Characterizations of Proper Actions

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Abstract

Three kinds of proper actions of increasing strength are defined. We prove that the three definitions specialize to the definitions by Bourbaki, by Palais, and by Baum, Connes, and Higson in their respective settings. The third of these, which thus turns out to be the strongest, originally only concerns actions of second countable locally compact groups on metrizable spaces. In this situation, it is shown to coincide with the other two definitions if the total space locally has the Lindelöf property and the orbit space is regular.[1] [2]

Introduction

Proper actions are an important generalization of actions of compact groups. They were introduced by Palais [16] as continuous actions of locally compact groups on completely regular spaces such that certain neighbourhoods of stabilizers are compact. Bourbaki [5] calls a continuous action of an arbitrary topological group proper if all stabilizers are compact and a certain map is closed. The Baum–Connes conjecture for second countable locally compact groups G, as formulated by Baum, Connes, and Higson [2], states that a certain map from the equivariant K-homology of a classifying space for proper G-actions to the topological K-theory of the reduced C^*-algebra of G is an isomorphism. In this context, a continuous action of G on a metrizable space is defined to be proper if each point lies in a slice and the orbit space is metrizable.

The relations between these three definitions of a proper action are obscured by the fact that they are given in three different settings and in terms of different concepts. This note starts with the single notion of "Cartan actions" of general topological groups. It defines three kinds of proper actions as Cartan actions for which the orbit space is a Hausdorff space, regular,

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and paracompact, respectively. These three definitions are equivalent to the concepts due to Bourbaki, to Palais, and to Baum, Connes, and Higson in their respective settings. In the course of the proof of this equivalence, we characterize the three kinds of proper actions for locally compact groups in the spirit of Palais [16]. This characterization differs considerably from our definition, so that it becomes even more transparent how the three concepts increase in strength.

If G is a topological group and Y is a topological space then the action of G on $G \times Y$ by $g.(g', y) := (gg', y)$ is a Cartan action. The orbit space of this action is homeomorphic to Y. Hence for actions on general spaces, any two of the three concepts of a proper action are inequivalent. Moreover, there are Cartan actions of $\mathbb{R}$ on G_δ-subsets of $\mathbb{R}^2$ for which the orbit space is not a Hausdorff space, or a non-regular Hausdorff space. However, under suitable hypotheses on a group G and a space X, if the orbit space of a Cartan action of G on X is regular then it is even paracompact. Extending an observation by Abels [1], we prove this for Cartan actions of locally compact Lindelöf groups G on paracompact locally Lindelöf spaces X. In particular, all three concepts of a proper action are equivalent for actions of second countable locally compact groups on second countable locally compact spaces. This special case has recently been proved by Chabert, Echterhoff, and Meyer [7].

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1 Notions of proper actions

If a group G acts on a set X and $A, B \subseteq X$ are subsets, set

$$\sigma_G(A, B) := \{g \in G \mid g.A \cap B \neq \emptyset\}.$$

For $a, b \in X$, write $\sigma_G(a, b) := \sigma_G(\{a\}, \{b\}) = \{g \in G \mid g.a = b\}$. Note that this set is either empty or a coset of a stabilizer.

The symbol $\mathcal{U}(\cdot)$ denotes the neighbourhood filter of a point or of a subset of a topological space.

We follow Bourbaki [5] in calling a topological space quasi-compact if every open covering has a finite subcovering. Compactness, local compactness, and paracompactness include the Hausdorff separation property.

1.1 Definition. A continuous action of a topological group G on a topological space X is called a *Cartan action* if all stabilizers are quasi-compact and for each point $x \in X$ and each neighbourhood $U \subseteq G$ of the stabilizer G_x, there is a neighbourhood $V \subseteq X$ of x such that every group element which maps a point of V into V belongs to U, i.e.

$$\forall\, x \in X\ \forall\, U \in \mathcal{U}(G_x)\ \exists\, V \in \mathcal{U}(x)\colon \sigma_G(V, V) \subseteq U.$$

A *proper action* is a Cartan action with a Hausdorff orbit space.

A *Palais-proper action* is a Cartan action with a regular orbit space.

A *strongly proper action* is a Cartan action with a paracompact orbit space.

