Flux homomorphisms and principal bundles over infinite dimensional manifolds

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Abstract. Flux homomorphisms for closed vector-valued differential forms on infinite dimensional manifolds are defined. We extend the relation between the kernel of the flux for a 2-form ω and Kostant's exact sequence associated to a principal bundle with curvature ω to the context of infinite-dimensional fiber and base space. We then use these results to construct central extensions of infinite dimensional Lie groups.

In finite-dimensional symplectic geometry one frequently encounters actions $\alpha: G\times M \to M$ of a connected Lie group G by automorphism on a connected symplectic manifold (M,ω). This means that for all vector fields $\dot\alpha(X)$, $X \in \mathfrak{g}$, of the derived action the 1-forms $i_{\dot\alpha(X)}\omega$ are closed. The action is called hamiltonian if all these 1-forms are exact, which in turn is equivalent to the existence of a moment map $M \to \mathfrak{g}^*$. Passing to a central extension $\widehat{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ of G, the moment map can be made equivariant. This means we have the following commutative diagram linking the two exact sequences of Lie algebras:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{R} & \to & C^\infty(M,\mathbb{R}) & \to & \mathfrak{X}_{\mathrm{ham}}(M) & \to & 0 \\
 & & \uparrow{=} & & \uparrow & & \uparrow & & \\
0 & \to & \mathbb{R} & \to & \widehat{\mathfrak{g}} & \to & \mathfrak{g} & \to & 0,
\end{array}$$

where $C^\infty(M,\mathbb{R})$ denotes the Lie algebra of smooth functions on M endowed with the Poisson bracket. If, in addition, (P,θ) is a $\mathbb{T}$-bundle over M with connection 1-form θ and curvature ω, then $\mathfrak{X}_\theta(P) \cong C^\infty(M)$, so the action of the simply connected Lie group $\widehat{G}$ with Lie algebra $\widehat{\mathfrak{g}}$ on M can be lifted to an action on P preserving the connection 1-form θ. In the infinite-dimensional setting, things become more complicated for several reasons, one crucial point being that not every topological Lie algebra belongs to a Lie group.

In this paper we study actions $\alpha: G \times M \to M$ of infinite-dimensional Lie groups on infinite-dimensional manifolds preserving a closed 2-form with values in a sequentially complete locally convex space $\mathfrak{z}$. If $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup, $Z := \mathfrak{z}/\Gamma$, and there is a principal Z-bundle P over M with curvature ω and connection 1-form θ, then we give conditions for lifting the action of G on M to an action of a central Lie group extension $\widehat{G}$ of G by θ-preserving automorphisms of the Z-bundle P. All manifolds and Lie groups considered in this paper are modeled over locally convex spaces ([Mil83]) which are not necessarily assumed to be sequentially complete ([Gl01]). The assumption of sequential completeness is only needed for $\mathfrak{z}$ to ensure the existence of integrals leading to parallel transport, local exactness of closed differential forms etc. We do also not assume that the manifold M is smoothly paracompact which is usually done to classify principal bundles over M (cf. [Bry93]).

The structure of the paper is as follows. In the first section we define a flux homomorphism for a closed p-form ω on an infinite-dimensional manifold M. Here the main point is to see that the approach, as f.i. in Banyaga's book [Ban97], can also be used in the context of infinite-dimensional manifolds. Having applications to principal bundles for infinite-dimensional abelian Lie groups in mind, we even work with vector-valued differential forms, which causes no serious additional difficulites. Let ω be a closed p-form on the manifold M with values in the sequentially complete locally convex (s.c.l.c.) space $\mathfrak{z}$ and $\Gamma_\omega \subseteq \mathfrak{z}$ be the group of periods obtained by integration of ω over piecewise smooth cycles. Let $H^{p-1}_{\rm dR}(M,\mathfrak{z},\Gamma_\omega)$ denote the quotient of the space of closed $\mathfrak{z}$-valued $(p-1)$-forms modulo those with periods contained in Γ_ω and for a curve $\psi\colon[0,1]\to\mathcal{D}_\omega(M):=\{\varphi\in \mathrm{Diff}(M)\colon\varphi^*\omega=\omega\}$ which is smooth in the appropriate sense, let $\delta^l(\psi)(t)\in\mathfrak{X}(M)$ denote its left logarithmic derivative in t. For a curve ψ from id_M to φ we then define the *flux homomorphism associated to* ω by

$$S_\omega\colon\mathcal{D}_\omega(M)_0\to H^{p-1}_{\rm dR}(M,\mathfrak{z},\Gamma_\omega),\quad S_\omega(\varphi):=\Big[\int_0^1 i_{\delta^l(\psi)(t)}\omega\,dt\Big].$$

Here $\mathcal{D}_\omega(M)_0$ denotes the group of all those ω-preserving diffeomorphisms of M which can be reached from id_M by a curve which is smooth in a natural sense.

Let $\Gamma\subseteq\mathfrak{z}$ be a discrete subgroup and $Z:=\mathfrak{z}/\Gamma$ the corresponding quotient Lie group. Then the case $p=2$ of the flux homomorphism is used in Section II to analyse which diffeomorphisms of M can be lifted to an automorphism of a Z-principal bundle over M with curvature $\omega\in\Omega^2(M,\mathfrak{z})$. In this case $\Gamma_\omega\subseteq\Gamma$ is automatically discrete. The main point of Section II is a generalization of a result of B. Kostant from [Ko70], characterizing automorphisms of (M,ω) which can be lifted to automorphisms of (P,θ) as those which do not change the holonomy of closed curves in M. From that we derive further that for smooth one-parameter curves in $\mathcal{D}_\omega(M)_0$ the condition from Kostant's Theorem is equivalent to the curve lying in the kernel of a modified flux homomorphism $\mathcal{D}_\omega(M)_0\to H^{p-1}_{\rm dR}(M,\mathfrak{z},\Gamma)$. Eventually we arrive at the result that the smooth path-component of the identity in $\mathrm{Aut}(P,\theta)$ is a central Z-extension of the kernel of the modified flux homomorphism. In [RS81], using Sobolev technics, T. Ratiu and R. Schmid obtained a similar result for compact symplectic manifolds (M,ω) which admit quantizing manifolds (P,θ).

A similar result in the particular case of a compact quantizable symplectic manifold (M,ω) was obtained by T. Ratiu and R. Schmid using Sobolev technics in [RS81].

In Section III we consider a principal Z-bundle $q\colon P\to M$ on a connected manifold M with a connection 1-form θ and a smooth action $\alpha\colon G\times M\to M$ of a connected Lie group G on M which is hamiltonian with respect to the curvature form $\omega\in\Omega^2(M,\mathfrak{z})$ of (P,θ). Then the results of Section II implies that all maps α_g can be lifted to automorphisms of (P,θ), and we thus obtain a central group extension $\widehat{G}$ of G by Z. The main result of Section III is that if we endow $\widehat{G}$ with the manifold structure obtained from pulling back the bundle P via orbit maps of G on M, then $\widehat{G}$ is a Lie group acting smoothly by automorphisms on (P,θ).

The ideas for the main constructions of this paper are not new (see f.i. [PS86] and [Bry93]). Our main point is that we show how they can be extended to infinite-dimensional manifolds and Lie groups without the burden of artificial assumptions such as smooth paracompactness which is first not so easy to verify and on the other hand not even satisfied for many Banach manifolds ([KM97]).

I. Flux homomorphisms for infinite-dimensional manifolds

In this section M denotes a possibly infinite-dimensional smooth manifold modeled over a locally convex space and $\mathfrak{z}$ a sequentially complete locally convex (s.c.l.c.) space ([Gl01]). We write $\Omega^p(M,\mathfrak{z})$ for the space of $\mathfrak{z}$-valued smooth p-forms on M, $Z^p(M,\mathfrak{z})$ for the subspace of closed forms, $B^p(M,\mathfrak{z})\subseteq Z^p(M,\mathfrak{z})$ for the space of exact forms, and $H^p_{\rm dR}(M,\mathfrak{z}):=Z^p(M,\mathfrak{z})/B^p(M,\mathfrak{z})$ for the corresponding de Rham cohomology space. We do not consider any topology on all these spaces.

Vector fields and differential forms

Definition I.1. (a) We write $\mathcal{D}(M) := \mathrm{Diff}(M)$ for the group of all diffeomorphisms of M and $\mathfrak{X}(M)$ for the Lie algebra of vector fields on M. For $\omega \in \Omega^p(M, \mathfrak{z})$ we write

$$\mathcal{D}_\omega(M) := \{\varphi \in \mathcal{D}(M) : \varphi^*\omega = \omega\} \quad \text{and} \quad \mathfrak{X}_\omega(M) := \{\varphi \in \mathfrak{X}(M) : \mathcal{L}_X\omega = 0\},$$

where

$$\mathcal{L}_X := i_X \circ d + d \circ i_X.$$

(b) Let $I \subseteq \mathbb{R}$ be an interval. A curve $\varphi: I \to \mathcal{D}(M)$ is called *smooth* if the corresponding map $I \times M \to M, (t,x) \mapsto \varphi(t).x$ is smooth. Then we obtain for each $t \in I$ a vector field

$$\delta^l(\varphi)(t)(x) := \varphi(t)^{-1}.\frac{d}{d\tau}\Big|_{\tau=t} \varphi(\tau).x$$

which is called the *left logarithmic derivative of φ in t*. Here $\psi.v := T(\psi).v$ refers to the action of $\mathcal{D}(M)$ on the tangent bundle $T(M)$ of M. The corresponding right logarithmic derivative is given by

$$\delta^r(\varphi)(t)(x) := \frac{d}{d\tau}\Big|_{\tau=t} \varphi(\tau).(\varphi(t)^{-1}.x)$$

and satisfies

$$\delta^r(\varphi)(t) = \varphi(t)_*.\delta^l(\varphi)(t), \quad t \in I. \tag{1.1}$$

(c) We call $\varphi, \psi \in \mathcal{D}(M)$ *smoothly homotopic* if there exists a smooth curve $\gamma: [0,1] \to \mathcal{D}(M)$ with $\gamma(0) = \varphi$ and $\gamma(1) = \psi$. We write $\mathcal{D}(M)_0$ for the normal subgroup of those diffeomorphisms which are smoothly homotopic to the identity and likewise $\mathcal{D}_\omega(M)_0$ for the normal subgroup of those elements of $\mathcal{D}_\omega(M)$ which can be connected to id_M by a smooth curve in $\mathcal{D}_\omega(M)$.
(d) For a smooth manifold M and a Lie group K we define for a smooth function $f \in C^\infty(M,K)$ the *left logarithmic derivative of f* by

$$\delta^l(f)(x) := d\lambda_{f(x)^{-1}}(f(x))df(x): T_m(M) \to \mathfrak{k} \cong T_e(K).$$

This derivative can be viewed as a $\mathfrak{k}$-valued 1-form on M. We also write simply $\delta^l(f) = f^{-1}.df$ and observe the product rule

$$\delta^l(f_1f_2) = \mathrm{Ad}(f_2)^{-1}.\delta^l(f_1) + \delta^l(f_2)$$

([KM97, 38.1]). ■

Lemma I.2. *For $\omega \in \Omega^p(M,\mathfrak{z})$ and a smooth curve $\varphi:]-\varepsilon,\varepsilon[\to \mathcal{D}(M)$ with $\varphi(0) = \mathrm{id}_M$ and $\varphi'(0) = X \in \mathfrak{X}(M)$ we have*

$$\mathcal{L}_X\omega = \frac{d}{dt}\Big|_{t=0} \varphi(t)^*\omega,$$

where the limit is considered pointwise on p-tuples of tangent vectors of M.

