Weighted L^q–Helmholtz Decompositions in Infinite Cylinders and in Infinite Layers

Reinhard Farwig

Department of Mathematics
Darmstadt University of Technology,
Schlossgartenstr. 7, D-64289 Darmstadt
farwig@mathematik.tu-darmstadt.de

Dedicated to Prof. Wolf von Wahl on the occasion of his 60th birthday

The aim of this paper is to present a new joint approach to the Helmholtz decomposition in infinite cylinders and in infinite layers $\Omega = \Sigma \times \mathbb{R}^k$ in the function space $L^q(\mathbb{R}^k; L^r(\Sigma))$ using even arbitrary Muckenhoupt weights with respect to $x' \in \Sigma \subset \mathbb{R}^{n-k}$ and, if possible, exponential weights with respect to $x'' \in \mathbb{R}^k$, $1 \le k \le n-1$, $n \ge 2$. For $n = 2$ we get the Helmholtz decomposition for a strip, for $n = 3$ in an infinite cylinder or an infinite layer and for $n > 3$ in some (non–physical) unbounded domains of cylinder or layer type. The proof based on a weak Neumann problem uses a partial Fourier transform and operator–valued multiplier functions, the $\mathcal{R}$–boundedness of the family of multiplier operators and an extrapolation property in weighted L^q–spaces.

1 Introduction

The Helmholtz decomposition of vector fields into a solenoidal and a gradient part is an important tool in the analysis of instationary Stokes and Navier–Stokes equations via analytic semigroup theory [29]. Besides L^2–theory which is available in any domain [23] and numerous L^q–results on bounded and

[0]Mathematical Subject Classification 2000: 35J20, 35J25, 35Q35,76D99

exterior domains, see e.g. [13, 16, 21, 25, 30] and [9, 10] for an approach in weighted spaces, there are only few L^q–results on domains with noncompact boundaries. So–called aperture domains have been considered in [7, 8, 17].

In an infinite layer such as $(-1,1)\times\mathbb{R}^2$ a more or less explicit representation and a multiplier technique can be used [1, 32]; for other approaches see [18, 19, 20, 22] and also [17]. However, the Helmholtz decomposition for an infinite cylinder $\Omega=\Sigma\times\mathbb{R}$ where $\Sigma\subset\mathbb{R}^{n-1}$ is a bounded domain requires a more refined analysis [22, 24, 27, 28]. To get the decomposition

$$u = u_0 + \nabla p$$

of a given vector field u such that $\operatorname{div} u_0 = 0$ and the normal component $\nu\cdot u_0$ vanishes on $\partial\Omega$ in the weak sense, the weak Neumann problem

$$\Delta p = \operatorname{div} u \ \text{ in } \ \Omega, \quad \nu\cdot(\nabla p - u) = 0 \ \text{ on } \ \partial\Omega$$

has to be solved; then the Helmholtz projection P is defined by $Pu = u_0 := u - \nabla p$. In [24] the solution of the weak Neumann problem is based on the splitting $-\Delta = -(\partial_1^2 + \ldots + \partial_{n-1}^2) - \partial_n^2$ and the existence of bounded imaginary powers of $-\Delta'$ and of $-\partial_n^2$ in suitable function spaces. A more classical analysis has been used in [22, 27, 28]; standard L^2–estimates are refined by exponential weights via St. Venant's principle to yield an L^q–theory, $q\neq 2$, of the weak Neumann problem. This approach allows to consider also domains of cylinder or layer type with varying cross section, see [18, 19, 20].

In this paper we present an operator-theoretical approach for both infinite cylinders and infinite layers. Using a partial Fourier transform $\mathcal{F}$ w.r.t. $x''\in\mathbb{R}^k$ we solve the weak Neumann problem $-\Delta p = F$, i.e.,

$$\int_\Omega \nabla p\cdot\nabla\varphi\,dx = \langle F,\varphi\rangle \quad \forall\varphi\in C_0^\infty(\overline{\Omega}).$$

For every Fourier variable $\xi\in\mathbb{R}^k\setminus\{0\}$ there exists a solution operator $M(\xi)$ leading to a Fourier multiplier operator $\mathcal{F}^{-1}M(\cdot)\mathcal{F}$ with an operator–valued multiplier function $M(\cdot)$ on $L^r(\Sigma)$–valued functions, $1<r<\infty$. To prove the continuity of $\mathcal{F}^{-1}M\mathcal{F}$ on $L^q(\mathbb{R}, L^r(\Sigma))$ we need the $\mathcal{R}$–boundedness of the operator family $\{M(\xi) : 0\neq\xi\in\mathbb{R}^k\}$, see Section 2. For these reasons we extend the Neumann problem to weighted L^r–spaces using arbitrary Muckenhoupt weights. Then results from harmonic analysis will prove the

$\mathcal{R}$–boundedness of $\{M(\cdot)\}$, and a recent multiplier theorem [26] will complete the proof in case of L^q–estimates without weights w.r.t. $x'' \in \mathbb{R}^k$. To extend the results to exponentially weighted L^q–estimates it suffices to consider a perturbation of the original Neumann problem.

This paper is organized as follows. In Section 2 we start with the definition of several important function spaces and of Muckenhoupt weights. Then we present the main theorems on the weak Neumann problem (Theorem 2.1) and the Helmholtz decomposition (Theorem 2.2). The section ends with some results for weighted function spaces. Section 3 deals with the weak Neumann problem after applying a partial Fourier transform, and Section 4 introduces the main tools from harmonic analysis and multiplier theory to complete the proof of Theorems 2.1 and 2.2. The paper ends with a result on exponentially weighted estimates in case of infinite layers (Theorem 4.4).

2 Main Theorems and Preliminaries

Let $\Omega = \Sigma \times \mathbb{R}^k$ be an infinite cylinder or layer with constant cross section $\Sigma \subset \mathbb{R}^{n-k}$, $1 \le k \le n-1$, where Σ is a bounded domain with C^1–boundary. The exterior normal vector on $\partial\Omega = \partial\Sigma \times \mathbb{R}^k$ and also on $\partial\Sigma \subset \mathbb{R}^k$ will be denoted by ν. Due to the product structure of Ω the canonical splitting $x = (x', x'') \in \Omega$ with $x' \in \Sigma$ and $x'' \in \mathbb{R}^k$ will analogously be applied to vector fields $u = (u', u'')$ and to differential operators. In particular $\nabla = (\nabla', \nabla'')$, and $\Delta = \Delta' + \Delta''$. In order to describe our main theorems on the weak Neumann problem and the Helmholtz decomposition we have to introduce several function spaces.

We use standard notation for classical Lebesgue spaces such as $L^q(\Omega) \cong L^q(\mathbb{R}^k; L^q(\Sigma))$ with norm $\|\cdot\|_{q,\Omega} = \|\cdot\|_q$, $1 \le q < \infty$, for local Lebesgue spaces $L^q_{\text{loc}}(\overline{\Omega}$ and for Sobolev spaces $H^1_q(\Omega)$. The same notation will be used for spaces of vector fields or matrix fields when confusion can be excluded. In particular we use the short notation $\|u, v\|_q$ for $\|u\|_q + \|u\|_q$, even if u and v are tensors of different order. Furthermore we need the homogeneous Sobolev space

$$\hat{H}^1_q(\Omega) = \{u \in L^1_{\text{loc}}(\overline{\Omega})/\mathbb{R} : \nabla u \in L^q(\Omega)\}$$

of equivalence classes of functions equipped with the norm $\|\nabla u\|_q$. The dual space $[(\hat{H}^1_{q'}(\Omega))]^*$ of $\hat{H}^1_{q'}(\Omega)$ where $q' = \frac{q}{q-1}$ is denoted by $\hat{H}^{-1}_q(\Omega)$, its norm by $\|\cdot\|_{-1,q}$. Note that $\hat{H}_q^{-1}(\Omega)$ is *not* the dual space of a space of functions with vanishing trace on the boundary $\partial\Sigma$ of Σ.

Given $1 < r < \infty$ a function $0 \le \omega \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-k})$ is called an A_r*–weight* (*Muckenhoupt weight of class* A_r) on $\mathbb{R}^{n-k}$ iff

$$\mathcal{A}_r(\omega) := \sup_Q \Big(\frac{1}{|Q|} \int_Q \omega \, dx' \Big) \cdot \Big(\frac{1}{|Q|} \int_Q \omega^{-1/(r-1)} dx' \Big)^{r-1} < \infty \, .$$

Here Q runs through the set of all bounded cubes $Q \subset \mathbb{R}^{n-k}$ with edges parallel to the main axes of $\mathbb{R}^{n-k}$ and $|Q|$ denotes the Lebesgue measure of Q. The real number $\mathcal{A}_r(\omega)$ is called the A_r–constant of the weight ω. Note that $\omega \in A_r = A_r(\mathbb{R}^{n-k})$ yields $\omega' := \omega^{-1/(r-1)} \in A_{r'}$ with $\mathcal{A}_{r'}(\omega') = \mathcal{A}_r(\omega)^{r'/r}$.

Given $\omega \in A_r$, $1 < r < \infty$, and an arbitrary domain $\Sigma \subset \mathbb{R}^{n-k}$ let

$$L^r_\omega(\Sigma) = \Big\{ u \in L^1_{\mathrm{loc}}(\overline{\Sigma}) : ||u||_{r,\omega} = \Big(\int_\Sigma |u|^r \omega \, dx' \Big)^{1/r} < \infty \Big\} \, .$$

It is well-known [9] that $L^r_\omega(\Sigma)$ is a separable reflexive Banach space with dense subspace $C_0^\infty(\Sigma)$. In particular $L^r_\omega(\Sigma)^* = L^{r'}_{\omega'}(\Sigma)$. In addition to the weighted Sobolev space $H^1_{r,\omega}(\Sigma)$ with norm $\|\nabla u, \, u\|_{r,\omega}$ we introduce the homogeneous Sobolev space

$$\hat{H}^1_{r,\omega}(\Sigma) = \big\{ u \in L^1_{\mathrm{loc}}(\overline{\Sigma})/\mathbb{R} : \ \nabla' u \in L^r_\omega(\Sigma) \big\}$$

equipped with the norm $\|\nabla u\|_{r,\omega}$. The dual space of $\hat{H}^1_{r',\omega'}(\Sigma)$ will be denoted by $\hat{H}^{-1}_{r,\omega}(\Sigma)$; the norm of a functional $F \in \hat{H}^{-1}_{r,\omega}(\Sigma)$ is defined by

$$||F||_{-1,r,\omega} = \sup_\varphi \frac{|\langle F, \varphi \rangle|}{\|\nabla \varphi\|_{r',\omega'}} \, ,$$

where the supremum is taken over all nonconstant $\varphi \in \hat{H}^1_{r',\omega'}(\Sigma)$. Since $\hat{H}^1_{r,\omega}(\Sigma)$ can be considered as a closed subspace of $L^r_\omega(\Sigma)^{n-k}$, Hahn-Banach's theorem easily implies that $F \in \hat{H}^{-1,r}_\omega(\Sigma)$ can be written in the form

$$F = f \cdot \nabla', \quad \text{i.e.,} \quad \langle F, \varphi \rangle = \int_\Sigma f \cdot \nabla' \varphi \, dx'$$

for all $\varphi \in \hat{H}^1_{r',\omega'}(\Sigma)$; here $f \in L^r_\omega(\Sigma)^{n-k}$ and $\|F\|_{-1,r,\omega} = \|f\|_{r,\omega}$.

On an infinite cylinder or layer $\Omega = \Sigma \times \mathbb{R}^k$ where $\Sigma \subset \mathbb{R}^{n-k}$ is a bounded C^1–domain we introduce the function space

$$L^q(L^r_\omega) := L^q\big(\mathbb{R}^k; L^r_\omega(\Sigma)\big)$$

of Bochner-integrable $L^r_\omega(\Sigma)$–valued L^q–functions, $1 < q < \infty$, with norm

$$\|u\|_{q;r,\omega} = \Big(\int_{\mathbb{R}^k} \|u(\cdot, x'')\|^q_{r,\omega}\, dx''\Big)^{1/q}.$$

Furthermore we need the homogeneous Sobolev space

$$\hat{H}^1_{q;r,\omega}(\Omega) = \big\{u \in L^1_{\mathrm{loc}}(\overline{\Omega})/\mathbb{R} :\ \nabla u \in L^q(L^r_\omega)\big\}$$

with norm $\|\nabla u\|_{q;r,\omega}$ and the dual space $\hat{H}^{-1}_{q;r,\omega}(\Omega)$ of $\hat{H}^1_{q';r',\omega'}(\Omega)$ with norm $\|\cdot\|_{-1,q;r,\omega}$.