1.2 Remark (Relations to established definitions). The most widespread concept of a proper action probably is the one to be found in Bourbaki [5, III.4, Def. 1]. By Proposition 1.4, a continuous action of a topological group on a topological space is proper in the sense of Bourbaki if and only if it is proper in the sense of our definition.

Palais [16] only considers actions of locally compact groups on completely regular spaces. Theorem 1.6 shows that his concepts of a Cartan G-space and of a proper G-space [16, 1.1.2 and 1.2.2] are equivalent to our definitions of a Cartan action and of a Palais-proper action.

Baum, Connes, and Higson [2] restrict themselves to actions of second countable locally compact groups on metrizable spaces. As we will see in Remark 3.6, such an action is proper in their sense if and only if it is strongly proper in our sense.

1.3 Remark (Implications and separating examples). Let G be a topological group, and let H be a subgroup. If the natural action of G on G/H is a Cartan action then H is quasi-compact. Conversely, assume that H is quasi-compact. Choose $g \in G$, and let $U \subseteq G$ be a neighbourhood of the subgroup gHg^{-1}, which is the stabilizer of $gH \in G/H$ under the natural action of G. By continuity of the group multiplication and quasi-compactness of H, there is a neighbourhood $V \subseteq G$ of g such that $VHV^{-1} \subseteq U$. The neighbourhood $V' := VH/H$ of gH in G/H satisfies $\sigma_G(V', V') \subseteq U$. We conclude that the natural action of G on the coset space G/H is a Cartan action if and only if the subgroup H is quasi-compact. Assume that this is the case, and let Y be an arbitrary topological space. The action of G on $G/H \times Y$ by $g(g'H, y) := (gg'H, y)$ is a Cartan action with orbit space homeomorphic to Y. For an arbitrary topological group G and suitable topological spaces Y, this construction yields Cartan actions of G which are not proper, proper actions which are not Palais-proper, and Palais-proper actions which are not strongly proper.

Here the orbit space lacks a certain topological property which the total space of the action also lacks. Thus the following examples may be more surprising. Palais [16, p. 298] describes a Cartan action of $\mathbb{R}$ on $[-1, 1] \times \mathbb{R}$ which is not proper. An example of a proper action of $\mathbb{R}$ on a G_δ-subset of $\mathbb{R}^2$ which is not Palais-proper is due to Bebutov (see Hájek [12, p. 79], cf. Bhatia and Szegö [3, IV.1.5.5] and Abels [1, 1.6]).

However, a Palais-proper action of $\mathbb{R}$ on a separable metric space is automatically strongly proper. More generally, let G be a locally compact group such that the quotient of G by its identity component is compact.

(Such a group is called *almost connected*.) A result sketched by Abels in the introduction to [1] asserts, translated into our terminology, that every Palais-proper action of G on a paracompact locally Lindelöf space is strongly proper. Corollary 3.5 below extends this result to actions of locally compact Lindelöf groups. (A locally compact group is a Lindelöf space if and only if it has an open almost connected subgroup of countable index.) Weaker hypotheses may entail that Palais-properness of an action implies strong properness. For instance, Abels [1] conjectured that every Palais-proper action of a connected locally compact group on a paracompact space is strongly proper.

By Part (b) of Corollary 3.5, every proper action of a locally compact Lindelöf group on a paracompact locally compact space is strongly proper.

1.4 Proposition. *A continuous action of a topological group G on a topological space X is proper if and only if the map $(g,x) \mapsto (x, g.x)\colon G \times X \to X \times X$ is proper (i.e. closed with quasi-compact fibres).*

In other words, the present definition of a proper action is equivalent to Bourbaki's [5, III.4, Def. 1].