Proof. This is proved exactly as [Ne00, Lemma A.2.4]. ■

Lemma I.3. *Let M and N be manifolds. If the smooth maps $\varphi, \psi\colon N \to M$ are smoothly homotopic, then for each $\omega \in Z^p(M,\mathfrak{z})$ the form $\varphi^*\omega - \psi^*\omega$ is exact.*

Proof. This can be obtained with the same arguments as Lemma 34.2 in [KM97]. Because it will be instructive for the following, we give a direct proof for the case $M = N$, where $\varphi, \psi \in \mathcal{D}(M)$ are smoothly homotopic in $\mathcal{D}(M)$.

First we note that for any $X \in \mathfrak{X}(M)$ the closedness of ω implies

$$\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega) = d(i_X\omega).$$

Now let $\sigma\colon [0,1] \to \mathcal{D}(M)$ be a smooth curve connecting φ to ψ. Then

$$\psi^*\omega - \varphi^*\omega = \int_0^1 \frac{d}{d\tau}\Big|_{\tau=t} \sigma(\tau)^*\omega\, dt,$$

and Lemma I.2 implies that this integral equals

$$\int_0^1 \mathcal{L}_{\delta^l(\sigma)(t)}(\sigma(t)^*\omega)\, dt = \int_0^1 d\Big(i_{\delta^l(\sigma)(t)}(\sigma(t)^*\omega)\Big)\, dt = d\Big[\int_0^1 i_{\delta^l(\sigma)(t)}(\sigma(t)^*\omega)\, dt\Big].$$

In view of the sequential completeness of $\mathfrak{z}$, the integral $\int_0^1 i_{\delta^l(\sigma)(t)}(\sigma(t)^*\omega)\, dt$ exists as a Riemann integral and defines an element of $\Omega^{p-1}(M,\mathfrak{z})$. This completes the proof. ■

Lemma I.4. *If $\varphi\colon I \to \mathcal{D}_\omega(M)$ is a smooth curve, then for each $t \in I$ the vector fields $\delta^l(\varphi)(t), \delta^r(\varphi)(t)$ are contained in $\mathfrak{X}_\omega(M)$.*

Proof. In view of $\varphi(t)^*\omega = \omega,\ t \in I$, Lemma I.2 shows that the left logarithmic derivative satisfies

$$0 = \frac{d}{d\tau}\Big|_{\tau=t} \varphi(\tau)^*\omega = \mathcal{L}_{\delta^l(\varphi)(t)}\varphi(t)^*\omega = \mathcal{L}_{\delta^l(\varphi)(t)}\omega$$

and the right logarithmic derivative satisfies

$$0 = \frac{d}{d\tau}\Big|_{\tau=t} \varphi(\tau)^*\omega = \varphi(t)^*.\mathcal{L}_{\delta^r(\varphi)(t)}\omega$$

which implies $\mathcal{L}_{\delta^r(\varphi)(t)}\omega = 0$. ■

The flux homomorphism associated to ω

For $\omega \in Z^p(M,\mathfrak{z})$ the map

$$\gamma\colon \mathfrak{X}_\omega(M) \to H^{p-1}(M,\mathfrak{z}), \quad X \mapsto [i_X\omega]$$

is a homomorphism of Lie algebras because

$$i_{[X,Y]}\omega = [\mathcal{L}_X, i_Y]\omega = \mathcal{L}_X i_Y\omega = d(i_X i_Y\omega) + i_X d(i_Y\omega) = d(i_X i_Y\omega) \in B^{p-1}(M,\mathfrak{z}). \tag{1.2}$$

Its kernel $\mathfrak{X}_\omega(M)^{\rm ex}$ consists of those vector fields for which $i_X\omega$ is exact. For $\varphi \in \mathcal{D}_\omega(M)_0$ and $X \in \mathfrak{X}_\omega(M)$ Lemma I.3 shows that

$$i_{\varphi_*X}\omega - i_X\omega = (\varphi^{-1})^*(i_X\omega) - i_X\omega$$

is exact. Therefore

$$[i_{\varphi_*X}\omega] = [i_X\omega]$$

in $H^{p-1}_{\rm dR}(M,\mathfrak{z})$. It follows in particular from (1.2) that for a smooth curve $\varphi\colon I \to \mathcal{D}_\omega(M)$ we have

$$[i_{\delta^l(\varphi)(t)}\omega] = [i_{\delta^r(\varphi)(t)}\omega].$$

Although the group $\mathcal{D}_\omega(M)$ is far from having a reasonable Lie group structure if M is infinite-dimensional, we will see in this subsection how the Lie algebra homomorphism γ can be "integrated" to a group homomorphism on $\mathcal{D}_\omega(M)_0$.

Definition I.5. (a) For $\omega \in \Omega^p(M,\mathfrak{z})$, N a triangulated compact p-dimensional oriented manifold, and $\sigma\colon N \to M$ a piecewise smooth map, the integral

$$\int_\sigma \omega := \int_N \sigma^*\omega \in \mathfrak{z}$$

is defined because $\mathfrak{z}$ is sequentially complete. The subgroup $\Gamma_\omega \subseteq \mathfrak{z}$ generated by the images of these integrals is called the *ps-period group of* ω ("ps" stands for piecewise smooth).

If M is smoothly paracompact, then de Rham's Theorem holds for M ([KM97, Thm. 34.7]), so that

$$H^p_{\rm dR}(M,\mathfrak{z}) \cong H^p_{\rm sing}(M,\mathfrak{z}) \cong \mathrm{Hom}(H_p(M,\mathbb{Z}),\mathfrak{z}).$$

Therefore each closed p-form ω corresponds to a unique homomorphism $H_p(M,\mathbb{Z}) \to \mathfrak{z}$ whose image is called the *period group of* ω. It seems that this period group might be larger than the ps-period group. For $p = 1$ this is not the case because every singular 1-cycle has piecewise smooth representatives. The fact that the cone over a connected one-dimensional manifold is homeomorphic to the 2-dimensional disc with boundary implies that singular 2-cycles can also be represented by piecewise smooth maps $N \to M$, but a similar argument does not work for higher dimensions. This motivates our definition of Γ_ω, because this subgroup is better accessible than the full period group of ω. It also has the advantage that our constructions remain valid for manifolds which are not smoothly paracompact, which already includes Banach manifolds modeled on spaces like $l^1(\mathbb{N},\mathbb{R})$ ([KM97, 14.11]).

(b) For a subgroup $\Gamma \subseteq \mathfrak{z}$ we define

$$B^p(M,\mathfrak{z},\Gamma) := \{\omega \in Z^p(M,\mathfrak{z})\colon \Gamma_\omega \subseteq \Gamma\}, \quad H^p_{\rm dR}(M,\Gamma) := B^p(M,\mathfrak{z},\Gamma)/B^p(M,\mathfrak{z})$$

and

$$H^p_{\rm dR}(M,\mathfrak{z},\Gamma) := Z^p(M,\mathfrak{z})/B^p(M,\mathfrak{z},\Gamma) \cong H^p_{\rm dR}(M,\mathfrak{z})/H^p_{\rm dR}(M,\Gamma).$$

For elements of the space $H^p_{\rm dR}(M,\mathfrak{z},\Gamma)$ we define the integral over piecewise smooth maps $\sigma\colon N \to M$ (N compact, oriented, triangulated manifold) via

$$\int_\sigma [\omega] := \int_\sigma \omega + \Gamma \in Z := \mathfrak{z}/\Gamma.$$

This integral has to be interpreted as an element of the quotient group Z. ■

Remark I.6. If $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup and $Z := \mathfrak{z}/\Gamma$ is the corresponding quotient Lie group, then we can identify $B^1(M,\mathfrak{z},\Gamma)$ in a natural way with $\delta^l(C^\infty(M,Z))$ (cf. [Ne00, Sect. III]). ■

We will integrate γ to a group homomorphism $S_\omega\colon \mathcal{D}_\omega(M)_0 \to H^{p-1}(M,\mathfrak{z},\Gamma_\omega)$.

Lemma I.7. *Let $\varphi\colon [a,b] \to \mathcal{D}_\omega(M)$ be a piecewise smooth closed curve. Then*

$$\int_a^b i_{\delta^l(\varphi)(t)}\omega\, dt \in B^{p-1}(M,\mathfrak{z},\Gamma_\omega).$$

Proof. It is easy to see that the integral $\int_a^b i_{\delta^l(\varphi)(t)}\omega\, dt$ defines an element of $Z^{p-1}(M,\mathfrak{z})$. One only has to use that for a smooth map $[a,b] \times X \to \mathfrak{z}$, X a manifold, integration over $[a,b]$ yields a smooth function $X \to \mathfrak{z}$.

