Integral operators with shifts on homogeneous groups

Vladimir S. Rabinovich*, Steffen Roch

Dedicated to G. S. Litvinchuk on the occasion of his 70th birthday.

Abstract

We study Fredholm properties of integral operators with shifts on homogeneous groups. This investigation is based on the limit operators method which allows us to reduce the problem of Fredholmness of convolution operators with variable coefficients and with variable shifts to the problem of invertibility of convolution operators with constant coefficients and constant shifts. For the invertibility of these operators, methods of harmonic analysis on noncommutative groups are available.

1 Introduction

Let $\mathbb{X}$ be a homogeneous group (see, for instance, [18], and also Section 3.1). We consider the C^*-algebra of operators acting on $L^2(\mathbb{X})$ which is generated by the operators of the form

$$\gamma I + \sum_{i=1}^{N} \prod_{j=1}^{M} a_{ij} K_{ij} T_{ij} \tag{1}$$

where $\gamma \in \mathbb{C}$, the a_{ij} are operators of multiplication by functions $a_{ij} \in L^\infty(\mathbb{X})$, the K_{ij} are operators of right convolution on the group $\mathbb{X}$ with kernels k_{ij} in $L^1(\mathbb{X})$, and the T_{ij} are operators of right shift by functions $g_{ij} : \mathbb{X} \to \mathbb{X}$,

$$(T_{ij})u(x) = u(x \cdot g_{ij}(x)), \quad x \in \mathbb{X}.$$

The functions g_{ij} will be specified later such that the operators $T_{ij} : L^2(\mathbb{X}) \to L^2(\mathbb{X})$ become bounded.

*Supported by the German Research Foundation (DFG) under Grant Nr. 436 RUS 17/24/01.

The goal of this paper is to examine the Fredholm and semi-Fredholm properties of operators of the form (1) and of their limits with respect to the norm topology of $L(L^2(\mathbb{X}))$.

A well-known and archetypical example of a noncommutative homogeneous group is the Heisenberg group. Singular integral operators and pseudodifferential operators on the Heisenberg group have been intensively studied by many authors (see, for example, the monographs [18, 12, 21, 22] which also contain extensive bibliographies). The Fredholm property of operators in some algebras generated by convolution operators and operators of multiplication by bounded functions on general noncommutative locally compact groups was the subject of [19, 20]. Integral operators with constant coefficients and constant shifts (i.e., $a_{ij} \in \mathbb{C}$ and $g_{ij} \in \mathbb{R}$) on the real half-line are considered in [4]. There is also an extensive bibliography devoted to singular integral and pseudodifferential operators with shifts (see, for instance, [1, 3, 6, 7, 5] and the references therein).

Our approach is essentially different from the approaches of the cited papers. It is based on the limit operators method which has been developed in [8, 9, 10, 13, 14, 15] to study the Fredholm properties of large classes of pseudodifferential operators and convolution operators on $\mathbb{R}^N$ and $\mathbb{Z}^N$. Here we apply this method to investigate the Fredholm and semi-Fredholm properties of integral operators with shifts. We employ an axiomatic scheme for the limit operators method which has been proposed in [17]. It should be mentioned that the results of this paper are new even for the operators with nonconstant shifts on the group $\mathbb{R}^N$.

2 The limit operators method

We start with recalling the axiomatic scheme for the application of the limit operators method developed in [17]. Let H be a Hilbert space and $L(H)$ the C^*-algebra of all bounded linear operators acting on H. Suppose that we are given

(A1) operators P, $\hat{P} \in L(H)$ with $P\hat{P} = \hat{P}P = P$.

(A2) a countable set $\{U_\alpha\}_{\alpha\in\Lambda}$ of unitary operators on H such that, with $P_\alpha := U_\alpha P U_\alpha^*$ and $\hat{P}_\alpha := U_\alpha \hat{P} U_\alpha^*$,

$$\sum_{\alpha\in\Lambda} \|P_\alpha u\|^2 = \|u\|^2 \quad \text{and} \quad \sum_{\alpha\in\Lambda} \|\hat{P}_\alpha u\|^2 \leq C\|u\|^2 \tag{2}$$

for all $u \in H$ with a constant C independent of u.

(A3) a sequence $(W_k)_{k\in\mathbb{N}}$ of unitary operators on H and an associated sequence $(D_k)_{k\in\mathbb{N}}$ of mappings from Λ into itself such that $W_k U_\alpha = U_{D_k(\alpha)} W_k$ for all $\alpha \in \Lambda$ and $k \in \mathbb{N}$, and such that the operators $\hat{P}^{(k)} := W_k \hat{P} W_k^*$ converge strongly to the identity operator on H. We also set $P^{(k)} := W_k P W_k^*$ and $P_{k,\alpha} := W_k P_\alpha W_k^*$ as well as $\hat{P}_{k,\alpha} := W_k \hat{P}_\alpha W_k^*$.

(A4) a bounded sequence $(Q_r)_{r\in\mathbb{N}}$ of operators in $L(H)$ such that

- there is a distinguished set $\mathfrak{B}$ of sequences in Λ which contains all sequences (β_m) for which there exist a $k \in \mathbb{N}$ and a sequence (r_m) in $\mathbb{N}$ tending to infinity such that

$$P_{k,\beta_m} Q_{r_m} \neq 0 \quad \text{for all } m \in \mathbb{N}, \tag{3}$$

- every subsequence of a sequence in $\mathfrak{B}$ belongs to $\mathfrak{B}$,
- the set $\mathfrak{B}$ is invariant with respect to each of the mappings D_k, i.e. if $(\beta_m) \in \mathfrak{B}$, then $(D_k\beta_m) \in \mathfrak{B}$ for every k,
- for each $r \in \mathbb{N}$ and each sequence $(\beta_m) \in \mathfrak{B}$,

$$\text{s-lim}_{m\to\infty} U^*_{\beta_m} Q_r U_{\beta_m} = I. \tag{4}$$

Since both the U_α and the W_k are unitary operators, one also has

$$\sum_{\alpha\in\Lambda} \|P_{k,\alpha} u\|^2 = \|u\|^2 \quad \text{and} \quad \sum_{\alpha\in\Lambda} \|\hat{P}_{k,\alpha} u\|^2 \leq C\|u\|^2$$

for all $u \in H$ and $k \in \mathbb{N}$ and

$$P_{k,\alpha}\hat{P}_{k,\alpha} = \hat{P}_{k,\alpha}P_{k,\alpha} = P_{k,\alpha}$$

for all $\alpha \in \Lambda$ and $k \in \mathbb{N}$.