Proof. It was proved in [4, 2.1] that the action of G on X is proper in the sense of Bourbaki if and only if all stabilizers are quasi-compact and

$$\forall\, x, x' \in X\ \forall\, U \in \mathcal{U}\big(\sigma_G(x,x')\big)\ \exists\, V \in \mathcal{U}(x), V' \in \mathcal{U}(x')\colon \sigma_G(V,V') \subseteq U.$$

It is easy to see that this condition is equivalent to the present definition of a proper action. $\square$

This paper follows Bourbaki [5] in considering topological groups which may not be Hausdorff spaces. However, it should be emphasized that the relation between topological groups and Hausdorff groups is much closer than that between topological spaces and Hausdorff spaces in general. For a topological group G, this relation is described via the closure N of $\{1\}$ in G, which is the smallest closed normal subgroup of G. The quotient G/N is the universal Hausdorff group associated to G (in the sense of categories). A subset $A \subseteq G$ is open (respectively closed) if and only if $AN = A$ and A/N is an open (respectively closed) subset of G/N. In this sense, the topology of G is derived from the topology of the Hausdorff group G/N. A subset $A \subseteq G$ is quasi-compact if and only if AN/N is a compact subset of G/N. If this is the case then the quasi-compact set AN is the closure of A in G. A coset space of G is a T_0-space if and only if it is a completely regular space if and only if it is the coset space of a closed subgroup of G (see Hewitt and Ross [13, 8.14(a)]). In particular, every action of G on a T_0-space X factors through an action of G/N on X.

Let a topological group G act continuously on a topological space X. Following Palais [16, 1.2.1], we will say that a subset $A \subseteq X$ is *G-small* if every point $x \in X$ has a neighbourhood V such that $\sigma_G(A, V)$ is contained in a quasi-compact subset of G. Equivalently, the closure of $\sigma_G(A, V)$ in G is quasi-compact. This is because every quasi-compact subset of a topological group has quasi-compact closure. Note that the closure of a G-small subset is G-small.

1.5 Lemma. *Let a topological group G act continuously on a topological space X, and let $A \subseteq X$ be a closed G-small subset. Then the following assertions hold:*

(a) *The restriction $\omega\colon G \times A \to X$, $(g, a) \mapsto g.a$ of the action to $G \times A$ is a proper map.*

(b) *The restriction $p\colon A \to G\backslash X$, $a \mapsto G.a$ of the orbit projection to A is a proper map.*

Proof. (a) If $x \in X$ then $\omega^{-1}(x) = \{(g, g^{-1}.x) \mid g \in \sigma_G(A, \{x\})\}$ is a continuous image of the set $\sigma_G(A, \{x\})$. This set is closed because A is closed, so that it is quasi-compact because A is G-small. Hence the fibres of ω are quasi-compact. Let $C \subseteq G \times A$ be a closed subset, and let $x \in X \setminus \omega(C)$. Choose a neighbourhood V of x such that the closure K of $\sigma_G(A, V)$ in G is quasi-compact. The restriction of ω to $K \times A$ is a proper map (see Bourbaki [5, III.4, Prop. 1]). As $V \cap G.A = V \cap K.A$, we conclude that $V \setminus \omega(C) = V \setminus \omega(C \cap (K \times A))$ is a neighbourhood of x which does not meet $\omega(C)$.

(b) The fibre of p through $x \in A$ is $G.x \cap A = \sigma_G(A, \{x\})^{-1}.x$. Being a continuous image of the quasi-compact set $\sigma_G(A, \{x\})$, it is quasi-compact. If $B \subseteq A$ is a closed subset then B is also G-small. Hence the orbit saturation $G.B$ of B in X is closed by part (a), so that $p(B)$ is closed in the orbit space $G\backslash X$. □

1.6 Theorem (Proper actions of locally compact groups). *Let a locally compact group G act continuously on a topological space X.*

(a) *The action of G on X is a Cartan action if and only if any two distinct points on an orbit have disjoint neighbourhoods in X and*

$$\forall\, x \in X\ \exists\, V \in \mathcal{U}(x)\colon \overline{\sigma_G(V, V)} \text{ is compact.}$$

(b) *The action of G on X is proper if and only if X is a Hausdorff space and*

$$\forall\, x, x' \in X\ \exists\, V \in \mathcal{U}(x), V' \in \mathcal{U}(x')\colon \overline{\sigma_G(V, V')} \text{ is compact.}$$

(c) *The action of G on X is Palais-proper if and only if X is regular and each point of X has a G-small neighbourhood, i.e.*

$$\forall x \in X\ \exists V \in \mathcal{U}(x)\ \forall x' \in X\ \exists V' \in \mathcal{U}(x')\colon \overline{\sigma_G(V,V')} \text{ is compact.}$$