It is classical to consider exponential weights $e^{\alpha x_n}$, $\alpha \in \mathbb{R}$, w.r.t. $x_n \in \mathbb{R}$ for an infinite cylinder, see [27, 28]. For the more general domain $\Omega = \Sigma \times \mathbb{R}^k$, $\Sigma \subset \mathbb{R}^{n-k}$, $1 \le k \le n-1$, to be considered in this paper we introduce exponential weights w.r.t. $x'' \in \mathbb{R}^k$ via the exponential $e^{\alpha\cdot x''}$ where $\alpha \in \mathbb{R}^k$. Then we define the weighted spaces

$$L^q_\alpha(L^r_\omega) = \big\{u \in L^1_{\mathrm{loc}}(\overline{\Omega}) :\ \|u\|_{q,\alpha;r,\omega} := \|e^{\alpha\cdot x''} u\|_{q;r,\omega} < \infty\big\}$$

and

$$\hat{H}^1_{q,\alpha;r,\omega}(\Omega) = \big\{u \in L^1_{\mathrm{loc}}(\overline{\Omega})/\mathbb{R} :\ \|\nabla u\|_{q,\alpha;r,\omega} < \infty\big\}\,;$$

the dual space of $\hat{H}^1_{q',-\alpha;r',\omega'}(\Omega)$ will be denoted by $\hat{H}^{-1}_{q,\alpha;r,\omega}(\Omega)$ with norm $\|\cdot\|_{-1,q,\alpha;r,\omega}$.

Finally we define the space of solenoidal vector fields

$$L^q_\alpha(L^r_\omega)_\sigma := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{q,\alpha;r,\omega}}$$

where $C^\infty_{0,\sigma}(\Omega) = \big\{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\big\}$. By Proposition 2.8 below

$$L^q_\alpha(L^r_\omega)_\sigma = \big\{u \in L^q_\alpha(L^r_\omega)^n :\ \operatorname{div} u = 0,\ u\cdot\nu = 0 \text{ on } \partial\Omega\big\}$$

where $\operatorname{div} u = 0$ and the vanishing normal component $u \cdot \nu = 0$ on $\partial\Omega$ have to be interpreted in the weak sense, i.e., $\int_\Omega u \cdot \nabla\varphi = 0$ for all $\varphi \in C^\infty_0(\overline{\Omega})$.

Note that a vector field $u \in L^q_\alpha(L^r_\omega)_\sigma$ has a vanishing flux in the following sense: For every ball $B''_r(x'') \subset \mathbb{R}^k$ with center $x'' \in \mathbb{R}^k$

$$\int_{\Sigma\times\partial B''_r(x'')} u''\cdot\nu\,do = \int_{\Sigma\times B''_r(x'')} \operatorname{div} u\,dx = 0.$$

For a cylinder $\Omega = \Sigma\times\mathbb{R}$ we conclude that the classical flux

$$\phi(u)(x_n) := \int_\Sigma u_n(x',x_n)\,dx' = 0 \quad \text{for a.a. } x_n \in \mathbb{R}.$$

Actually for every slab $\Sigma\times(x_n,y_n)\subset\Omega$

$$0 = \int_{\Sigma\times(x_n,y_n)} \operatorname{div} u\,dx = \phi(u)(y_n) - \phi(u)(x_n)$$

where by Hölder's inequality $|\phi(u)(x_n)| \le c\,\|u(\cdot,x_n)\|_{r,\omega}$; thus the constant $\phi(u)(x_n)$ is L^q_α–integrable on $\mathbb{R}$.

Now the main theorems are as follows.

Theorem 2.1 *Let $\Sigma \subset \mathbb{R}^{n-k}$ be a bounded domain with C^1–boundary, let $1<q,r<\infty$ and $\omega\in A_r$.*

(1) Given $F \in \hat{H}^{-1}_{q;r,\omega}(\Omega)$ the weak Neumann problem

$$\int_\Omega \nabla p\cdot\nabla\varphi\,dx = \langle F,\varphi\rangle \quad \forall\varphi\in\hat{H}^{1,q'}(\Omega). \tag{2.1}$$

has a unique solution $u \in \hat{H}^1_{q;r,\omega}(\Omega)$; furthermore

$$\|\nabla u\|_{q;r,\omega} \le C\|F\|_{-1,q;r,\omega}$$

with a constant $C = C\big(\Omega,q,r,\mathcal{A}_r(\omega)\big)$.

(2) Let $\alpha_1>0$ denote the smallest positive eigenvalue of the Neumann-Laplacian in $\hat{H}^{1,2}(\Sigma)$ and let $\Omega=\Sigma\times\mathbb{R}\subset\mathbb{R}^n$ be an infinite cylinder. Then for every $\alpha\in(-\sqrt{\alpha_1},\sqrt{\alpha_1})$ the assertion (1) extends to every $F\in\hat{H}^{-1}_{q,\alpha;r,\omega}(\Omega)$; moreover the solution $u\in\hat{H}^1_{q,\alpha;r,\omega}(\Omega)$ satisfies the estimate

$$\|\nabla u\|_{q,\alpha;r,\omega} \le C(\alpha)\,\|F\|_{-1,q,\alpha;r,\omega}\,.$$

(3) *If in (2) the functional F satisfies the estimate* $|\langle F, \varphi\rangle| \leq c(F)\|e^{\alpha|x_n|}\nabla\varphi\|_{q';r',\omega'}$ *for some* $\alpha \in (0, \sqrt{a_1})$, *then also* $\|e^{\alpha|x_n|}\nabla u\|_{q;r,\omega} \leq c\, c(F)$.

For details of exponentially weighted estimates in infinite layers $\Omega = \Sigma \times \mathbb{R}^k$, $2 \leq k \leq n-1$, see Theorem 4.4 in Section 4.

Theorem 2.2 *(1) Given* $\Omega = \Sigma \times \mathbb{R}^k$ *as in Theorem 2.1 fix* $1 < q,\, r < \infty$ *and* $\omega \in A_r$. *Then there exists a unique continuous linear projection*

$$P = P_{q;r,\omega} : L^q(L^r_\omega)^n \;\rightarrow\; L^q(L^r_\omega)_\sigma \subset L^q_\alpha(L^r_\omega)^n$$

with range $L^q(L^r_\omega)_\sigma$ *such that* $\operatorname{Ker} P_{q;r,\omega} = \nabla \hat{H}^1_{q;r,\omega}(\Omega)$. *Every* $u \in L^q(L^r_\omega)^n$ *has the unique decomposition*

$$u = u_0 + \nabla p\,, \quad u_0 = Pu\,, \quad \nabla p = (I-P)u$$

satisfying

$$\|u_0, \nabla p\|_{q;r,\omega} \leq C\|u\|_{q;r,\omega}$$

where $C = C\big(\Omega, q, r, A_r(\omega)\big)$.

(2) *Concerning the duality product on* $L^q(L^r_\omega)$

$$P^*_{q;r,\omega} = P_{q';r',\omega'}\,, \quad [L^q(L^r_\omega)_\sigma]^* = L^{q'}(L^{r'}_{\omega'})_\sigma\,.$$

(3) *Let* $\Omega = \Sigma \times \mathbb{R} \subset \mathbb{R}^n$ *be an infinite cylinder and let* $\alpha \in (-\sqrt{\alpha_1}, \sqrt{\alpha_1})$. *Then there exists a projection*

$$P = P_{q,\alpha;r,\omega} : L^q_\alpha(L^r_\omega)^n \;\rightarrow\; L^q_\alpha(L^r_\omega)_\sigma$$

such that $\operatorname{Ker} P_{q,\alpha;r,\omega} = \nabla \hat{H}^1_{q,\alpha;r,\omega}(\Omega)$. *Every* $u \in L^q_\alpha(L^r_\omega)^n$ *has the unique decomposition* $u = u_0 + \nabla p$, $u_0 = Pu$ *satisfying*

$$\|u_0, \nabla p\|_{q,\alpha;r,\omega} \leq C\|u\|_{q,\alpha;r,\omega}$$

where $C = C\big(\Omega, q, \alpha, r, A_r(\omega)\big)$. *Moreover,* $P^*_{q,\alpha;r,\omega} = P_{q',-\alpha;r',\omega'}$ *and* $[L^q_\alpha(L^r_\omega)_\sigma]^* = L^{q'}_{-\alpha}(L^{r'}_{\omega'})_\sigma$. *If even* $\|e^{\alpha|x_n|}u\|_{q;r,\omega} < \infty$ *for some* $\alpha \in (0, \sqrt{\alpha_1})$, *then also* $\|e^{\alpha|x_n|}(u_0, \nabla p)\|_{q;r,\omega} \leq c\|e^{\alpha|x_n|}u\|_{q;r,\omega}$.

Remark 2.3 The different constants $C = C(\omega)$ in Theorems 2.1 – 2.2 do *not* depend on the explicit form of the weight $\omega \in A_r$, but only on the A_r–constant $\mathcal{A}_r(\omega)$. Moreover it is important to note that even for every $d \geq 1$

$$\sup \big\{ C(\omega) : \ \omega \in A_r, \ \mathcal{A}_r(\omega) \leq d \big\} < \infty \, .$$

A constant $C = C(\omega) : A_r \to \mathbb{R}_+$ with this property is called A_r*–consistent.* In all subsequent proofs we will check that the crucial constants $C = C(\omega)$ are A_r–consistent.

Before coming to the proof of Theorems 2.1 and 2.2 in Section 3 and 4 below we prove several results for Muckenhoupt weights and for weighted function spaces.

Lemma 2.4 *Let $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-k})$.*

(1) Let $T : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be a bijective, bi-Lipschitz vector field. Then also $\omega \circ T \in A_r$ and $\mathcal{A}_r(\omega \circ T) \leq c\, \mathcal{A}_r(\omega)$ with a constant $c = c(T, r) > 0$ independent of ω.

(2) Define the weight $\tilde{\omega}(x') = \omega(\tilde{x}, |x_{n-k}|)$ for $x' = (\tilde{x}, x_{n-k}) \in \mathbb{R}^{n-k}$. Then $\tilde{\omega} \in A_r$ and $\mathcal{A}_r(\tilde{\omega}) \leq 2^r \mathcal{A}_r(\omega)$.

(3) Let $\Sigma \subset \mathbb{R}^{n-k}$ be a bounded domain. Then there exist $s_1, s_2 \in (1, \infty)$ such that the continuous embeddings

$$L^{s_1}(\Sigma) \subset L^r_\omega(\Sigma) \subset L^{s_2}(\Sigma)$$

hold. Here s_1 and $\frac{1}{s_2}$ are A_r–consistent. Moreover, if $Q \subset \mathbb{R}^{n-k}$ denotes a cube with $\overline{\Sigma} \subset Q$, the embedding constants do not depend on the weights $\omega \in W \subset A_r$ provided that

$$\sup_{\omega \in W} \mathcal{A}_r(\omega) < \infty \, , \qquad \int_Q \omega \, dx' = 1 \quad \textit{for all } \omega \in W \, . \tag{2.2}$$

Proof For the elementary properties (1), (2) see [11, 12]. The embeddings (3) are based on the Reverse Hölder Inequality [14] for the weight ω and on the classical Hölder Inequality; for details see Lemma 2.2 in [12]. ∎

Proposition 2.5 *Let $\Sigma \subset \mathbb{R}^{n-k}$ be a bounded Lipschitz domain and let $1 < r < \infty$.*

(1) For every $\omega \in A_r$ the embedding $H^1_{r,\omega}(\Sigma) \subset L^r_\omega(\Sigma)$ is compact.