Definition 2.1 *We say that the operator A_β is the* limit operator *of $A \in L(H)$ with respect to the sequence $\beta = (\beta_j) \in \mathfrak{B}$ if, for every $k \in \mathbb{N}$,*

$$\lim_{j\to\infty} \|(U^*_{\beta_j} A U_{\beta_j} - A_\beta)\hat{P}^{(k)}\| = \lim_{j\to\infty} \|(\hat{P}^{(k)})^*(U^*_{\beta_j} A U_{\beta_j} - A_\beta)\| = 0.$$

The set of all limit operators of A with respect to sequences in $\mathfrak{B}$ will be denoted by $\lim_{\mathfrak{B}}(A)$.

The following proposition describes some elementary properties of limit operators.

Proposition 2.2 *Let $\beta \in \mathfrak{B}$, and let A, $B \in L(H)$ be operators for which the limit operators A_β and B_β exist. Then*

(a) $\|A_\beta\| \leq \|A\|$.

(b) $(A+B)_\beta$ *exists and* $(A+B)_\beta = A_\beta + B_\beta$.

(c) $(A^*)_\beta$ *exists and* $(A^*)_\beta = (A_\beta)^*$.

(d) if C, $C_n \in L(H)$ are operators with $\|C - C_n\| \to 0$, and if the limit operators $(C_n)_\beta$ exist for all sufficiently large n, then C_β exists and $\|C_\beta - (C_n)_\beta\| \to 0$.

Definition 2.3 *Let $\mathcal{A}_0(H)$ denote the set of all operators $A \in L(H)$ with the following properties*

(a) $\lim_{k\to\infty} \|[P_{k,\alpha}, A]\| = 0$ *and* $\lim_{k\to\infty} \|[P_{k,\alpha}, A^*]\| = 0$ *uniformly with respect to $\alpha \in \Lambda$,*

(b) *every sequence in $\mathfrak{B}$ possesses a subsequence β for which the limit operator A_β exists,*

(c) *there is a $k_0 \in \mathbb{N}$ such that $P_{k,\alpha}A = P_{k,\alpha}A\hat{P}_{k,\alpha}$ for all $k \geq k_0$.*

Further, let $\mathcal{A}(H)$ denote the closure of $\mathcal{A}_0(H)$ in $L(H)$.

It is easy to check that $\mathcal{A}_0(H)$ and $\mathcal{A}(H)$ are linear spaces. Moreover, every operator A in $\mathcal{A}(H)$ also satisfies conditions (a) and (b) (the latter follows from Proposition 2.2), and if A and B are operators which satisfy (a) and (b), then their product also satisfies these conditions. On the other hand, condition (c) (which is the abstract analogue of the band property) is not stable with respect to norm limits and products of operators.

Let $\nu(A) := \inf_{\|f\|=1} \|Af\|$ refer to the lower norm of the operator $A \in L(H)$. It is well-known that A is invertible from the left if and only if $\nu(A) > 0$ and invertible from the right if and only if $\nu(A^*) > 0$. Thus, A is invertible if and only if both $\nu(A) > 0$ and $\nu(A^*) > 0$.

For every non-zero (but not necessarily closed) subspace L of H we also consider the lower norm of the restriction $A|_L$ of A onto L. If, in particular, L is the range of a non-zero operator $P \in L(H)$, then we call

$$\nu(A|_{P(H)}) = \inf_{\|Pf\|=1} \|APf\|$$

the lower norm of A relative to P. The lower norms of A relative to the Q_r are closely related to the Fredholm properties of A (see the proof of Theorem 4.5 below).

The following result has been proved in [17].

Theorem 2.4 *Let $A \in \mathcal{A}(H)$. Then*

$$\liminf_{r\to\infty} \nu(A|_{Q_r(H)}) = \inf_{A_\beta \in \lim_{\mathfrak{B}}(A)} \nu(A_\beta). \tag{5}$$

3 Operators on homogeneous groups

3.1 Homogeneous groups

Following [18], Chapter XIII, Section 5, we cite some facts on homogeneous groups which are needed in what follows. Homogeneous groups $\mathbb{X}$ arise by equipping $\mathbb{R}^m$ with a Lie group structure and with a family of dilations that act as group automorphisms on $\mathbb{X}$. To be precise, assume we are given smooth mappings

$(x,\, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ from $\mathbb{R}^m$ to $\mathbb{R}^m$ which provide $\mathbb{R}^m$ with a Lie group structure, and assume that the origin of $\mathbb{R}^m$ is the identity element of the associated Lie group. Further we suppose that $a_1 \leq \ldots \leq a_m$ are positive integers such that the dilations

$$D_\delta : (x_1,\, \ldots,\, x_m) \mapsto (\delta^{a_1} x_1,\, \ldots,\, \delta^{a_m} x_m)$$

are group automorphisms for every $\delta > 0$, i.e. that

$$D_\delta(x \cdot y) = D_\delta x \cdot D_\delta y \quad \text{for all } x,\, y \in \mathbb{R}^m.$$

It follows from these assumptions that the group operation is necessarily of the form

$$x \cdot y = x + y + Q(x, y)$$

where $Q : \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$Q(0, 0) = Q(x, 0) = Q(0, x) = 0 \quad \text{for every } x \in \mathbb{R}^m.$$

Moreover, if one writes $Q = (Q_1,\, \ldots,\, Q_m)$, then each Q_r is a polynomial in $2m$ real variables which is homogeneous of degree a_r. Thus, Q contains no pure monomials in x or y.

The Euclidean measure dx is both left and right invariant with respect to the group multiplication, hence, it is the Haar measure on $\mathbb{X}$. Note also that $d(D_\delta x) = \delta^a dx$ where $a := a_1 + \ldots + a_m$.