(d) *The action of G on X is strongly proper if and only if X is paracompact and there is an open G-small subset V of X such that $G.V = X$.*

Proof. (a) Assume that the action is a Cartan action. Let $x \in X$. If $g \in G$ is a group element such that $g.x \neq x$ then $G \setminus \{g\}$ is a neighbourhood of the stabilizer G_x, whence we may choose a neighbourhood V of x such that $g \notin \sigma_G(V,V)$. This means that V and $g.V$ are disjoint neighbourhoods of x and of $g.x$. Let U be a compact neighbourhood of the stabilizer G_x. Choose a neighbourhood V' of x in X such that $\sigma_G(V',V') \subseteq U$. The closure of $\sigma_G(V',V')$ in G is compact.

Conversely, assume that the condition in (a) is satisfied. Choose a point $x \in X$. If $g.x \neq x$ for an element $g \in G$ then there are neighbourhoods U of g and V, V' of x, $g.x$ such that $V \cap V' = \emptyset$ and $U.V \subseteq V'$. Hence U is disjoint from $\sigma_G(V,V)$. This proves that

$$G_x = \bigcap_{V \in \mathcal{U}(x)} \overline{\sigma_G(V,V)}.$$

This intersection is compact by hypothesis. Let U be an open neighbourhood of G_x. Then we find neighbourhoods $V_1, \dots, V_n$ of x such that

$$U \supseteq \bigcap_{j=1}^{n} \overline{\sigma_G(V_j,V_j)} \supseteq \sigma_G(V_1 \cap \dots \cap V_n, V_1 \cap \dots \cap V_n).$$

(b) If X is a Hausdorff space and $x, x' \in X$ then we find as above that

$$\sigma_G(x,x') = \bigcap \{\, \overline{\sigma_G(V,V')} \mid V \in \mathcal{U}(x), V' \in \mathcal{U}(x')\}.$$

The arguments from part (a) yield that the condition in (b) is equivalent to the characterization of proper actions used in the proof of Proposition 1.4. Alternatively, Proposition 1.4 implies that part (b) is the well-known characterization of proper actions of locally compact groups given by Bourbaki [5, III.4, Prop. 3 and Prop. 7].

(c) Assume that the action of G on X is Palais-proper, and let $x \in X$. Choose an open neighbourhood W_1 of x such that $\sigma_G(W_1,W_1)$ has compact closure in G. As the orbit space is regular, we may choose a closed G-invariant neighbourhood W_2 of x in X which is contained in $G.W_1$. We claim that the neighbourhood $V := W_1 \cap W_2$ of x is G-small. Indeed, if $x' \in G.W_1$, say $x' \in g.W_1$, then $g.W_1$ is a neighbourhood of x' for which $\sigma_G(V, g.W_1) \subseteq g.\sigma_G(W_1,W_1)$ has compact closure in G. If $x' \in X \setminus G.W_1$ then $X \setminus W_2$ is a neighbourhood of x' such that $\sigma_G(V, X \setminus W_2) = \emptyset$. (This part of the proof has been adapted from Palais [16, 1.2.5].) The closure of V in X is a G-small subset of X, and it is regular since it is a Hausdorff space and admits a proper map onto a regular space (Lemma 1.5 and Engelking [10, 3.7.23]). As every point of X has a closed regular neighbourhood, we conclude that X is regular.

Conversely, assume that X is regular and that each point of X has a G-small neighbourhood. Choose $x \in X$, and let W be a G-invariant neighbourhood of x. Choose a G-small closed neighbourhood W_1 of x which is contained in W. Lemma 1.5 shows that the G-invariant neighbourhood $G.W_1 \subseteq W$ of x is closed in X. Hence the orbit space $G\backslash X$ is regular. Moreover, we may choose a neighbourhood W_2 of x such that $\sigma_G(W_1, W_2)$ has compact closure in G. Then $V := W_1 \cap W_2$ is a neighbourhood of x such that $\sigma_G(V, V)$ has compact closure. Part (a) shows that the action is a Cartan action.

