Global in time L^p-estimates for the instationary Stokes- and Navier-Stokes flow through an aperture

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Abstract

Using a characterisation of maximal L^p-regularity by $\mathcal{R}$-bounded operator families we prove global in time estimates in $L^p(\mathbb{R}_+; L^q(\Omega))$, $1 < p, q < \infty$, for solutions of the instationary Stokes system in an aperture domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with $\partial\Omega \in C^{1,1}$. The results are applied to obtain new global in time estimates for weak solutions of the Navier-Stokes equations with nonvanishing flux through the aperture.

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1 Introduction

For $n \geq 3$ and $d \geq 0$ let $\mathbb{R}^n_\pm = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \ : \ \pm x_n > d/2\}$. Then an open connected set $\Omega \subset \mathbb{R}^n$ is called an aperture domain if there is a bounded set $B \subset \mathbb{R}^n$ such that $\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B$. We assume $\partial\Omega \in C^{1,1}$.

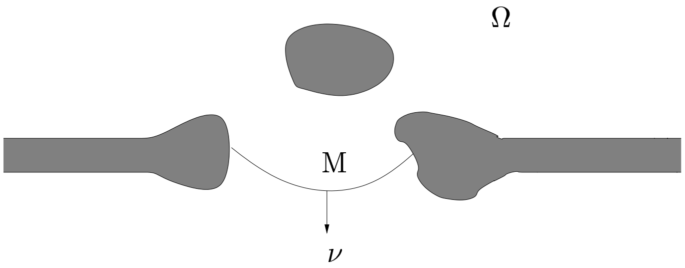

Figure 1: An aperture domain

For a smooth bounded $(n-1)$-dimensional manifold M with unit normal vector ν directed downwards dividing Ω into two connected components $\Omega_\pm$ and a solenoidal, sufficiently smooth vector field $u : \Omega \to \mathbb{R}^n$ the flux of u through the aperture is defined by $\Phi(u) = \int_M u \cdot \nu \, d\sigma$.

In an aperture domain we consider the instationary Navier-Stokes system

$$u_t + (u, \nabla u) - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \text{ in } \mathbb{R}_+ \times \Omega \tag{1}$$

$$u = 0 \text{ on } \partial\Omega, \quad u(x, 0) = u_0(x) \quad \text{in } \Omega. \tag{2}$$

In [5] M. Franzke constructed a global weak solution with prescribed flux $\Phi(u) = \alpha$ under suitable assumptions on the data in the case $n = 3$. Here we are interested in global estimates in $L^s(0,T;L^q(\Omega))-$spaces of $u_t, \nabla^2 u, \nabla p$ for such a weak solution. This kind of estimate was obtained in [10] for the whole space, the half space, bounded and exterior domains for exponents $1 < s,q < \infty$ satisfying $2/s + n/q = n+1$. However, if Ω is an aperture domain a new effect occurs in the case $\Phi(u) \neq 0$: The conditions $2/s+n/q = n+1$ and $s > 1$ yield $q < n' := n/(n-1)$, but for $q \le n'$ a solenoidal vector field $u \in L^q(\Omega)^n$ necessarily has vanishing flux (see [2], [5]). Therefore in the case $\Phi(u) \neq 0$ the regularity result can not be the same as for the domains considered in [10]. We have to use the sum of spaces $L^s(\mathbb{R}_+;L^q) + L^r(\mathbb{R}_+;L^\rho)$ for $1 < s,q,r,\rho < \infty$, $2/s + n/q = n+1$, $\rho > n'$.

Writing for simplicity $L^s(\mathbb{R}_+;L^q)$ for $L^s(\mathbb{R}_+;L^q(\Omega)^N)$ regardless of $N \in \mathbb{N}$ and denoting the respective norm by $\|\cdot\|_{s,q}$ the result reads as follows:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, be an aperture domain with boundary of class $C^{1,1}$, let $1 < s,q,r < \infty$, $2/s + n/q = n+1$ and assume $f \in L^s(\mathbb{R}_+;L^q)$, $u_0 \in L_\Sigma^{[s,q]}(\Omega)$ and $\alpha \in W^{1,r}(\mathbb{R}_+)$ with $\Phi(u_0) = \alpha(0)$. Let u be a weak solution of the Navier-Stokes equations (1)-(2) with flux $\Phi(u) = \alpha$ and let ∇p be the associated pressure gradient. Then*

$$u_t,\ \nabla^2 u, \nabla p \in L^s(\mathbb{R}_+;L^q) + \bigcap_{\rho > n'} L^r(\mathbb{R}_+;L^\rho). \tag{3}$$

The respective norms of u_t, $\nabla^2 u, \nabla p$ in $L^s(\mathbb{R}_+;L^q)+L^r(\mathbb{R}_+;L^\rho)$ for $\rho > n'$ can be estimated by a constant $C_{s,r,q,\rho} > 0$ times

$$\|f\|_{s,q} + \|u\|_{2,\infty}^{2-2/s}\|\nabla u\|_{2,2}^{2/s} + \|u_0\|_{[s,q],\chi} + \|\alpha\|_{W^{1,r}(\mathbb{R}_+)}. \tag{4}$$

If $\alpha = 0$ it holds $u_t, \nabla^2 u, \nabla p \in L^s(\mathbb{R}_+;L^q)$.

Here $L_\Sigma^{[s,q]}(\Omega)$ denotes an appropriate space for the initial value with norm $\|\cdot\|_{[s,q],\chi}$ defined in Section 4 below. For the precise definition of a weak solution of the Navier-Stokes equations with flux α see Section 5 below.

As in [10] our proof of global $L^s(\mathbb{R}_+;L^q)$-estimates for the Navier Stokes equations rests on $L^s(\mathbb{R}_+;L^q)$-estimates of the instationary Stokes system

$$u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}_+ \times \Omega \tag{5}$$

$$u\big|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad \Phi(u) = \alpha. \tag{6}$$

In [8] the a priori estimate

$$\int_0^T \{\, \|u_t\|_q^s + \|\nabla^2 u\|_q^s + \|\nabla p\|_q^s \,\}\, dt \le C \Big(\int_0^T \|f\|_q^s \, dt + \|u_0\|_{[s,q]}^s + \|\alpha\|_{W^{1,s}(0,T)}^s \Big), \tag{7}$$

for $0 < T < \infty$ with $C = C_T > 0$ is proved. But it has remained open if the constant $C = C_T$ in (7) can be chosen independently of $T \in (0,\infty)$. The aim of this paper is not only to give a positive answer to this question but also to improve the new method of proof from [8] which rests on resolvent estimates in weighted L^q-spaces for Muckenhoupt weights.

It is well known from semigroup theory that the behaviour of the solution of the instationary Stokes system for large times corresponds to the behaviour of the solution of the Stokes resolvent system

$$\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{8}$$

for small values of the modulus of $\lambda \in \Sigma_\varepsilon := \{\lambda \in \mathbb{C} \setminus \{0\} : |arg\,\lambda| < \pi - \varepsilon\}$, $0 < \varepsilon < \frac{\pi}{2}$.

The idea used in [8] to prove (7) is as follows: An abstract theorem on maximal L^p-regularity (see [11], [12] or Theorem 4.3 below) is combined with the fact that for L^q-spaces the assumptions of this theorem can be verified by using weighted L^q-estimates of the corresponding resolvent problem. To be more specific, for $1 < q < \infty$ and a weight $\omega \in A_q$ in the Muckenhoupt class (see Defintion 2.1) define the weighted L^q-space

$$L^q_\omega(\Omega) = \{u \in L^1_{loc}(\Omega) \ : \ \|u\|^q_{q,\omega} := \int_\Omega |u|^q \omega \, dx < \infty\}.$$

Then by [8] the validity of the estimate

$$\|\lambda u\|_{q,\omega} \leq C\,\|f\|_{q,\omega}, \quad \lambda \in \Sigma_\varepsilon,\ |\lambda| \geq \delta > 0 \tag{9}$$

in (8) for *all* weights $\omega \in A_q$ together with the fact that $C = C_{\varepsilon,\delta}(\omega)$ depends only on the A_q-constant (see Definition 2.1) of the weight $\omega \in A_q$ implies (7). The dependence of C in (9) on $\delta > 0$ corresponds to the (possible) dependence of the constant C in (7) on T.