An archetypical example of a homogeneous non-commutative group is the Heisenberg group $\mathbb{H}^n$ which can be identified with the product $\mathbb{C}^n \times \mathbb{R}$, provided with the group operation

$$(w,\, s) \cdot (z,\, t) := (w + z,\, s + t + 2\, \Im\langle w,\, z\rangle)$$

where $\langle w,\, z\rangle := \sum_{j=1}^n w_j \bar{z}_j$. Consider the norm function ρ on $\mathbb{R}^m$, defined by

$$\rho(x) := \max_{1 \leq j \leq m} \{|x_j|^{1/a_j}\}.$$

Note that $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$. Also, $\rho(D_\delta x) = \delta\rho(x)$, and there is a constant $c > 0$ such that

$$\rho(x \cdot y) \leq c(\rho(x) + \rho(y)) \quad \text{and} \quad \rho(x^{-1}) \leq c\rho(x).$$

Set $\rho(x,\, y) := \rho(x^{-1} \cdot y)$. Then the collection of all balls

$$B(x,\, \varepsilon) := \{y \in \mathbb{X} : \rho(x, y) < \varepsilon\}, \quad \varepsilon > 0,$$

forms an open neighborhood basis of the point $x \in \mathbb{X}$.

3.2 Multiplication operators on $\mathbb{X}$

Throughout what follows, let $\mathbb{X}$ be a homogeneous group. By $C_b(\mathbb{X})$ we denote the C^*-algebra of all continuous functions on $\mathbb{X}$ with $\|f\|_\infty := \sup_{x\in\mathbb{X}} |f(x)| < \infty$, and we let $BUC(\mathbb{X})$ stand for the C^*-subalgebra of $C_b(\mathbb{X})$ which consists of the uniformly continuous functions, i.e. $f \in C_b(\mathbb{X})$ belongs to $BUC(\mathbb{X})$ if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $\rho(x, y) < \delta$.

Further, we let $Q(\mathbb{X})$ refer to set of all measurable bounded functions a on $\mathbb{X}$ such that

$$\limsup_{y\to\infty} \int_\Omega |a(y^{-1} \cdot x)|\, dx = 0$$

for every compact $\Omega \subset \mathbb{X}$. It is easy to check that $BUC(\mathbb{X})$ is a C^*-subalgebra of $L^\infty(\mathbb{X})$ and that $Q(\mathbb{X})$ is a closed ideal of $L^\infty(\mathbb{X})$.

3.3 Convolution operators on $\mathbb{X}$

Given $k \in L^1(\mathbb{X})$ we define the operator of right convolution by k by

$$(C_{k,r}u)(x) := \int_{\mathbb{R}^m} k(x^{-1} \cdot y)u(y)dy = \int_{\mathbb{R}^m} k(z)u(x \cdot z)dz, \quad x \in \mathbb{R}^m.$$

It is well-known that $C_{k,r}$ is bounded on $L^2(\mathbb{R}^m)$ and invariant with respect to the left shift,

$$U_{l,g}C_{k,r} = C_{k,r}U_{l,g} \quad \text{where} \quad (U_{l,g}f)(x) := f(g \cdot x) \quad \text{for } g \in \mathbb{X}.$$

We denote by $V_r(\mathbb{X})$ the set of all operators $C_{k,r}$ of right convolution by a function $k \in L^1(\mathbb{R}^m)$. Note that, if $a \in Q(\mathbb{X})$ and $T \in V_r(\mathbb{X})$, then aT and TaI are compact operators on $L^2(\mathbb{X})$ (see [20]).

Let $\mathbb{Y}$ be a discrete subgroup of the group $\mathbb{X}$ which acts freely on $\mathbb{X}$ such that $\mathbb{X}/\mathbb{Y}$ is a compact manifold. Let M be a fundamental domain of $\mathbb{X}$ with respect to the action of $\mathbb{Y}$ on $\mathbb{X}$ by left shift, i.e., M is a bounded domain in $\mathbb{X}$ such that

$$\mathbb{X} = \bigcup_{\alpha\in\mathbb{Y}} \alpha \cdot \overline{M}.$$

Let M' be an open neighborhood of $\overline{M}$ such that the family $\{\alpha M'\}_{\alpha\in\mathbb{Y}}$ provides a covering of $\mathbb{X}$ of finite multiplicity. Let $f : \mathbb{X} \to [0, 1]$ be a continuous function with $f(x) = 1$ if $x \in \overline{M}$ and $f(x) = 0$ outside M', and let φ be the non-negative function which satisfies

$$\varphi^2(x) := \frac{f(x)}{\sum_{\beta\in\mathbb{Y}} f(\beta \cdot x)}.$$

For $\alpha \in \mathbb{Y}$, set $\varphi_\alpha(x) := \varphi(\alpha \cdot x)$. Evidently, $0 \le \varphi_\alpha(x) \le 1$ and

$$\sum_{\alpha\in\mathbb{Y}} \varphi_\alpha^2(x) = 1, \quad x \in \mathbb{X}. \tag{6}$$

In that sense, the family $\{\varphi_\alpha^2\}_{\alpha\in\mathbb{Y}}$ forms a partition of unity on $\mathbb{X}$. For $\delta > 0$, set $\varphi_{\delta,\alpha}(x) := \varphi_\alpha(D_\delta x)$. The following is proved in [17].

Proposition 3.1 *Let $K \in V_r(\mathbb{X})$. Then $\lim_{\delta\to 0} \|[\varphi_{\delta,\alpha} I,\, K]\| = 0$ uniformly with respect to $\alpha \in \mathbb{Y}$.*

3.4 Shift operators on $\mathbb{X}$

Let $g = (g_1, \ldots, g_m) : \mathbb{X} \to \mathbb{X}$. We consider the shift operators of the form

$$(T_g u)(x) := u(x \cdot g(x))$$

where

(α) $g_j \in C_b^1(\mathbb{X})$ for all j.

(β) The mapping $F_g : \mathbb{X} \to \mathbb{X},\ x \mapsto x \cdot g(x)$ is invertible.

(γ) $\lim_{x\to\infty} \det(dF_g(x)) = 1$ were df refers to the derivative of the function $f : \mathbb{R}^m \to \mathbb{R}^m$.

Proposition 3.2 *If g satisfies the conditions $(\alpha) - (\gamma)$, then the operator T_g is bounded on $L^2(\mathbb{X})$.*

Proof. We have

$$\|T_g u\|^2 = \int_{\mathbb{R}^m} |u(F_g(x))|^2\, dx = \int_{\mathbb{R}^m} |u(y)|^2 |\det dF_g^{-1}(y)|\, dy \le C\|u\|^2$$

where $C := \sup_{y\in\mathbb{R}^m} |\det dF_g^{-1}(y)| < \infty$ due to conditions (β) and (γ). ∎

We call the function g *slowly oscillating* if, in addition to the conditions $(\alpha) - (\gamma)$,

(δ) $\lim_{x\to\infty} \|dg(x)\| = 0$.