(d) Suppose that the action of G on X is strongly proper. Then it is Palais-proper, so that part (c) yields a covering $\mathfrak{U}$ of X by G-small open sets. Let $\mathfrak{W}$ be a locally finite open refinement of the open covering $\{\mathrm{pr}(U) \mid U \in \mathfrak{U}\}$ of $G\backslash X$, where pr denotes the orbit projection. For each $W \in \mathfrak{W}$, choose $U_W \in \mathfrak{U}$ such that $W \subseteq \mathrm{pr}(U_W)$, and set $V_W := U_W \cap \mathrm{pr}^{-1}(W)$, so that $\mathrm{pr}(V_W) = W$. Let V be the union of the G-small open sets V_W, where W ranges over $\mathfrak{W}$. Then V is an open subset of X with $G.V = X$. We claim that V is G-small. Let $x \in X$. Choose a neighbourhood V' of x in X such that $\mathrm{pr}(V')$ meets only finitely many elements $W_1, \ldots, W_n$ of $\mathfrak{W}$. For each $j \in \{1, \ldots, n\}$, let $V_j \subseteq V'$ be a neighbourhood of x such that $\sigma_G(V_{W_j}, V_j)$ has compact closure in G. Then the closure of

$$\sigma_G(V, V_1 \cap \cdots \cap V_n) \subseteq \bigcup_{j=1}^{n} \sigma_G(V_{W_j}, V_j)$$

is compact. Hence V is indeed G-small. Let A be the closure of V in X, which is G-small. Then A is paracompact because it admits a proper map onto the paracompact space $G\backslash X$, see Lemma 1.5 and Dugundji [9, XI.5.3]. Locally compact groups are paracompact (Bourbaki [5, III.4, Prop. 13]). The product of a paracompact locally compact space with a paracompact space is paracompact (use Engelking [10, 5.1.34 and 5.1.36], cf. [10, 5.5.5]). Therefore, the product $G \times A$ is paracompact. Lemma 1.5 yields a proper map from $G \times A$ onto X. Hence X is paracompact [9, VIII.2.6].

Conversely, part (c) shows that the action is Palais-proper if the condition in (d) is satisfied. Lemma 1.5 yields a proper map from the closure of V in X onto the orbit space, whence $G\backslash X$ is paracompact [9, VIII.2.6]. □

1.7 Proposition. *The orbit space of a strongly proper action of a locally compact group G on a metrizable space X is metrizable.*

Proof. Theorem 1.6 yields a G-small subset $V \subseteq X$ such that $G.V = X$. Let A be the closure of V in X. Then A is a G-small subset of X, and the restriction of the orbit projection to A is a proper map onto $G\backslash X$ by Lemma 1.5. The image of a proper map with metrizable domain is metrizable (see Dugundji [9, XI.5.2]). $\square$

2 Slices

If G is a group and $H \leq G$ is a subgroup which acts on a set S then H acts on $G \times S$ by $h.(g, s) := (gh^{-1}, h.s)$. The orbit space of this action is called the *twisted product* $G \times_H S$. The H-orbit of $(g, s) \in G \times S$ is written as $[g, s] \in G \times_H S$. The full group G acts on $G \times_H S$ by $g.[g', s] := [gg', s]$. For elementary properties of twisted products, see Bredon [6, I.6] or tom Dieck [8, I.4].

2.1 Definition. Let a Hausdorff group G act continuously on a Hausdorff space X, and let H be a closed subgroup of G. An *H-slice in X for the action of* G is an H-invariant subset $S \subseteq X$ such that the continuous G-equivariant map

$$[g, s] \longmapsto g.s \colon G \times_H S \to X$$

is an open embedding.

We say that the action *has enough slices* if for each point $x \in X$, there is a compact subgroup $K \leq G$ such that x is contained in an K-slice.

2.2 Remark. If S is an H-slice for an action of G on X then $G.S$ is an open subset of X and $g.s \mapsto gH \colon G.S \to G/H$ is a continuous G-equivariant surjection.