In the case without weights in [2] the resolvent estimate (9) was proved with a constant C independent of $\delta > 0$. The proof rests on a uniqueness assertion for the stationary Stokes system and on Sobolev imbedding inequalities. In order to transfer this approach to the weighted situation we need weighted Sobolev inequalities, which require additional restrictions on the class of weight functions; therefore we are not able to verify the estimate (9) with a constant independent of $\delta > 0$ for *all* Muckenhoupt weights $\omega \in A_q$.

Actually a smaller class of weights turns out to be sufficient: Let $\bar{A}_q = A_1 \cup \{\omega^{-1/(q'-1)} : \omega \in A_1\} \subset A_q$ for $1 < q < \infty$, $1/q + 1/q' = 1$. We observe that (7) follows for all $1 < s, q < \infty$ from the resolvent estimate (9) for $q = 2$ and for all weights $\omega \in \bar{A}_2$ with a constant C independent of $\delta > 0$ and depending only on the A_1-constant of ω (see Theorem 4.2 below). Indeed, we have weighted Sobolev inequalities at hand to prove (9) with a constant independent of $\delta > 0$ for $n' < q < n$ and all weight functions in $\bar{A}_q$.

2 Preliminaries

A cube Q is a subset of $\mathbb{R}^n$ of the form $\Pi_{j=1}^n I_j$, where $I_1, \dots, I_n \subset \mathbb{R}$ are bounded intervals of the same length.

Definition 2.1 *Let $1 < q < \infty$. A function $0 \leq \omega \in L^1_{loc}(\mathbb{R}^n)$ is called A_q-weight, if*

$$A_q(\omega) := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{q-1}}\,dx\right)^{q-1} < \infty, \tag{10}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $|Q|$ is the Lebesgue measure of Q. A function $0 \leq \omega \in L^1_{loc}(\mathbb{R}^n)$ is called A_1-weight, if

$$A_1(\omega) := \sup_Q \left\{ \left(\frac{1}{|Q|}\int_Q \omega\,dx\right) \operatorname*{ess\,sup}_{x \in Q} \frac{1}{\omega(x)} \right\} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For $1 \leq q < \infty$ the value $A_q(\omega)$ is called the A_q-constant of ω.

Finally we set $A_\infty := \bigcup_{1 \leq q < \infty} A_q$.

Obviously cubes may be replaced by balls in this definition. We use the abbreviations $\omega(S)$ for $\int_S \omega(x)\,dx$ for a measurable subset $S \subset \mathbb{R}^n$, $q' := \frac{q}{q-1}$ and $\omega' := \omega^{-\frac{1}{q-1}}$ if $1 < q < \infty$ is fixed. For $1 < q < \infty$, $\omega \in A_q$ and an open set $U \subset \mathbb{R}^n$ note that with respect to the dual product $(f,g) = \int_U f\,g\,dx$ the dual space of $L^q_\omega(U)$ can be identified with $L^{q'}_{\omega'}(U)$.

In the sequel constants $C = C(\omega) > 0$ in weighted L^q-estimates depending on the weight $\omega \in A_q$, $1 \le q < \infty$, will appear. We call such a constant *A_q-consistent* if for every $c_0 > 0$ it can be chosen uniformly for all $\omega \in A_q$ with $A_q(\omega) \le c_0$.

Lemma 2.1 (Properties of Muckenhoupt weights)

(i) *For* $1 < q < \infty\ :\ \omega \in A_q \iff \omega' = \omega^{-\frac{1}{q-1}} \in A_{q'}$. *It holds* $A_q(\omega) = A_{q'}(\omega')^{q-1}$.

(ii) $A_p \subset A_q$ *and* $A_q(\omega) \le A_p(\omega)$ *for* $1 \le p \le q \le \infty$.

(iii) *For every* $\omega \in A_\infty$ *there are constants* $C > 0$ *and* $\delta > 0$ *such that for every ball* B *and every subset* $A \subset B$

$$\frac{\omega(A)}{\omega(B)} \le C\left(\frac{|A|}{|B|}\right)^\delta.$$

(iv) $\omega \in A_\infty \Rightarrow \omega(\mathbb{R}^n) = \infty$.

(v) $|x|^\alpha$ *and* $(1+|x|)^\alpha$ *are* A_q*-weights for* $-n < \alpha < n(q-1)$.

(vi) *For* $1 < q < \infty$ *and* $\omega \in A_q$ *there is a* $1 < p_0 < q$ *such that* $\omega \in A_p$ *for all* $p_0 < p < q$.

(vii) *For all* $\omega \in A_q$ *and* $1 < q < \infty$ *it holds* $\int_{\mathbb{R}^n} \omega(x)(1+|x|)^{-nq}\,dx < \infty$.

Proof: (i) follows immediately from the definition.
(ii) [9], Chapter IV, Theorem 1.14.
(iii) See [9], Chapter IV, Theorem 2.9.
(iv) By iii) there is for $\omega \in A_\infty$ some $\delta > 0$ such that $\omega(B_R) \ge R^{n\delta}\omega(B_1) \to \infty$ for $R \to \infty$.
(v) See [4], Lemma 2.3 v).
(vi) [9], Chapter IV, Theorem 2.6.
(vii) [4], Lemma 2.2 (iii). $\square$

The following nice property of the class of weighted L^q-spaces equipped with Muckenhoup weights will be useful:

Lemma 2.2 *Let* $1 < r, q < \infty$, $v \in A_r$ *and* $\omega \in A_q$. *Then there exist* $s > 1$ *and a weight* $\rho \in A_s$ *such that*

$$L^r_v(\mathbb{R}^n) + L^q_\omega(\mathbb{R}^n) \subset L^s_\rho(\mathbb{R}^n).$$

Here ρ *can be chosen in the form* $\rho(x) = (1+|x|)^{-\alpha}$ *with* $0 < \alpha < n$.

Proof: By Lemma 2.1 (vi) there exist $p < q$ and $\tau < r$ such that $\omega \in A_p$ and $v \in A_r$ and $p/q = r/\tau =: s$. By (i) and (vi) of the same lemma there are $\varepsilon, \delta > 0$ such that $\omega^{-\frac{1}{p-1}} \in A_{p'-\varepsilon}$ and $v^{-\frac{1}{\tau-1}} \in A_{\tau'-\delta}$. Let $\max\{\frac{p'-\varepsilon}{p'}, \frac{\tau'-\varepsilon}{\tau'}\} < t < 1$ and $\alpha = nt$. Then with $\rho(x) := (1+|x|)^{-\alpha} \in A_1 \subset A_s$ we get by Hölder's inequality

$$\int_{\mathbb{R}^n} |f|^s \rho\,dx \le \|f\|^s_{q,\omega}\Big(\int_{\mathbb{R}^n} \omega(x)^{-\frac{1}{p-1}}(1+|x|)^{-\alpha p'}\,dx\Big)^{1/p'}.$$

Since $\alpha p' \geq n(p'-\varepsilon)$ and $\omega^{-\frac{1}{p-1}} \in A_{p'-\varepsilon}$ the last factor on the left hand side is finite by Lemma 2.1 (vii). Analogously

$$\int_{\mathbb{R}^n} |f|^s \rho \, dx \leq \|f\|^s_{r,v} \Big(\int_{\mathbb{R}^n} v(x)^{-\frac{1}{\tau-1}} \, (1+|x|)^{-\alpha\tau'} \, dx \Big)^{1/\tau'}.$$

Since $\alpha\tau' \geq n(p'-\delta)$ and $v^{-\frac{1}{\tau-1}} \in A_{\tau'-\delta}$ an application of Lemma 2.1 (vii) completes the proof. □