Let $\sigma\colon N \to M$ be piecewise smooth map, where N is a triangulated oriented $(p-1)$-dimensional manifold. Then the map

$$h\colon [a,b] \times N \to M, \quad (t,x) \mapsto \varphi(t).\sigma(x)$$

is smooth and we have for $v_1, \ldots, v_{p-1} \in T_x(N)$:

$$\begin{aligned}
&(h^*\omega)(t,x)\Big(\frac{d}{dt}, v_1, \ldots, v_{p-1}\Big) \\
&= \omega(h(t,x))\big(\varphi(t).\delta^l(\varphi)(t)(\sigma(x)), \varphi(t).d\sigma(x)(v_1, \ldots, v_{p-1})\big) \\
&= (\varphi(t)^*\omega)(\sigma(x))\big(\delta^l(\varphi)(t)(\sigma(x)), d\sigma(x)(v_1, \ldots, v_{p-1})\big) \\
&= \omega(\sigma(x))\big(\delta^l(\varphi)(t)(\sigma(x)), d\sigma(x)(v_1, \ldots, v_{p-1})\big) \\
&= (i_{\delta^l(\varphi)(t)}\omega)(\sigma(x))\big(d\sigma(x)(v_1, \ldots, v_{p-1})\big) \\
&= \sigma^*(i_{\delta^l(\varphi)(t)}\omega)(x)(v_1, \ldots, v_{p-1}).
\end{aligned}$$

We thus obtain

$$\int_{[a,b]\times N} h^*\omega = \int_a^b \int_N \sigma^*(i_{\delta^l(\varphi)(t)}\omega)\,dt = \int_a^b \int_\sigma i_{\delta^l(\varphi)(t)}\omega\,dt = \int_\sigma \Big(\int_a^b i_{\delta^l(\varphi)(t)}\omega\,dt\Big). \tag{1.3}$$

As φ is a closed curve, we may consider h as a piecewise smooth map $\mathbb{S}^1 \times N \to M$. Therefore $\int_{[a,b]\times N} h^*\omega \in \Gamma_\omega$, which proves the lemma. ∎

Proposition I.8. *Let $\varphi \in \mathcal{D}_\omega(M)_0$ and $\psi\colon [0,1] \to \mathcal{D}_\omega(M)$ a piecewise smooth curve with $\varphi = \psi(1)$ and $\psi(0) = \mathrm{id}_M$. Then*

$$S_\omega\colon \mathcal{D}_\omega(M)_0 \to H^{p-1}(M, \mathfrak{z}, \Gamma_\omega), \quad \varphi = \psi(1) \mapsto \Big[\int_0^1 i_{\delta^l(\psi)(t)}\omega\,dt\Big] \tag{1.4}$$

is a well defined group homomorphism.

Proof. If $\eta\colon [0,1] \to \mathcal{D}_\omega(M)$ is another smooth curve from id_M to φ, then concatenation yields a closed curve in $\mathcal{D}_\omega(M)$, so that Lemma I.7 implies that

$$\int_0^1 i_{\delta^l(\psi)(t)}\omega\,dt \in \int_0^1 i_{\delta^l(\eta)(t)}\omega\,dt + B^{p-1}(M, \mathfrak{z}, \Gamma_\omega)$$

Therefore S_ω is well defined by (1.4). To see that S_ω is a group homomorphism, suppose that $\alpha\colon [0,1] \to \mathcal{D}_\omega(M)$ is a piecewise smooth curve connecting id_M to φ, and that $\beta\colon [0,1] \to \mathcal{D}_\omega(M)$ is a piecewise smooth curve connecting id_M to ψ. Then

$$\xi\colon [0,1] \to \mathcal{D}_\omega(M), \quad \xi(t) := \begin{cases} \alpha(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \varphi \circ \beta(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

is a piecewise smooth curve from id_M to $\varphi\psi$. Then

$$\delta^l(\xi)(t) = \begin{cases} 2\delta^l(\alpha)(2t) & \text{for } t \in [0, \frac{1}{2}] \\ 2\delta^l(\beta)(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

implies

$$\begin{aligned}
S_\omega(\varphi \circ \psi) &= \int_0^1 i_{\delta^l(\xi)(t)}\omega\,dt = \int_0^{\frac{1}{2}} 2i_{\delta^l(\alpha)(2t)}\omega\,dt + \int_{\frac{1}{2}}^1 2i_{\delta^l(\beta)(2t-1)}\omega\,dt \\
&= \int_0^1 i_{\delta^l(\alpha)(t)}\omega\,dt + \int_0^1 i_{\delta^l(\beta)(t)}\omega\,dt = S_\omega(\varphi) + S_\omega(\psi).
\end{aligned}$$

∎

The homomorphism S_ω in (1.4) is called the *flux homomorphism* associated to the closed p-form $\omega \in Z^p(M, \mathfrak{z})$.

Remark I.9. In [Ban97, p.56] the flux homomorphism for a closed p-form $\omega \in Z^p(M,\mathbb{R})$ on a finite-dimensional manifold M is defined on the simply connected covering group $\widetilde{\mathcal{D}}_{\omega,c}(M)_0$ of the identity component $\mathcal{D}_{\omega,c}(M)_0$ of the group $\mathcal{D}_{\omega,c}(M)$ of ω-preserving diffeomorphisms with compact support as follows. An element g of this covering group is represented by a piecewise smooth curve $\varphi\colon[0,1]\to\mathcal{D}_{\omega,c}(M)$ from e to the image of g in $\mathcal{D}_{\omega,c}(M)_0$. Then

$$\widetilde{\mathcal{S}}_\omega\colon \widetilde{\mathcal{D}}_{\omega,c}(M)_0 \to H^{p-1}_{\mathrm{dR,c}}(M,\mathbb{R}), \quad \widetilde{\mathcal{S}}_\omega(g) := \Big[\int_0^1 i_{\delta^l(\varphi)(t)}\omega\,dt\Big]$$

defines a group homomorphism which descends to a group homomorphism

$$\mathcal{S}_\omega : \mathcal{D}_{\omega,c}(M)_0 \to H^{p-1}_{dR,c}(M,\mathbb{R})/\Pi, \tag{1.5}$$

where $\Pi := \widetilde{\mathcal{S}}_\omega(\pi_1(\mathcal{D}_{\omega,c}(M)))$.

Applying Lemma I.7 to $\mathfrak{z}=\mathbb{R}$, we see that $\Pi\subset H^{p-1}_{\mathrm{dR}}(M,\Gamma_\omega)$. Let

$$\pi : H^{p-1}_{\mathrm{dR}}(M,\mathfrak{z})/\Pi \to H^{p-1}_{\mathrm{dR}}(M,\mathfrak{z})/H^{p-1}_{\mathrm{dR}}(M,\Gamma_\omega) = H^{p-1}_{\mathrm{dR}}(M,\mathfrak{z},\Gamma_\omega)$$

denote the projection. Then the relation between the two flux homomorphisms in (1.4) and (1.5) is

$$S_\omega = \pi\circ\mathcal{S}_\omega.$$

■

Remark I.10. Suppose that $p=2$. Let $\psi\colon[0,1]\to\mathcal{D}_\omega(M)$ be a piecewise smooth curve with $\psi(0)=\mathbf{1}$, $\psi(1)=\varphi$ and $\ell\colon\mathbb{S}^1\to M$ a piecewise smooth loop. Then (1.3) implies that we have in the sense of Definition I.5

$$\int_\ell S_\omega(\varphi) = \int_\sigma \omega, \tag{1.6}$$

where $\sigma\colon[0,1]\times\mathbb{S}^1\to M$ is given by $\sigma(t,s)=\psi(t).\ell(s)$ (cf. [MDS98, p. 317]). This means that the flux homomorphism is the "symplectic area" of surface swept out by a closed curve which is moved by a curve of symplectomorphisms, hence the name "flux homomorphism." ■

Group actions and the flux homomorphism

Let $\alpha\colon G\times M\to M$ be a smooth left action of the connected Lie group G on M. We write $\alpha(g):=\alpha_g:=\alpha(g,\cdot)$ for the diffeomorphism of M defined by the element $g\in G$. The corresponding Lie algebra homomorphism $\dot\alpha\colon\mathfrak{g}\to\mathfrak{X}(M)$ is given by

$$\dot\alpha(X)(p) := T\alpha(\mathbf{1},p)(-X,0).$$

Let $\omega\in Z^p(M,\mathfrak{z})$. Then [Ne00, Lemma A.2.5] (see also Lemma I.4) implies that $\alpha(G)\subseteq\mathcal{D}_\omega(M)$ is equivalent to $\dot\alpha(\mathfrak{g})\subseteq\mathfrak{X}_\omega(M)$. We call the action α *hamiltonian* if all 1-forms $i_{\dot\alpha(X)}\omega$, $X\in\mathfrak{g}$, are exact. One would like to take this as the infinitesimal criterion for $\alpha(G)\subseteq\ker S_\omega$, but the situation is a bit subtle because in general S_ω can not be viewed as a homomorphism of Lie groups.

Proposition I.11. *Let $\omega\in Z^p(M,\mathfrak{z})$ be a closed p-form and $\alpha\colon G\times M\to M$ a smooth action of the connected Lie group G on M. For the assertions*
(1) $\alpha(G)\subseteq\ker S_\omega$.
(2) *All 1-forms $i_{\dot\alpha(X)}\omega$ are exact.*
we then have (2) ⇒ (1) and the converse holds if $p=2$ and Γ_ω is discrete.