(2) Consider a sequence of weights $(\omega_j) \subset A_r$ satisfying (2.2) for $W = \{\omega_j : j \in \mathbb{N}\}$ and a fixed cube $Q \subset \mathbb{R}^{n-k}$ with $\overline{\Sigma} \subset Q$. Further let (u_j) be a sequence of functions on Σ satisfying

$$\sup_j \|u_j\|_{H^1_{r,\omega_j}(\Sigma)} < \infty \quad \text{and} \quad u_j \rightharpoonup 0 \quad \text{in } H^1_s(\Sigma)$$

for $j \to \infty$ where $s \in (1, \infty)$ is given by Lemma 2.4(3). Then

$$\|u_j\|_{r,\omega_j} \to 0 \quad \text{for } j \to \infty\,.$$

(3) Under the same assumptions on $(\omega_j) \subset A_r$ as in (2) consider a sequence of functions (v_j) on Σ satisfying

$$\sup_j \|v_j\|_{r,\omega_j} < \infty \quad \text{and } v_j \rightharpoonup 0 \ \text{ in } L^s(\Sigma)$$

for $j \to \infty$. Then considering v_j as functionals on $H^1_{r',\omega'_j}(\Sigma)$

$$\|v_j\|_{[H^1_{r',\omega'_j}(\Sigma)]^*} \to 0 \quad \text{for } j \to \infty\,.$$

Proof For (1), (2) see Theorems 2.3, 2.4 in [12]. Note that in a first step of the proof of (2) the uniformly bounded embeddings $H^1_{r,\omega_j}(\Sigma) \subset H^1_s(\Sigma)$, see Lemma 2.4(3), allow to find $u \in H^1_s(\Sigma)$ such that w.l.o.g. $u_j \rightharpoonup u$ in $H^1_s(\Sigma)$ and $u_j \to u$ in $L^s(\Sigma)$. The second step in [12] yields for every $\varepsilon > 0$ a linear operator $T_\varepsilon : L^s(\Sigma) \to C^\infty(\overline{\Omega})$ such that

$$\|u_j - T_\varepsilon(u)\|_{r,\omega_j} \le \varepsilon \quad \text{for all sufficiently large } j\,. \tag{2.3}$$

Under the given assumption $u_j \rightharpoonup 0$ in $H^1_s(\Sigma)$ we conclude that even $u = 0$, $T_\varepsilon(u) = 0$ and consequently that $\|u_j\|_{r,\omega_j} \to 0$.

To prove (3) find for every $j \in \mathbb{N}$ a $\varphi_j \in H^1_{r',\omega'_j}(\Sigma)$ such that

$$\|v_j\|_{[H^1_{r',\omega'_j}(\Sigma)]^*} = \int_\Sigma v_j \varphi_j \, dx', \quad \|\varphi_j\|_{H^1_{r',\omega'_j}(\Sigma)} = 1\,.$$

By the definition of $\mathcal{A}_r(\omega_j)$ and Hölder's inequality

$$1 \le \Big(\frac{1}{|Q|}\int_Q \omega_j \, dx'\Big) \cdot \Big(\frac{1}{|Q|}\int_Q \omega'_j \, dx'\Big)^{r-1} \le \mathcal{A}_r(\omega_j) < \infty\,.$$

Hence due to (2.2) also ω_j' satisfies a uniform integrability condition on Q; w.l.o.g. we may assume that $\int_Q \omega_j'\,dx' = 1$. Applying (2.3) to (φ_j) we find for $\varepsilon > 0$ a function $\phi_\varepsilon \in C^\infty(\overline{\Omega})$ such that w.l.o.g. $\|\varphi_j - \phi_\varepsilon\|_{r',\omega_j'} \le \varepsilon$ for all large j. Thus

$$\|v_j\|_{H^1_{r',\omega_j'}(\Sigma)^*} \le \left| \int_\Sigma v_j \phi_\varepsilon\,dx' \right| + \left| \int_\Sigma v_j(\varphi_j - \phi_\varepsilon)dx' \right|,$$

where the first term on the right-hand side converges to 0 for $j \to \infty$ by assumption; the second term is bounded by $C\varepsilon$ uniformly in $j \in \mathbb{N}$. Now (3) is proved. ∎

Corollary 2.6 (Poincaré Inequality) *Let $1 < r < \infty$, $\omega \in A_r$ and $\Sigma \subset \mathbb{R}^{n-k}$ be a bounded Lipschitz domain. Then there exists an A_r–consistent constant $c > 0$ such that*

$$\|u\|_{r,\omega} \le c\|\nabla' u\|_{r,\omega}$$

for all $u \in H^1_{r,\omega}(\Sigma)$ with vanishing integral mean $\int_\Sigma u\,dx' = 0$.

Proof The proof is based on Proposition 2.5; for details see [12]. ∎

Lemma 2.7 *Under the assumptions of Theorem 2.1 $C_0^\infty(\overline{\Omega})/\mathbb{R}$ is dense in $\hat{H}^1_{q,\alpha;r,\omega}(\Omega)$ for every $1 < q,\ r < \infty$, $\omega \in A_r$ and $\alpha \in \mathbb{R}^k$.*

Proof Given $u \in \hat{H}^1_{q,\alpha;r,\omega}(\Omega)$ define the decomposition

$$u = u_\Sigma + v, \quad u_\Sigma(x'') := \frac{1}{|\Sigma|} \int_\Sigma u(x', x'')dx'$$

such that v has vanishing means on Σ for a.a. $x'' \in \mathbb{R}^k$. Concerning the approximation of u_Σ in

$$\hat{H}^1_{q,\alpha}(\mathbb{R}^k) = \{h \in L^1_{\mathrm{loc}}(\mathbb{R}^k)/\mathbb{R} : \int_{\mathbb{R}^k} e^{q\alpha\cdot x''} |\nabla'' h(x'')|^q\,dx'' < \infty\} \hookrightarrow \hat{H}^1_{q,\alpha;r,\omega}(\Omega)$$

consider a functional $F \in \hat{H}^1_{q,\alpha}(\mathbb{R}^k)^*$ vanishing on $C_0^\infty(\mathbb{R}^k)/\mathbb{R}$. We may assume w.l.o.g. that $F \in L^{q'}_{-\alpha}(\mathbb{R}^k)^k$ and that

$$0 = \int_{\mathbb{R}^k} \nabla'' v(x'') \cdot F(x'')\,dx'' \quad \text{for all} \quad v \in C_0^\infty(\mathbb{R}^k).$$

Thus $\nabla'' F = 0$ in the sense of distributions. Hence $F \equiv$ const and even $F \equiv 0$, since $F \in L^{q'}_{-\alpha}(\mathbb{R}^k)^k$. Now Hahn–Banach's Theorem implies that $C_0^\infty(\mathbb{R}^k)/\mathbb{R}$ is dense in $\hat{H}^1_{q,\alpha}(\mathbb{R}^k)$.

Concerning v let $\varphi \in C_0^\infty(\mathbb{R}^k)$ be equal to 1 for $x'' \in B''_{1/2}(0)$ and vanish for $x'' \notin B''_1(0)$. Then for $N \in \mathbb{N}$

$$\begin{aligned}
&\|\nabla\Big(\varphi(\frac{x''}{N})v(x)\Big) - \nabla v\|_{q,\alpha;r,\omega} \\
&\leq \; c\|\Big(\varphi(\frac{x''}{N}) - 1\Big)\nabla v\|_{q,\alpha;r,\omega} + \frac{c}{N}\Big(\int_{B''_N(0)} e^{q\alpha\cdot x''}\|v(\cdot,x'')\|^q_{r,\omega}\,dx''\Big)^{1/q}
\end{aligned}$$

where the first term on the right–hand side converges to 0 for $N \to \infty$ by Lebesgue's Theorem on Dominated Convergence. Since $\int_\Sigma v(x',x'')dx' = 0$ and consequently $\|v(\cdot,x'')\|_{r,\omega} \leq c\,\|\nabla' v(\cdot,x'')\|_{r,\omega}$ for a.a. $x'' \in \mathbb{R}^k$ with a constant $c = c(r,\omega,\Sigma)$ due to Poincaré's inequality, see Corollary 2.6, we conclude that v is approximated in $\hat{H}^1_{q,\alpha;r,\omega}(\Omega)$ by functions with compact support and vanishing mean on Σ for a.a. $x'' \in \mathbb{R}$. Ignoring the weight $e^{\alpha\cdot x''}$ on $B''_N(0)$, using the above Poincaré inequality on Σ and the classical Poincaré inequality on $B''_N(0)$ we may assume that v is contained in a non-homogeneous Sobolev space $H^1_{q;r,\omega}(\Omega)$ and has compact support in $\overline{\Sigma}\times B''_N(0)$.

By Theorem 1.1 in [4] there exists a linear bounded extension operator E' : $H^1_{r,\omega}(\Sigma) \to H^1_{r,\omega}(\mathbb{R}^{n-k})$. Applying E' to $v(\cdot,x'')$ for a.a. $x'' \in B''_N(0)$ we get a bounded linear extension operator E such that Ev is weakly differentiable, $\operatorname{supp} Ev \subset \mathbb{R}^{n-k} \times B''_N(0)$ and $Ev \in H^1_{q;r,\omega}(\mathbb{R}^n)$. Choose $\rho \in C_0^\infty(\mathbb{R}^{n-k})$ with $\int_{\mathbb{R}^{n-k}} \rho(x')dx' = 1$, let $\rho_\varepsilon(x') = \varepsilon^{k-n}\,\rho\big(\frac{x'}{\varepsilon}\big)$ and $J_\varepsilon w(x') = \rho_\varepsilon * w(x')$. Then the family $\{J_v e : \varepsilon > 0\}$ of Friedrichs' mollifier operators is uniformly bounded on $L^r_\omega(\mathbb{R}^{n-k})$ such that $J_\varepsilon w \to w$ in $L^r_\omega(\mathbb{R}^{n-k})$ as $\varepsilon \to 0+$, see Remark 3.4 in [9]. Thus

$$\int_{\mathbb{R}^k} \big(\|(J_\varepsilon E - I)\,v(\cdot,x'')\|^q_{r,\omega,\Sigma} + \|(\nabla J_\varepsilon E - \nabla)\,v(\cdot,x'')\|^q_{r,\omega,\Sigma}\big)dx'' \to 0$$

as $\varepsilon \to 0+$ due to Lebesgue's theorem. Then a further k–dimensional mollification process on $L^q(L^r_\omega)$ with respect to the x''–variables proves that v may be approximated in $H^1_{q,\alpha;r,\omega}(\Omega)$ by $C_0^\infty(\overline{\Omega})$. ∎

Proposition 2.8 *Under the assumptions of Theorem 2.1*

$$L^q_\alpha(L^r_\omega)_\sigma = \big\{u \in L^q_\alpha(L^r_\omega)^n : \operatorname{div} u = 0\,,\ \nu\cdot u = 0 \text{ on } \partial\Omega\big\}\,.$$

Proof Since by definition $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^q(L^r_\omega)_\sigma$, the inclusion "$\subset$" is obvious. To prove the opposite inclusion let $F \in L^{q'}_{-\alpha}(L^{r'}_{\omega'})^n$ be a functional vanishing on $C^\infty_{0,\sigma}(\Omega)$, i.e., $\int_\Omega F \cdot \varphi \, dx = 0$ for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$. Since $F \in L^1_{\mathrm{loc}}(\Omega)^n$, de Rham's argument (for an elementary proof see Theorem 1.1 in [21]) yields a $p \in L^1_{\mathrm{loc}}(\Omega)$ such that $F = \nabla p$. Consequently $p \in \hat{H}^1_{q',-\alpha;r',\omega'}(\Omega)$, and the density of $C^\infty_0(\overline{\Omega})/\mathbb{R}$ in $\hat{H}^1_{q',-\alpha;r',\omega'}(\Omega)$ shows that

$$\int_\Omega F \cdot u \, dx = \int_\Omega \nabla p \cdot u \, dx = 0$$

for all $u \in L^q_\alpha(L^r_\omega)^n$ with $\operatorname{div} u = 0$, $\nu \cdot u = 0$ on $\partial\Omega$. Now the theorem of Hahn-Banach completes the proof.