Lemma 2.3 *Let $n' < q < \infty$ and $\omega \in A_1$. Then $(1+|\cdot|)^q \, \omega \in A_q$ or equivalently $(1+|\cdot|)^{-q'} \omega' \in A_{q'}$. Furthermore $A_q((1+|\cdot|)^q \, \omega) \leq c \, A_1(\omega)$.*

Proof: Using the notation $\fint_M g := \frac{1}{|M|} \int_M g \, dx$ we have for $x_0 \in \mathbb{R}^n$ and $R > 0$

$$\begin{aligned}
&\Big(\fint_{B_R(x_0)} (1+|x|)^q \, \omega \Big) \, \Big(\fint_{B_R(x_0)} (1+|x|)^{-q'} \, \omega^{-\frac{1}{q-1}} \Big)^{q-1} \\
&\leq \Big(\fint_{B_R(x_0)} (1+|x|)^q \, \omega \Big) \ \operatorname*{ess\,sup}_{x \in B_R(x_0)} \frac{1}{\omega(x)} \ \Big(\fint_{B_R(x_0)} (1+|x|)^{-q'} \Big)^{q-1}.
\end{aligned}$$

Since $q' < n$ we can estimate this quantity from above as follows (cf. [4], p. 260)

$$\begin{aligned}
&\leq c \ \max\{(1+|x_0|)^q, (1+R)^q\} \Big(\fint_{B_R(x_0)} \omega \Big) \operatorname*{ess\,sup}_{x \in B_R(x_0)} \frac{1}{\omega(x)} \ \min\{(1+R)^{-q}, (1+|x_0|)^{-q}\} \\
&\leq c \Big(\fint_{B_R(x_0)} \omega \Big) \ \operatorname*{ess\,sup}_{x \in B_R(x_0)} \frac{1}{\omega(x)} \leq c \, A_1(\omega) < \infty.
\end{aligned}$$

□

For $1 < q < \infty$, $\omega \in A_q, k \geq 1$ and a domain $\Omega \subset \mathbb{R}^n$ we define the weighted Sobolev spaces

$$\begin{aligned}
W^{k,q}_\omega(\Omega) &= \{u \in L^q_\omega(\Omega) \ : \ D^\alpha u \in L^q_\omega(\Omega), \ |\alpha| \leq k\}, \\
\widehat{W}^{1,q}_\omega(\Omega) &= \{u \in W^{1,1}_{loc}(\Omega) \ : \ \nabla u \in L^q_\omega(\Omega)^n\},
\end{aligned}$$

equipped with their respective norm $\|\cdot\|_{k,q,\omega}$ and seminorm $\|\nabla \cdot\|_{q,\omega}$. The subspace of functions $u \in W^{1,q}_\omega(\Omega)$ and $u \in \widehat{W}^{1,q}_\omega(\Omega)$ with trace $u\big|_{\partial\Omega} = 0$ is denoted by $W^{1,q}_{0,\omega}(\Omega)$ and $\widehat{W}^{1,q}_{0,\omega}(\Omega)$ respectively (see [6], [8] for the definition of the trace). The dual spaces of $W^{1,q'}_{0,\omega'}(\Omega)$, $\widehat{W}^{1,q'}_{0,\omega'}(\Omega)$ and $\widehat{\mathcal{W}}^{1,q'}_{\omega'}(\Omega) := \widehat{W}^{1,q'}_{\omega'}(\Omega)/\mathbb{C}$ are denoted by $W^{-1,q}_\omega(\Omega)$, $\widehat{W}^{-1,q}_{0,\omega}(\Omega)$ and $\widehat{\mathcal{W}}^{-1,q}_\omega(\Omega)$ respectively. The norm of $\widehat{\mathcal{W}}^{-1,q}_\omega(\Omega)$ is denoted by $\|\cdot\|_{-1,q,\omega}$. If $\omega \equiv 1$ we simply write $L^q(\Omega)$, $W^{1,q}(\Omega)$, $\widehat{\mathcal{W}}^{-1,q}(\Omega), \ldots$

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^n$ be an aperture domain with Lipschitz boundary.*

i) Let $1 < q < \infty$, $\omega \in A_q$ and $(1+|\cdot|)^{-q}\omega \in A_q$. Then there is a constant $C > 0$ such that for every $u \in \widehat{W}^{1,q}_\omega(\Omega)$ there are constants $K^\pm$ with

$$\|u - K^\pm\|_{q,(1+|\cdot|)^{-q}\omega,\Omega_\pm} \leq C \, \|\nabla u\|_{q,\omega,\Omega_\pm}.$$

The constant $C = C(\omega) > 0$ can be chosen as an increasing function of $A_q(\omega) A_q((1+|\cdot|)^{-q}\omega)$.

ii) Let $n' < q < \infty$, $\omega \in A_1$. Then there is an A_1-consistent constant $C > 0$ such that for every $u \in \widehat{W}^{1,q'}_{\omega'}(\Omega)$ there are constants $K^{\pm}$ with

$$\|u - K^{\pm}\|_{q',(1+|\cdot|)^{-q'}\omega',\Omega_\pm} \le C\,\|\nabla u\|_{q',\omega',\Omega_\pm}.$$

Proof: i) The estimate for the whole space $\mathbb{R}^n$ is a special case of Corollary 3.7 in [4]. The constant appearing in this result is an increasing function of $A_{q'}(\omega')A_{q'}((1+|\cdot|)^{-q'}\omega')$. Since $\Omega_\pm$ and $\omega' \in A_{q'}$ satisfy the assumptions in [1] for the existence of bounded linear extension operators $E_\pm : \widehat{W}^{1,q}_\omega(\Omega_\pm) \to \widehat{W}^{1,q}_\omega(\mathbb{R}^n)$ with A_q-consistent norms assertion i) is clear.

ii) Since ω', $(1+|\cdot|)^{-q'}\omega' \in A_{q'}$ by Lemma 2.3, we can apply i). It follows from Lemma 2.3 that $A_{q'}(\omega')A_{q'}(\omega'(1+|\cdot|)^{-q'}) \le c\,A_1(\omega)^{2/(q-1)}$. Hence by the properties of the constant in i) the constant $C > 0$ is A_1-consistent. $\square$

Remark: Note that the imbeddings $\widehat{\mathcal{W}}^{1,q}_\omega(\mathbb{R}^n) \hookrightarrow L^q_{(1+|x|)^{-q}\omega}(\mathbb{R}^n)/\mathbb{C}$ and $\widehat{\mathcal{W}}^{1,q}_\omega(\mathbb{R}^n) \hookrightarrow L^r_\omega(\Omega)/\mathbb{C}$, $1 < r < \infty$, are not true for general Muckenhoupt weights $\omega \in A_q$: Let $1 < q < n$ and $\omega(x) = (1+|x|)^{-\alpha}$ with $n-1 < \alpha < n$ yielding $\omega \in A_q$. Further consider a sequence $(u_k) \subset C_0^\infty(\mathbb{R}^n)$ such that u_k is equal to 1 for $|x| < k$, equal to 0 for $|x| > k+1$ and $|\nabla u_k|$ is bounded independent of k. Then $\|\nabla u_k\|_{q,\omega} \to 0$ for $k \to \infty$ while the sequences $\|u_k\|_{(1+|x|)^{-q}\omega}$ and $\|u_k\|_{r,\omega}$ are increasing.

For $n < q < \infty$ one may take even $\omega \equiv 1$ to get a counter-example.