This property of an H-slice can be used as a definition. Consider a continuous action of a locally compact group G on a Hausdorff space X, and let H be a closed subgroup of G. Assume that $S \subseteq X$ is an H-slice in the sense of Palais [16, 2.1.1], which means that S is H-invariant, the set $Y := G.S$ is open in X, and there is a continuous G-equivariant map from Y onto G/H which maps S to the base-point $H \in G/H$. Then the map

$$[g, s] \longmapsto g.s \colon G \times_H S \longrightarrow Y$$

is a G-equivariant homeomorphism (see [4, 3.2]). Therefore, our definition of an H-slice is equivalent to the definition by Palais.

2.3 Proposition. *Let G be a topological group, and let $H \leq G$ be a subgroup which acts continuously on a topological space S. Then the natural action of G on the twisted product $Y := G \times_H S$ is a Cartan action if and only if the action of H on S is a Cartan action. Moreover, the map*

$$\varphi \colon H\backslash S \longrightarrow G\backslash Y, \quad H.s \longmapsto G.[1, s]$$

is a homeomorphism.

In particular, the action of G on $G \times_H S$ is proper, Palais-proper, or strongly proper if and only if so is the action of H on S.

Proof. Using the notation of Definition 1.1, observe that all $g, g' \in G$ and all $s, s' \in S$ satisfy

$$\sigma_G([g,s],[g',s']) = g' \ \sigma_H(s,s') \ g^{-1}.$$

In particular, this yields $G_{[g,s]} = gH_s g^{-1}$. Therefore, all stabilizers of the action of G on Y are quasi-compact if and only if all stabilizers of the action of H on S are quasi-compact.

Assume that the action of G on Y is a Cartan action, and let $U \subseteq H$ be an open neighbourhood of the stabilizer H_s. Let $U' \subseteq G$ be an open subset with $U' \cap H = U$. Then U' is an open neighbourhood of $G_{[1,s]} = H_s$. Hence there is a neighbourhood $V \subseteq Y$ of $[1,s]$ such that $\sigma_G(V,V) \subseteq U'$. The pre-image W of V under the continuous injection $s \mapsto [1,s] \colon S \to Y$ is a neighbourhood of s in S such that $\sigma_H(W,W) \subseteq U$.

Conversely, assume that the action of H on S is a Cartan action, and let $U \subseteq G$ be a neighbourhood of the stabilizer $G_{[g,s]} = gH_s g^{-1}$. By continuity of the group multiplication and quasi-compactness of H_s, there are neighbourhoods $V \subseteq G$ of g and $V' \subseteq H$ of H_s such that $VV'V^{-1} \subseteq U$. Choose a neighbourhood $W \subseteq S$ of s such that $\sigma_H(W,W) \subseteq V'$. Let W' the image of $V \times W$ under the H-orbit projection from $G \times S$ onto Y. Then W' is a neighbourhood of $[g,s]$ which satisfies $\sigma_G(W',W') \subseteq U$.

It is easy to see that φ is a continuous bijection. For a subset U of S, the image of $H\backslash H.U \subseteq H\backslash S$ under φ is the image of $G \times U \subseteq G \times S$ under the natural projection of $G \times S$ onto $G\backslash Y$, which is an open map. This implies that φ is open. $\square$

2.4 Lemma. *A continuous action of a Hausdorff group G on a Hausdorff space X which has enough slices is a Cartan action.*

Proof. Choose a point $x \in X$. Let $K \leq G$ be a compact subgroup such that x is contained in a K-slice $S \subseteq X$. As the stabilizer G_x is contained in K, it is a compact subgroup. Since K is compact, Theorem 1.6 implies that the action of this group on S is a Cartan action. Proposition 2.3 yields that the action of G on the neighbourhood $G.S$ of x is a Cartan action. Therefore, the action of G on X is a Cartan action. $\square$

2.5 Theorem. *A continuous action of a locally compact group G on a completely regular space X has enough slices if and only if it is a Cartan action.*

Proof. Lemma 2.4 asserts that the action is a Cartan action if it has enough slices. Conversely, assume that the action of G on X is a Cartan action. By

