A NOTE ON μ-STABLE SURFACES WITH PRESCRIBED CONSTANT MEAN CURVATURE

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Using a generalized stability condition we give an upper bound of the principle curvatures of certain constant mean curvature surfaces which implies a theorem of Bernstein type.

1. Introduction

Let $B := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ be the open unit disc, $\overline{B} \subset \mathbb{R}^2$ its topological closure. We consider immersions $X \in C^{3+\alpha}(B,\mathbb{R}^3) \cap C^0(\overline{B},\mathbb{R}^3)$, $\alpha \in (0,1)$, of prescribed constant mean curvature $H(X) \equiv h_0 \in [0,+\infty)$. Introducing conformal parameters $(u,v) \in B$, such an immersion satisfies the nonlinear elliptic system

$$\begin{aligned} &\triangle X(u,v) = 2h_0(X_u \wedge X_v), \\ &|X_u|^2 = W = |X_v|^2\,, \quad X_u \cdot X_v^t = 0 \quad \text{in } B, \end{aligned}$$

where $W := |X_u \wedge X_v| > 0$ denotes the surface element with the usual cross product $\wedge$ between two vectors in $\mathbb{R}^3$. By

$$N(u,v) := \frac{X_u(u,v) \wedge X_v(u,v)}{|X_u(u,v) \wedge X_v(u,v)|}$$

we denote the spherical map of the surface $X = X(u,v)$.

Definition 1.1. The immersion $X \in C^{3+\alpha}(B,\mathbb{R}^3) \cap C^0(\overline{B},\mathbb{R}^3)$, $\alpha \in (0,1)$, of prescribed mean curvature $H(X) \equiv h_0 \in [0,+\infty)$ is of class $\mathcal{C}(B,\mathbb{R}^3)$, if it has finite Dirichlet integral

$$\iint\limits_B |\nabla X(u,v)|^2\,dudv < +\infty$$

and satisfies the above nonlinear elliptic system.

Furthermore, the immersion $X = X(u,v)$ is called *stable* if

$$\iint\limits_B |\nabla\varphi(u,v)|^2\,dudv \geq 2\iint\limits_B (2h_0^2 - K)W\varphi(u,v)^2\,dudv$$

holds true for all test functions $\varphi \in C_0^\infty(B,\mathbb{R})$. Assuming stability, the oscillation of the spherical image can be controlled in the following way:

Let $Z \in \mathbb{R}^3$ be an arbitray unit vector, and $\omega \in (0, 4\pi)$. Then we define the spherical cap $S^2_\omega(Z) := \{X \in \mathbb{R}^3 \,:\, X \cdot Z^t \geq \cos\omega\}$ of polar angle $\omega \in (0, 4\pi)$ and center $Z \in S^2$, $S^2 := \{X \in \mathbb{R}^3 \,:\, |X| = 1\}$. For $w_0 \in B$ and real $\nu \in (0, 1-|w_0|)$ we consider the open disc $B_\nu(w_0) := \{w \in B \,:\, |w-w_0| < \nu\}$. If the spherical image fulfills

$$N(\partial B_\nu(w_0)) \subset S^2_{\frac{\pi}{2}}(Z), \quad w_0 \in B,\ \nu \in (0, 1-|w_0|),$$

then it follows that $N(\overline{B}_\nu(w_0)) \subset S^2_{\frac{\pi}{2}}(Z)$ (cp. [Sa2]).

Proving a modulus of continuity of the spherical image is a crucial step in estimating the first and second derivatives of the immersion. In the next section we give a variant of this result for immersions of prescribed constant mean curvature which are *μ-stable* in the sense of

Definition 1.2. The immersion $X \in \mathcal{C}(B, \mathbb{R}^3)$ with prescribed constant mean curvature $h_0 \in [0, +\infty)$ is called μ-stable with a real number $\mu > 0$ if

$$\iint\limits_B |\nabla\varphi(u,v)|^2 \, dudv \geq \mu \iint\limits_B (2h_0^2 - K) W \varphi(u,v)^2 \, dudv$$

holds true for all test functions $\varphi \in C_0^\infty(B, \mathbb{R})$.

As shown in section 3, μ-stability can be realized by assuming

$$\iint\limits_B (2h_0^2 - K) W \, dudv < \omega_0$$

with a real constant $\omega_0 \in (0, 4\pi)$. In section 4 we apply our result to give an upper bound for the principle curvatures of these immersions and derive a result of Bernstein type.