The class of all shifts T_g with g slowly oscillating will be denoted by $\mathcal{R}(\mathbb{X})$.

Proposition 3.3 *Let $T_g \in \mathcal{R}(\mathbb{X})$. Then*

$$\lim_{\delta\to 0} \|[\varphi_{\delta,\alpha} I,\, T_g]\| = 0 \quad \textit{uniformly with respect to } \alpha \in \mathbb{Y}.$$

Proof. For every $u \in L^2(\mathbb{X})$, one has

$$\begin{aligned} \|[\varphi_{\delta,\alpha} I,\, T_g]\, u\| &\le \sup_{x\in\mathbb{X}} |\varphi_{\delta,\alpha}(x) - \varphi_{\delta,\alpha}(x \cdot g(x))|\, \|T_g u\| \\ &\le C \sup_{x\in\mathbb{X}} |\varphi_{\delta,\alpha}(x) - \varphi_{\delta,\alpha}(x \cdot g(x))|\, \|u\|. \end{aligned}$$

Let us estimate

$$|\varphi_{\delta,\alpha}(x) - \varphi_{\delta,\alpha}(x\cdot g(x))| = |\varphi(D_\delta(\alpha\cdot x)) - \varphi(D_\delta(\alpha\cdot x)\cdot D_\delta(g(x)))|.$$

The function g is bounded due to assumption (α). Hence,

$$\|D_\delta(g(x))\|_\infty \le \max_{1\le j\le m} \delta^{a_j}\, \|g\|_\infty.$$

Since φ is uniformly continuous on $\mathbb{X}$ we obtain that, given $\varepsilon > 0$, one finds a $\delta_0 > 0$ such that, if $\max_{1\le j\le m} \delta^{a_j}\, \|g\|_\infty < \delta_0$, then

$$\sup_{x\in\mathbb{X}} |\varphi_{\delta,\alpha}(x) - \varphi_{\delta,\alpha}(x\cdot g(x))| < \varepsilon$$

uniformly with respect to $\alpha \in \mathbb{Y}$. This implies the assertion. ■

Here are a few instances where the requirements $(\alpha) - (\delta)$ are satisfied.

Example A. If g is a constant function then, evidently, T_g belongs to $\mathcal{R}(\mathbb{X})$. ■

Example B. Let $\mathbb{X}$ be the commutative group $\mathbb{R}^m$ where

$$(T_g u)(x) = u(x + g(x)),$$

and let the conditions (α) and (δ) be fulfilled. If one of the conditions

$$\max_{1\le j\le m} \sum_{k=1}^{m} \sup_x \left|\frac{\partial g_j(x)}{\partial x_k}\right| < 1, \quad \max_{1\le k\le m} \sum_{j=1}^{m} \sup_x \left|\frac{\partial g_j(x)}{\partial x_k}\right| < 1 \tag{7}$$

is satisfied, then $T_g \in \mathcal{R}(\mathbb{X})$. Indeed, the conditions (7) imply that

$$F_g : \mathbb{R}^m \to \mathbb{R}^m, \quad x \mapsto x + g(x)$$

is a contraction. Thus, by the Banach fixed point theorem, F_g is invertible, and it follows from condition (δ) that

$$\lim_{x\to\infty} \det(dF_g(x)) = 1$$

whence condition (γ). ■

Example C. Let $\mathbb{H}^n$ be the Heisenberg group with coordinates $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Consider the function $g(x, y, t) := (p(x, y), q(x, y), \tau(x, y, t))$ where the mapping

$$(x, y) \mapsto (p(x, y), q(x, y)) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

is subject to a condition analogous to (7) and where, consequently, the mapping

$$(x, y) \mapsto \Phi(x, y) := (x + p(x, y), y + q(x, y))$$

is invertible. Thus, the system

$$x' = x + p(x,\, y), \quad y' = y + q(x,\, y)$$

possesses a unique solution

$$x = f(x',\, y'), \quad y = \varphi(x',\, y').$$

Moreover we suppose that g is slowly oscillating in the sense that

$$\lim_{(x,y)\to\infty} d_x p(x,y) = \lim_{(x,y)\to\infty} d_y p(x,y) = 0, \tag{8}$$

$$\lim_{(x,y)\to\infty} d_x q(x,y) = \lim_{(x,y)\to\infty} d_y q(x,y) = 0, \tag{9}$$

$$\lim_{(x,y,t)\to\infty} d_x \tau(x,y,t) = \lim_{(x,y,t)\to\infty} d_y \tau(x,y,t) = \lim_{(x,y,t)\to\infty} d_t \tau(x,y,t) = 0. \tag{10}$$

Let, moreover,

$$\sup_{(x,y,t)\in\mathbb{R}^{2n+1}} |d_t \tau(x,y,t)| < 1. \tag{11}$$

Then the mapping $F_g : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$, which sends (x,y,t) to

$$\begin{aligned}(x,y,t)\cdot g(x,y,t) \;&=\; (x + p(x,\, y),\, y + q(x,\, y),\\ &\qquad t + \tau(x,\, y,\, t) + 2\,(\langle (x,\, q(x,\, y)\rangle - \langle y,\, p(x,y)\rangle)\end{aligned}$$

is invertible. Indeed, for arbitrary $(x',\, y',\, t') \in \mathbb{R}^{2n+1}$, the equation

$$t' = t + \tau(f(x',\, y'), \varphi(x',\, y'),\, t) + 2\,\Psi(x',\, y')$$

where

$$\Psi(x',\, y') := \langle f(x',\, y'),\, q(f(x',\, y'),\, \varphi(x',\, y'))\rangle - \langle \varphi(x',\, y'),\, p(f(x',\, y'),\, \varphi(x',\, y'))\rangle$$

has a unique solution t due to (11). This proves condition (β), and condition (γ) follows since (8) $-$ (10) imply that

$$\lim_{(x,\,y,\,t)\to\infty} \det(d_{(x,\,y,\,t)} F_g(x,\, y,\, t)) = 1.$$

Consequently, under the above assumptions, $T_g \in \mathcal{R}(\mathbb{H}^{2n+1})$. ∎

4 Fredholmness of convolution operators with shifts

We denote by $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ the smallest C^*-subalgebra of $L(L^2(\mathbb{X}))$ which contains all operators of the form

$$A = \gamma I + \sum_{j=1}^{N} \prod_{k=1}^{L} a_{jk} K_{jk} b_{jk} T_{jk} \tag{12}$$

where $\gamma \in \mathbb{C}$, a_{jk}, $b_{jk} \in BUC(\mathbb{X})$, $K_{jk} \in V_r(\mathbb{X})$ and $T_{jk} \in \mathcal{R}(\mathbb{X})$. Further we write $\mathcal{B}_0(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ for the smallest symmetric (but not necessarily closed) subalgebra of $L(L^2(\mathbb{X}))$ which contains all operators of the form (12) where the kernel functions of the convolution operators K_{jk} are compactly supported.