■

3 The Weak Neumann Problem in Fourier Space

Given an infinite cylinder or layer $\Omega = \Sigma \times \mathbb{R}^k$ where $\Sigma \subset \mathbb{R}^{n-k}$ is a bounded domain with C^1–boundary, fixed $1 < q, r < \infty$ and a weight $\omega \in A_r = A_r(\mathbb{R}^{n-k})$ we consider the weak Neumann problem

$$-\Delta u = F \quad \text{ with } \quad u \in \hat{H}^1_{q;r,\omega}(\Omega), \quad F \in \hat{H}^{-1}_{q;r,\omega}(\Omega)\,.$$

To be more precise, we are looking for the unique solution $u \in \hat{H}^1_{q;r,\omega}(\Omega)$ of the variational problem

$$\int_\Omega \nabla u \cdot \nabla\varphi \, dx = \langle F, \varphi\rangle \quad \forall \varphi \in \hat{H}^1_{q';r',\omega'}(\Omega)\,. \tag{3.1}$$

Since F can be written in the form $F = f \cdot \nabla$ with $f = (f', f'') \in L^q(L^r_\omega)^n$ and $\|F\|_{-1,q;r,\omega} = \|f', f''\|_{q;r,\omega}$, (3.1) is restated in the form

$$\begin{aligned} &\int_{\mathbb{R}^k}\int_\Sigma (\nabla' u' \cdot \nabla'\varphi + \nabla'' u \cdot \nabla''\varphi) dx' dx'' \\ &= \int_{\mathbb{R}^k}\int_\Sigma (f' \cdot \nabla'\varphi + f'' \cdot \nabla''\varphi) dx' dx''\,. \end{aligned} \tag{3.2}$$

Under suitable assumptions on u and F, f we use a partial Fourier transform $\mathcal{F} = \wedge$ with respect to $x'' \in \mathbb{R}^k$ and with Fourier variable $\xi \in \mathbb{R}^k$. Then Parseval's formula yields the variational problem

$$\int_{\mathbb{R}^k}\int_\Sigma (\nabla'\hat{u} \cdot \overline{\nabla'\hat{\varphi}} + \xi^2 \hat{u}\overline{\hat{\varphi}}) \, dx' d\xi = \int_{\mathbb{R}^k}\int_\Sigma (\hat{f}' \cdot \overline{\nabla'\hat{\varphi}} + \hat{f}'' \cdot \overline{i\xi\hat{\varphi}}) \, dx' d\xi$$

for all suitable complex–valued functions φ. Here for short $\xi^2 = |\xi|^2$. Considering test functions of tensor product type $\varphi(x', x'') = \psi(x')\Phi(x'')$ where $\psi \in C^\infty(\overline{\Sigma}), \Phi \in C_0^\infty(\mathbb{R}^k)$, we are led to the variational problem

$$\int_\Sigma (\nabla'\hat{u} \cdot \overline{\nabla'\psi} + \xi^2 \hat{u}\overline{\psi})\, dx' = \int_\Sigma (\hat{f}' \cdot \overline{\nabla'\psi} + \hat{f}'' \cdot \overline{i\xi\psi})\, dx' \tag{3.3}$$

for all $\psi \in C^\infty(\overline{\Sigma})$ or even for all $\psi \in H^1_{r',\omega'}(\Sigma)$ and for all $\xi \in \mathbb{R}^k$. Shortly we write

$$\nabla'\hat{u} \cdot \nabla' + \xi^2 \hat{u} = \hat{f}' \cdot \nabla' - i\xi \cdot \hat{f}''$$

or equivalently

$$\begin{aligned} (-\Delta' + \xi^2)\hat{u} &= \hat{f}' \cdot \nabla' - i\xi \cdot \hat{f}'' && \text{in } \Sigma \\ \nu \cdot (\nabla'\hat{u} - \hat{f}') &= 0 && \text{on } \partial\Sigma \end{aligned} \tag{3.4}$$

for all $\xi \in \mathbb{R}^k$. Here for fixed $\xi \in \mathbb{R}^k$ the right–hand side functions $\hat{f}', \hat{f}''$ are in $L^r_\omega(\Sigma)$ and the Neumann boundary condition has to be understood in a formal sense. Before solving (3.4) in Σ and above all in $\mathbb{R}^{n-k}$ or $\mathbb{R}^{n-k}_+$ we cite the Hörmander–Michlin Multiplier Theorem in weighted spaces.

Theorem 3.1 *Let $m \in C^{n-k}(\mathbb{R}^{n-k}\backslash\{0\})$ admit a constant $M \in \mathbb{R}$ such that*

$$|\mu|^\gamma \big|\partial^\gamma m(\mu)\big| \le M \quad \textit{for all} \quad \mu \in \mathbb{R}^{n-k}\backslash\{0\} \tag{3.5}$$

and multi–indices $\gamma \in \mathbb{N}_0^{n-k}$ with $|\gamma| \le n-k$. Then for all $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-k})$ the multiplier operator $Tf = \mathcal{F}^{-1}m(\cdot)\mathcal{F}f$ defined for all rapidly decreasing functions $f \in \mathcal{S}(\mathbb{R}^{n-k})$ can be uniquely extended to a bounded linear operator from $L^r_\omega(\mathbb{R}^{n-k})$ to $L^r_\omega(\mathbb{R}^{n-k})$. Moreover there exists an A_r–consistent constant $C = C(r, \mathcal{A}_r(\omega))$ such that

$$\|T\|_{r,\omega} \le CM\|f\|_{r,\omega}.$$

For the proof see Chapter IV, Theorem 3.9 in [14]. The A_r–consistency of the constant C can be checked by carefully examining the proof in [14].

Theorem 3.2 *Let Σ denote the whole space $\mathbb{R}^{n-k}$ or the half space $\mathbb{R}^{n-k}_+$, let $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-k})$. Then for every $\xi \in \mathbb{R}^k_* = \mathbb{R}^k\backslash\{0\}$ and $\hat{f} = (\hat{f}', \hat{f}'') \in L^r_\omega(\Sigma)^n$ problem (3.4) has a unique solution $\hat{u} \in H^1_{r,\omega}(\Sigma)$. This solution satisfies the a priori estimate*

$$\|\nabla'\hat{u}, i\xi\hat{u}\|_{r,\omega} \le c\|\hat{f}\|_{r,\omega} \tag{3.6}$$

with an A_r–consistent constant $c = c(A_r(\omega))$ independent of $\xi \in \mathbb{R}^k_*$.

Proof In the proof we will omit the notation $\hat{}$ for the original partial Fourier transform $\mathcal{F}$ which led from (3.2) to (3.3), since we have to introduce a further partial Fourier transform.

First let $\Sigma = \mathbb{R}^{n-k}$. Then we apply the $(n-k)$–dimensional Fourier transform $\mathcal{G} = \tilde{}$ with the Fourier variable $\mu \in \mathbb{R}^{n-k}$ to (3.3), (3.4) to get the problem

$$(\mu^2 + \xi^2)\tilde{u}(\mu) = -i\mu \cdot \tilde{f}' - i\xi \cdot \tilde{f}'' \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^{n-k})$$

where $\mu^2 = \mu \cdot \mu$; no boundary condition is needed in this case. For its explicit solution $\tilde{u}(\mu) = (\mu^2 + \xi^2)^{-1}(-i\mu \cdot \tilde{f}' - i\xi \cdot \tilde{f}'')$ we have

$$\begin{pmatrix} \widetilde{\nabla' u} \\ \widetilde{i\xi u} \end{pmatrix} = \begin{pmatrix} \dfrac{\mu \otimes \mu}{\mu^2 + \xi^2} & \dfrac{\mu \otimes \xi}{\mu^2 + \xi^2} \\ \dfrac{\xi \otimes \mu}{\mu^2 + \xi^2} & \dfrac{\xi \otimes \xi}{\mu^2 + \xi^2} \end{pmatrix} \cdot \begin{pmatrix} \tilde{f}' \\ \tilde{f}'' \end{pmatrix}.$$

All matrix elements satisfy the Hörmander–Michlin multiplier condition (3.5) with a constant M independent of $\xi \in \mathbb{R}^k_*$. Thus Theorem 3.1 yields the a priori estimate (3.6) with an A_r–consistent constant c independent of $\xi \in \mathbb{R}^k_*$. If $f = 0$ and consequently $\tilde{f} = 0$, also $\tilde{u} = 0$ and $u = 0$ proving the uniqueness assertion for every $\xi \in \mathbb{R}^k_*$.

Next let $\Sigma = \mathbb{R}^{n-k}_+ = \{x' = (x^*, x_{n-k}) : x^* \in \mathbb{R}^{n-k-1},\, x_{n-k} > 0\}$ and fix $f = (f', f'') = (f^*, f_{n-k}, f'') \in L^r_\omega(\Sigma)^n$, $\omega \in A_r(\mathbb{R}^{n-k})$. At this moment it is convenient to assume w.l.o.g. that ω is even w.r.t. x_{n-k}, see Lemma 2.4(2). Due to the formal boundary condition $\partial_{n-k} u - f_{n-k} = 0$ on $\partial\Sigma \cong \mathbb{R}^{n-k-1}$ we extend f_{n-k} in an odd way to $f_{n-k,o} \in L^r_\omega(\mathbb{R}^{n-k})$ and f^*, f'' in an even way to f^*_e, $f''_e \in L^r_\omega(\mathbb{R}^{n-k})$. By the results proved just before there exists a unique $u \in H^1_{r,\omega}(\mathbb{R}^{n-k})$ such that

$$(-\Delta' + \xi^2)u = f^*_e \cdot \nabla^* + f_{n-k,o} \cdot \partial_{n-k} - i\xi \cdot f''_e \quad \text{on } \mathbb{R}^{n-k}\,.$$

Since f^*_e is even w.r.t. x_{n-k} etc., also $u(x^*, -x_{n-k})$ solves this equation. Hence the uniqueness assertion proves that u is even w.r.t. x_{n-k}. Given $\psi \in C^\infty_0(\overline{\Sigma})$ let $\psi_e \in H^1_{r',\omega'}(\mathbb{R}^{n-k})$ be its even extension to $\mathbb{R}^{n-k}$. Then

$$\begin{aligned} \int_\Sigma f^* \cdot \nabla^* \psi\, dx' &= \frac{1}{2} \int_{\mathbb{R}^{n-k}} f^*_e \cdot \nabla^* \psi_e\, dx'\,, \\ \int_\Sigma f_{n-k}\, \partial_{n-k} \psi\, dx' &= \frac{1}{2} \int_{\mathbb{R}^{n-k}} f_{n-k,o}\, \partial_{n-k} \psi_e\, dx'\,; \end{aligned}$$

similar identities hold for the integrals involving f'' and u since u is even. Hence u actually solves (3.3). Finally

$$\begin{aligned}\|\nabla' u, i\xi u\|_{r,\omega,\Sigma} &\leq \|\nabla' u, i\xi u\|_{r,\omega,\mathbb{R}^{n-k}} \\ &\leq c\big(\mathcal{A}_r(\omega)\big)\,\|f_e^*, f_{n-k,o}, f_e''\|_{r,\omega,\mathbb{R}^{n-k}} \leq c_r c\big(\mathcal{A}_r(\omega)\big)\,\|f\|_{r,\omega,\Sigma}\,.\end{aligned}$$

If $f = 0$ and $u \in H^1_{r,\omega}(\Sigma)$ is a solution of (3.3) on Σ, then $u_e \in H^1_{r,\omega}(\mathbb{R}^{n-k})$ solves (3.3) on $\mathbb{R}^{n-k}$ with a vanishing right-hand side. Thus $u_e = 0$, $u = 0$ proving the uniqueness assertion. ∎

Remark 3.3 Assume $\hat{f} \in L^{r_1}_{\omega_1}(\Sigma)^n \cap L^{r_2}_{\omega_2}(\Sigma)^n$ for exponents $1 < r_i < \infty$ and weights $\omega_i \in A_{r_i}$, $i = 1, 2$. Then the unique solution $\hat{u} \in H^{r_1}_{\omega_1}(\Sigma)$ of (3.4) also satisfies $\hat{u} \in H^{r_2}_{\omega_2}(\Sigma)$. For the proof in the case $\Sigma = \mathbb{R}^{n-k}$ note that the solution is uniquely defined by $\tilde{u}(\mu) = (\mu^2 + \xi^2)^{-1}(-i\mu \cdot \tilde{f}' - i\xi \cdot \tilde{f}'')$ in $\mathcal{S}'(\mathbb{R}^{n-k})$. If $\Sigma = \mathbb{R}^{n-k}_+$, the extension techniques in the proof of Theorem 3.2 prove the uniqueness of u.

Next we consider the Neumann problem (3.4) in a bended half space $\Sigma_\sigma \subset \mathbb{R}^{n-k}$,

$$\Sigma_\sigma = \{x' = (x^*, x_{n-k}) \in \mathbb{R}^{n-k} : \; x_{n-k} > \sigma(x^*)\}\,,$$

where $\sigma \in C^{0,1}(\mathbb{R}^{n-k-1})$.