3 Resolvent estimates for the Stokes system

We need a uniqueness assertion for the stationary Stokes system

$$-\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{11}$$

$$u\big|_{\partial\Omega} = 0, \quad \Phi(u) = 0. \tag{12}$$

To reduce this problem to perturbed Stokes problems on half spaces $\mathbb{R}^n_\pm$ and on bounded domains we introduce a localisation technique:
Let B be an open ball centered at 0 such that $\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B$. Let B' be another open ball B' with $\overline{B'} \subset B$ such that $\Omega \cup B' = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B'$. Furthermore we may assume that there is a bounded domain $G \subset \Omega$ with $B \cap \Omega \subset G$ and with boundary of class C^1 or $C^{1,1}$, if $\partial\Omega \in C^1$ or $\partial\Omega \in C^{1,1}$, respectively. We define cut-off functions $\eta \in C^\infty(\mathbb{R}^n)$ and $\eta+, \eta_- \in C^\infty(\mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B')$ by the following properties:

$$\begin{gathered} 0 \le \eta, \eta_+, \eta_- \le 1,\ \eta_+ + \eta_- + \eta = 1 \text{ on } \Omega \\ \eta = 0 \text{ on } \mathbb{R}^n \setminus B,\ \eta = 1 \text{ on } \overline{B'}, \\ \eta_+ = 1 \text{ on } \mathbb{R}^n_+ \setminus B,\ \eta_+ = 0 \text{ on } \mathbb{R}^n_- \cup \overline{B'}, \\ \eta_- = 1 \text{ on } \mathbb{R}^n_- \setminus B,\ \eta_- = 0 \text{ on } \mathbb{R}^n_+ \cup \overline{B'}. \end{gathered}$$

Lemma 3.1 *Let $1 < q < \infty$, $\omega \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be an aperture domain with $\partial\Omega \in C^{1,1}$. Furthermore let $u \in \widehat{W}^{2,q}_\omega(\Omega) \cap \widehat{W}^{1,q}_\omega(\Omega)$ and p be a solution of the Stokes system (11), (12). Assume that there are constants $K^\pm$ such that $p - K^\pm \in L^q_\omega(\Omega_\pm)$. Then $u = 0$ and $\nabla p = 0$.*

Proof: For $U = \mathbb{R}^n_\pm, G$ and $\phi = \eta_\pm, \eta$ consider the local equations

$$-\Delta(\phi u) + \nabla(\phi p) = \tilde{f}, \quad \operatorname{div}(\phi u) = \tilde{g}, \quad (\phi u)\big|_{\partial U} = 0,$$

where $\tilde{f} = -2(\nabla\phi)\nabla u - (\Delta\phi)u + (\nabla\phi)p$ and $\tilde{g} = (\nabla\phi) \cdot u$. Here we replace p by $p - K^\pm$ if $U = \mathbb{R}^n_\pm$.

Let $U = \mathbb{R}^n_\pm$: Note that because of the bounded support of $\nabla\phi$ we have for some $1 < s < \infty$ that $\tilde{f} \in \widehat{W}_0^{-1,s}(U)^n :=$ and $\tilde{g} \in L^s(U)$ (see [7]), Lemma 2.2) and that $(\phi u, \phi(p - K^\pm)) \in \widehat{W}^{1,q}_\omega(U)^n \times L^q_\omega(U)$ is a weak solution of the stationary Stokes system in $U = \mathbb{R}^n_\pm$ with right hand sides $\tilde{f}$ and $\tilde{g}$. Hence Theorem 5.2 (II) in [6] yields $(\phi u, \phi(p - K_\pm)) \in \widehat{W}^{1,s}(\mathbb{R}^n_\pm)^n \times L^s(\mathbb{R}^n_\pm)$. Now we are in the case without weights and one can conclude by a procedure using Sobolev's imbedding exactly as in [2] p. 20 that $\nabla(\phi u) \in L^2(\mathbb{R}^n_\pm)$.

Let $U = G$: Noting $\tilde{f} \in L^s(G)^n$ and $\tilde{g} \subset W^{1,s}(G) \cap \widehat{W}^{-1,s}(G)$ Theorem 3.3 ii) in [7] implies that $(\phi u, \phi p) \in W^{2,s}(G)^n \times W^{1,s}(G)$; again a bootstrapping argument yields $\nabla(\phi u) \in L^2(G)^n$.

We get that $\nabla u \in L^2(\Omega)^{n^2}$. Following the arguments in [2] p. 20 we conclude that $u = 0$ and $\nabla p = 0$. □

Lemma 3.2 *Let $\omega \in A_1$, $n' < q < \infty$, $0 < \varepsilon < \frac{\pi}{2}$ and let $\Omega \subset \mathbb{R}^n$ be an aperture domain with $\partial\Omega \in C^{1,1}$. Then for every $f \in L^{q'}_{\omega'}(\Omega)^n$ and every $\lambda \in \Sigma_\varepsilon$ there is a unique solution $(u, p) \in W^{2,q'}_{\omega'}(\Omega)^n \times \widehat{W}^{1,q'}_{\omega'}(\Omega)$ of the Stokes resolvent system*

$$\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0, \quad \Phi(u) = 0. \tag{13}$$

Furthermore there is an A_1-consistent constant $C = C(\omega, \varepsilon) > 0$ such that

$$\|\lambda u\|_{q',\omega'} + \|\nabla^2 u\|_{q',\omega'} + \|\nabla p\|_{q',\omega'} \le C\,\|f\|_{q',\omega'} \tag{14}$$

for all $\lambda \in \Sigma_\varepsilon$.

Proof: Since $\omega' \in A_{q'}$ by Lemma 2.1 (i), the existence and uniqueness assertion as well as the estimate (14) for $|\lambda| \ge \delta > 0$ with a constant $C = C(\delta)$ depending $A_{q'}$-consistently on ω' (and therefore depending A_1-consistently on ω) is proved in [8]. So it remains to prove that under the assumptions of the lemma C is independent of $\delta > 0$ and A_1-consistent. Assume this was not true. Then there is a $c_0 > 0$ a sequence $(\omega_j) \subset A_1$ with $\sup_j A_1(\omega_j) \le c_0$ and a sequence $(\lambda_j) \subset \Sigma_\varepsilon$ with $\lambda_j \to 0$ for $j \to \infty$ and $(u_j, p_j) \in W^{2,q'}_{\omega'_j}(\Omega)^n \times \widehat{W}^{1,q'}_{\omega'_j}(\Omega)$ solving (13) with respect to $\lambda = \lambda_j$ and $f = f_j$ such that

$$\|(\lambda_j u_j, \nabla^2 u_j, \nabla p_j)\|_{q',\omega'_j} = 1, \tag{15}$$

$$\|f_j\|_{q',\omega'_j} \to 0. \tag{16}$$

A standard cut-off technique [8] and Theorem 1.1 in [7] yield the preliminary estimate:

$$\begin{aligned}\|(\lambda_j u_j, \nabla^2 u_j, \nabla p_j)\|_{q',\omega'_j} \le C\,(\,&\|f_j\|_{q',\omega'_j} + \|u_j\|_{1,q',\omega'_j,G} \\ &+ \|\lambda_j u_j\|_{[W^{1,q}_{\omega_j}(G)]'} + \|p_j\|_{q',\omega'_j,G}\,).\end{aligned} \tag{17}$$

By Lemma 2.4 and (15) there is a constant $C > 0$ and a constant $(n \times n)$-Matrix $K_j^\pm$ such that $\|\nabla u_j - K_j^\pm\|_{q',(1+|\cdot|)^{-q'}\omega',\Omega_\pm} \le C\,\|\nabla^2 u_j\|_{q',\omega'_j,\Omega} \le C$. Note that $\nabla u_j \in L^{q'}_{\omega'_j}(\Omega_\pm)^{n^2}$ and

$K_j^{\pm} - \nabla u_j \in L^{q'}_{(1+|\cdot|)^{-q'}\omega_j'}(\Omega_\pm)^{n^2}$, thus $K_j^{\pm} \in L^{q'}_{(1+|\cdot|)^{-q'}\omega_j'}(\Omega_\pm)^{n^2}$. Since $(1+|\cdot|)^{-q'}\omega_j' \in A_{q'}$ by Lemma 2.3, this implies $K_j^{\pm} = 0$ by Lemma 2.1 iv). Hence with a constant $C > 0$ independent of j

$$\|\nabla u_j\|_{q',\omega_j'(1+|\cdot|)^{-q'},\Omega} + \|\nabla^2 u_j\|_{q',\omega_j'(1+|\cdot|)^{-q'},\Omega} \leq C\,\|\nabla^2 u_j\|_{q',\omega_j',\Omega} \leq C.$$

Since due to the proof of Lemma 2.3 $A_{q'}((\omega_j'(1+|\cdot|)^{-q'}) \leq c\,A_1(\omega_j)^{\frac{1}{q-1}} \leq c\,c_0^{\frac{1}{q-1}}$, by Lemma 2.2 in [8] there is an $1 < s < n'$ and a weight function $\rho = (1+|\cdot|)^{-\alpha} \in A_1$, $0 < \alpha < n$, such that all the spaces $L^{q'}_{\omega_j'(1+|\cdot|)^{-q'}}(\Omega)$ are continuously imbedded into $L^s_\rho(\Omega)$, where the embedding constant can be chosen uniformly with respect to j. Hence $L^{q'}_{\omega_j'}(\Omega)$ is continuously imbedded into $L^s_v(\Omega), v = (1+|\cdot|)^s\rho$ with imbedding constant independent of j. Assuming w.l.o.g. that $s < \alpha < n$ we get that $v = (1+|\cdot|)^{s-\alpha} \in A_1$.