Proof. Assume that $\alpha(G) \subseteq \mathcal{D}_\omega(M)$. Let $g \in G$ and $\psi\colon [0,1] \to G$ a smooth curve from $\mathbf{1}$ to g. Then $\varphi(t) := \alpha_{\psi(t)}$ is a smooth curve from $\mathbf{1}$ to α_g in $\mathcal{D}(M)$. We then have

$$\begin{aligned}\delta^l(\varphi)(t)(y) &= \varphi(t)^{-1}.\frac{d}{d\tau}\Big|_{\tau=t}\varphi(\tau).y = \alpha(\psi(t))^{-1}.\frac{d}{d\tau}\Big|_{\tau=t}\alpha(\psi(\tau)).y \\ &= T\alpha(\mathbf{1},y)\Big(\psi(t)^{-1}.\frac{d}{d\tau}\Big|_{\tau=t}\psi(\tau).y\Big) = T\alpha(\mathbf{1},y)\big(\delta^l(\psi)(t),0\big) = -\dot\alpha(\delta^l(\psi)(t))(y).\end{aligned}$$

Therefore

$$S_\omega(\alpha_g) = \Big[\int_0^1 i_{\delta^l(\varphi)(t)}\omega\,dt\Big] = -\Big[\int_0^1 i_{\dot\alpha(\delta^l(\psi)(t))}\omega\,dt\Big].$$

(2) $\Rightarrow$ (1): If all $(p-1)$-forms $i_{\dot\alpha(X)}\omega$ are exact, then their periods vanish. Hence all periods of

$$\int_0^1 i_{\dot\alpha(\delta^l(\psi)(t))}\omega\,dt$$

vanish, and therefore $\alpha(G) \subseteq \ker S_\omega$.
(1) $\Rightarrow$ (2): Suppose, conversely, that $\alpha(G) \subseteq \ker S_\omega$. Then we obtain for each smooth curve $\psi\colon [0,1] \to G$ with $\psi(0) = \mathbf{1}$ that

$$\int_0^T i_{\dot\alpha(\delta^l(\psi)(t))}\omega\,dt \in B^{p-1}(M,\mathfrak{z},\Gamma_\omega).$$

If Γ_ω is discrete, then this implies for each piecewise smooth map $\sigma\colon N \to M$ (N a $(p-1)$-dimensional, compact, oriented and triangulated manifold), that

$$\int_0^T \int_\sigma i_{\dot\alpha(\delta^l(\psi)(t))}\omega\,dt \in \Gamma_\omega$$

for each $T \in [0,1]$, and therefore

$$\int_0^T \int_\sigma i_{\dot\alpha(\delta^l(\psi)(t))}\omega\,dt = 0$$

because Γ_ω is discrete. Taking the derivative in $T = 0$, we thus obtain

$$i_{\dot\alpha(\delta^l(\psi)(0))}\omega \in B^{p-1}(M,\mathfrak{z},\{0\})$$

and since $\delta^l(\psi)(0)$ can be any element of $\mathfrak{g}$, we see that

$$i_{\dot\alpha(X)}\omega \in B^{p-1}(M,\mathfrak{z},\{0\}), \quad X \in \mathfrak{g}.$$

If, in addition, $p = 2$, then [Ne00, Th. III.6] implies that $B^1(M,\mathfrak{z},\{0\})$ coincides with the space of exact 1-forms, and we conclude that $i_{\dot\alpha(X)}\omega$ is exact for each $X \in \mathfrak{g}$. ■

II. Lifting diffeomorphisms to principal bundles

In this section $\mathfrak{z}$ denotes a s.c.l.c. space, $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup, and $Z := \mathfrak{z}/\Gamma$ the abelian quotient Lie group with the exponential function $\exp_Z(z) := z + \Gamma$. Let $q\colon P \to M$ be a principal Z-bundle over the connected manifold M with principal right action $\rho : P \times Z \to P$. Let $\theta \in \Omega^1(P,\mathfrak{z})$ be a *connection 1-form*, i.e. $\theta(\dot\rho(X)) = X$ for each $X \in \mathfrak{z}$ and $\rho_z^*\theta = \theta$ for each $z \in Z$. If $\theta_Z = \delta^l(\mathrm{id}_Z)$ is the *Maurer–Cartan form* on Z, then the connection form condition is equivalent to $\eta_y^*\theta = \theta_Z$ for each orbit map $\eta_y\colon Z \to P, z \mapsto y.z$. The curvature form $\omega \in \Omega^2(M,\mathfrak{z})$ is defined by $q^*\omega = d\theta$.

Parallel transport and holonomy

Remark II.1. (a) Although the existence of solutions of ordinary differential equations in infinite-dimensional vector spaces is problematic if they are not Banach spaces, the assumption of sequential completeness guarantees the existence of Riemann integrals. Hence there exists for each continuous curve $\eta\colon[a,b]\to\mathfrak{z}$ and each $X_0\in\mathfrak{z}$ a differentiable curve $\psi\colon[a,b]\to\mathfrak{z}$ with $\psi(a)=X_0$ and $\psi'=\eta$. Then the curve $\psi_G:=\exp_Z\circ\psi$ satisfies $\delta^l(\psi_G):=\psi_G^{-1}.d\psi_G=\psi'=\eta$.
(b) The remarks under (a) imply that each piecewise differentiable curve $\eta\colon[a,b]\to M$ can be lifted to a curve $\widetilde\eta\colon[a,b]\to P$ with horizontal tangent vectors, i.e., those in the kernel of θ. We then call $\widetilde\eta$ a *horizontal lift* of η. In fact, since this is a local assertion, we may assume that the bundle $q\colon P\to M$ is trivial, i.e., $P=M\times Z$ with $q(x,z)=x$. Then $\sigma\colon M\to P, x\mapsto(x,\mathbf{1})$ is a smooth section and we obtain a gauge potential $A:=\sigma^*\theta\in\Omega^1(M,\mathfrak{z})$ and $\theta=q^*A+p_Z^*\theta_Z$, where $p_Z\colon P\to Z$ is the projection onto the second factor. To obtain a lift of η, we have to look for a piecewise differentiable function $h\colon[a,b]\to Z$ for which the curve $\widetilde\eta(t):=(\eta(t),h(t))$ has horizontal tangent vectors. This means that for each t we have

$$0=\theta(\widetilde\eta(t))(\widetilde\eta'(t))=A(\eta(t))(\eta'(t))+h(t)^{-1}.h'(t)=A(\eta(t))(\eta'(t))+\delta^l(h)(t).$$

Since the differential equation

$$\delta^l(h)(t)=-A(\eta(t))(\eta'(t))$$

has a unique solution on $[a,b]$ for each initial value in Z, the assertion follows.
(c) For any piecewise differentiable curve $\eta\colon[a,b]\to M$ we then obtain a *parallel transport map*

$$\mathrm{Pt}(\eta)\colon q^{-1}(\eta(a))\to q^{-1}(\eta(b))$$

by assigning to an element $y\in q^{-1}(\eta(a))$ the value $\widetilde\eta(b)$ of the unique continuous horizontal lift $\widetilde\eta\colon[a,b]\to P$ of η with $\widetilde\eta(a)=y$. It is easy to see that the parallel transport maps are Z-equivariant.

If η is a loop, then $\mathrm{Pt}(\eta)$ maps $q^{-1}(\eta(a))$ into itself and commutes with Z, hence is given by the action of an element $h(\eta)\in Z$, called the *holonomy of the loop* η. This means that each horizontal lift $\widetilde\eta$ of η satisfies

$$\widetilde\eta(b)=\widetilde\eta(a).h(\eta).$$

(d) If $\psi\colon[0,1]\to M$ is a piecewise smooth path, then we write $\psi^\sharp\colon[0,1]\to M$ for the piecewise smooth path given by $\psi^\sharp(t):=\psi(1-t)$. If $\eta\colon[0,1]\to M$ is another piecewise smooth path, then we define the composition of ψ and η by

$$(\psi*\eta)(t):=\begin{cases}\eta(2t) & \text{for } t\in[0,\frac12]\\ \psi(2t-1) & \text{for } t\in[\frac12,1].\end{cases}$$

Then it is easy to verify that

$$\mathrm{Pt}(\psi)^{-1}=\mathrm{Pt}(\psi^\sharp)\quad\text{and}\quad\mathrm{Pt}(\psi*\eta)=\mathrm{Pt}(\psi)\circ\mathrm{Pt}(\eta).$$ ∎

Proposition II.2. *Let $\Delta_2:=\{(x,y)\in\mathbb{R}^2\colon 0\le x,y,x+y\le1\}$, $\psi\colon\partial\Delta_2\to M$ be a piecewise smooth loop, and $\sigma\colon\Delta_2\to M$ a piecewise smooth extension of ψ. Then*

$$h(\psi)=\exp_Z\Big(-\int_{\Delta_2}\sigma^*\omega\Big).$$

Proof. (cf. [Bry93, Prop. 2.4.6]) Using simplicial subdivisions of Δ_2 and observing that both sides are the sums of the contributions of all small smooth singular simplices, we may w.l.o.g. assume that $\mathrm{im}(\sigma)$ lies in an open set U over which the bundle is trivial. We may therefore assume that $P = M \times Z$ with $q(x,z) = x$ is a trivial bundle. Let $\sigma\colon M \to P, x \mapsto (x, \mathbf{1})$ be the canonical global section and $A := \sigma^*\theta$. Then we can write θ as

$$\theta = q^*A + p_Z^*\theta_Z,$$

where $p_Z\colon P \to Z$ is the projection. We parametrize $\partial\Delta_2$ by a piecewise smooth map $[0,1] \to \partial\Delta_2$, so that we may consider ψ as a piecewise smooth map $[0,1] \to M$. Let $\widetilde{\psi}(t) = (\psi(t), z(t))$ be a piecewise smooth horizontal lift of ψ. Then z satisfies the differential equation

$$\delta^l(z)(t) = -A(\psi(t)).\psi'(t).$$

We now calculate

$$\begin{aligned} h(\psi) &= z(1)z(0)^{-1} = \exp_Z\Big(\int_0^1 \delta^l(z)(t)\,dt\Big) = \exp_Z\Big(-\int_0^1 A(\psi(t)).\psi'(t)\,dt\Big) \\ &= \exp_Z\Big(-\int_{[0,1]}\psi^*A\Big) = \exp_Z\Big(-\int_\psi A\Big) = \exp_Z\Big(-\int_\sigma dA\Big) = \exp_Z\Big(-\int_\sigma \omega\Big) \\ &= \exp_Z\Big(-\int_{\Delta_2}\sigma^*\omega\Big). \end{aligned}$$

■

For smoothly paracompact manifolds the following corollary follows also from the classification of smooth principal Z-bundles by their Chern classes. In our setting it is a more or less direct consequence of Proposition II.2.

Corollary II.3. *Let $q : P \to M$ be a principal $Z = \mathfrak{z}/\Gamma$-bundle with principal connection form θ. Then the group Γ_ω of periods of the curvature form ω is a subgroup of Γ.*

Proof. The abelian group $Z_{1,pw}(M,\mathbb{Z})$ of piecewise smooth 1-cycles in M consists of formal linear combinations of $\psi_1 * \psi_2 * \ldots * \psi_n$ with $\psi_1, \ldots, \psi_n$ smooth curves in M closing to a cycle. The holonomy along piecewise smooth loops defines a group homomorphism

$$h : Z_{1,pw}(M,\mathbb{Z}) \to Z.$$

Let $\partial : C_{2,pw}(M,\mathbb{Z}) \to Z_{1,pw}(M,\mathbb{Z})$ be the boundary map for piecewise smooth 2-chains in M. Then Proposition II.2 implies that

$$h(\partial\sigma) = \exp_Z\Big(-\int_\sigma \omega\Big).$$

It follows that for 2-cycles $\sigma \in Z_{2,pw}(M,\mathbb{Z})$ the integral $\int_\sigma \omega \in \Gamma$. This proves $\Gamma_\omega \subset \Gamma$. ■

Remark II.4. Proposition II.2. is very close to the construction of a global group cocycle for a central extension of a simply connected Lie group G by Z ([Ne00]).