To study the Fredholmness of operators in $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ by the limit operators method, we specify the axioms (A1) – (A4) as follows.

(A1) Let $\hat{M}$ be an open set which contains the closure $\overline{M'}$ of M' and for which the covering $\{\alpha\hat{M}\}_{\alpha\in\mathbb{Y}}$ of $\mathbb{X}$ has a finite multiplicity. Then we let P be the operator of multiplication by φ and $\hat{P}$ be the operator of multiplication by the characteristic function of $\hat{M}$.

(A2) We choose $\Lambda := \mathbb{Y}$ and let U_α, $\alpha \in \Lambda$, be the operator of left shift by α,

$$(U_\alpha u)(x) := (U_{l,\alpha} u)(x) = u(\alpha \cdot x).$$

Observe that then P_α is the operator of multiplication by φ_α. Hence, the first condition in (2) follows from (6)

$$\sum \|P_\alpha u\|^2 = \sum \langle \varphi_\alpha u,\, \varphi_\alpha u\rangle = \sum \langle \varphi_\alpha^2 u,\, u\rangle = \langle u,\, u\rangle = \|u\|^2,$$

and the second one follows similarly due to the finite multiplicity of the covering $\{\alpha\hat{M}\}$.

(A3) We choose a sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive numbers with $\delta_k \to 0$ as $k \to \infty$ and such that $D_{\delta_k^{-1}}\mathbb{Y} \subseteq \mathbb{Y}$, and we define W_k

$$(W_k u)(x) := \delta_k^{a/2} u(D_{\delta_k} x)$$

with $a := a_1 + \ldots + a_m$. Then

$$W_k U_\alpha = U_{D_{\delta_k^{-1}}\alpha} W_k \quad \text{for all } k \in \mathbb{N} \text{ and } \alpha \in \mathbb{Y},$$

and the operators $\hat{P}^{(k)}$ converge strongly to the identity.

(A4) For $r \in \mathbb{N}$, let Q_r be the operator of multiplication by the characteristic function of $\{x \in \mathbb{X} : \rho(x,\, 0) > r\}$, and let $\mathfrak{B}$ be the set of all sequences in $\mathbb{Y}$ which tend to infinity. Then conditions (3) and (4) are fulfilled.

We show that, under these assumptions, the algebra $\mathcal{B}_0(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ is a subset of $\mathcal{A}_0$ and, hence, its closure $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ is a subset of $\mathcal{A}$.

Proposition 4.1 *Let $A \in \mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$. Then*

$$\lim_{k\to\infty} \|[A,\, P_{k,\alpha}]\| = 0 \quad \textit{uniformly with respect to } \alpha \in \mathbb{Y}. \tag{13}$$

Proof. For operators of the form (12), the proof follows immediately from Propositions 3.1 and 3.3 in connection with Proposition 2.2. Since the set of all operators A which satisfy (13) is a C^*-subalgebra of $L(L^2(\mathbb{X}))$, the result holds for all operators in $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$. ■

Proposition 4.2 *Let $A \in \mathcal{B}_0(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$. Then, for all $\alpha \in \mathbb{Y}$ and for all sufficiently large k,*

$$P_{k,\alpha}A = P_{k,\alpha}A\hat{P}_{k,\alpha}.$$

Proof. Given an open set $N \subseteq \mathbb{X}$, let χ^N refer to the characteristic function of N and, for $k \in \mathbb{N}$ and $\alpha \in \mathbb{Y}$, define $\chi^N_{k,\alpha}$ in analogy to $\varphi_{k,\alpha}$. Thus, $\hat{P}_{k,\alpha}$ is the operator of multiplication by $\chi^{\hat{M}}_{k,\alpha}$. We consider the set $\mathcal{C}$ of all operators B on $L^2(\mathbb{X})$ with the following property. If N_1, N_2 are open sets with $\overline{N_1} \subset N_2$ then, for every $\alpha \in \mathbb{Y}$ and every sufficiently large k,

$$\chi^{N_1}_{k,\alpha}B = \chi^{N_1}_{k,\alpha}B\chi^{N_2}_{k,\alpha}I.$$

It is evident that $\mathcal{C}$ contains the operators aI of multiplication by bounded functions on $\mathbb{X}$, the operators K of right convolution by compactly supported measurable functions on $\mathcal{X}$, and all shift operators $T \in \mathcal{R}(\mathbb{X})$. The set $\mathcal{C}$ also contains the adjoints of these operators. This is again evident for the adjoints of aI and K (the adjoint of aI is the operator of multiplication by the complex conjugate of a, and K^* is the operator of right convolution by the function $x \mapsto \overline{k(x^{-1})}$ which is also compactly supported). Let now $T = T_g$ be a shift operator in $\mathcal{R}(\mathbb{X})$. The substitution rule shows that the adjoint of T_g is the operator $T_h bI$ where

$$h(y) = (g(F_g^{-1}(y)))^{-1} \quad \text{and} \quad b = |\det dF_g^{-1}|.$$

The mapping h is continuous and bounded, and b is a bounded and uniformly continuous function, which follows from property (γ). Thus, the inclusion $T^* \in \mathcal{C}$ will follow once we have shown that $\mathcal{C}$ is an algebra.

Let A, B in $\mathcal{C}$. Then, clearly, $A + B \in \mathcal{C}$. To prove that $AB \in \mathcal{C}$, let N_1, N_2 be open sets with $\overline{N_1} \subset N_2$, and choose an open set N' such that

$$\overline{N_1} \subset N' \subset \overline{N'} \subset N_2.$$

Then

$$\chi^{N_1}_{k,\alpha}AB = \chi^{N_1}_{k,\alpha}A\chi^{N'}_{k,\alpha}B\chi^{N_2}_{k,\alpha}I = \chi^{N_1}_{k,\alpha}AB\chi^{N_2}_{k,\alpha}I,$$

which implies that $AB \in \mathcal{C}$. Consequently, $\mathcal{B}_0(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X})) \subseteq \mathcal{C}$, whence the assertion. ■

Proposition 4.3 *Let $A \in \mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$, and let $\beta = (\beta_k)$ be a sequence in $\mathfrak{B}$. Then there is a subsequence $\tilde{\beta}$ of β for which the limit operator $A_{\tilde{\beta}}$ exists.*

Proof. We start with verifying the assertion for the operators in the algebra $\mathcal{B}_0 := \mathcal{B}_0(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$. Let $\mathcal{D}$ denote the set of all operators $A \in \mathcal{B}_0$ having the property that every sequence in $\mathfrak{B}$ has a subsequence $\tilde{\beta} = (\tilde{\beta}_k)$ such that the operators $U_{\tilde{\beta}_k}^{-1} A U_{\tilde{\beta}_k}$ converge *-strongly as $k \to \infty$.