Thus the sequences (∇u_j), $(\nabla^2 u_j)$ and (∇p_j) are bounded in $L^s_\rho(\Omega)^{n^2}$, $L^s_v(\Omega)^{n^3}$ and $L^s_v(\Omega)^n$, respectively. Therefore, suppressing the notation of subsequences, we get the weak convergences $\nabla u_j \rightharpoonup \nabla u \in L^s_\rho(\Omega)^{n^2}$, $\nabla^2 u_j \rightharpoonup \nabla^2 u \in L^s_v(\Omega)^{n^3}$ and $\nabla p_j \rightharpoonup \nabla p \in L^s_v(\Omega)^n$, $\int_G p\,dx = 0$. Since $v \in A_s$ and $\rho = (1+|\cdot|)^{-s}v \in A_s$ we get from Lemma 2.4 i) constants $K_\pm$ such that $p - K_\pm \in L^s_\rho(\Omega_\pm)$. Furthermore $\operatorname{div} u = 0$. Since $u_k\big|_{\partial\Omega} = 0$ and because of the special form of ρ we conclude by Poincaré's inequality that $u_k \rightharpoonup u$ weakly in $L^s(\Omega \cap \hat{B})$ for all balls $\hat{B}$ with $\Omega \cap \hat{B} \neq 0$. Hence $u \in L^s_{loc}(\overline{\Omega})^n$, $u\big|_{\partial\Omega} = 0$ and $-\Delta u + \nabla p = 0$ in the distributional sense. As in [2] p. 22 one shows $\Phi(u) = \int_M u\cdot\nu\,ds = 0$. Therefore Lemma 3.1 can be applied to $u \in \widehat{W}^{2,s}_v(\Omega)^n \cap \widehat{W}^{1,s}_\rho(\Omega)^n \subset \widehat{W}^{2,s}_\rho(\Omega)^n \cap \widehat{W}^{1,s}_\rho(\Omega)^n$ and $\nabla p \in L^s_v(\Omega)^n \subset L^s_\rho(\Omega)^n$ with $p - K_\pm \in L^s_\rho(\Omega_\pm)$ to conclude that $u = 0$ and $\nabla p = 0$.

Note that $A_{q'}(\omega_j') \leq A_1(\omega_j)^{\frac{1}{q-1}} \leq c_0^{\frac{1}{q-1}}$. Thus we may use a compactness argument (Lemma 2.3 in [8]) to show exactly as in the proof of Theorem 1.1 in [8] that (after choosing a subsequence) all the terms on the right hand side of (17) converge to 0. This contradicts (15). $\square$

4 Maximal L^p-Regularity for the instationary Stokes system

To apply an abstract result on maximal L^p-regularity stated as Theorem 4.3 below, we introduce the Stokes operator in weighted L^q-spaces. In [8] the Helmholtz decomposition of weighted L^q-spaces in aperture domains $\Omega \subset \mathbb{R}^n$ with boundary of class C^1

$$L^q_\omega(\Omega)^n = L^q_{\omega,\sigma}(\Omega) \oplus G^q_\omega(\Omega)$$

is proved for $1 < q < \infty$ and arbitrary $\omega \in A_q$. Here $L^q_{\omega,\sigma}(\Omega)$ denotes the closure in $L^q_\omega(\Omega)^n$ of the space $C^\infty_{0,\sigma}(\Omega)$ of smooth, solenoidal vector fields with compact support in Ω and $G^q_\omega(\Omega)$ are the gradient fields in $L^q_\omega(\Omega)^n$. The corresponding bounded projection operator from $L^q_\omega(\Omega)^n$ onto $L^q_{\omega,\sigma}(\Omega)$ with kernel $G^q_\omega(\Omega)$ is denoted by $P_{q,\omega}$.

Then the Stokes operator $\mathcal{A}_{q,\omega}$ is defined as follows:

$$\begin{aligned} \mathcal{D}(\mathcal{A}_{q,\omega}) &= W^{2,q}_\omega(\Omega)^n \cap W^{1,q}_{0,\omega}(\Omega)^n \cap L^q_{\omega,\sigma}(\Omega) \\ \mathcal{A}_{q,\omega}u &= -P_{q,\omega}\Delta u \qquad \text{for } u \in \mathcal{D}(\mathcal{A}_{q,\omega}). \end{aligned}$$

Lemma 6.2 in [8] shows that $\mathcal{D}(\mathcal{A}_{q,\omega}) = \{u \in W^{2,q}_\omega(\Omega)^n \cap W^{1,q}_{0,\omega}(\Omega)^n : \operatorname{div} u = 0,\ \Phi(u) = 0\}$. If $\omega \equiv 1$ we simply write $L^q_\sigma(\Omega)$, $\mathcal{A}_q, \cdots$.

Theorem 4.1 *Let $\omega \in A_1, 1 < q < \infty$, $0 < \varepsilon < \frac{\pi}{2}$ and let $\Omega \subset \mathbb{R}^n$ be an aperture domain with $\partial\Omega \in C^{1,1}$. Then for all $\lambda \in \Sigma_\varepsilon$ the resolvents $(\lambda + \mathcal{A}_{q,\omega})^{-1}$ and $(\lambda + \mathcal{A}_{q',\omega'})^{-1}$ exist.*

For $1 < q < n$ there is an A_1-consistent constant $C = C(\varepsilon, \omega) > 0$ such that the estimate

$$\|\lambda\,(\lambda + \mathcal{A}_{q,\omega})^{-1} f\|_{q,\omega} \leq C\,\|f\|_{q,\omega} \quad \forall f \in L^q_{\omega,\sigma}(\Omega). \tag{18}$$

holds uniformly for all $\lambda \in \Sigma_\varepsilon$.

For $n' < q < \infty$ there is an A_1-consistent constant $C = C(\varepsilon, \omega) > 0$ such that the estimate

$$\|\lambda\,(\lambda + \mathcal{A}_{q',\omega'})^{-1} f\|_{q',\omega'} \leq C\,\|f\|_{q',\omega'} \quad \forall f \in L^{q'}_{\omega',\sigma}(\Omega) \tag{19}$$

holds uniformly for all $\lambda \in \Sigma_\varepsilon$.