Let $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ be a continuous Lie algebra cocycle and Ω the corresponding left invariant closed $\mathfrak{z}$-valued 2-form on G with $\Omega_{\mathbf{1}} = \omega$. We assume that $\Gamma_\omega \subseteq \Gamma$ and that $Z := \mathfrak{z}/\Gamma$.

On the Lie group G we choose a left invariant system $\alpha_{g,h}$ of smooth paths from g to h. This means that $\alpha_{g,h} = \lambda_g \circ \alpha_{\mathbf{1},g^{-1}h}$ and that the paths from $\mathbf{1}$ to a group element may be chosen freely. From the simple connectedness of G we can then derive for each triple (g,h,u) of group elements the existence of a piecewise smooth map $\alpha_{g,h,u}\colon \Delta_2 \to G$ with respect to a simplicial subdivision of Δ_2 whose boundary values are given by the cycle $\alpha_{g,h} + \alpha_{h,u} - \alpha_{g,u}$. Then the global cocycle is obtained by

$$f(g,h) := \exp_Z\Big(\int_{\alpha_{\mathbf{1},g,gh}} \Omega\Big).$$

In view of Proposition II.2, we have $f(g,h) = h(\partial\alpha_{\mathbf{1},g,gh})$. ■

Lifting diffeomorphisms to principal bundles

In the following we write $C^\infty_{pw}(\mathbb{S}^1, M)$ for the set of maps $\mathbb{S}^1 \to M$ which are piecewise smooth with respect to a finite subdivision of $\mathbb{S}^1$ into intervals and $\mathcal{H}$ for the group of "holonomy preserving" diffeomorphisms:

$$\mathcal{H} := \{\varphi \in \mathcal{D}_\omega(M) \colon (\forall \ell \in C^\infty_{pw}(\mathbb{S}^1, M)) h(\varphi \circ \ell) = h(\ell)\}.$$

Remark II.5. From Proposition II.2 we obtain for each $\varphi \in \mathcal{D}(M)$ with

$$(\forall \ell \in C^\infty_{pw}(\mathbb{S}^1, M)) \ h(\varphi \circ \ell) = h(\ell)$$

and for each piecewise smooth map $\sigma \colon \Delta_2 \to M$ the relation

$$\int_\sigma \varphi^*\omega - \int_\sigma \omega \in \Gamma.$$

Pick $m \in M$ and let $\psi \colon U \to M$ be a chart of M with $m = \psi(0) \in \psi(U)$ and U open and convex in the locally convex space V. For $v, w \in U$ and $h > 0$ we consider the map

$$\sigma_h \colon \Delta_2 \to U, \quad (x, y) := xhv + yhw.$$

Then

$$\lim_{h \to 0} \frac{2}{h^2} \int_{\varphi \circ \sigma_h} \omega = \lim_{h \to 0} \frac{2}{h^2} \int_{\Delta_2} \sigma_h^* \psi^* \omega = (\psi^*\omega)(0)(v, w)$$

and likewise

$$\lim_{h \to 0} \frac{2}{h^2} \int_{\varphi \circ \sigma_h} \varphi^*\omega = (\psi^*\varphi^*\omega)(0)(v, w).$$

In particular we have

$$\lim_{h \to 0} \int_{\varphi \circ \sigma_h} \omega = 0 \quad \text{and} \quad \lim_{h \to 0} \int_{\varphi \circ \sigma_h} \varphi^*\omega = 0,$$

and since Γ is discrete, we get

$$\int_{\psi \circ \sigma_h} \varphi^*\omega - \int_{\psi \circ \sigma_h} \omega = 0$$

for h sufficiently small. Eventually this implies that $\psi^*\omega = \psi^*\varphi^*\omega$ on U, and hence that $\varphi^*\omega = \omega$. Therefore we also have

$$\mathcal{H} = \{\varphi \in \mathcal{D}(M) \colon (\forall \ell \in C^\infty_{pw}(\mathbb{S}^1, M)) h(\varphi \circ \ell) = h(\ell)\}. \qquad \blacksquare$$

For finite dimensional manifolds M and ω real-valued, B. Kostant shows in [Ko70] that $\mathcal{H}$ is the group of those diffeomorphisms on the basis M which can be lifted to automorphisms of (P, θ). For compact symplectic manifolds (M, ω), T. Ratiu and R. Schmid show in [RS81] that the identity component of $\mathcal{H}^{s+1}$, the group of elements in $\mathcal{H}$ which are of Sobolev class H^{s+1}, equals the kernel of the flux homomorphism, i.e. the group of Hamiltonian diffeomorphisms of Sobolev class H^{s+1}. In this subsection we show similar results in our infinite dimensional setting.

For a subgroup $\Pi \subseteq \Gamma_\omega$ let

$$\pi_\Pi : H^1_{\rm dR}(M, \mathfrak{z}, \Gamma_\omega) \twoheadrightarrow H^1_{\rm dR}(M, \mathfrak{z}, \Pi)$$

be the canonical projection and define the *flux homomorphism corresponding to* Π by

$$S_\Pi := \pi_\Pi \circ S_\omega \colon \mathcal{D}_\omega(M)_0 \to H^1_{\rm dR}(M, \mathfrak{z}, \Pi).$$

Proposition II.6. *Let S_Γ denote the flux homomorphism associated to Γ whose existence follows from $\Gamma_\omega \subseteq \Gamma$. Then*

$$\ker(S_\Gamma) = \{\varphi \in \mathcal{D}_\omega(M)_0 : (\forall \ell \in C^\infty_{pw}(\mathbb{S}^1, M)) h(\varphi \circ \ell) = h(\ell)\}.$$

Proof. Let $\varphi: [0,1] \to \mathcal{D}_\omega(M)$ be piecewise smooth with $\varphi(0) = \mathbf{1}$, $\ell: [0,1] \to M$ a piecewise smooth loop in M, and define a piecewise smooth map

$$\sigma: [0,1] \times [0,1] \to M, \quad (t,s) \mapsto \varphi(t).\ell(s).$$

Then

$$c_0: [0,1] \to M, \quad c_0(t) := \varphi(t).\ell(0) = \varphi(t).\ell(1)$$

is a piecewise smooth curve. Let $\widetilde{c}_0: [0,1] \to P$ denote a horizontal lift of c and $\widetilde{\ell}_t$ denote the unique horizontal lift of the curve $\ell_t(s) := \varphi(t).\ell(s)$ with $\widetilde{\ell}_t(0) = \widetilde{c}_0(t)$. Then

$$\widetilde{\sigma}: [0,1] \times [0,1] \to P, \quad (t,s) \mapsto \widetilde{\ell}_t(s)$$

is a piecewise smooth lift of σ. We define $\widetilde{c}_1(t) := \widetilde{\ell}_t(1)$. Both curves $\widetilde{c}_0$ and $\widetilde{c}_1$ lie over c_0, and

$$\widetilde{c}_1(t) = \mathrm{Pt}(\ell_t)(\widetilde{c}_0(t)) = \widetilde{c}_0(t).h(\ell_t).$$

The condition $\varphi(1) \in \ker S_\Gamma$ is equivalent to the relation $\int_\ell S_\omega(\varphi(1)) \in \Gamma$ for all piecewise smooth loops ℓ in M. According to formula (1.6), we have

$$\int_\ell S_\omega(\varphi(1)) = \int_\sigma \omega.$$

Further

$$\int_\sigma \omega = \int_{\widetilde{\sigma}} q^*\omega = \int_{\widetilde{\sigma}} d\theta = \int_{\partial\widetilde{\sigma}} \theta = \int_{\widetilde{c}_0} \theta + \int_{\widetilde{\ell}_1} \theta - \int_{\widetilde{c}_1} \theta - \int_{\widetilde{\ell}_0} \theta$$
$$= -\int_{\widetilde{c}_1} \theta = -\int_0^1 \theta(\widetilde{c}_1(t))(\widetilde{c}_1'(t))\, dt$$

because the curves $\widetilde{c}_0$, $\widetilde{\ell}_0$ and $\widetilde{\ell}_1$ are horizontal. We have $\widetilde{c}_1(t) = \widetilde{c}_0(t).h(\ell_t)$. Therefore

$$\widetilde{c}_1'(t) = \widetilde{c}_0'(t).h(\ell_t) + \dot{\rho}(\delta^l(h(\ell_t)))(\widetilde{c}_0(t)),$$

where

$$\delta^l(h(\ell_t)) := h(\ell_t)^{-1}.\frac{d}{dt}h(\ell_t) \in \mathfrak{z}.$$

Since the vectors $\widetilde{c}_0'(t)$ are horizontal, the same holds for $\widetilde{c}_0'(t).h(\ell_t)$, and we obtain

$$\int_\sigma \omega = -\int_0^1 \delta(h(\ell_t))\, dt.$$

To evaluate this integral, let $H: [0,1] \to \mathfrak{z}$ be a lift of the curve $[0,1] \to Z, t \mapsto h(\ell_t)$. Then

$$\int_\sigma \omega = -\int_0^1 H'(t)\, dt = H(0) - H(1).$$

This integral is in Γ if and only if $h(\varphi(1) \circ \ell) = h(\ell_1) = h(\ell)$. This completes the proof. ∎

Let

$$\mathrm{Aut}(P) := \{\varphi \in \mathcal{D}(P) : (\forall z \in Z) \varphi \circ \rho_z = \rho_z \circ \varphi\} \quad \text{and} \quad \mathrm{Aut}(P,\theta) := \mathrm{Aut}(P) \cap \mathcal{D}_\theta(P).$$

Then we have two group homomorphisms

$$q_*: \mathrm{Aut}(P) \to \mathcal{D}(M) \quad \text{and} \quad q_*^\theta := q_*|_{\mathrm{Aut}(P,\theta)}: \mathrm{Aut}(P,\theta) \to \mathcal{D}_\omega(M)$$

which are defined by $q_*(\varphi) \circ q = q \circ \varphi$. It is easy to see that $\ker q_* \cong C^\infty(M, Z)$, where we associate to a smooth function $f: M \to Z$ the diffeomorphism ρ_f of P given by $\rho_f(y) := \rho_{f(q(y))}(y)$. For each such operator we have

$$\rho_f^*\theta = \theta + q^*\delta^l(f).$$

Therefore $\rho_f \in \mathrm{Aut}(P,\theta)$ is equivalent to $df = 0$, which, in view of the connectedness of M, means that f is constant. Therefore $\ker(q_*^\theta) \cong Z$. We want to describe the image of q_*^θ in $\mathcal{D}_\omega(M)$.