We claim that $\mathcal{D}$ contains the generating operators of $\mathcal{B}_0$. Since $\mathcal{D}$ is an algebra (which can be easily shown), this implies that $\mathcal{B}_0 \subseteq \mathcal{D}$.

If aI is the operator of multiplication by the function $a \in BUC(\mathbb{X})$, then $U_{\beta_k}^{-1} a U_{\beta_k}$ is the operator of multiplication by the function $x \mapsto a(\beta_k \cdot x)$. The functions in $BUC(\mathbb{X})$ are bounded and uniformly continuous by definition. Hence, by the Arzelà-Ascoli theorem, the sequence β possesses a subsequence $\tilde{\beta}$ such that the functions $x \mapsto a(\tilde{\beta}_k \cdot x)$ tend uniformly on compact subsets of $\mathbb{X}$ to a certain bounded function $a_{\tilde{\beta}}$ as $k \to \infty$. Consequently, the operators $U_{\tilde{\beta}_k}^{-1} a U_{\tilde{\beta}_k}$ converge strongly to $a_{\tilde{\beta}} I$, and the strong convergence of the adjoint sequence follows analogously.

If $A = K$ is a convolution operator, then there is nothing to prove because A commutes with the U_k.

Next we consider the operator $T = T_g \in \mathcal{R}(\mathbb{X})$ of shift by the function g. Then one has

$$(U_{\beta_k}^{-1} T_g U_{\beta_k} u)(x) = u(x \cdot g(\beta_k \cdot x)).$$

Since the functions $x \mapsto g(\beta_k \cdot x)$ are uniformly bounded with respect to $k \in \mathbb{N}$ and equicontinuous on compact subsets of $\mathbb{X}$, the Arzelà-Ascoli theorem implies the existence of a subsequence $\tilde{\beta}$ of β such that the functions $x \mapsto g(\tilde{\beta}_k \cdot x)$ converge uniformly on compacts in $\mathbb{X}$ to a certain bounded function $g_{\tilde{\beta}}$. Since g is slowly oscillating, the function $g_{\tilde{\beta}}$ is constant.

We proceed with showing that the strong limit of the operators $U_{\tilde{\beta}_k}^{-1} T_g U_{\tilde{\beta}_k}$ as $k \to \infty$ exists and that

$$\text{s-lim}_{k\to\infty} U_{\tilde{\beta}_k}^{-1} T_g U_{\tilde{\beta}_k} = T_{g_{\tilde{\beta}}}. \tag{14}$$

Let u be a compactly supported continuous function on $\mathbb{X}$. Thus, u is uniformly continuous on $\mathbb{X}$, and there exists a compact subset Ω of $\mathbb{X}$ such that

$$u(x \cdot g(\tilde{\beta}_k \cdot x)) - u(x \cdot g_{\tilde{\beta}}) = 0 \quad \text{whenever} \quad x \notin \Omega$$

(recall that g is bounded). It is further evident from the definition of $g_{\tilde{\beta}}$ that, for arbitrary $\delta > 0$, there exists a $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$ and all $x \in \Omega$, $\rho(g(\tilde{\beta}_k \cdot x),\, g_{\tilde{\beta}}) < \delta$. Since u is uniformly continuous, this implies that for each $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that

$$\sup_{x\in\Omega} |u(x \cdot g(\tilde{\beta}_k \cdot x)) - u(x \cdot g_{\tilde{\beta}})| < \varepsilon \quad \text{for all } k \geq k_0.$$

Thus,

$$\lim_{k\to\infty} U_{\tilde{\beta}_k}^{-1} T_g U_{\tilde{\beta}_k} u = T_{g_{\tilde{\beta}}} u$$

for every continuous and compactly supported function u on $\mathbb{X}$. Since these functions form a dense subset of $L^2(\mathbb{X})$, this implies (14). The strong convergence of the adjoint sequence follows from the representation of T_g^* derived in the previous proof, from the above results, and from the fact that $\mathcal{D}$ is an algebra.

The remaining part of the proof makes use of the elementary fact that, if $A_n \to A$ and $B_n^* \to B^*$ strongly, and if K is compact, then $\|A_n K B_n - AKB\| \to 0$.

Every operator in $\mathcal{B}_0$ can be written as a sum of operators of the form ACB where C is the operator of convolution by a compactly supported function, and where A, $B \in \mathcal{B}_0$. Let $\beta \in \mathfrak{B}$ and $m \in \mathbb{N}$. We choose a subsequence $\tilde{\beta}$ of β such that

$$U_{\tilde{\beta}_k}^{-1} A U_{\tilde{\beta}_k} \to A_{\tilde{\beta}} \quad \text{and} \quad U_{\tilde{\beta}_k}^{-1} B U_{\tilde{\beta}_k} \to B_{\tilde{\beta}}$$

*-strongly as $k \to \infty$. It follows from the proof of the previous proposition that, if N is sufficiently large,

$$\begin{aligned} U_{\tilde{\beta}_k}^{-1} ACB U_{\tilde{\beta}_k} \hat{P}^{(m)} &= U_{\tilde{\beta}_k}^{-1} A U_{\tilde{\beta}_k} C U_{\tilde{\beta}_k}^{-1} B U_{\tilde{\beta}_k} \hat{P}^{(m)} \\ &= U_{\tilde{\beta}_k}^{-1} A U_{\tilde{\beta}_k} C \hat{P}^{(N)} U_{\tilde{\beta}_k}^{-1} B U_{\tilde{\beta}_k} \hat{P}^{(m)}. \end{aligned}$$

Since the operator $C\hat{P}^{(N)}$ is compact, it follows from the fact just mentioned that

$$\|(U_{\tilde{\beta}_k}^{-1} ACB U_{\tilde{\beta}_k} - A_{\tilde{\beta}} C B_{\tilde{\beta}}) \hat{P}^{(m)}\| \to 0$$

as $k \to \infty$. The dual condition follows analogously. Thus, $A_{\tilde{\beta}} C B_{\tilde{\beta}}$ is a limit operator of ACB. This yields the assertion for operators in $\mathcal{B}_0$ and, employing Proposition 2.2 (d), also for operators in $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$. ■

The following theorem is a corollary of the general Theorem 2.4.