Theorem 3.4 *Let $n \geq 3$, $1 < r < \infty$, $\omega \in A_r(\mathbb{R}^{n-k})$, $1 \leq k \leq n-2$ and $\sigma \in C^{0,1}(\mathbb{R}^{n-k-1})$. There exists an A_r–consistent constant $K = K(r,\omega) > 0$ with the following property: Assume that $\|\nabla'\sigma\|_\infty \leq \frac{1}{K}$. Then for every $\xi \in \mathbb{R}^k_*$ and for every $\hat{f} = (\hat{f}', \hat{f}'') \in L^r_\omega(\Sigma_\sigma)^n$ problem (3.4) has a unique solution $\hat{u} \in H^1_{r,\omega}(\Sigma_\sigma)$. This solution satisfies the a priori estimate (3.6) with an A_r–consistent constant $c = c\big(\mathcal{A}_r(\omega), K\big)$ independent of $\xi \in \mathbb{R}^k_*$.*

Proof For notational convenience we omit the symbol ^ and write u instead of $\hat{u}$ etc. The problem (3.3) in Σ_σ is reduced to the half space problem via the coordinate transform $\phi : \Sigma_\sigma \to \mathbb{R}^{n-k}_+, \tilde{x}' = (\tilde{x}^*, \tilde{x}_{n-k}) = \phi(x') := (x^*, x_{n-k} - \sigma(x^*))$. Obviously ϕ is a bijection with Jacobian equal to 1. For a function u on Σ_σ we define $\tilde{u}(\tilde{x}') = u(\phi^{-1}(\tilde{x}'))$ and denote by $\tilde{\partial}_j, \tilde{\nabla}'$ etc. the derivatives w.r.t. the variable $\tilde{x} \in \mathbb{R}^{n-k}_+$. In particular, using $\partial_{n-k}\sigma = 0$,

$$\begin{aligned}\partial_j u(x') &= \big(\tilde{\partial}_j - (\partial_j\sigma)\tilde{\partial}_{n-k}\big)\tilde{u}(\tilde{x}'), \quad 1 \leq j \leq n-k\,, \\ \|u\|_{r,\omega,\Sigma_\sigma} &= \|\tilde{u}\|_{r,\tilde{\omega},\mathbb{R}^{n-k}_+}, \quad \|\nabla u\|_{r,\omega,\Sigma_\sigma} \leq c(1 + \|\nabla'\sigma\|_\infty)\|\tilde{\nabla}\tilde{u}\|_{r,\tilde{\omega},\mathbb{R}^{n-k}_+}.\end{aligned}$$

Here the modified weight $\tilde{\omega}(\tilde{x}') = \omega(\phi^{-1}(\tilde{x}'))$ satisfies $\tilde{\omega} \in A_r$ and $\mathcal{A}_r(\tilde{\omega}) \le c\mathcal{A}_r(\omega)$ where $c = c(\phi)$ is independent of ω, see Lemma 2.8 (1).

Given $\psi \in H^1_{r'\omega'}(\Sigma_\sigma)$ the variational problem (3.3) yields

$$\begin{aligned}\int_{\mathbb{R}^{n-k}_+} (\tilde{\nabla}'\tilde{u} \cdot \overline{\tilde{\nabla}'\tilde{\psi}} + \xi^2 \tilde{u}\overline{\tilde{\psi}}) d\tilde{x}' \\ = \int_{\mathbb{R}^{n-k}_+} (\tilde{f}' \cdot \tilde{\nabla}'\tilde{\psi} + \tilde{f}'' \cdot \overline{i\xi\tilde{\psi}}) d\tilde{x}' + R(\tilde{\psi}, \tilde{u}, \tilde{f}, \sigma) \qquad (3.7)\end{aligned}$$

with the remainder

$$\begin{aligned} R(\tilde{\psi}) \;=\; & \int_{\mathbb{R}^{n-k}_+} \Big((\nabla'\sigma \cdot \tilde{\nabla}'\tilde{\psi})\tilde{\partial}_{n-k}\tilde{u} + (\nabla'\sigma \cdot \tilde{\nabla}'\tilde{u})\tilde{\partial}_{n-k}\tilde{\psi} \\ & - |\nabla'\sigma|^2 \tilde{\partial}_{n-k}\tilde{u}\tilde{\partial}_{n-k}\tilde{\psi} - (\tilde{f}' \cdot \nabla'\sigma)\tilde{\partial}_{n-k}\tilde{\psi}\Big) d\tilde{x}'. \end{aligned}$$

Thus $\tilde{u}$ can be considered as a solution of (3.3) on $\mathbb{R}^{n-k}_+$ with modified right–hand side where $\tilde{f}''$ is unchanged, but $\tilde{f}'$ has to be replaced by

$$\tilde{f}' + \big(\tilde{\partial}_{n-k}\tilde{u}\big)\nabla'\sigma + (\nabla'\sigma \cdot \tilde{\nabla}'\tilde{u})e_{n-k} - |\nabla'\sigma|^2\big(\tilde{\partial}_{n-k}\tilde{u}\big)e_{n-k} - (\tilde{f}' \cdot \nabla'\sigma)e_{n-k}$$

with the unit vector $e_{n-k} = (0, \ldots, 0{,}1) \in \mathbb{R}^{n-k}$.

If $\|\nabla'\sigma\|_\infty$ is sufficiently small, Kato's perturbation method implies that the implicit problem (3.7) has a unique solution $\tilde{u}$. For more details of this standard argument see e.g. [8]. Actually, the a priori estimate (3.6) for $\tilde{u}$ on $\mathbb{R}^{n-k}_+$ with an A_r–consistent constant c proves that there exists an A_r–consistent constant K such that for σ with $\|\nabla'\sigma\|_\infty \le \frac{1}{K}$ the solution $\tilde{u}$ satisfies

$$\|\tilde{\nabla}'\tilde{u}, i\xi\tilde{u}\|_{r,\tilde{\omega},\mathbb{R}^{n-k}_+} \le \tilde{C}\|\tilde{f}\|_{r,\tilde{\omega},\mathbb{R}^{n-k}_+}$$

where again $\tilde{C}$ is A_r–consistent. This estimate yields (3.6) for u on Σ_σ with an A_r–consistent constant $c > 0$. Furthermore the uniqueness assertion for $\tilde{u}$ in (3.7) proves the uniqueness of u. ■

Remark 3.5 Assume $\hat{f} \in L^{r_1}_{\omega_1}(\Sigma_\sigma) \cap L^{r_2}_{\omega_2}(\Sigma_\sigma)$ for exponents $1 < r_i < \infty$ and weights $\omega_i \in A_{r_i}$, $i = 1, 2$. Then for $\sigma \in C^{0,1}(\mathbb{R}^{n-k-1})$ satisfying $\|\nabla'\sigma\|_\infty \le \min(\frac{1}{K_1}, \frac{1}{K_2})$ where $K_i = K(r_i, \omega_i)$ the unique solution $\hat{u} \in H^1_{r_1,\omega_1}(\Sigma_\sigma)$ of (3.4) also satisfies $\hat{u} \in H^1_{r_2,\omega_2}(\Sigma_\sigma)$. The proof is based on the construction of $\hat{u}$ in the proof of Theorem 3.4 and on Remark 3.3.

Next we will consider a bounded domain $\Sigma \subset \mathbb{R}^{n-k}$ with boundary of class C^1. It is well–known that the *Neumann eigenvalue problem*

$$-\Delta' \hat{u} = \alpha \hat{u} \text{ in } \Sigma, \quad \frac{\partial \hat{u}}{\partial \nu} = 0 \text{ on } \Sigma \tag{3.8}$$

has a sequence of nonnegative eigenvalues $0 = \alpha_0 < \alpha_1 \leq \alpha_2 \leq \dots$ and corresponding eigenfunctions $\hat{u}_j \in W^{1,r}(\Sigma) \cap C^\infty(\Sigma)$. These eigenvalues and eigenfunctions do not depend on the exponent $r \in (1, \infty)$ of the L^r–space in which (3.8) is analyzed. It is even allowed to consider (3.8) in weighted spaces L^r_ω, $\omega \in A_r$. The proof is based on standard elliptic regularity techniques and on the embeddings $L^{s_1}(\Sigma) \subset L^r_\omega(\Sigma) \subset L^{s_2}(\Sigma)$, see Lemma 2.4(3).

To avoid difficulties originating from the eigenvalue $\alpha_0 = 0$ of (3.8) when solving (3.4) in bounded domains $\Sigma \subset \mathbb{R}^{n-k}$ we introduce the spaces

$$\begin{aligned} L^{r,0}_\omega(\Sigma) &= \{u \in L^r_\omega(\Sigma) : \int_\Sigma u \, dx' = 0\}, \\ H^{1,0}_{r,\omega}(\Sigma) &= \{u \in L^{r,0}_\omega(\Sigma) : \nabla u' \in L^r_\omega(\Sigma)\} \end{aligned}$$

of functions with vanishing integral mean on Σ. Note that $H^{1,0}_{r,\omega}(\Sigma)$ is compactly embedded into $L^{r,0}_\omega(\Sigma)$ and that $\|\nabla u\|_{r,\omega}$ is a norm on $H^{1,0}_{r,\omega}(\Sigma)$, see Proposition 2.5 and Corollary 2.6.

To extend $L^q(L^r_\omega)$-estimates without weights w.r.t. $x'' \in \mathbb{R}^k$ to exponentially weighted L^q–estimates we consider (3.3),(3.4) also for complex ζ in the strip

$$S_\beta = \{\zeta = \xi + i\alpha \in \mathbb{C}^k : \xi, \alpha \in \mathbb{R}^k, |\alpha| < \beta\}, \quad \beta > 0.$$

For $\zeta \in S_\beta$ (3.4) has the form

$$\begin{aligned} (-\Delta' + \xi^2 + 2i\alpha \cdot \xi - \alpha^2)\hat{u} &= \hat{f}' \cdot \nabla' - i(\xi + i\alpha) \cdot \hat{f}'' && \text{in } \Sigma \\ \nu \cdot (\nabla' \hat{u} - \hat{f}') &= 0 && \text{on } \partial\Sigma \end{aligned} \tag{3.9}$$

which formally is the partial Fourier transform of the equation

$$(-\Delta + 2\alpha \cdot \nabla'' - \alpha^2)u = f' \cdot \nabla' + f'' \cdot \nabla'' + \alpha \cdot f''$$

in Ω together with the boundary condition $\partial u / \partial \nu = f' \cdot \nu$ on $\partial\Omega$.

Theorem 3.6 *Let $\Sigma \subset \mathbb{R}^{n-k}$ be a bounded domain of class C^1, let $1 < r < \infty$ and $\omega \in A_r$. Then for every $\zeta \in S_\beta$, $0 < \beta < \sqrt{\alpha_1}$, and*

$$\hat{f} = (\hat{f}', \hat{f}'') \in L^r_\omega(\Sigma)^n \quad \textit{with} \quad \hat{f}'' \in L^{r,0}_\omega(\Sigma)^k$$

problem (3.9) has a unique solution $\hat{u} \in H^{1,0}_{r,\omega}(\Sigma)$ *satisfying the a priori estimate*

$$\|\nabla'\hat{u}, \zeta\hat{u}\|_{r,\omega} \le c\|\hat{f}\|_{r,\omega} \tag{3.10}$$

with an A_r*–consistent constant* c *independent of* $\zeta \in S_\beta$*.*

Note that due to the condition $\hat{f}'' \in L^{r,0}_{\omega}(\Sigma)^k$ and its implication $\hat{u} \in H^{1,0}_{r,\omega}(\Sigma)$ it suffices to consider only test functions $\psi \in H^{1,0}_{r',\omega'}(\Sigma)$ in the variational formulation of (3.9). In the first step of the proof of Theorem 3.6 we prove a preliminary estimate.

Lemma 3.7 *In the setting of Theorem 3.6 a solution* $\hat{u} \in H^{1,0}_{r,\omega}(\Sigma)$ *of* (3.9) *satisfies the a priori estimate*

$$\|\nabla'\hat{u}, \zeta\hat{u}, \hat{u}\|_{r,\omega} \le c\big(\|\hat{f}\|_{r,\omega} + \|\hat{u}\|_{r,\omega} + \|\nabla'\hat{u}\|_{[H^1_{r'\omega'}(\Sigma)^n]^*}\big) \tag{3.11}$$

with an A_r*–consistent constant* $c > 0$ *independent of* $\zeta \in S_\beta$*.*

Proof The closure of the bounded domain Σ can be covered by a finite number of balls $B_1, \dots, B_m \subset \mathbb{R}^{n-k}$. Furthermore there are cut-off functions $0 \le \varphi_1, \dots, \varphi_m \in C^\infty_0(\mathbb{R}^{n-k})$ with $\operatorname{supp}\varphi_j \subset B_j$ and $\sum_{j=1}^m \varphi_j = 1$ in Σ. Since $\partial\Sigma \subset C^1$, for every j with $B_j \cap \partial\Sigma \ne \emptyset$ there exists a perturbation $\sigma_j \in C^1(\mathbb{R}^{n-k-1})$ such that (after a suitable translation and rotation T_j of the coordinate system) $B_j \cap \Sigma \subset \Sigma_j := \Sigma_{\sigma_j}$ and $B_j \cap \partial\Sigma \subset \partial\Sigma_j$. Assume that each $B_j \cap \Sigma$ is a Lipschitz domain. Looking at the A_r–consistent constant $K = K(r,\omega)$ in Theorem 3.4 we can even choose a fixed, sufficiently large and A_r–consistent number $m = m(K) \in \mathbb{N}$ such that each function σ_j satisfies $\|\nabla'\sigma_j\|_\infty \le \frac{1}{K}$. Thus we may use the same partition of unity $\{\varphi_j\}_{j=1}^m$ for every $\omega \in A_r$, $\mathcal{A}_r(\omega) \le d$. Since the coordinate transform T_j does not essentially affect the subsequent estimates, e.g. $\mathcal{A}_r(\omega) \sim \mathcal{A}_r(\omega \circ T_j)$ by Lemma 2.4(1), we suppress this transform in the following. If $B_j \cap \partial\Sigma = \emptyset$ it will be convenient to define $\Sigma_j := \mathbb{R}^{n-k}_+$.