Theorem II.7. *For each principal Z-bundle $q\colon P \to M$ with connection 1-form θ and curvature $\omega \in \Omega^2(M,\mathfrak{z})$ we have a short exact sequence of groups*

$$\mathbf{1} \to Z \hookrightarrow \mathrm{Aut}(P,\theta) \twoheadrightarrow \mathcal{H} \to 1.$$

Since Z is central in $\mathrm{Aut}(P,\theta)$, this sequence defines a central extension of $\mathcal{H}$ by Z.

Proof. It remains to be shown that $\mathrm{im}(q_*^\theta) = \mathcal{H}$.

First we show that $\mathrm{im}(q_*^\theta) \subseteq \mathcal{H}$. So let $\ell\colon[0,1] \to M$ be a piecewise smooth loop and $\widetilde{\ell}\colon[0,1] \to P$ a horizontal lift. Let $\widetilde{\varphi} \in \mathrm{Aut}(P,\theta)$ and $\varphi = q_*(\widetilde{\varphi})$. Then $\widetilde{\varphi} \circ \widetilde{\ell}$ is a horizontal lift of $\varphi \circ \ell$, and therefore

$$\widetilde{\varphi}(\widetilde{\ell}(0)).h(\varphi \circ \ell) = \widetilde{\varphi}(\widetilde{\ell}(1)) = \widetilde{\varphi}(\widetilde{\ell}(0).h(\ell)) = \widetilde{\varphi}(\widetilde{\ell}(0)).h(\ell)$$

implies that $h(\varphi \circ \ell) = h(\ell)$.

Now we assume $\varphi \in \mathcal{H}$ and construct $\psi \in \mathrm{Aut}(P,\theta)$ with $q_*^\theta(\psi) = \varphi$. Let $x_0 \in M$. We start with a Z-equivariant isomorphism

$$\psi_{x_0}\colon q^{-1}(x_0) \to q^{-1}(\varphi(x_0)).$$

Let $y \in P$ and $x := q(y)$. For a piecewise smooth path $\gamma\colon[0,1] \to M$ from x_0 to x we define

$$\psi(y) := \mathrm{Pt}(\varphi \circ \gamma) \circ \psi_{x_0} \circ \mathrm{Pt}(\gamma)^{-1}(y) \in q^{-1}(\varphi(x)).$$

We claim that ψ_γ is independent of γ. So let $\eta\colon[0,1] \to M$ be another piecewise smooth path from x_0 to x. Then $\gamma^{-1} * \eta$ is a loop in x_0. Our claim is equivalent to the relation

$$\mathrm{Pt}(\varphi \circ \gamma) \circ \psi_{x_0} \circ \mathrm{Pt}(\gamma)^{-1} = \mathrm{Pt}(\varphi \circ \eta) \circ \psi_{x_0} \circ \mathrm{Pt}(\eta)^{-1}$$

on $q^{-1}(x)$, which follows from

$$\begin{aligned}\mathrm{Pt}(\varphi \circ (\gamma^{-1} * \eta)) \circ \psi_{x_0} \circ \mathrm{Pt}(\gamma^{-1} * \eta)^{-1} &= \rho_{h(\varphi\circ(\gamma^{-1}*\eta))} \circ \psi_{x_0} \circ \rho^{-1}_{h(\gamma^{-1}*\eta)} \\ = \rho_{h(\varphi\circ(\gamma^{-1}*\eta))} \circ \rho^{-1}_{h(\gamma^{-1}*\eta)} \circ \psi_{x_0} &= \rho_{h(\gamma^{-1}*\eta)} \circ \rho^{-1}_{h(\gamma^{-1}*\eta)} \circ \psi_{x_0} = \psi_{x_0}.\end{aligned}$$

We now obtain a well defined map

$$\psi\colon P \to P, \quad \psi(y) := \psi_\gamma(y) \quad \text{for} \quad \gamma(1) = q(y).$$

Now we investigate the properties of ψ. Eventually we will show that $\psi \in \mathrm{Aut}(P,\theta)$ with $q_*^\theta(\psi) = \varphi$.

(1) We obviously have $q \circ \psi = \varphi \circ q$ because ψ_γ maps $q^{-1}(x)$ to $q^{-1}(\varphi(x))$.

(2) ψ commutes with the action of Z on P because this is true for ψ_{x_0} and all parallel transport maps.

(3) ψ intertwines parallel transport maps in the sense that

$$\psi \circ \mathrm{Pt}(\eta) = \mathrm{Pt}(\varphi \circ \eta) \circ \psi.$$

In fact, suppose that $\eta\colon[0,1] \to M$ is a piecewise smooth path from x_1 to x_2. Then we choose a path γ from x_0 to x_1, so that $\eta * \gamma$ connects x_0 to x_2. Therefore we have on $q^{-1}(x_1)$

$$\begin{aligned}\psi \circ \mathrm{Pt}(\eta) = \psi_{\eta*\gamma} \circ \mathrm{Pt}(\eta) &= \mathrm{Pt}(\varphi \circ (\eta * \gamma)) \circ \psi_{x_0} \circ \mathrm{Pt}(\eta * \gamma)^{-1} \circ \mathrm{Pt}(\eta) \\ &= \mathrm{Pt}(\varphi \circ \eta)\,\mathrm{Pt}(\varphi \circ \gamma) \circ \psi_{x_0} \circ \mathrm{Pt}(\gamma)^{-1} \\ &= \mathrm{Pt}(\varphi \circ \eta)\psi_\gamma = \mathrm{Pt}(\varphi \circ \eta)\psi.\end{aligned}$$

(4) ψ is a smooth map. Let U be a neighborhood of $x \in M$ which is diffeomorphic to an open convex subset of a locally convex space, so that we may view U as such a set. Then we have

for each $x_1 \in U$ a canonical path from x to x_1 given by $\eta_{x_1}(t) := x + t(x_1 - x)$. From the discussion in Remark II.1(c) it then follows easily that the parallel transport map

$$U \times q^{-1}(x) \to q^{-1}(U), \quad (x_1, z) \mapsto \mathrm{Pt}(\eta_{x_1}).z$$

is smooth, and similarly, its inverse map

$$q^{-1}(U) \to U \times q^{-1}(x), \quad y \mapsto \mathrm{Pt}(\eta_{q(y)})^{-1}.y$$

is smooth.

To see that ψ is smooth, we now choose a fixed path γ from x_0 to x, so that we have on $q^{-1}(x_1)$, $x_1 \in U$, the relation

$$\psi = \mathrm{Pt}(\varphi \circ \eta_{x_1}) \, \mathrm{Pt}(\varphi \circ \gamma) \psi_{x_0} \, \mathrm{Pt}(\gamma)^{-1} \, \mathrm{Pt}(\eta_{x_1})^{-1}.$$

Therefore the preceding remarks imply that ψ is smooth on $q^{-1}(U)$, and since x was arbitrary, the smoothness of ψ follows. We likewise see that ψ^{-1} is smooth.
(5) Next we show that $\psi \in \mathrm{Aut}(P, \theta)$. It remains to see that $\psi^*\theta = \theta$. Let $v \in T_y(P)$ be a horizontal tangent vector. Then there exists a path $\gamma: [0,1] \to M$ and a horizontal lift $\widetilde{\gamma}: [0,1] \to P$ with $\widetilde{\gamma}'(0) = v$. For each $t \in [0,1]$ we now have

$$\widetilde{\gamma}(t) = \mathrm{Pt}(\gamma|_{[0,t]})\widetilde{\gamma}(0),$$

and hence

$$\psi(\widetilde{\gamma}(t)) = \mathrm{Pt}(\varphi \circ \gamma|_{[0,t]})\psi(\widetilde{\gamma}(0)),$$

showing that $\psi \circ \widetilde{\gamma}$ is a horizontal lift of the path $\varphi \circ \gamma$. Taking derivatives in $t = 0$, we see that

$$d\psi(y).v = (\psi \circ \widetilde{\gamma})'(0)$$

is horizontal. Moreover, (2) implies that $\psi \circ \rho_z = \rho_z \circ \psi$ for all $z \in Z$, so that

$$\psi_* \dot{\rho}(X) = \dot{\rho}(X), \quad X \in \mathfrak{z}.$$

For $X \in \mathfrak{z}$ we now obtain

$$\begin{aligned}(\psi^*\theta)(y)(v + \dot{\rho}(X)(y)) &= \theta(\psi(y))(d\psi(y).v + d\psi(y).\dot{\rho}(X)(y)) = \theta(\psi(y)).\dot{\rho}(X)(\psi(y)) \\ &= X = \theta(y)(v + \dot{\rho}(X)(y)).\end{aligned}$$

This proves that $\psi^*\theta = \theta$, and the proof is complete. ∎

In the following corollary the connected component G_0 of a subgroup $G \subseteq \mathcal{D}(M)$ is considered with respect to piecewise smooth curves in G (cf. Definition I.1).

Corollary II.8. *For each principal Z-bundle $q: P \to M$ with connection 1-form θ and curvature $\omega \in \Omega^2(M, \mathfrak{z})$ we have a short exact sequence of groups*

$$\mathbf{1} \to Z \hookrightarrow \mathrm{Aut}(P, \theta)_0 \twoheadrightarrow (\ker S_\Gamma)_0 \to \mathbf{1},$$

where $S_\Gamma: \mathcal{D}_\omega(M)_0 \to H^1_{\mathrm{dR}}(M, \mathfrak{z}, \Gamma)$ is the flux homomorphism for the subgroup Γ of Γ_ω.