2. Projectivity

Lemma 2.1. *Let $X \in \mathcal{C}(B, \mathbb{R}^3)$ be μ-stable with a real constant $\mu \in (1, 2]$. For $w_0 \in B$ and real $\nu \in (0, 1-|w_0|)$ we assume*

$$N(u,v) \cdot (0,0,1)^t > \frac{2}{\mu} - 1 \quad \textit{for all } (u,v) \in \partial B_\nu(w_0).$$

Statement: Then the inequality

$$N(u,v) \cdot (0,0,1)^t > \frac{2}{\mu} - 1 \quad \textit{for all } (u,v) \in \overline{B}_\nu(w_0)$$

holds true. In particular, $X|_{\overline{B}_\nu(w_0)}$ represents a graph over the plane perpendicular to the vector $(0,0,1)$.

Remark 2.2. This result as well as its method of proof are motivated by Hilfssatz 6 from [Sa1].

Proof of Lemma 1: Let $\psi^* := N \cdot (0,0,1)^t$. We consider the function

$$\psi(u,v) := \psi^*(u,v) - \omega, \quad (u,v) \in \overline{B}_\nu(w_0),\ \omega := \frac{2}{\mu} - 1.$$

Since $\triangle N + 2qN = 0$, $q := (2h_0^2 - K)W > 0$, holds true for the normal mapping of the surface, by multiplication with the vector $(0,0,1)$ we obtain

$$\triangle \psi^* = -2q\psi^* = -2q\psi - 2q\omega = \triangle \psi \quad \text{in } B_\nu(w_0).$$

We define

$$\psi^-(u,v) := \min\left(\psi(u,v), 0\right) \in H_1^2(\overline{B}_\nu(w_0), \mathbb{R}) \cap C^0(\overline{B}_\nu(w_0), \mathbb{R}),$$

and it remains to prove $\psi^- \equiv 0$. On account of $\psi|_{\partial B} > 0$ there exists a radius $\varrho \in (0,\nu)$ with $\operatorname{supp}(\psi^-) \subset B_\varrho(w_0)$. That means $\psi^- \in H^{1,2}(B_\varrho(w_0), \mathbb{R}) \cap C_0^0(B_\varrho(w_0), \mathbb{R})$ together with

$$\nabla \psi^- = \begin{cases} 0, & \text{if } \psi \geq 0 \\ \nabla \psi, & \text{if } \psi < 0 \end{cases} .$$

Partial integration yields (we set $B^* := B_\varrho(w_0)$ and omit $dudv$)

$$\iint_{B^*} |\nabla \psi^-|^2 = -\iint_{B^*} \psi^- \triangle \psi = \mu \iint_{B^*} q|\psi^-|^2 + (2-\mu) \iint_{B^*} q|\psi^-|^2 + 2\omega \iint_{B^*} q\psi^- .$$

For the admissable test function

$$\varphi(u,v) := \psi^-(u,v) + \varepsilon\chi(u,v), \quad \chi \in C_0^\infty(B^*, \mathbb{R}),\ \varepsilon \in \mathbb{R},$$

the μ-stability condition implies

$$\begin{aligned} &\iint_{B^*} |\nabla \psi^-|^2 + 2\varepsilon \iint_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint_{B^*} |\nabla\chi|^2 \\ &\qquad \geq \mu \iint_{B^*} q|\psi^-|^2 + 2\mu\varepsilon \iint_{B^*} q\psi^-\chi + \mu\varepsilon^2 \iint_{B^*} q\chi^2 . \end{aligned}$$

Therefore we have

$$\begin{aligned} &2\varepsilon \iint_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint_{B^*} |\nabla\chi|^2 \\ &\geq (\mu - 2) \iint_{B^*} q|\psi^-||\psi^-| + 2\omega \iint_{B^*} q|\psi^-| + 2\mu\varepsilon \iint_{B^*} q\psi^-\chi + \mu\varepsilon^2 \iint_{B^*} q\chi^2 . \end{aligned}$$

Since $-1-\omega \le \psi^- \le 0$ and $\mu - 2 \le 0$ we deduce the estimate

$$
\begin{aligned}
&2\varepsilon \iint\limits_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint\limits_{B^*} |\nabla\chi|^2 \, dudv \\
&\ge \ (1+\omega)(\mu-2) \iint\limits_{B^*} q|\psi^-| + 2\omega \iint\limits_{B^*} q|\psi^-| + 2\mu\varepsilon \iint\limits_{B^*} q\psi^-\chi + \mu\varepsilon^2 \iint\limits_{B^*} q\chi^2 \\
&= \ 2\mu\varepsilon \iint\limits_{B^*} q\psi^-\chi + \mu\varepsilon^2 \iint\limits_{B^*} q\chi^2
\end{aligned}
$$

taking $(1+\omega)(\mu-2)+2\omega = 0$ into account (note $\omega = 2\mu^{-1}-1$). Therefore