Again, for notational convenience, we write u instead of $\hat{u}$, etc. We start with a solution u of (3.9) when $\zeta = \xi \in \mathbb{R}^k_*$. Given a test function $\psi_j \in C^\infty_0(\overline{\Sigma}_j)$, $1 \le j \le m$, we will use $\varphi_j\psi_j - d_j \in H^{1,0}_{r',\omega'}(\Sigma)$ with $d_j = \frac{1}{|\Sigma|}\int_\Sigma \varphi_j\psi_j\,dx'$ as an admissable test function in (3.3) on Σ. Note that the constant d_j drops out in (3.3) since $u, f'' \in L^{r;0}_{\omega}(\Sigma)$. Then an elementary calculation yields the identity

$$\begin{aligned}\int_{\Sigma_j} &\big(\nabla'(u\varphi_j)\cdot\overline{\nabla'\psi_j}+\xi^2(u\varphi_j)\overline{\psi_j}\big)dx'\\ &=\int_{\Sigma_j}\big((f'\varphi_j)\cdot\overline{\nabla'\psi_j}+(f''\varphi_j)\cdot\overline{i\xi\psi_j}\big)dx'+R^j(\psi_j)\end{aligned}$$

with the remainder term $R^j = R^j_1 + R^j_2 + R^j_3$, where

$$\begin{aligned}R^j_1(\psi_j) &= \int_{\Sigma_j} u\nabla'\varphi_j\cdot\overline{\nabla'\psi_j}\,dx',\\ R^j_2(\psi_j) &= \int_{\Sigma_j}(f'-\nabla'u)(\overline{\psi_j}-\overline{c_j})\cdot\nabla'\varphi_j\,dx',\\ R^j_3(\psi_j) &= \int_{\Sigma_j} g_j\cdot\overline{i\xi\psi_j}\,dx';\end{aligned}$$

here $c_j = \frac{1}{|B_j\cap\Sigma|}\int_{B_j\cap\Sigma}\psi_j\,dx'$ is used to guarantee that $\psi_j - c_j \in L^{r';0}_{\omega'}(B_j\cap\Sigma)$ and to define

$$g_j(x') = \frac{i\xi}{\xi^2}\,\frac{1}{|B_j\cap\Sigma|}\Big(\int_\Sigma (f'-\nabla'u)\cdot\nabla'\varphi_j\,dy'\Big)\,\chi_{B_j\cap\Sigma}(x')\,.$$

Besides the trivial estimate $|R^j_1(\psi)| \le c\|u\|_{r,\omega}\,\|\nabla'\psi\|_{r',\omega',\Sigma_j}$ Poincaré's inequality on $B_j\cap\Sigma$ implies that

$$|R^j_2(\psi)| \le c\big(\|f\|_{r,\omega}+\|\nabla'u\|_{[H^1_{r'\omega'}(\Sigma)^n]^*}\big)\,\|\nabla'\psi\|_{r',\omega',\Sigma_j}\,.$$

Obviously $R^j_3(\psi)$ satisfies the estimate

$$|R^j_3(\psi)| \le \|g_j\|_{r,\omega}\,\|\xi\psi\|_{r',\omega',\Sigma_j}$$

with

$$\|g_j\|_{r,\omega} \le \frac{c}{|\xi|}\,\Big(\|f\|_{r,\omega}+\|\nabla'u\|_{r,\omega}\Big)\,; \tag{3.12}$$

this inequality will be used for $|\xi|$ sufficiently large, say for $|\xi| \ge M$. For small $|\xi|$ we exploit the fact that u solves (3.3) in Σ. Replacing φ_j in the definition of g_j by $\varphi_j - \varphi_{j\Sigma} \in H^{1,0}_{r'\omega'}(\Sigma)$ with $\varphi_{j\Sigma} = \frac{1}{|\Sigma|}\int_\Sigma \varphi_j dx'$ we rewrite g_j in the form

$$g_j(x') = \frac{i\xi}{\xi^2}\,\frac{1}{|B_j \cap \Sigma|}\Big(\int_\Sigma (\xi^2 u + i\xi\cdot f'')\varphi_j\, dy'\Big)\,\chi_{B_j\cap\Sigma}(x').$$

Consequently

$$\|g_j\|_{r,\omega} \le c(\|f\|_{r,\omega} + \|\xi u\|_{r,\omega})\,. \tag{3.13}$$

Now apply the a priori estimate (3.6) to $u\varphi_j$ and sum up for $j = 1, \dots, m$ to get that

$$\|\nabla' u, \xi u\|_{r,\omega} \le c\big(\|f\|_{r,\omega} + \|u\|_{r,\omega} + \|\nabla' u\|_{[H^1_{r'\omega'}(\Sigma)^n]^*}\big)$$

for all $\xi \in \mathbb{R}^k_*$. There we used that the term $\frac{c}{|\xi|}\|\nabla' u\|_{r,\omega}$, which, to begin with, appears on the right-hand side of this estimate, see (3.12), can be absorbed by the term $\|\nabla' u\|_{r,\omega}$ on the left-hand side for $|\xi| \ge M$; then for $|\xi| < M$ we use (3.13) to estimate g_j and $R^j_3(\psi)$. Note that all constants are A_r–consistent due to the corresponding assertions in Theorems 3.2 and 3.4 and in Poincaré's inequality; in particular the bound M is A_r–consistent.

To extend (3.11) to complex $\zeta = \xi + i\alpha \in S_\beta$, $\xi \neq 0$, we write (3.9) in the form

$$(-\Delta' + \xi^2)u \;=\; f'\cdot\nabla' - i\xi\cdot(f'' + 2\alpha u) + \alpha^2 u + \alpha\cdot f''. \tag{3.14}$$

For a test function $\psi \in H^1_{r',\omega'}(\Sigma)$ the crucial term $\alpha^2 u + \alpha\cdot f''$ satisfies the estimate

$$\Big|\int_\Sigma (\alpha^2 u + \alpha\cdot f'')\psi\, dx'\Big| \le c\|u, f''\|_{r,\omega}\|\nabla'\psi\|_{r',\omega'}$$

due to the vanishing means of u, f'' on Σ and Poincaré's inequality. Thus the functional $\alpha^2 u + \alpha\cdot f''$ may be rewritten in the form $h\cdot\nabla'$ where $\|h\|_{r,\omega} \le c\|u, f''\|_{r,\omega}$. Therefore the first part of the proof, i.e. the case $\zeta = \xi \in \mathbb{R}^k_*$, completes the proof when $\xi \neq 0$. If $\xi = 0$ or even $\zeta = 0$, we may add u on both sides of (3.9) to get (3.11). ∎

Proof of Theorem 3.6 Assume that (3.10) is not satisfied with an A_r–consistent constant c. Thus there exist sequences $(\zeta_j) \subset S_\beta$ where $\zeta_j = \xi_j + i\alpha_j$, $\xi_j, \alpha_j \in \mathbb{R}^k$, $(\omega_j) \subset A_r$ with $\mathcal{A}_r(\omega_j) \le d$, $\hat f_j \in L^r_{\omega_j}(\Sigma)^n$ with

$\hat{f}_j'' \in L^{r;0}_{\omega_j}(\Sigma)^k$ and corresponding solutions $\hat{u}_j \in H^{1,0}_{r,\omega_j}(\Sigma)$ of (3.9) such that, omitting the symbol ^, w.l.o.g.

$$1 = \|\nabla' u_j, \zeta_j u_j\|_{r,\omega_j} \geq j\|f_j\|_{r,\omega_j} \quad \text{for every} \quad j \in \mathbb{N}. \tag{3.15}$$

By Lemma 2.4 there exists an $s \in (1,\infty)$ *not* depending on $j \in \mathbb{N}$ such that

$$(\nabla' u_j), (\zeta_j u_j), (u_j) \subset L^s(\Sigma) \quad \text{are bounded.}$$

Hence these sequences will admit weakly convergent subsequences in $L^s(\Sigma)$. Omitting an additional subindex for subsequences we have to distinguish 3 cases concerning the behavior of (ζ_j).

First Case $\zeta_j \to \zeta_0 \in \overline{S}_\beta \backslash \{0\}$: We may assume that $u_j \rightharpoonup u$ in $H^1_s(\Sigma)$ for $j \to \infty$ and that u satisfies

$$(-\Delta' + \zeta_0^2)u = 0 \text{ in } \Sigma, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma$$

in the weak sense. Since $-\zeta_0^2$ differs from every eigenvalue α_l of the Neumann eigenvalue problem (3.8) we conclude that $u = 0$. In particular $u_j \rightharpoonup 0$, $\nabla' u_j \rightharpoonup 0$ in $L^s(\Sigma)$ for $j \to \infty$. Then the compactness assertions of Proposition 2.5(2),(3) imply that

$$\|u_j\|_{r,\omega_j} + \|\nabla' u_j\|_{[H^1_{r',\omega_j'}(\Sigma)^n]^*} \to 0 \quad \text{for} \quad j \to \infty.$$

But this convergence yields a contradiction to (3.11) and (3.15).

Second Case $\zeta_j \to 0$. In this case $u_j \rightharpoonup u$ in $H^1_s(\Sigma)$ where u solves $-\Delta' u = 0$, $\partial u/\partial \nu = 0$. But since $\int_\Sigma u_j dx' = 0$ for every $j \in \mathbb{N}$, also $\int_\Sigma u\, dx' = 0$ yielding $u = 0$. Thus we will arrive at the same contradiction as before.

Third Case $|\zeta_j| \to \infty$: Obviously $u_j \rightharpoonup 0$ and consequently also $\nabla' u_j \rightharpoonup 0$ yielding the same contradiction as above.

Up to now we proved the a priori estimate (3.10) for every $\zeta \in S_\beta$,

$$f \in X := L^r_\omega(\Sigma)^{n-k} \times L^{r;0}_\omega(\Sigma)^k$$

and a given solution $u \in H^{1,0}_{r,\omega}(\Sigma)$; the constant c in (3.10) is A_r–consistent. In particular the uniqueness of a solution is guaranteed. To prove the solvability of the Neumann problem fix $\zeta \in S_\beta$ and consider the bounded linear operator

$$T_{r,\omega} : H^{1,0}_{r,\omega}(\Sigma) \to H^{1,0}_{r',\omega'}(\Sigma)^*, \quad T_{r,\omega} u = \nabla' u \cdot \nabla' + \zeta^2 u,$$

where $\zeta^2 = \xi^2 + 2i\alpha\cdot\xi - \alpha^2$. Obviously $T_{r,\omega}$ is injective and its dual operator $(T_{r,\omega})'$ equals $T_{r',\omega'}$. To prove the surjectivity of $T_{r,\omega}$ it suffices due to the Closed Range Theorem to show that $T_{r,\omega}$ has a closed range.

For these reasons we introduce the closed subspace

$$Y = \{f = (f', f'') \in X : \quad f'\cdot\nabla' - i\zeta\cdot f'' = 0\}$$

of X. Then we consider the linear operators

$$T_1 : H^{1,0}_{r,\omega}(\Sigma) \to X/Y, \quad T_1 u = \big[(\nabla' u, i\zeta u)\big],$$

where $[f] \in X/Y$ denotes the equivalence class in the quotient space X/Y represented by $f \in X$, and

$$T_2 : X/Y \to H^{1,0}_{r',\omega'}(\Sigma)^*, \quad T_2[f] = f'\cdot\nabla' - i\zeta\cdot f''.$$

Obviously both operators are bounded and $T_{r,\omega} = T_2 \circ T_1$. Moreover T_2 is injective, surjective and consequently, due to the Open Mapping Theorem, an isomorphism. Hence there exists a constant $c_1 > 0$ such that

$$\|T_2[f]\|_{[H^{1,0}_{r',\omega'}(\Sigma)]^*} \geq c_1 \inf_{h\in[f]} \|h\|_{r,\omega} = c_1\|f\|_{X/Y}.$$

Concerning T_1 the a priori estimate (3.10) yields a constant $c > 0$ such that

$$\|\nabla' u, \zeta u\|_{r,\omega} \leq c \ \inf\big\{\|f\|_{r,\omega} : f \in \big[(\nabla' u, i\zeta u)\big]\big\} = c\|T_1 u\|_{X/Y};$$

note that every $f \in [(\nabla' u, i\zeta u)]$ is an admissable right-hand side in (3.4) with solution u. Combining the previous estimates leads to the inequality

$$\|T_2\circ T_1 u\|_{[H^{1,0}_{r',\omega'}(\Sigma)]^*} \geq c_1\|T_1 u\|_{X/Y} \geq \frac{c_1}{c}\,\|\nabla' u, \zeta u\|_{r,\omega}.$$

Thus $T_{r,\omega} = T_2 \circ T_1$ has closed range. Now the proof of Theorem 3.6 is complete. ■

Next we extend Theorem 3.6 in a certain sense from $\hat f \in X$ to all $\hat f \in L^r_\omega(\Sigma)^n$ and rewrite the result in a more operator–theoretical way. For $\zeta = \xi + i\alpha \in S_\beta$ let

$$M_\alpha(\xi) : L^r_\omega(\Sigma)^n \to L^r_\omega(\Sigma)^n, \quad M_\alpha(\xi)(\hat f) = (\nabla' \hat u, \xi\hat u),$$

denote the bounded linear *solution operator* of 3.9; however $\hat{f}$ is replaced by

$$\hat{f}' \cdot \nabla' - i(\xi + i\alpha) \cdot \left(\hat{f}'' - \hat{f}''_\Sigma\right) \quad \text{where} \quad \hat{f}''_\Sigma(x'') = \frac{1}{|\Sigma|} \int_\Sigma \hat{f}''(x', x'')\, dx'.$$

Let $|||T|||$ denote the operator norm for a linear map $T \in \mathcal{L}(L^r_\omega(\Sigma)^n)$.