Proof: Since $\omega \in A_1 \subset A_q$ and $\omega' \in A_{q'}$ the existence of the resolvents $(\lambda + \mathcal{A}_{q,\omega})^{-1}$ and $(\lambda + \mathcal{A}_{q',\omega'})^{-1}$ for $\lambda \in \Sigma_\varepsilon$ follows from Theorem 1.1 in [8]. The estimate (19) follows immediately from Lemma 3.2. The estimate (18) is proved by duality: Let $g \in L^q_{\omega,\sigma}(\Omega)$, $f \in L^{q'}_{\omega'}(\Omega)^n$ and let u, p be the solution of the Stokes resolvent system corresponding to λ, f from Lemma 3.2. Then for $v = (\lambda + \mathcal{A}_{q,\omega})^{-1} g \in L^q_\omega(\Omega)^n$

$$|(v,f)| = |(v, \lambda u - \Delta u + \nabla p)| = |(\lambda v - \Delta v, u)| = |(g,u)| \leq \frac{1}{|\lambda|}\|g\|_{q,\omega}\|f\|_{q',\omega'}.$$

Since $f \in L^{q'}_{\omega'}(\Omega)^n = [L^q_\omega(\Omega)^n]'$ was arbitrary the estimate (18) follows. □

To apply a characterisation of maximal regularity due to L. Weis ([11], [12]) we introduce the notion of $\mathcal{R}$-bounded operator families. In the sequel $\mathcal{L}(X)$ denotes the space of bounded linear operators on a Banach space X.

Definition 4.1 *Let X be a Banach space. A subset $\mathcal{T} \subset \mathcal{L}(X)$ is called $\mathcal{R}$-bounded if there exists a constant $C \in \mathbb{R}$ such that*

$$\int_0^1 \Big\| \sum_{j=1}^N r_j(u)\, T_j x_j \Big\|_X \, du \leq C \int_0^1 \Big\| \sum_{j=1}^N r_j(u)\, x_j \Big\|_X \, du \tag{20}$$

for all $T_1, \dots, T_N \in \mathcal{T}$, $x_1, \dots, x_N \in X$ and $N \in \mathbb{N}$, where (r_j) is a sequence of independent, symmetrically distributed, $\{-1,1\}$-valued random variables defined on $[0,1]$, e.g. the Rademacher functions. The smallest constant C such that (20) holds is called $\mathcal{R}$-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$.

Lemma 4.1 *Let (Ω, Σ, μ) be a measure space, $1 < q < \infty$ and $X = L^q(\Omega, \mu)$. Then $\mathcal{T} \subset L(X)$ is $\mathcal{R}$-bounded if and only if there is a constant $C \in \mathbb{R}$ such that*

$$\Big\| \Big(\sum_{j=1}^N |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\Omega,\mu)} \leq C \Big\| \Big(\sum_{j=1}^N |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\Omega,\mu)} \tag{21}$$

for all $T_1, \dots, T_N \in \mathcal{T}$, $f_1, \dots, f_N \in X$ and $N \in \mathbb{N}$.

Proof: See e.g. [7], Lemma 4.2. $\square$

Note that for $q = 2$ we have $\omega' = \omega^{-1/(q-1)} = \omega^{-1}$ for a weight function $\omega \in A_2$, i.e., ω^{-1} means $\frac{1}{\omega}$; this notation should not cause confusion with an inverse function.

The following Theorem is a consequence of the proof of Theorem 6.4., Chapter V, in [9], although the statement given there is weaker. We give the proof with slight modifications for the convenience of the reader.

Theorem 4.2 *Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Further let $\mathcal{T} \subset \mathcal{L}(L^q(\Omega))$ be a family of linear operators with the property that for all $\omega \in A_1$ there is a constant $C = C(\omega) > 0$ depending only on the A_1-constant of ω such that the following estimates hold*

$$\|Tf\|_{2,\omega} \le C\,\|f\|_{2,\omega} \qquad \forall f \in L^q(\Omega) \cap L^2_\omega(\Omega), \tag{22}$$

$$\|Tf\|_{2,\omega^{-1}} \le C\,\|f\|_{2,\omega^{-1}} \qquad \forall f \in L^q(\Omega) \cap L^2_{\omega^{-1}}(\Omega). \tag{23}$$

Then $\mathcal{T}$ is $\mathcal{R}$-bounded in $L^q(\Omega)$.

Proof: First, let $\Omega = \mathbb{R}^n$ and $q > 2$. Set $s = \left(\frac{q}{2}\right)'$ and $1 < \sigma := (1+s)/2 < s$. Then given $0 \le w \in L^s(\mathbb{R}^n)$ let $W(x) = (Mw^\sigma(x))^{\frac{1}{\sigma}}$ and observe that

i) $w(x) \le W(x)$

ii) $\|W\|_s \le C_s\,\|w\|_s$.

iii) $W \in A_1$ with A_1-constant depending only on s (see [9], Theorem 3.4, Chapter II).

It follows from i), iii) and (22)

$$\|Tf\|_{2,w} \le \|Tf\|_{2,W} \le C\,\|f\|_{2,W},$$

where C does not depend on the choice of w but only on s. Given sequences $(T_j) \subset \mathcal{T}$ and $(f_j) \subset L^q(\mathbb{R}^n)$ there exists $0 \le w \in L^s(\mathbb{R}^n)$ with norm $\|w\|_s = 1$ such that

$$\begin{aligned}
\Big\|\Big(\sum_j |T_jf_j|^2\Big)^{\frac12}\Big\|_q &= \Big\|\sum_j |T_jf_j|^2\Big\|_{s'}^{\frac12} \\
&= \Big(\int_{\mathbb{R}^n} \sum_j |T_jf_j|^2 w\,dx\Big)^{\frac12} \le C\Big(\int_{\mathbb{R}^n} \sum_j |f_j|^2 W\,dx\Big)^{\frac12} \\
&\le C\,\Big\|\sum_j |f_j|^2\Big\|_{s'}^{\frac12}\,\|W\|_s^{\frac12} \le C\,\Big\|\Big(\sum_j |f_j|^2\Big)^{\frac12}\Big\|_q
\end{aligned}$$

where C depends only on s, i.e., $\mathcal{T}$ is $\mathcal{R}$-bounded in $L^q(\mathbb{R}^n)$.

Let $q < 2$. Choose $g = (g_1, \dots, g_N) \in L^{q'}(\mathbb{R}^n)^N$ with norm $\|g\|_{q'} = 1$ such that

$$\begin{aligned}
\Big\|\Big(\sum_{j=1}^N |T_jf_j|^2\Big)^{\frac12}\Big\|_q &= \sum_{j=1}^N \int_{\mathbb{R}^n} T_jf_j\,g_j\,dx \\
&= \sum_{j=1}^N \int_{\mathbb{R}^n} f_j\,(T_j)'g_j\,dx \le \Big\|\Big(\sum_j |f_j|^2\Big)^{\frac12}\Big\|_q \Big\|\Big(\sum_j |(T_j)'g_j|^2\Big)^{\frac12}\Big\|_{q'}.
\end{aligned}$$

By (23) and a duality argument using $(L^2_\omega(\mathbb{R}^n))' = L^2_{\omega^{-1}}(\mathbb{R}^n)$ there is an A_1-consistent constant $C > 0$ such that $\|(T)'f\|_{2,\omega} \le C\,\|f\|_{2,\omega}$ for all $\omega \in A_1$, all $T \in \mathcal{T}$ and all $f \in L^{q'}(\mathbb{R}^n) \cap L^2_\omega(\mathbb{R}^n)$. Since $q' > 2$ the first step of the proof yields a constant C independent of N such that

$$\Big\|\Big(\sum_{j=1}^N |(T_j)'g_j|^2\Big)^{\frac{1}{2}}\Big\|_{q'} \le C\,\Big\|\Big(\sum_{j=1}^N |g_j|^2\Big)^{\frac{1}{2}}\Big\|_{q'} = 1.$$

This proofs the claim in the case $q < 2$ and $\Omega = \mathbb{R}^n$.

The case $q = 2$ follows from interpolation.

The assertion for a general open set $\Omega \subset \mathbb{R}^n$ can be reduced to $\Omega = \mathbb{R}^n$ by extending the functions by 0 to $\mathbb{R}^n$. See [7], proof of Theorem 4.3, for details. □

Let A be the generator of an analytic semigroup in the Banach space X. We consider the abstract Cauchy problem

$$u_t - Au = f, \qquad u(0) = 0. \tag{24}$$

For $f \in L^1_{loc}([0,\infty); X)$ the mild solution on $J = \mathbb{R}_+$ is given by $u(t) = \int_0^t e^{(t-s)A} f(s)ds$.