Proof. Let $(\varphi_t)_{0 \le t \le 1}$ be a smooth curve in $\mathcal{H}$ with $\varphi_0 = \mathrm{id}_M$ and $\varphi_1 = \varphi$. We fix a point $x_0 \in M$. Then we have a smooth curve $\zeta: [0,1] \to M, t \mapsto \varphi_t(x_0)$. We define the maps

$$\psi_{t,x_0} := \mathrm{Pt}(\zeta|_{[0,t]}): q^{-1}(x_0) \to q^{-1}\big(\varphi_t(x_0)\big)$$

and write ψ_t for the unique extension of ψ_{t,x_0} to an element of $\mathrm{Aut}(P, \alpha)$ satisfying $q \circ \psi_t = \varphi_t \circ q$ (Theorem II.7). It remains to see that $(\psi_t)_{0 \le t \le 1}$ is a smooth curve of diffeomorphisms of P in the sense of Definition I.1.

We proceed as in Step (4) of the proof of Theorem II.7. Let $x \in M$ and U be a neighborhood of x which is diffeomorphic to an open convex subset of a locally convex space, so that we may view U as such a set. Then we have for each $x_1 \in U$ a canonical path from x to x_1 given by $\eta_{x_1}(t) := x + t(x_1 - x)$. To see that $(t, x_1) \mapsto \psi_t(x_1)$ is smooth, we now choose a fixed path γ from x_0 to x, so that we have on $q^{-1}(x_1)$, $x_1 \in U$, the relation

$$\psi_t = \mathrm{Pt}(\varphi_t \circ \eta_{x_1}) \, \mathrm{Pt}(\varphi_t \circ \gamma) \psi_{t,x_0} \, \mathrm{Pt}(\gamma)^{-1} \, \mathrm{Pt}(\eta_{x_1})^{-1}.$$

Here the product of the rightmost two factors does not depend on t, and its smoothness follows as in the proof of Theorem II.7. Therefore it remains to consider the map

$$[0,1] \times q^{-1}(x_0) \to P, \quad (t, z) \mapsto \mathrm{Pt}(\varphi_t \circ \eta_{x_1}) \, \mathrm{Pt}(\varphi_t \circ \gamma) \psi_{t,x_0}.z,$$

whose smoothness follows from the observation that if a curve depends smoothly on one parameter, then the parallel transport along that curve depends smoothly on that parameter, too. This in turn follows from the discussion in Remark II.1 and the corresponding statement on the smooth dependence of integrals of smooth functions on parameters. ■

III. Application to central extensions of Lie groups

Let $q: P \to M$ be a principal Z-bundle with connection 1-form θ and curvature ω. Further let G be a connected Lie group and $\alpha: G \times M \to M$ be a smooth Lie group action which is hamiltonian, which is equivalent to

$$\alpha(G) \subseteq \ker S_\omega \subseteq \mathcal{D}_\omega(M)$$

because the period group of ω is contained in Γ by Corollary II.3, hence discrete.

As we have seen in Corollary II.8, the bundle P defines a central Z-extension of the group $\mathcal{H} \supseteq (\ker S_\omega)_0$. Viewing α as a homomorphism $G \to \mathcal{H}$, we can pull back this central extension to a central Z-extension $\widehat{G}$ of G. The main result of this section will be the observation that $\widehat{G}$ is a Lie group, that the natural projection $q_G: \widehat{G} \to G$ is a Z-principal bundle, and that $\widehat{G}$ acts smoothly on P by automorphisms of (P, θ).

We start by defining the abstract group $\widehat{G}$ as the pull back

$$\widehat{G} := \{(g, \psi) \in G \times \mathrm{Aut}(P, \theta) : q_*^\theta(\psi) = \alpha_g\} \quad \text{and} \quad q_G: \widehat{G} \to G, \quad (g, \psi) \mapsto g.$$

In view of Theorem II.7, $\alpha(G) \subseteq \mathcal{H}$ implies that $q_G: \widehat{G} \to G$ is a surjective group homomorphism. Its kernel is isomorphic to $Z \cong \ker q_*^\theta$ (Corollary II.8), and this is a central subgroup of $\widehat{G}$. Therefore q_G defines a central Z-extension of G.

Proposition III.1. *Let $q: P \to M$ be a principal Z-bundle with connection form θ, N_2 a connected manifold, N_1 a manifold and*

$$\varphi: N_1 \times N_2 \to M$$

a smooth map. Let $\varphi_t(x) := \varphi(x, t)$. Then all the bundles $P_t := \varphi_t^* P \to N_1$ are isomorphic as principal Z-bundles.

Proof. Since N_2 is connected, it suffices to prove the assertion for $N_2 = [0,1]$ and a family of maps $\varphi_t := \varphi(\cdot, t): N := N_1 \to M$ which is smooth in the sense that the corresponding map $N \times [0,1] \to M$ is smooth.

To obtain a bundle isomorphism $\psi: P_0 \to P_1$, we consider $P_t := \varphi_t^* P$ as the manifold

$$P_t := \{(x, y) \in N \times P : \varphi_t(x) = q(y)\}$$

with the bundle projection $q_{P_t}(x, y) = x$. Then we define the map

$$\psi: P_0 \to P_1, \quad \psi(x, y) := (x, \mathrm{Pt}(\gamma_x).y), \quad \gamma_x: [0,1] \to M, \gamma_x(t) := \varphi_t(x).$$

It is easy to see that ψ is a smooth bundle isomorphism. ■

Lemma III.2. *For $m_o \in M$ let $\varphi_{m_o}: G \to M, g \mapsto g.m_o$ be the orbit map and $P_{m_o} := \varphi_{m_o}^* P$ the corresponding pullback Z-bundle over G. Pick an element $y_o \in P$ with $q(y_o) = m_o$. Then*

$$\Phi: \widehat{G} \to P_{m_o} \subseteq G \times P, \quad (g, \psi) \mapsto (g, \psi.y_o)$$

is a bijection. We thus obtain on $\widehat{G}$ a smooth manifold structure. If M is connected, this manifold structure does not depend on the choice of m.

Proof. If $\Phi(g, \psi_1) = \Phi(g, \psi_2)$, then $\psi_1.y_o = \psi_2.y_o$ and

$$\alpha(g) \circ q = q \circ \psi_2 = q \circ \psi_1$$

leads to $\psi_2 = \psi_1$. Hence Φ is injective.

To see that Φ is surjective, let $(g, y) \in P_{m_o}$. Then $q(y) = g.m_o$. Let $(g, \psi) \in \widehat{G}$. Then $q(\psi.y_o) = g.m_o$ and since $(g, \psi Z) \subseteq \widehat{G}$, there also exists a $z \in Z$ with $y_o.z = y$. Therefore Φ is surjective. We now transport the manifold structure from P_{m_o} to $\widehat{G}$ to obtain a natural manifold structure on the group $\widehat{G}$, such that the quotient homomorphism $q_G: \widehat{G} \to G$ defines on $\widehat{G}$ the structure of a principal Z-bundle. The independence of the differentiable structure on $\widehat{G} \cong P_{m_o}$ from m_o follows from Proposition III.1. ■

Lemma III.3. *Let G be a group endowed with the structure of a connected manifold such that*
(1) *all left multiplication maps $\lambda_g: G \to G, x \mapsto gx$ are smooth, and*
(2) *multiplication and inversion are smooth on identity neighborhoods.*
Then G is a Lie group.

Proof. We have to show that the map $\mu: G \times G \to G, (x, y) \mapsto xy^{-1}$ is smooth. That it is smooth in an identity neighborhood follows from (2). Moreover, (2) implies that there exists an open symmetric identity neighborhood V such that for all elements $g \in V$ the right multiplication map $\rho_g(x) = xg$ is smooth in an identity neighborhood. Now $\rho_g \lambda_h = \lambda_h \rho_g$ for $g, h \in G$, together with (1) implies that the maps ρ_g, $g \in V$, are smooth on all of G. Since all left multiplications on G are smooth, all sets aV, $a \in G$, are open, which implies that $H := \bigcup_n V^n$ is an open subgroup of G. Its other cosets gH are also open, so that H is closed, and the connectedness of G implies that $H = G$. Therefore all right multiplications ρ_g, $g \in G$, are smooth.

Fix $(g, h) \in G \times G$. We then have

$$\mu \circ (\lambda_g \times \lambda_h)(x, y) = \mu(gx, hy) = gxy^{-1}h^{-1} = (\lambda_g \circ \rho_{h^{-1}} \circ \mu)(x, y),$$

showing that μ is smooth in a neighborhood of (g, h). As g and h were arbitrary, the smoothness of μ follows. ■

In view of Lemma III.2, we obtain for each $m_o \in M$ on the group $\widehat{G}$ a natural manifold structure. We have to show that the group multiplication on $\widehat{G}$ is smooth with respect to this manifold structure.

Theorem III.4. *Let $q: P \to M$ be a principal Z-bundle with connection 1-form θ and curvature $\omega \in Z^2(M, \mathfrak{z})$. Further let $\alpha: G \times M \to M$ be a hamiltonian action on (M, ω). Then there exists a central Lie group extension $q_G: \widehat{G} \to G$ and a smooth action $\widehat{\alpha}$ of $\widehat{G}$ on P by automorphisms of (P, θ) such that $q \circ \widehat{\alpha} = \alpha \circ (\mathrm{id}_G \times q)$.*

Proof. Let $\widehat{G}$ be the pull back of the central Z-extension $\mathrm{Aut}(P, \theta) \to \mathcal{H}$ by the homomorphism $\alpha: G \to \mathcal{H} \subseteq \mathcal{D}_\omega(M)$. We consider the action

$$\widehat{\alpha}: \widehat{G} \times P \to P, \quad ((g, \psi), y) \mapsto \psi(y)$$

of $\widehat{G}$ on P. Since all elements of $\mathrm{Aut}(P, \theta)$ are smooth, the group $\widehat{G}$ acts on P by smooth maps. Therefore it remains to endow $\widehat{G}$ with a Lie group structure for which $q_G: \widehat{G} \to G$ is a central

Lie group extension and to show that for this manifold structure on G the map $\widehat{\alpha}$ is smooth in a neighborhood of each pair $(\mathbf{1}, y_o) \in \widehat{G} \times P$.

To this end, we first give a local description of the action. Let $U \subseteq M$ be an open subset of M on which the bundle $P|_U$ is trivial. Let $\sigma_U : U \to P$ be a smooth section of this bundle, and $A_U := \sigma^*\theta$ the corresponding local gauge potential. We identify $q^{-1}(U)$ with $U \times Z$ in such a way that $\sigma(x) = (x, \mathbf{1})$ for $x \in U$.