$$
2\varepsilon \iint\limits_{B^*} (\nabla\psi^- \cdot \nabla\chi - \mu q\psi^-\chi) + \varepsilon^2 \iint\limits_{B^*} (|\nabla\chi|^2 - \mu q\chi^2) \ge 0
$$

holds true for all $\varepsilon \in \mathbb{R}$, and we find

$$
\iint\limits_{B^*} (\nabla\psi^- \cdot \nabla\chi - \mu q\psi^-\chi) = 0
$$

for all $\chi \in C_0^\infty(B^*, \mathbb{R})$. Since $q \in C^\alpha(\overline{B}^*, \mathbb{R})$, the well known Lemma of Weyl yields $\psi^- \in C_0^2(B^*, \mathbb{R})$ (cp. [Hl], chapter 4.2). Now

$$
\triangle\psi + 2q\psi = -2q\omega \le 0 \quad \text{in } B_\nu(w_0)\,, \quad \psi > 0 \quad \text{on } \partial B_\nu(w_0)
$$

holds true. We set

$$
\psi^+(u,v) := \max\,(\psi(u,v), 0), \quad (u,v) \in \overline{B}_\nu(w_0),
$$

where we note $\psi(u,v) = \psi^+(u,v) + \psi^-(u,v)$, $(u,v) \in \overline{B}_\nu(w_0)$. Here the functions are continued by 0. We get $\psi^+ \in C^2(\overline{B}_\nu(w_0), \mathbb{R})$, since $\psi \in C^2(\overline{B}_\nu(w_0), \mathbb{R})$ and $\psi^- \in C^2(\overline{B}_\nu(w_0), \mathbb{R})$, and arrive at

$$
\triangle\psi^+ + 2q\psi^+ \le 0 \quad \text{in } B_\nu(w_0), \quad \psi^+ \ge 0 \quad \text{on } \overline{B}_\nu(w_0).
$$

Assuming $\psi^- \not\equiv 0$, there exists a point $w^* \in B^*$, such that $\psi^-(w^*) < 0$, and therefore $\psi^+(w^*) = 0$ holds true. We conclude $\psi^+ \equiv 0$ by [He], Lemma 6, and this contradicts $\psi > 0$ on $\partial B_\nu(w_0)$. q.e.d.

3. A result of Ruchert-type

By $\triangle^*$ we denote the Laplace-Beltrami operator on S^2. The proof of the next result follows the lines of [Ru].

Lemma 3.1. *Let $X \in \mathcal{C}(B, \mathbb{R}^3)$. Assume that*

$$
Q := \iint\limits_{B} (2h_0^2 - K) W \, dudv < \omega_0
$$

holds true, where $\omega_0 \in (0, 4\pi)$ *is a real positive constant. Finally, let* $S^2_\omega \subset S^2$ *be a spherical cap with the property* $\operatorname{Area} S^2_\omega = \omega_0$*, and* $\mu > 0$ *be its first eigenvalue of the spherical Laplacian* $\triangle^*$ *w.r.t. the Dirichlet-problem*

$$\triangle^* \psi + \lambda \psi = 0 \quad \textit{on } S^2_\omega\,, \quad \psi = 0 \quad \textit{on } \partial S^2_\omega\,.$$

Statement: Then the surface is μ*-stable with this number* $\mu > 0$.

Proof: Using conformal parameters, the Gaussian curvature $K = K(u, v)$ of the surface satisfies

$$K(u, v) = -\frac{1}{W}\,\triangle(\log\sqrt{W}).$$

We set $\chi := 2h_0^2 - K$, and for the Gaussian curvature $\widehat{K} = \widehat{K}(u, v)$ w.r.t. the metric $(\widehat{g}_{ij})_{i,j=1,2}$ with $\widehat{g}_{11} = \chi W = \widehat{g}_{22}$, $\widehat{g}_{12} = 0 = \widehat{g}_{21}$, one finds

$$\chi\widehat{K} = K - \frac{1}{2W}\,\triangle(\log\chi).$$

Furthermore, we have $\widehat{K} \le 1$ in B (cp. [Ru], Lemma 2.3). Now, let $\widehat{\triangle}$ denote the Laplace-Beltrami operator w.r.t. the metric given by the $\widehat{g}_{ij}$, $i, j = 1, 2$, and by $\widehat{\lambda}_1 > 0$ we mean the first eigenvalue of the problem

$$\widehat{\triangle}\varphi + \lambda\varphi = 0 \quad \text{in } B, \quad \varphi = 0 \quad \text{on } \partial B.$$