Definition 4.2 *Let $1 < s < \infty$ and $f \in L^s(J, X)$. We say that A has the property $\mathcal{MR}$ of maximal regularity, if for every $0 < T < \infty$ and every $f \in L^s(0,T;X)$ the mild solution u of (24) belongs to $W^{1,s}(0,T;X) \cap L^s(0,T;D(A))$ and if there is a constant $C \in \mathbb{R}$ such that the estimate*

$$\|u_t\|_{L^s(0,T;X)} + \|Au\|_{L^s(0,T;X)} \le C\,\|f\|_{L^s(0,T;X)} \tag{25}$$

holds.

A generator A of a bounded analytic semigroup is said to have the property $\mathcal{MR}_\infty$ of maximal regularity if A has the property $\mathcal{MR}$ and the estimate (25) holds with a constant C independent of $T \in (0,\infty)$, i.e., $(0,T)$ can be replaced by the infinite interval $(0,\infty)$ in (25).

For an aperture domain with boundary of class $C^{1,1}$ the property $\mathcal{MR}$ was proved for the Stokes operator $-\mathcal{A}_{q,\omega}$ in $L^q_{\omega,\sigma}(\Omega)$ even for general Muckenhoupt weights, see [8]. In the sequel we show in the case without weights that the Stokes operator $\mathcal{A}_q$ has the property $\mathcal{MR}_\infty$.

The following characterisation of maximal regularity by $\mathcal{R}$-bounded operator families is due to L. Weis [11], [12]. Recall that a UMD space X is defined to be a Banach space such that the Hilbert transform is bounded from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};X)$ for $1 < p < \infty$. It is well known that $L^q(\Omega,\mu)$ is an UMD space for $1 < q < \infty$.

Theorem 4.3 *Let A be the generator of a bounded analytic semigroup in a UMD space X. Then A has the property of maximal regularity $\mathcal{MR}_\infty$, iff the operator family*

$$\{\lambda\,(\lambda - A)^{-1} \,:\, \lambda \in i\mathbb{R},\ \lambda \neq 0\}$$

is $\mathcal{R}$-bounded.

Theorem 4.4 *Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, be an aperture domain with boundary of class $C^{1,1}$. Then the Stokes operator $-\mathcal{A}_q$ generates a bounded analytic semigroup in $L^q_\sigma(\Omega)$, which has the property $\mathcal{MR}_\infty$ of maximal regularity.*

Proof: We can write $\lambda(\lambda+\mathcal{A}_q)^{-1}P_q$ for $\lambda \in i\mathbb{R}\setminus\{0\}$ as an $n\times n$-martix $(A^\lambda_{i,j})_{i,j=1,\dots n}$ of linear bounded operators on $L^q(\Omega)$.

Recall from Lemma 2.2 that for $1<q<\infty$ and $v \in A_2$ there exist $s\in(1,\infty)$ and $\rho \in A_s$ such that $L^2_v(\Omega)+L^q(\Omega)\subset L^s_\rho(\Omega)$. Applying the uniqueness assertion of Theorem 1.1 in [8] to the pair (s,v) we get $(\lambda+\mathcal{A}_q)^{-1}P_q f = (\lambda+\mathcal{A}_{s,\rho})^{-1}P_{s,\rho}f = (\lambda+\mathcal{A}_{2,v})^{-1}P_{2,v}f$ for $f\in L^q(\Omega)^n\cap L^2_v(\Omega)^n$. We will use this fact with $v=\omega$ and $v=\omega^{-1}$ in the following arguments.

Since furthermore $n'<2<n$ for $n\geq 3$ and since $\omega\in A_1$ implies $\omega\in A_2$ as well as $\omega^{-1}\in A_2$, the operators $A^\lambda_{i,j}$ $i,j=1,\dots,n$, extend by Theorem 4.1 and the boundedness properties of the Helmholtz projection (see Theorem 3.1 in [8]) to linear bounded operators on $L^2_\omega(\Omega)$ and on $L^2_{\omega^{-1}}(\Omega)$ for all $\omega\in A_1$ with A_1-consistent norm bound independent of $\lambda\in i\mathbb{R}\setminus\{0\}$. Hence Theorem 4.2 yields that the operator families $\{A^\lambda_{i,j} : \lambda \in i\mathbb{R}\setminus\{0\}\}$ are $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\Omega))$ for $i,j=1,\dots,n$. This implies the $\mathcal{R}$-boundedness of $\{\lambda(\lambda+\mathcal{A}_q)^{-1} : \lambda\in i\mathbb{R}\setminus\{0\}\}$ in $\mathcal{L}(L^q_\sigma(\Omega))$. Thus Theorem 4.3 completes the proof. □

To formulate the global in time result for the instationary Stokes system following from Theorem 4.4 with a nonvanishing flux and a nonzero inital value we introduce some notation:

For $n'<q<\infty$ and an aperture domain $\Omega\subset\mathbb{R}^n$ with Lipschitz boundary there exists a flux function $\chi\in W^{2,q}(\Omega)^n$ satisfying

$$\chi\big|_{\partial\Omega}=0, \quad \operatorname{div}\chi=0, \quad \Phi(\chi)=1, \tag{26}$$

see [5], Theorem 1.4.

We introduce some appropriate spaces for the initial value:

$$\begin{aligned}
&L^{[s,q]}_\sigma(\Omega) := \{u\in L^q_\sigma(\Omega) \ : \ \|u\|_{[s,q]}<\infty\},\\
&\|u\|_{[s,q]} := \|u\|_q + \|\mathcal{A}_q e^{-t\mathcal{A}_q}u\|_{L^s(\mathbb{R}_+,L^q_\sigma(\Omega))}.\\
&L^{[s,q]}_\Sigma(\Omega) := L^{[s,q]}_\sigma(\Omega)\oplus \operatorname{span}\{\chi\}\\
&\|u+\alpha\chi\|_{[s,q],\chi} := \|u\|_{[s,q]}+|\alpha|
\end{aligned}$$

for $1<s,q<\infty$. Note that the flux $\Phi(u)=\int_M u\cdot\nu\,d\sigma$ initially defined for $u\in W^{1,q}_0(\Omega)^n$ extends uniquely to a bounded linear functional on the subspace $\{u\in L^q(\Omega)^n : \operatorname{div}u=0,\ u\cdot\nu\big|_{\partial\Omega}=0\}$ of $L^q(\Omega)^n$ (see [5], Theorem 1.7).

Theorem 4.5 *Let $1<s,q<\infty$ and let Ω be as in Theorem 4.4. Let $f\in L^s(\mathbb{R}_+;L^q(\Omega)^n)$.*

i) For $n'<q<\infty$, $\alpha\in W^{1,s}(\mathbb{R}_+)$ and $u_0\in L^{[s,q]}_\Sigma(\Omega)$ with $\Phi(u_0)=\alpha(0)$ there exists a unique solution $(u,\nabla p)$ of the Stokes system (5), (6) with flux $\Phi(u)=\alpha$ satisfying

$$\nabla^2 u\in L^s(\mathbb{R}_+;L^q(\Omega)^{n^2}), \quad u_t,\ \nabla p\in L^s(\mathbb{R}_+;L^q(\Omega)^n), \tag{27}$$

$$\int_0^\infty \{\,\|u_t\|^s_q+\|\nabla^2u\|^s_q+\|\nabla p\|^s_q\,\}\,dt \leq C\Big(\int_0^\infty \|f\|^s_q\,dt+\|u_0\|^s_{[s,q],\chi}+\|\alpha\|^s_{W^{1,s}}\Big) \tag{28}$$

with a constant $C=C(s,q,\Omega)>0$.