Theorem 4.4 *Let $A \in \mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$. Then*

$$\liminf_{r \to \infty} \nu(A|_{Q_r L^2(\mathbb{X})}) = \inf_{A_\beta \in \sigma_{\mathfrak{B}}(A)} \nu(A_\beta).$$

As a corollary of the previous theorem we derive the desired criteria of semi-Fredholmness and Fredholmness.

Theorem 4.5 *Let $A \in \mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$. Then*
(a) A is a Φ_+-operator if and only if

$$\inf_{A_\beta \in \sigma_{\mathfrak{B}}(A)} \nu(A_\beta) > 0. \tag{15}$$

(b) A is a Φ_--operator if and only if

$$\inf_{A_\beta \in \sigma_{\mathfrak{B}}(A)} \nu(A_\beta^*) > 0. \tag{16}$$

(c) A is a Fredholm operator if and only if all limit operators of A are invertible and if the norms of their inverses are uniformly bounded.

Proof. We will show assertion (a) only. The proof of (b) proceeds similarly, and (c) is a consequence of (a) and (b).

Let (15) be satisfied. Then, by Theorem 4.4, there exist an $r \in \mathbb{N}$ and a constant $C > 0$ such that

$$|\langle Q_r A^* A Q_r f,\, Q_r f\rangle| \geq C\|Q_r f\|^2 \quad \text{for all } f \in L^2(\mathbb{X}). \tag{17}$$

This implies that the operator $Q_r A^* A Q_r$ is invertible on $L^2(Q_r\mathbb{X})$, i.e. there is an operator B such that

$$B Q_r A^* A Q_r = Q_r. \tag{18}$$

It follows from the inverse closedness of C^*-algebras that the operator B belongs to the smallest C^*-subalgebra $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}),\, Q_r)$ of $L(L^2(\mathbb{X}))$ which contains the algebra $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ and the operator Q_r.

Let J_0 refer to the closed ideal of $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}),\, Q_r)$ which is generated by the operators $I - Q_r$, $r \in \mathbb{N}$, and let J_1 stand for the smallest closed ideal of that algebra which contains all operators in $V_r(\mathbb{X})$ and all compact operators. It is evident from the definition of the algebra $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}),\, Q_r)$ that for every operator G in this algebra, there is a (uniquely determined) complex number γ_G such that $G - \gamma_G I \in J_1$. Clearly, the mapping $G \mapsto \gamma_G$ is a continuous algebra homomorphism. Since $\gamma_{Q_r} = 1$, it follows from (18) that $\gamma_A \neq 0$.

The equality (18) further implies that there is an operator R' in the algebra $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}),\, Q_r)$ such that $R'A - I \in J_0$. If we set $R := \gamma_A R' - AR' + I$, then

$$RA - \gamma_A I = \gamma_A R'A - AR'A + A - \gamma_A I = (\gamma_A I - A)(R'A - I).$$

Since $T_0 := R'A - I \in J_0$ and $T_1 := \gamma_A I - A \in J_1$, the operator $RA - \gamma_A I = T_1 T_0$ is compact. Hence, and because of $\gamma_A \neq 0$, A is a Φ_+-operator.

Conversely, let A be a Φ_+-operator. Then there is a compact operator T as well as a positive constant C such that the a priori estimate

$$C\|Au\| \geq \|u\| - \|Tu\|, \quad u \in L^2(\mathbb{X}),$$

holds (see [11], I, Lemma 2.1). This estimate yields

$$C\|AQ_r u\| \geq \|Q_r u\| - \|TQ_r u\|$$

for all $u \in L^2(\mathbb{X})$ and $r \in \mathbb{N}$. Due to the strong convergence of the operators Q_r to 0, there is an $r_0 \in \mathbb{N}$ such that $\|TQ_{r_0}\| \leq C/2$. Thus,

$$\|AQ_{r_0} u\| \geq \frac{C}{2}\,\|Q_{r_0} u\| \quad \text{for all } u \in L^2(\mathbb{X})$$

whence $\liminf_{r\to\infty} \nu(A|_{Q_r L^2(\mathbb{X})}) > 0$. This implies (15) via Theorem 4.4. ■

Finally, we are going to specialize the results of the previous theorem to a class of operators for which the invertibility of their limit operators can be effectively checked. A function $a \in C_b(\mathbb{X})$ is called *slowly oscillating at infinity* if, for every compact $\Omega \subset \mathbb{X}$,

$$\lim_{x\to\infty} \sup_{y\in\Omega} |a(x\cdot y) - a(x)| = 0.$$

For example, if $a \in C_b^1(\mathbb{X})$ and

$$\lim_{x\to\infty} \frac{\partial a(x)}{\partial x_j} = 0, \quad 1 \le j \le m,$$

then a is slowly oscillating at infinity. We write $SO(\mathbb{X})$ for the class of all slowly oscillating functions on $\mathbb{X}$ and set $W(\mathbb{X}) := SO(\mathbb{X})+Q(\mathbb{X})$. Let further the algebra $\mathcal{B}(W(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ be defined in analogy to $\mathcal{B}(BUC(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$. We claim that all limit operators of operators in $\mathcal{B}(W(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ are invariant with respect to left shifts.

Let $a \in Q(\mathbb{X})$ and $K \in V_r(\mathbb{X})$. Then the operators aK and KaI are compact (see [20]). Hence, the limit operators of these operators exist with respect to every sequence $\beta \in \mathfrak{B}$, and they are equal to zero.

Further, let a be slowly oscillating, $\beta = (\beta_m)_{m\in\mathbb{N}} \in \mathfrak{B}$, and let $a_{\tilde{\beta}}$ be as in the proof of Proposition 4.3. Then, evidently,

$$a_{\tilde{\beta}}(x) - a_{\tilde{\beta}}(y) = \lim_{m\to\infty} (a(\beta_m \cdot x) - a((\beta_m \cdot x)\cdot(x^{-1}\cdot y))) = 0$$

for arbitrary $x,\, y \in \mathbb{X}$. Thus, $a_{\tilde{\beta}}$ is indeed a constant function.