Let $x_o \in U$ and $y_o := \sigma(x_o)$. In view of the continuity of the action of G on M, there exists an open identity neighborhood V_G in G and an open neighborhood V_M of x_o with $V_G.V_M \subseteq U$. Here we may w.l.o.g. assume that V_M is diffeomorphic to an open convex subset of a locally convex space.

We endow $\widehat{G}$ with the manifold structure obtained from Lemma III.2. Pulling back the smooth section σ_U to a smooth local section of the bundle $q_G : \widehat{G} \to G$, we obtain a smooth local section

$$\sigma_G : V_G \subseteq \varphi_{x_o}^{-1}(U) \to \widehat{G}, \quad g \mapsto (g, \psi_g) \quad \text{with} \quad \psi_g.y_o = \sigma_U(g.x_o)$$

(see the proof of Lemma III.2). Now the action map

$$\widehat{\alpha} : \widehat{G} \times P \to P, \quad ((g,\psi), y) \mapsto \psi(y)$$

restricts to a map

$$(V_G \times Z) \times (V_M \times Z) \to U \times Z, \quad ((g,z),(x,z')) \mapsto (g.x, f(g,x)zz'),$$

where $f : V_G \times V_M \to Z$ is a function for which all the partial maps $f_g := f(g,\cdot)$ are smooth with $f_g(x_o) = \mathbf{1}$ and $f_{\mathbf{1}} = \mathbf{1}$. This means that

$$\psi_g(x,z') = (g.x, f_g(x)z'), \quad g \in V_G, x \in V_M, z' \in Z.$$

In product coordinates the connection 1-form θ can be written as

$$\theta = q^*A_U + p_Z^*\theta_Z,$$

where $p_Z : U \times Z \to Z$ is the Z-projection. Therefore

$$\theta = \psi_g^*\theta = q^*\alpha_g^*A_U + q^*\delta^l(f_g) + p_Z^*\theta_Z$$

leads to

$$A_U = \alpha_g^*A_U + \delta^l(f_g).$$

From $dA_U = \omega$ we derive that the 1-form $A_U - \alpha_g^*A_U$ on V_M is closed, hence exact by the assumption that V_M is diffeomorphic to a convex set and the Poincaré Lemma (cf. [Ne00, Lemma III.3]). Now there exists a unique smooth function

$$f_g^{\mathfrak{z}} : V_M \to \mathfrak{z} \quad \text{with} \quad f_g^{\mathfrak{z}}(x_o) = 0, \quad df_g^{\mathfrak{z}} = (A_U - \alpha_g^*A_U)|_{V_M}.$$

Moreover, the explicit formula in the Poicaré-Lemma implies that the function

$$f^{\mathfrak{z}} : V_G \times V_M \to \mathfrak{z}, \quad (g,x) \mapsto f_g^{\mathfrak{z}}(x)$$

is smooth. Let $f^Z := \exp_Z \circ f^{\mathfrak{z}}$. Then

$$f^Z : V_G \times V_M \to Z$$

is a smooth function, and the functions $f_g^Z := f^Z(g,\cdot)$ satisfy $f_g^Z(x_o) = \mathbf{1}$ and $\delta^l(f_g^Z) = (A_U - \alpha_g^*A_U)|_{V_M}$. We conclude that $f_g = f_g^Z$ for each $g \in V_G$. It follows in particular that the action map $\widehat{\alpha}$ is smooth on $(V_G \times Z) \times (V_M \times Z)$.

If $\widehat{G}$ carries a Lie group structure for which the natural map $V_G \times Z \hookrightarrow \widehat{G}$ is a local diffeomorphism, the fact that $\widehat{G}$ acts by smooth maps on P implies that $\widehat{\alpha}$ is a smooth action because $x_o \in M$ was arbitrary.

It therefore remains to see that $\widehat{G}$ is a Lie group. Let us assume, in addition, that V_G is symmetric with $V_G^2.V_M \subseteq U$. We then have

$$\sigma_G(g_1)\sigma_G(g_2) = (g_1g_2, \psi_{g_1}\psi_{g_2})$$

and

$$\psi_{g_1}\psi_{g_2}.(x_0, \mathbf{1}) = \psi_{g_1}(g_2.x_o, \mathbf{1}) = (g_1g_2.x_o, f(g_1, g_2.x_0)).$$

Therefore

$$\psi_{g_1}\psi_{g_2} = \psi_{g_1g_2} f(g_1, g_2.x_o).$$

Hence the local group cocycle corresponding to the section σ_G is given by

$$f_{\sigma_G}(g_1, g_2) = f(g_1, g_2.x_o).$$

The smoothness of this function $V_G \times V_G \to Z$ implies that multiplication and inversion on $\widehat{G}$ are smooth in an identity neighborhood. That the left multiplications are also smooth follows from the fact that $\widehat{G}$ acts by smooth maps on the bundle P, hence on the pull back bundles obtained from orbit maps $G \to M, g \mapsto g.x_o$, and therefore also on $\widehat{G}$. Now Lemma III.3 implies that $\widehat{G}$ is a Lie group. ∎

Remark III.5. To obtain the Lie algebra cocycle corresponding to the Lie group structure on $\widehat{G}$ obtained from the point $x_o \in M$, we use a bundle chart on an open neighborhood U of x_o. With the notation from above, we then have

$$\alpha_g^* A_U - A_U = f_g^{-1}.df_g$$

on V_M. For the left invariant 2-form $\Omega := \varphi_{x_o}^*\omega$ on G and the local 1-form $\theta_U := \varphi_{x_o}^* A_U$ we then have $d\theta_U = \Omega$, and the local functions $f_g^G := f_g \circ \varphi_{x_o}$ defining the local group cocycle of $\widehat{G}$ satisfy

$$\lambda_g^*\theta_U - \theta_U = \delta^l(f_g^Z) = df_g^{\mathfrak{z}}.$$

The condition $\theta_U(x_o) = 0$ means that the image of the differential of the local section in x_o is the horizontal space, and this can be assumed, as can be seen from Step 4 in Theorem II.7. We therefore see with [Ne00, Th. II.7] that the Lie algebra cocycle corresponding to the central extension $\widehat{G}$ of G is given by

$$\Omega_{\mathbf{1}}(X, Y) = \omega_{x_o}(d\varphi_{x_o}(\mathbf{1})(X), d\varphi_{x_o}(\mathbf{1})(Y)) = \omega_{x_o}(\dot{\alpha}(X)(x_o), \dot{\alpha}(Y)(x_o)).$$

As all local group cocycles obtained by the construction in the proof of Theorem III.4 correspond to equivalent central Lie group extensions $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$, the group cocycles are equivalent, which implies in particular that the corresponding Lie algebra cocycles are equivalent because M is assumed to be connected.

A direct argument for that can be given as follows. Let $x_0, x_1 \in M$ and γ a curve in M joining x_0 and x_1. Then $\beta : \mathfrak{g} \to \mathbb{R}$ given by

$$\beta(X) = \int_\gamma i_{\dot{\alpha}(X)}\omega$$

has the property that $\Omega_{\mathbf{1}}^{x_0} - \Omega_{\mathbf{1}}^{x_1} = d\beta$. Indeed, formula (1.2) implies that

$$\begin{aligned} -\beta([X,Y]) &= -\int_\gamma i_{\dot{\alpha}([X,Y])}\omega = \int_\gamma d\big(\omega(\dot{\alpha}(X), \dot{\alpha}(Y))\big) \\ &= \omega(\dot{\alpha}(X), \dot{\alpha}(Y))(x_1) - \omega(\dot{\alpha}(X), \dot{\alpha}(Y))(x_0). \end{aligned}$$

∎

In [Bry93, 2.4.8] and [PS86] one also finds a construction of a "Lie group" $\widehat{G}$ via the pullback of the group extension $\mathrm{Aut}(P, \theta) \to \mathcal{D}_\omega(M)$, but no argument is given for the Lie group structure on $\widehat{G}$ and the independence of the point one uses to pull back the bundle $P \to M$ to a Z-bundle over G.

Remark III.6. If the action of G on M has a fixed point x_0, then the central Lie group extension $\widehat{G} \to G$ is trivial (cf. [Bry93, Th. 2.4.12]). In fact, we have a natural lift $\widetilde{\sigma}: G \to \widehat{G}$ given by $\widetilde{\sigma}(g)|_{q^{-1}(x_0)} = \mathrm{id}_{q^{-1}(x_0)}$ given by taking $\psi_{x_0} = \mathrm{id}_{q^{-1}(x_0)}$ in the proof of Theorem II.7. The construction above shows that we thus obtain a smooth section $G \to \widehat{G}$ which is group homomorphism. ■

Remark III.7. In [Bry93] and [We89] one also finds a discussion of the situation for $\mathbb{C}/\Lambda$-bundles over M, where $\Lambda \subseteq \mathbb{C}$ is a countable subgroup which need not be discrete. This should generalize to the case where $\Gamma \subseteq \mathfrak{z}$ is an arbitrary subgroup.

The bundles considered in this section should be constructed as follows. Let $Z := \mathfrak{z}/\Gamma$. On M we consider the sheaf $\mathcal{S}_\mathfrak{z}$ with $\mathcal{S}_\mathfrak{z}(U) = C^\infty(U, \mathfrak{z})$, the constant sheaf $\mathcal{S}_\Gamma$, and the quotient sheaf $\mathcal{S}_Z := \mathcal{S}_\mathfrak{z}/\mathcal{S}_\Gamma$.

Suppose that M is smoothly paracompact. Then the sheaf cohomologies

$$H^1(M, \mathcal{S}_\mathfrak{z}) \quad \text{and} \quad H^2(M, \mathcal{S}_\mathfrak{z})$$

vanish, and the long exact sequence in sheaf cohomology yields an isomorphism

$$H^1(M, \mathcal{S}_Z) \cong H^2(M, \mathcal{S}_\Gamma).$$

As the group $H^2(M, \mathcal{S}_\Gamma)$ is isomorphic to the Čech cohomolgy group $\check{H}^2(M, \Gamma)$, we see in particular that if Γ is discrete and $Z \cong K(1, \Gamma)$ is a CW-complex, then $H^1(M, \mathcal{S}_Z)$ classifies the Z-bundles over M. Nevertheless, the group $H^1(M, \mathcal{S}_Z)$ is defined in all cases.

For relations to Souriau's concept of diffeological groups, we refer to [So85] and [DI85]. ■

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