Corollary 3.8 *Given* $1 < r < \infty$, $\omega \in A_r$ *and* $\alpha \in [-\beta, \beta]$, $0 < \beta < \sqrt{\alpha_1}$ *the operator family* $\{M_\alpha(\xi) : \xi \in \mathbb{R}^k_*\}$ *has the following properties:* $M_\alpha(\xi)$ *is Fréchet–differentiable w.r.t.* $\xi \in \mathbb{R}^k_*$ *and there exists an* A_r*–consistent constant* $c = c(\beta)$ *such that for every multi–index* $\gamma \in \{0,1\}^k$

$$|||M_\alpha(\xi)||| + |||\, |\xi|^\gamma \partial^\gamma M_\alpha(\xi)||| \le c \quad \textit{for all} \quad \xi \in \mathbb{R}^k_*.$$

Proof The uniform estimate of $|||M_\alpha(\cdot)|||$ is a consequence of (3.10). Since ξ enters (3.9) in a polynomial way it is easy to show that $M_\alpha(\cdot)$ is Fréchet differentiable. Given $\hat{u}(\xi) \in H^{1,0}_{r,\omega}(\Sigma)$ by $(\nabla'\hat{u}, \xi\hat{u}) = M_\alpha(\xi)(\hat{f})$ the Fréchet derivative $\hat{v}_j(\xi) = \partial\hat{u}(\xi)/\partial\xi_j$, $n-k+1 \le j \le n$, solves the Neumann problem

$$\nabla'\hat{v}_j \cdot \nabla' + (\xi^2 + 2i\alpha\cdot\xi - \alpha^2)\hat{v}_j = -i(\hat{f}''_j - \hat{f}''_{j,\Sigma}) - 2(\xi_j + i\alpha_j)\hat{u}.$$

Then $|\xi|\hat{v}_j$ solves a similar Neumann problem and Theorem 3.6 yields the estimate

$$\|\, |\xi|(\nabla'\hat{v}_j, \xi\hat{v}_j, \hat{v}_j)\|_{r,\omega} \le c\|\hat{f}, \xi\hat{u}, \hat{u}\|_{r,\omega} \le c\|\hat{f}\|_{r,\omega}$$

for every $n-k+1 \le j \le n$ with an A_r-consistent constant $c = c(\beta) > 0$. For the mixed second order derivative $\partial^2 M_\alpha/\partial\xi_j\partial\xi_l$, $n-k+1 \le j \ne l \le n$, we proceed in a similar way. The function $\hat{w}_{jl} = (\partial^2 M_\alpha/\partial\xi_j\partial\xi_l)\hat{f}$ satisfies the equation

$$\nabla'\hat{w}_{jl} \cdot \nabla' + (\xi^2 + 2i\alpha\xi - \alpha^2)\hat{w}_{jl} = -2(\xi_l + i\alpha_l)\hat{v}_j - 2(\xi_j + i\alpha_j)\hat{v}_l$$

admitting the estimate $\|\, |\xi|^2(\nabla'\hat{w}_{jl}, \xi\hat{w}_{jl})\|_{r,\omega} \le c\|\hat{f}\|_{r,\omega}$. Analogously we show that every set of partial derivatives $\{|\xi|^\gamma\partial^\gamma M_\alpha(\xi) : \xi \in \mathbb{R}^k_*\}, \gamma \in \{0,1\}^k$, is uniformly bounded in the operator norm $|||\cdot|||$. The generic constant c in these estimates is A_r–consistent and independent of $\alpha \in \mathbb{R}^k$, $|\alpha| < \beta$. ■

By Corollary 3.8 $M_\alpha(\cdot)$ satisfies the classical Hörmander–Michlin multiplier condition, cf. Theorem 3.1. However $M_\alpha(\xi)$ is *operator–valued* and will be applied to *Banach space–valued functions*, e.g. to $f \in L^q(\mathbb{R}^k; L^r_\omega(\Sigma))^n$. It is well–known, see e.g. [3], that in this setting the Hörmander–Michlin condition is not sufficient to guarantee the $L^q(\mathbb{R}^k; L^r_\omega(\Sigma))$–continuity of the map

$$f \mapsto (\nabla' u, \nabla'' u) = \mathcal{F}^{-1} M_\alpha(\cdot)\mathcal{F} f.$$

4 The Weak Neumann Problem and the Helmholtz Decomposition

To deal with the Fourier multiplier operator $\mathcal{F}^{-1} M_\alpha(\cdot)\mathcal{F}$ we refer to a recent multiplier theorem of Štrkalj–Weis [26] and introduce the definition of $\mathcal{R}$–bounded operator families. In that definition $\{r_j(\cdot)\}$ will denote a sequence of independent, symmetric, $\{-1, 1\}$–valued random variables on $[0, 1]$, e.g. the Rademacher functions

$$r_j(s) = \text{sign} \sin(2^j \pi s), \quad j \in \mathbb{N}.$$

Definition Let X be a Banach space. A subset $\mathcal{T} \subset \mathcal{L}(X)$ is called $\mathcal{R}$–bounded if there exists a constant $C > 0$ and a $p \in [1, \infty)$ such that

$$\int_0^1 \|\sum_{j=1}^N r_j(s) T_j x_j\|^p ds \le C \int_0^1 \|\sum_{j=1}^N r_j(s) x_j\|^p ds$$

for all $T_1, \dots, T_N \in \mathcal{T}$, $x_1, \dots, x_N \in X$ and $N \in \mathbb{N}$. The smallest constant C in this inequality is called the $\mathcal{R}$–bound $\mathcal{R}(\mathcal{T})$ of $\mathcal{T}$.

Due to Kahane's inequality [6] the definition of $\mathcal{R}$–boundedness does *not* depend on the choice of the exponent $p \in [1, \infty)$. Then Khinchin's inequality [6] and Fubini's Theorem easily yield the following equivalent definition for Lebesgue spaces $X = L^r(\Sigma, \mu)$ using square function estimates. For further details see also [5].

Lemma 4.1 *Let $(\Sigma, \mathfrak{A}, \mu)$ be a measure space, $1 < r < \infty$ and $X = L^r(\Sigma, \mu)$. Then $\mathcal{T} \subset \mathcal{L}(X)$ is $\mathcal{R}$–bounded iff there exists a constant $C > 0$ such that*

$$\|\Big(\sum_{j=1}^N |T_j f_j(\cdot)|^2\Big)^{1/2}\| \le C \|\Big(\sum_{j=1}^N |f_j(\cdot)|^2\Big)^{1/2}\|$$

for all $T_1, \dots, T_N \in \mathcal{T}$, $f_1, \dots, f_N \in X$ and $N \in \mathbb{N}$.

To state the multiplier theorem of Štrkalj–Weis (Theorem 4.4 in [26], see also Theorem 3.7 in [15] and, for the one–dimensional case, [31]) we need the notion of *UMD*–spaces, see [2, 3]. A Banach space X is called a *UMD*–space iff the Hilbert transform is continuous for functions $f \in L^p(\mathbb{R}; X)$, $1 < p < \infty$. It is well-known that every Lebesgue space $X = L^r(\Sigma, \mu)$, $1 < r < \infty$, is *UMD*.

Theorem 4.2 *Let X be a UMD–space and let $\{M(\xi) : \xi \in \mathbb{R}^k_*\} \subset \mathcal{L}(X)$ be a k–times Fréchet differentiable operator family on X such that the sets*

$$\big\{|\xi|^\gamma \partial^\gamma M(\xi) : \xi \in \mathbb{R}^k_*\big\},\ \gamma \in \{0,1\}^k, \quad \textit{are } \mathcal{R}\textit{–bounded.}$$

Then the operator $\mathcal{F}^{-1}M(\cdot)\mathcal{F}$ defined on $C_0^\infty(\mathbb{R}^k; X)$ extends to a bounded linear operator on $L^p(\mathbb{R}^k; X)$ for $1 < p < \infty$. Furthermore there exists a constant $c > 0$ independent of $M(\cdot)$ such that

$$|||\mathcal{F}^{-1}M\mathcal{F}||| \le c \sum_{\gamma \in \{0,1\}^k} \mathcal{R}\big(\big\{|\xi|^\gamma \partial^\gamma M(\xi)\ : \xi \in \mathbb{R}^k_*\big\}\big).$$

We note that the above estimate of $|||\mathcal{F}^{-1}M\mathcal{F}|||$ is easily obtained when examining the proof in [26]. To apply Theorem 4.2 to the operator family $\{M_\alpha(\cdot)\}$ we need an important extrapolation property of operators on weighted function spaces, see [14], and its consequence concerning $\mathcal{R}$–boundedness [12].

Theorem 4.3 *Let $1 < r,\ s < \infty$, $\omega \in A_r$ and let $\Sigma \subset \mathbb{R}^{n-k}$ be an open set. Furthermore let $\mathcal{T} \subset \mathcal{L}(L^r_\omega(\Sigma))$ satisfy the estimate*

$$\|Tf\|_{s,\nu} \le C\|f\|_{s,\nu} \quad \textit{for all} \quad T \in \mathcal{T},$$

for all $f \in L^r_\omega(\Sigma) \cap L^s_\nu(\Sigma)$ and for every weight $\nu \in A_s$ with a constant $C = C(\mathcal{A}_s(\nu))$. Then $\mathcal{T}$ is $\mathcal{R}$–bounded on $\mathcal{L}(L^r_\omega(\Sigma))$.

This result easily extends to $\mathcal{T} \subset \mathcal{L}(L^r_\omega(\Sigma)^n)$.

Proof of Theorem 2.1 By Theorems 4.2, 4.3 and by Corollary 3.8 $\mathcal{F}^{-1}M_\alpha(\cdot)\mathcal{F}$ defines a bounded linear operator on $L^q(\mathbb{R}^k; L^r_\omega(\Sigma)^n)$ for every $1 < q, r < \infty$ and $\omega \in A_r$. Looking at (3.9) we solved the variational problem

$$\nabla u \cdot \nabla + 2\alpha \cdot \nabla'' u - \alpha^2 u = f \cdot \nabla - f''_\Sigma \cdot \nabla'' + \alpha \cdot (f'' - f''_\Sigma) \tag{4.1}$$

with $\nabla u \in L^q(\mathbb{R}^k; L^r_\omega(\Sigma))$ due to the L^q–continuity of $\mathcal{F}^{-1}M_\alpha\mathcal{F}$. Moreover $u \in L^q(\mathbb{R}^k; L^{r,0}_\omega(\Sigma))$ and by Poincaré's inequality

$$\|\nabla u,\, u\|_{q;r,\omega} \le c\|f\|_{q;r,\omega} \tag{4.2}$$

with a constant $c = c(\alpha) > 0$ independent of $f \in L^r_\omega(\Sigma)$.

