{1}{r!}\begin{pmatrix} \cos(\pi r/2) - t^{(r)} & 0 & 0 \\ 0 & -\sin(\pi r/2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Matrix $A(t)$ has not a constant $rank$. In this case we have $rank(E + A_1) < \dim\ker A_0$. Condition (4) is not satisfied at $n = 1$. It's evident that solution of the system (3) is not defined uniquely by the initial vector and that's way we can't get asymptotic expansions for solutions of this system.

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