Palais's main result in [16, 2.3.3], if G is a Lie group and $x \in X$ then x is even contained in a G_x-slice. If G is an almost connected locally compact group then every identity neighbourhood of G contains a compact normal subgroup N such that G/N is a Lie group. (This was proved by Yamabe [17] and by Gluškov [11, Theorem 9], see also Montgomery and Zippin [15, Chapter IV] and Kaplansky [14, II.10, Theorem 18].) Applying this fundamental result to suitable almost connected open subgroups of an arbitrary locally compact group G, one can deduce from Palais's Slice Theorem that the action of G on X has enough slices. For proper actions, this was carried out in [4, 3.8]. The proof need hardly be changed for Cartan actions, but we can also apply the result [4, 3.8] directly if we cover X by G-invariant open sets on which G acts properly. Such a covering is provided by Theorem 1.6. Indeed, let $x \in X$, and choose an open neighbourhood $V \subseteq X$ of x such that $\sigma_G(V, V)$ has compact closure in G. Then the action of G on $G.V$ is proper (even Palais-proper). $\square$

3 Paracompactness and metrizability of orbit spaces

Recall that a Lindelöf space is a topological space with the property that every open covering contains a countable subcovering. Every regular Lindelöf space is paracompact (see Dugundji [9, VIII.6.5]). In particular, every regular locally Lindelöf space is completely regular, so that every Cartan action on such a space has enough slices.

3.1 Proposition. *The following statements about a regular locally Lindelöf space X are equivalent:*

(i) *X is paracompact.*

(ii) *Every open covering of X has a locally countable open refinement.*

(iii) *X is a topological sum of Lindelöf spaces.*

This observation and an indication of the following proof are due to Hájek [12, Prop. 13]. They generalize the well-known equivalence of statements (i) and (iii) for locally compact spaces X (cf. Dugundji [9, XI.7.3]).

Proof. (i) $\Rightarrow$ (ii) By definition, every open covering of a paracompact space has an open refinement which is even locally finite.

(ii) $\Rightarrow$ (iii) Let $\mathcal{U}$ be an open covering of X such that $\overline{U}$ is a Lindelöf space for each $U \in \mathcal{U}$. Let $\mathcal{V}$ be a locally countable open refinement of $\mathcal{U}$. Then each element of $\mathcal{V}$ meets at most countably many others. Assuming that $\emptyset \notin \mathcal{V}$, we call elements $V, V' \in \mathcal{V}$ equivalent if there is a finite chain $V_1, \ldots, V_n$ of elements of $\mathcal{V}$ such that $V = V_1$, $V' = V_n$, and $V_{j-1} \cap V_j \neq \emptyset$ for every $j \in \{1, \ldots, n\}$. For each element $V \in \mathcal{V}$, the equivalence class $[V]$ is countable. If $V, V' \in \mathcal{V}$ are not equivalent then the open sets $\bigcup[V]$ and $\bigcup[V']$ are disjoint. Hence X is the topological sum of its subspaces $\bigcup[V]$, where $[V]$ ranges over the equivalence classes in $\mathcal{V}$. As each subspace $\bigcup[V]$ is closed, we find that $\bigcup[V] = \bigcup_{V' \in [V]} \overline{V'}$ is a countable union of Lindelöf spaces and hence has the Lindelöf property.

(iii) $\Rightarrow$ (i) A regular Lindelöf space is paracompact (see Dugundji [9, VIII.6.5]). A topological sum of paracompact spaces is paracompact. $\square$

3.2 Proposition. *The following statements about a regular space X are equivalent:*

(i) *X is a metrizable locally Lindelöf space.*

(ii) *X has a locally countable basis.*

(iii) *X is a topological sum of second countable spaces.*

Proof. (i) $\Rightarrow$ (ii) The metrizable space X is paracompact. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be a locally finite open refinement of the covering $\{U_{1/n}(x) \mid x \in X\}$. Then $\mathcal{U} := \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ is a basis. Choose a point $x \in X$, and let $V \subseteq X$ be a Lindelöf neighbourhood of x. For each $n \in \mathbb{N}$, the neighbourhood V meets at most countably many sets from $\mathcal{U}_n$. Hence V meets at most countably many sets from $\mathcal{U}$.