ii) For $1 < q < n'$ and $u_0 \in L^{[s,q]}_\sigma(\Omega)$ there exists a unique solution $(u, \nabla p)$ of the Stokes system (5), (6) satisfying (27). Furthermore u has a vanishing flux $\Phi(u) = 0$ for all $t > 0$ and there is a constant $C = C(s, q, \Omega) > 0$ such that

$$\int_0^\infty \{\|u_t\|_q^s + \|\nabla^2 u\|_q^s + \|\nabla p\|_q^s\}\, dt \leq C\Big(\int_0^\infty \|f\|^s_{q,\omega}\, dt + \|u_0\|^s_{[s,q]}\Big).$$

Proof: This theorem is a consequence of Theorem 4.4. For further details we refer the reader to the proof of Theorem 6.1 in [8]. □

5 Weak solutions of the Navier-Stokes equations with non-vanishing flux

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an aperture domain with $C^{1,1}$-boundary. We consider the Navier-Stokes equations

$$u_t + u \cdot \nabla u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \text{ in } \mathbb{R}_+ \times \Omega \tag{29}$$

$$u = 0 \text{ on } \partial\Omega, \quad u(x,0) = u_0(x). \tag{30}$$

Then u is called a weak solution of the Navier-Stokes equations in Ω with flux α if

$$\begin{aligned} u \in L^\infty(\mathbb{R}_+, L^2(\Omega)^n), &\quad \nabla u \in L^2(\mathbb{R}_+, L^2(\Omega)^{n^2}), \\ \operatorname{div} u = 0 \text{ in } \mathbb{R}_+ \times \Omega \quad u = 0 \text{ on } \mathbb{R}_+ \times \partial\Omega, &\quad \Phi(u) = \alpha \text{ in } \mathbb{R}_+, \\ (u(t), v(t)) + \int_0^t \{-(u, v_t) + (\nabla u, \nabla v) &+ (u \cdot \nabla u, v)\}\, d\tau \\ &= (u_0, v(0)) + \int_0^t (f, v)\, d\tau \end{aligned}$$

for all $t > 0$ and all $v \in C^1_0([0,\infty); C^\infty_{0,\sigma}(\Omega))$.

For $n = 3$ in [5], Theorem 1.1, the existence of a global weak solution with prescribed flux α is shown under the assumptions

$$f \in L^1(\mathbb{R}_+; L^2(\Omega)^3) + L^2(\mathbb{R}_+; \widehat{W}^{-1,2}(\Omega)^3), \quad \alpha \in W^{1,1}(\mathbb{R}_+) \tag{31}$$

$$\text{and} \quad u_0 \in L^2(\Omega)^3 \text{ with } \operatorname{div} u_0 = 0,\ u_0 \cdot \nu\big|_{\partial\Omega} = 0,\ \Phi(u_0) = \alpha(0), \tag{32}$$

where $\alpha(0)$ denotes the trace of $\alpha \in W^{1,1}(\mathbb{R}_+)$ at 0.

Proof of Theorem 1.1: The function $v = u - \alpha\chi$ satisfies $\Phi(v) = 0$ and is a weak solution of

$$\begin{aligned} v_t - \Delta v + u \cdot \nabla u + \nabla p = f - \alpha'\chi + \alpha\Delta\chi, &\quad \operatorname{div} v = 0 \\ v\big|_{\partial\Omega} = 0, &\quad v(0) = u_0 - \alpha(0)\chi \in L^{[s,q]}_\sigma. \end{aligned} \tag{33}$$

As usual (cf. [10]) one gets the estimate

$$\|u \cdot \nabla u\|_{s,q} \leq C\, \|u\|^{2-2/s}_{2,\infty} \|\nabla u\|^{2/s}_{2,2}. \tag{34}$$

Define

$$V(t)=e^{-t\mathcal{A}_q}[u_0-\alpha(0)\chi]+\int_0^t e^{-(t-s)\mathcal{A}_q}P_q[f-(u\cdot\nabla u)](s)\,ds$$
$$W(t)=-\int_0^t e^{-(t-s)\mathcal{A}_\rho}P_\rho[\alpha'\chi-\alpha\Delta\chi](s)\,ds,\quad \rho>n'.$$

By (34), the properties $\chi,\Delta\chi\in L^\rho$ for $\rho>n'$, $\alpha\in W^{1,r}(\mathbb{R}_+)$, $u_0-\alpha(0)\chi\in L^{[s,q]}_\sigma(\Omega)$ and Theorem 4.4 we get

$$V_t,\,\mathcal{A}_qV\in L^s(\mathbb{R}_+;L^q),\quad W_t,\,\mathcal{A}_\rho W\in L^r(\mathbb{R}_+;L^\rho)\quad\forall\rho>n',\tag{35}$$

and the respective norms can be estimated by $\|f\|_{s,q}+\|u\|_{2,\infty}^{2-2/s}\|\nabla u\|_{2,2}^{2/s}+\|u_0\|_{[s,q],\chi}+\|\alpha\|_{W^{1,r}(\mathbb{R}_+)}$. Set $\tilde v:=V+W$ and $\tau:=\min\{r,s\}$. Note that $L^q+\bigcap_{\rho>n'}L^\rho\subset L^q+L^2\subset L^q_\omega$ where $\omega=(1+|\cdot|)^{-\gamma}\in A_1$, $n(2-q)/2<\gamma<n$. Hence $\tilde v\in L^\tau(0,T;D(\mathcal{A}_{q,\omega}))\cap W^{1,\tau}(0,T;L^q_\omega)$ for every $0<T<\infty$ and $\tilde v$ is a solution of

$$\tilde v_t+\mathcal{A}_{q,\omega}\tilde v=P_{q,\omega}[f-u\cdot\nabla u-\alpha'\chi+\alpha\Delta\chi]=:F,\quad \tilde v(0)=u_0-\alpha(0)\chi.\tag{36}$$

Applying the Yosida approximation $J_k=(I+\mathcal{A}_{q,\omega}/k)^{-1}$ (which exists by Theorem 1.1 in [8]) to v it follows from Theorem 1.2 in [8] that $J_kv\in L^\tau(0,T;D(\mathcal{A}_{q,\omega}))\cap W^{1,\tau}(0,T;L^q_\omega)$ for $0<T<\infty$ and

$$(J_kv)_t+\mathcal{A}_{q,\omega}(J_kv)=J_kF,$$

see also [10] for details. Theorem 1.1 and Theorem 1.2 in [8] imply that (J_kv) is uniformly bounded in $L^\tau(0,T;D(\mathcal{A}_{q,\omega}))\cap W^{1,\tau}(0,T;L^q_\omega)$. This yields $v\in L^\tau(0,T;D(\mathcal{A}_{q,\omega}))\cap W^{1,\tau}(0,T;L^q_\omega)$ for $0<T<\infty$ and $v_t+\mathcal{A}_{q,\omega}v=F$. Hence the uniqueness statement of Theorem 1.2 in [8] yields $v=\tilde v$ a.e. in $\mathbb{R}_+$.

Note that the conditions on s and q imply $q<n'<n$. Therefore (see [5], Theorem 2.8)

$$\|\nabla^2V\|_q\le C\,\|\mathcal{A}_qV\|_q,\quad \|\nabla^2W\|_\rho\le C\,\|\mathcal{A}_\rho W\|_\sigma,\quad n'<\rho<n.$$

Since $u=v+\alpha\chi=V+W+\alpha\chi$ and $\nabla p=f-u_t+\Delta u-u\cdot\nabla u$, the proof is complete. $\square$

Remark: Assume $s=r$ in Theorem 1.1 and note that for $\gamma>n(n'-q)/n'$ and $\omega=(1+|\cdot|)^{-\gamma}$ the embedding

$$L^q+\bigcap_{\rho>n'}L^\rho\subset L^q_\omega$$

holds. Therefore the norm of u_t, $\nabla^2u,\nabla p$ in $L^s(\mathbb{R}_+;L^q_\omega)$ can be estimated by (4). The exponents s,q satisfy the the condition $2/s+n/q=n+1$ as in the case of the whole space, the half space, a bounded or an exterior domain (see [10]) and the exponent $\gamma>0$ of the weight function tends to 0 if s tends to 1 and q tends to n'.

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