In particular, if $A \in \mathcal{B}(W(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$, then every limit operator of A belongs to the smallest C^*-subalgebra $\mathcal{B}(V_r(\mathbb{X}),\, \mathcal{R}^c(\mathbb{X}))$ of $L(L^2(\mathbb{X}))$ which contains all convolution operators in $V_r(\mathbb{X})$ and all shift operators in $\mathcal{R}(\mathbb{X})$ by a constant function (i.e. by an element of the group $\mathbb{X}$).

Thus, in this special setting, Theorem 4.5 reduces the problem of (semi-) Fredholmness for operators in $\mathcal{B}(W(\mathbb{X}),\, V_r(\mathbb{X}),\, \mathcal{R}(\mathbb{X}))$ to the problem of invertibility of operators in the algebra $\mathcal{B}(V_r(\mathbb{X}),\, \mathcal{R}^c(\mathbb{X}))$ which are invariant with respect to left shifts by elements in $\mathbb{X}$. To study this invertibility problem, methods of (non-commutative) harmonic analysis are available (cp. [22]). For example, in case of the commutative group $\mathbb{R}^n$, the operator

$$A := \gamma I + \sum_{j=1}^{N} K_j T_j$$

where $\gamma \in \mathbb{C}$, K_j is a convolution with kernel $k_j \in L^1(\mathbb{R}^n)$ and T_j is the shift by $g_j \in \mathbb{R}^n$, is invertible on $L^2(\mathbb{R}^n)$ if and only if

$$\inf_{\xi\in\mathbb{R}^n} |\gamma + \sum_{j=1}^{N} \hat{k}_j(\xi) e^{i\langle \xi, g_j\rangle}| > 0$$

where $\hat{k}_j$ refers to the Fourier transform of k_j.

References

[1] A. B. ANTONEVICH, Linear Functional Equations. Operator Approach. – Birkhäuser Verlag, Basel, Berlin 1995.

[2] A. B. ANTONEVICH, M. BELOUSOV, A. LEBEDEV, Functional Differential Equations II: C^*-applications. – Longman, Harlow 1998.

[3] A. ANTONEVICH, A. LEBEDEV, Functional Differential Equations I: C^*-theory. – Longman, Harlow 1994.

[4] I. GOHBERG, I. FELDMAN, Convolution Equations and Projection Methods for Their Solution. – Nauka, Moskva 1971 (Russian, Engl. transl.: Amer. Math. Soc. Transl. of Math. Monographs **41**, Providence, R. I., 1974).

[5] N. KARAPETIANTS, S. SAMKO, Equations with Involutive Operators. – Birkhäuser, Boston, Basel, Berlin, 2001.

[6] V. G. KRAVCHENKO, G. S. LITVINCHUK, Introduction to the Theory of Singular Integral Operators with Shifts. – Kluwer Academic Publisher, 1994.

[7] V. G. KURBATOV, Functional Differential Operators and Equations. – Math. and its Appl. **473**, Kluwer Academic Publisher, 1999.

[8] B. V. LANGE, V. S. RABINOVICH, On the Noether property of multidimensional discrete convolutions. – Mat. Zametki **37**(1985), 3, 407 – 421 (Russian).

[9] B. V. LANGE, V. S. RABINOVICH, On the Noether property of multidimensional operators of convolution type with measurable bounded coefficients. – Izv. Vyssh. Ucheb. Zaved., Mat. **6**(1985), 22 – 30 (Russian).

[10] B. V. LANGE, V. S. RABINOVICH, Pseudo-differential operators on $\mathbb{R}^n$ and limit operators. – Mat. Sb. **129**(1986), 2, 175 – 185 (Russian, Engl. transl.: Math. USSR Sbornik **577**(1987), 1, 183 – 194).

[11] S. G. MIKHLIN, S. PRÖSSDORF, Singular Integral Operators. – Springer-Verlag, Heidelberg 1986.

[12] A. NAGEL, E. M. STEIN, Lectures on Pseudo-Differential Operators: Regularity Theorems and Applications to Non-Elliptic Problems. – Math. Notes, Princeton Univ. Press, 1979.

[13] V. S. RABINOVICH, Fredholmness of pseudo-differential operators on $\mathbb{R}^n$ in the scale of $L_{p,q}$-spaces. – Sib. Mat. Zh. **29**(1988), 4, 635 –646 (Russian, Engl. transl.: Siberian Math. J. **29**(1988), 4, 635 – 646).

[14] V. S. Rabinovich, Criterion for local invertibility of pseudodifferential operators with operator symbols and some applications. – Proc. St. Petersburg Math. Soc. **V**, AMS Translation Series 2, Vol. **193**(1998), 239 – 260.

[15] V. S. Rabinovich, S. Roch, B. Silbermann, Fredholm theory and finite section method for band-dominated operators. – Integral Equations Oper. Theory **30**(1998), 4, 452 – 495.

[16] V. S. Rabinovich, S. Roch, B. Silbermann, Band-dominated operators with operator-valued coefficients, their Fredholm properties and finite section. – Integral Equations Oper. Theory **40**(2001), 3, 342 – 381.

[17] V. S. Rabinovich, S. Roch, An axiomatic approach to the limit operators method. – Preprint 2153 Technical University Darmstadt 2001. (To be published in Proc. Inter. Workshop Operator Theory and Applications, Faro 2000).

[18] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. – Princeton Univ. Press, Princeton, New Jersey, 1993.

[19] B. Ya. Steinberg, On convolution operators on locally compact groups. – Funkts. Anal. Prilozh. **15**(1981), 95 – 96 (Russian).

[20] B.Ya. Steinberg, Boundedness and compactness of convolution operators on locally compact groups. – Mat. Zametki **38**(1985), 2, 278 – 292 (Russian).

[21] M. E. Taylor, Noncommutative Microlocal Analysis, I. – Memoirs AMS **313**, 1984.

[22] M. E. Taylor, Noncommutative Harmonic Analysis. – Math. Surveys and Monographs **22**, AMS 1986.

Authors' addresses:

V. S. Rabinovich, Instituto Politechnico National, ESIME-Zacatenco, Ed.1, 2-do piso, Av.IPN, Mexico, D.F., 07738
email: rabinov@maya.esimez.ipn.mx

S. Roch, Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany
email: roch@mathematik.tu-darmstadt.de