(ii) $\Rightarrow$ (iii) Let $\mathcal{U}$ be a locally countable basis of X. Omitting some basis sets, we may assume that each element of $\mathcal{U}$ is non-empty and meets at most countably many others. Call basis sets $U, U' \in \mathcal{U}$ equivalent if there is a finite chain $U_1, \ldots, U_n$ of elements of $\mathcal{U}$ such that $U = U_1$, $U' = U_n$, and $U_{j-1} \cap U_j \neq \emptyset$ for each $j \in \{1, \ldots, n\}$. Let $[U]$ denote the equivalence class of $U \in \mathcal{U}$. If $U, U' \in \mathcal{U}$ are not equivalent then $\bigcup[U]$ and $\bigcup[U']$ are disjoint. Each equivalence class $[U]$ is countable, and it is a basis of the open subset $\bigcup[U]$ of X. Hence X is the topological sum of its second countable subspaces $\bigcup[U]$, where $[U]$ ranges over the equivalence classes in $\mathcal{U}$.

(iii) $\Rightarrow$ (i) A second countable regular space has the Lindelöf property, and it is metrizable by the Urysohn Metrization Theorem (cf. Dugundji [9, IX.9.2]). A topological sum of metrizable spaces is metrizable. $\square$

3.3 Lemma. *Let a locally compact Lindelöf group G act continuously on a regular locally Lindelöf space X. Let $\mathcal{F}$ be a locally countable family of subsets of X. Then the image of $\mathcal{F}$ under the orbit projection $X \to G\backslash X$ is a locally countable family.*

Proof. Let $x \in X$ be an arbitrary point. Choose a Lindelöf neighbourhood $U \subseteq X$ of x. It suffices to show that $G.U$ is a Lindelöf space, since this implies that $G.U$ meets at most countably many members of $\mathcal{F}$. The locally compact Lindelöf group G is the union of a countable family $(K_n)_{n \in \mathbb{N}}$ of compact subsets. The product $K_n \times U$ is a Lindelöf space for each $n \in \mathbb{N}$ (see Dugundji [9, XI.5.4]), whence so is its continuous image $K_n.U$ (see [9, VIII.6.6]). Therefore, the countable union $\bigcup_{n\in\mathbb{N}} K_n.U = G.U$ is a Lindelöf space. □

3.4 Theorem. *Let a locally compact Lindelöf group G act continuously on a regular locally Lindelöf space X. Suppose that the orbit space $G\backslash X$ is regular.*

(a) *If X is paracompact then so is $G\backslash X$.*

(b) *If X is metrizable then so is $G\backslash X$.*

Proof. (a) Since the orbit projection $\mathrm{pr}\colon X \to G\backslash X$ is a continuous open map, the orbit space $G\backslash X$ has the local Lindelöf property. Let $\mathcal{U}$ be an open covering of $G\backslash X$. Choose a locally finite open refinement $\mathcal{V}$ of the open covering $\{\mathrm{pr}^{-1}(U) \mid U \in \mathcal{U}\}$ of X. Then $\{\mathrm{pr}(V) \mid V \in \mathcal{V}\}$ is an open covering of $G\backslash X$ which refines $\mathcal{U}$ and which is locally countable by Lemma 3.3. Proposition 3.1 yields that $G\backslash X$ is paracompact.

(b) Proposition 3.2 shows that X has a locally countable basis, the image of which under the orbit projection is a basis of $G\backslash X$, and this basis is locally countable by Lemma 3.3. A second application of Proposition 3.2 yields that $G\backslash X$ is metrizable. □

3.5 Corollary. *Let G be a locally compact Lindelöf group.*

(a) *An action of G on a paracompact locally Lindelöf space is strongly proper if and only if it is Palais-proper.*

(b) *An action of G on a paracompact locally compact space is strongly proper if and only if it is proper.*

Proof. Assertion (a) follows immediately from Theorem 3.4. If G acts properly on a paracompact locally compact space X then the orbit space $G\backslash X$ is locally compact and hence regular. Theorem 3.4 shows that the action is strongly proper. □

3.6 Remark. Baum, Connes, and Higson [2] call a continuous action of a second countable locally compact group G on a metrizable space X proper if it has enough slices and $G\backslash X$ is metrizable. By Proposition 1.7 and Theorem 2.5, these conditions are satisfied if and only if the action is strongly proper. Under the hypotheses on X given by Corollary 3.5, it suffices to assume that the action is Palais-proper or just proper.

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