Let $S^2_\omega \subset S^2$, $\omega \in (0, 4\pi)$, be a spherical cap with the property $\operatorname{Area}(S^2_\omega) = Q$, and $\lambda_1^* > 0$ means the first eigenvalue of the spherical Laplacian of the problem

$$\triangle^*\varphi^* + \lambda_1^*\varphi^* = 0 \quad \text{in } S^2_\omega\,, \quad \varphi^* = 0 \quad \text{on } \partial S^2_\omega\,.$$

Since $\widehat{K} \le 1$, Proposition 3.3 and Proposition 3.16 of [BdC2] yields $\lambda_1^* \le \widehat{\lambda}_1$. By assumption, S^2_ω is contained in a spherical cap with first eigenvalue $\mu > 0$. By the monotonicity of the first eigenvalue we deduce $\mu < \lambda_1^*$, and therefore

$$\mu < \lambda_1^* \le \widehat{\lambda}_1 \le \frac{\displaystyle\iint\limits_B |\nabla\varphi|^2\,dudv}{\displaystyle\iint\limits_B \varphi^2(2h_0^2 - K)W\,dudv}$$

$$\text{for all } \varphi \in H^{1,2}(B, \mathbb{R}) \setminus \{0\},\ \varphi|_{\partial B} = 0.$$

The statement follows. q.e.d.

Remark 3.2. In the minimal surface case, the immersion is stable if the area of its spherical image is smaller than 2π (cp. [BdC1]).

4. An a priori bound for the principle curvatures

We assume that the immersion $X \in C^{2+\alpha}(B,\mathbb{R}^3) \cap C^0(\overline{B},\mathbb{R}^3)$, $\alpha \in (0,1)$, is a geodesic disc $\mathcal{B}_r(X_0)$ of radius $r > 0$ with the center $X_0 = X(0,0)$. In geodesic polar coordinates we have the representation $Z = Z(\varrho,\varphi) : [0,r] \times [0,2\pi] \to \mathbb{R}^3$. For its line element we find

$$ds_P^2 = |Z_\varrho|^2\,d\varrho + 2Z\varrho \cdot Z\varphi\,d\varrho d\varphi + |Z_\varphi|^2\,d\varphi = d\varrho^2 + P(\varrho,\varphi)\,d\varphi^2\,.$$

From [Sa2], Proof of Theorem 3, we obtain the following results.

Lemma 4.1. *Let $X \in \mathcal{C}(B,\mathbb{R}^3)$ be μ-stable, $\mu > \frac{1}{2}$, and let it represent a geodesic disc $\mathcal{B}_r(X_0)$.*

Statement: Then the estimate

$$\mathcal{A}(Z) := \int\limits_0^r \int\limits_0^{2\pi} \sqrt{P(\varrho,\varphi}\,d\varrho d\varphi \le \frac{2\pi\mu}{2\mu-1}\,r^2$$

for its area $\mathcal{A}(Z)$ holds true.

Lemma 4.2. *Let $X \in \mathcal{C}(B,\mathbb{R}^3)$ be μ-stable, and let $\nu \in (0,1)$.*

Statement: Then the energy of the spherical image satisfies the inequality

$$\iint\limits_{|w|\le 1-\nu} |\nabla N(u,v)|^2\,dudv \le \frac{8\pi}{\mu\nu^2}\,.$$

Now we can apply Theorem 1 from [Sa2] to obtain

Theorem 4.3. *Let $X \in \mathcal{C}(B,\mathbb{R}^3)$ be μ-stable, $\mu > \frac{1}{2}$, and let it represent a geodesic disc $\mathcal{B}_r(X_0)$ of radius $r > 0$ with center $X_0 = X(0,0)$.*

Statement: Then there exists a constant $\Theta = \Theta(h_0 r,\mu)$ such that the estimate

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \le \frac{1}{r^2}\,\Theta(h_0 r,\mu)$$

holds true for the principle curvatures κ_1 and κ_2 of $X = X(u,v)$.

Remark 4.4. In [Sa2], section 5, a curvature estimate for constant mean curvature surfaces is established under the integral condition

$$\iint\limits_B (h_0^2 - K)W\,dudv < 4\pi.$$

The method is based on a comparison surface of Bonnet type and an isoperimetric inequality.

In the case $h_0 = 0$ we immediately obtain the

Corollary 4.5. *Let the regular, complete and μ-stable minimal surface $X : \mathbb{R}^2 \to \mathbb{R}^3$, $\mu > \frac{1}{2}$, be given.*

Statement: Then the surface represents a plane in $\mathbb{R}^3$.

REMARK 4.6. In [Fr], an adequate μ-stability condition is applied to immersions of minimal surface type. Corollary 4.5 is then contained in the Bernstein results of that article.

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