To prove exponentially weighted estimates let $|\alpha| \le \beta < \sqrt{\alpha_1}$, $g \in C_0^\infty(\mathbb{R}^k; L^r_\omega(\Sigma)^n)$ and let v denote the solution of (4.1) with f replaced by g satisfying $\|\nabla v\|_{q;r,\omega} \le c\|g\|_{q;r,w}$. Then (4.1) may be rewritten as

$$\int_\Omega \nabla(ve^{-\alpha\cdot x''})\cdot\nabla(\varphi e^{\alpha\cdot x''})dx$$
$$= \int_\Omega \big((ge^{-\alpha\cdot x''})\cdot\nabla(\varphi e^{\alpha\cdot x''}) - (g''_\Sigma e^{-\alpha\cdot x''})\cdot\nabla''(\varphi e^{\alpha\cdot x''})\big)dx.$$

Thus $u = ve^{-\alpha\cdot x''}$ solves the Neumann problem $-\Delta u = f\cdot\nabla - f''_\Sigma\cdot\nabla''$ in Ω, $\nu\cdot(\nabla u - f) = 0$ on $\partial\Omega$, with the right–hand side defined by $f = ge^{-\alpha\cdot x''}$. Moreover, due to the estimate of v,

$$\|e^{\alpha\cdot x''}\nabla' u,\, e^{\alpha\cdot x''}\nabla'' u + \alpha e^{\alpha\cdot x''}u\|_{q;r,\omega} \le c\|e^{\alpha\cdot x''}f\|_{q;r,\omega}.$$

Since $v(\cdot, x'') \in L^{r,0}_\omega(\Sigma)$, Poincaré's inequality yields the estimate

$$\|\nabla u\|_{q,\alpha;r,\omega} \le c\|f\|_{q,\alpha;r,\omega}. \tag{4.3}$$

Due to the density of $C_0^\infty(\mathbb{R}^k; L^r_\omega(\Sigma))$ in $L^q_\alpha(L^r_\omega(\Sigma))$, these results extend to every $f \in L^q_\alpha(L^r_\omega(\Sigma)^n)$.

Since the solution constructed up to now solves the Neumann problem $\nabla u\cdot\nabla = f\cdot\nabla - f''_\Sigma\cdot\nabla''$ we still have to solve the equation

$$\nabla u\cdot\nabla = f''_\Sigma\cdot\nabla'' \tag{4.4}$$

and to find exponentially weighted estimates w.r.t. to x'', if possible. Since $f''_\Sigma = f''_\Sigma(x'')$, we find a solution of (4.4) by solving the Neumann problem

$$\nabla'' u\cdot\nabla'' = f''_\Sigma\cdot\nabla'' \quad \text{in} \quad \mathbb{R}^k.$$

Let E denote the fundamental solution of the Laplacian on $\mathbb{R}^k$. Then $u(x'') = E * \operatorname{div} f''_\Sigma(x'')$ solves (4.4) admitting the a priori estimate $||\nabla'' u||_q \le c||f''_\Sigma||_q$, $1 < q < \infty$. Since even for $f''_\Sigma \in C_0^\infty(\mathbb{R}^k)^k$ only $|\nabla'' u(x'')| \le c|x''|^{-k}$

can be guaranteed, in general there is no exponentially weighted estimate. But if f''_Σ is a potential field, i.e., $f''_\Sigma = \nabla'' h$ for a scalar–valued function h, then obviously $u = h$ yielding the a priori estimate

$$||\nabla'' u||_{q,\alpha} = ||\nabla'' h||_{q,\alpha} = ||f''_\Sigma||_{q,\alpha}.$$

In particular, the one–dimensional case $k = 1$ when Ω is an infinite cylinder always admits exponentially weighted estimates.

It remains to prove the uniqueness of solutions. Let $u \in \hat{H}^1_{q,\alpha;r,\omega}(\Omega)$ be a solution of the problem

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = 0 \quad \text{for all} \quad \varphi \in \hat{H}^1_{q',-\alpha;r',\omega'}(\Omega).$$

Looking at "test functions" φ not depending on $x' \in \Sigma$ we conclude that $\int_\Sigma u(x', x'')dx'$ is constant in $x'' \in \mathbb{R}^k$, say $\int_\Sigma u(x', \cdot)dx' \equiv 0$. Now the existence result proved just before for functionals $F \in \hat{H}^{-1}_{q',-\alpha;r',\omega'}(\Omega)$ easily implies that

$$\langle u, F \rangle = 0 \quad \text{for all} \quad F \in \hat{H}^{-1}_{q',-\alpha;r',\omega'}(\Omega).$$

Next choose $F \triangleq g(x', x'') \in C_0^\infty(\Omega)$ satisfying $\int_\Sigma g(x', \cdot)dx' \equiv 0$. The estimate

$$\begin{aligned} |\langle F, \psi \rangle| = |\int_\Omega g\psi dx| \ &\leq \ \int_{\mathbb{R}^k} \|g(\cdot, x'')\|_{r',\omega'} \|\nabla' \psi(\cdot, x'')\|_{r,\omega} dx'' \\ &\leq \ \|g\|_{q',-\alpha;r',\omega'} \|\nabla \psi\|_{q,\alpha;r,\omega} \end{aligned}$$

for $\psi \in H^1_{q,\alpha;r,\omega}(\Omega)$ shows that actually $F \in \hat{H}_{q,\alpha;r,\omega}(\Omega)$. Thus $\int_\Omega ug \, dx = 0$ for all $g \in C_0^\infty(\Omega)$ with $\int_\Sigma g(x', \cdot)dx' \equiv 0$. Since $\int_\Sigma u \equiv 0$, the restriction $\int_\Sigma g \equiv 0$ may be omitted, and a standard density argument yields $u \equiv 0$.

Assume that $|\langle F, \varphi \rangle| \leq c(F) \|e^{\alpha|x_n|} \nabla \varphi\|_{q';r'\omega'}$ for some $\alpha \in (0, \sqrt{\alpha_1})$. Then there exists $f \in L^1_{loc}(\overline{\Omega})$ such that $\langle F, \varphi \rangle = \int_\Omega f \cdot \nabla \varphi \, dx$ for all $\varphi \in C_0^\infty(\overline{\Omega})$ and $\|e^{\alpha|x_n|} f\|_{q';r'\omega'} \leq c(F)$. In particular $f \in L^q_{\pm\alpha}(L^r_\omega)$ and the unique solution $u \in \hat{H}^1_{q;r,\omega}(\Omega)$ satisfies the estimate

$$\int_{\mathbb{R}} e^{\alpha|x_n|} \|\nabla u(\cdot, x_n)\|_{r,\omega} \, dx_n \leq c \int_{\mathbb{R}} (e^{\alpha x_n} + e^{-\alpha x_n}) \|f\|^q_{r,\omega} \, dx_n \leq c \, c(F).$$

Now Theorem 2.1 is completely proved. ∎

Proof of Theorem 2.2 (1) Given $u \in L^q_\alpha(L^r_\omega)^n$ let $p \in \hat{H}^1_{q,\alpha;r,\omega}(\Omega)$ denote the unique solution of the weak Neumann problem

$$(\nabla p, \nabla \varphi) = (u, \nabla \varphi) \quad \text{for all} \quad \varphi \in \hat{H}^1_{q',-\alpha;r',\omega'}(\Omega).$$

By Theorem 2.1 we know that $\|\nabla p\|_{q,\alpha;r,\omega} \leq c\|u\|_{q,\alpha;r,\omega}$. Then the Helmholtz projection $P = P_{q,\alpha;r,\omega}$ is defined by $u_0 = Pu = u - \nabla p$. Obviously P is a bounded linear projection on $L^q_\alpha(L^r_\omega)$ with kernel $\nabla \hat{H}^1_{q,\alpha;r,\omega}(\Omega)$. Moreover Proposition 2.8 immediately implies that the range of P equals $L^q_\alpha(L^r_\omega)_\sigma$. Finally the uniqueness assertion of Theorem 2.1 yields the uniqueness of the Helmholtz decomposition.

(2) Using (1) standard duality arguments prove the assertion on P^* and $L^q_\alpha(L^r_\omega)^*$, see [8, 13, 16]. ■

By the previous analysis there are exponentially weighted estimates w.r.t. $x'' \in \mathbb{R}^k$ for an infinite layer $\Omega = \Sigma \times \mathbb{R}$ without any further restrictions. In the final theorem we summarize the results for arbitrary domains $\Omega = \Sigma \times \mathbb{R}^k$, $2 \leq k \leq n-1$. It is convenient to describe the result on the weak Neumann problem by using functions f and not functionals F.

Theorem 4.4 *Let $\Sigma \subset \mathbb{R}^{n-k}$, $n \geq 3$, $2 \leq k \leq n-1$, be a bounded domain with C^1–boundary, let $1 < q, r < \infty$, $\omega \in A_r(\mathbb{R}^{n-k})$ and $\alpha \in \mathbb{R}^k$, $|\alpha| < \sqrt{\alpha_1}$.*

(1) Assume that for $f \in L^q_\alpha(L^r_\omega)^n$ the k–dimensional field

$$f''_\Sigma(x'') = \frac{1}{|\Sigma|}\int_\Sigma f''(x', x'')\, dx'$$

is a potential field. Then the weak Neumann problem

$$\nabla u \cdot \nabla = f \cdot \nabla$$

has a unique solution $u \in \hat{H}^1_{q,\alpha;r,\omega}(\Omega)$ satisfying the estimate

$$\|\nabla u\|_{q,\alpha;r,\omega} \leq c\|f\|_{q,\alpha;r,\omega}.$$

(2) Assume that for $u \in L^q_\alpha(L^r_\omega)$ the k–dimensional field $u''_\Sigma = \frac{1}{|\Sigma|}\int_\Sigma u''(x', x'')\, dx'$ is a potential field. Then u admits a unique Helmholtz decomposition

$$u = u_0 + \nabla p \quad \textit{in} \quad L^q_\alpha(L^r_\omega)^n$$

and the exponentially weighted estimate

$$\|u_0, \nabla p\|_{q,\alpha;r,\omega} \le c\|u\|_{q,\alpha;r,\omega}$$

with a constant $c > 0$ *independent of* u.

(3) *If in (1) or (2)* $\|e^{\alpha|x''|}f\|_{q;r,\omega} < \infty$ *or* $\|e^{\alpha|x''|}u\|_{q;r,\omega} < \infty$, *resp., for some* $\alpha \in (0, \sqrt{\alpha_1})$, *then even*

$$\Big\|\frac{e^{\alpha|x''|}}{(1+\alpha|x''|)^{(k-1)/(2q)}}\nabla u\Big\|_{q;r,\omega} \le c\|e^{\alpha|x''|}f\|_{q;r,\omega} \tag{4.5}$$

or

$$\Big\|\frac{e^{\alpha|x''|}}{(1+\alpha|x''|)^{(k-1)/(2q)}}(u_0, \nabla p)\Big\|_{q;r,\omega} \le c\|e^{\alpha|x''|}u\|_{q;r,\omega}. \tag{4.6}$$

Proof It remains to prove (4.5). Since $f \in L^q_{\gamma\alpha}(L^r_\omega)$ for some $\alpha \in (0, \sqrt{\alpha_1})$ and for every $\gamma \in \mathbb{R}^k$, $|\gamma| = 1$, for short $\gamma \in S^{k-1}$, by (1) and Fubini's theorem

$$\begin{aligned}&\int_{S^{k-1}}\int_{\mathbb{R}^k} e^{q\alpha\gamma\cdot x''}\|\nabla u(\cdot, x'')\|^q_{r,\omega}\, dx''d\gamma\\ \le\;& c\int_{\mathbb{R}^k}\int_{S^{k-1}} e^{q\alpha\gamma\cdot x''}\|f(\cdot, x'')\|^q_{r,\omega}\, dx''d\gamma \le c\|e^{\alpha|x''|}f\|^q_{q;r,\omega}.\end{aligned}$$

To get a lower bound of the left–hand side we use for fixed $x'' \in \mathbb{R}^k$ with $\alpha|x''| \le 1$ the elementary estimate $\int_{S^{k-1}} e^{q\alpha\gamma\cdot x''}\, d\gamma \ge ce^{\alpha|x''|}$. For $\alpha|x''| > 1$ use polar coordinates on S^{k-1} to get

$$\begin{aligned}&\int_{S^{k-1}} e^{q\alpha\gamma\cdot x''}\, d\gamma \ge c\int_0^\pi (\sin\theta)^{k-2}e^{q\alpha|x''|\cos\theta}\, d\theta\\ \ge\;& ce^{q\alpha|x''|}\int_0^{\pi/2}\theta^{k-2}e^{-q\alpha|x''|\theta^2/2}\, d\theta \ge ce^{q\alpha|x''|}(\alpha|x''|)^{-(k-1)/2}.\end{aligned}$$

Thus the inequality

$$\int_{S^{k-1}}\int_{\mathbb{R}^k} e^{q\alpha\gamma\cdot x''}\|\nabla u(\cdot, x'')\|^q_{r,\omega}\, dx''d\gamma \ge c\int_{\mathbb{R}^k}\frac{e^{q\alpha|x''|}}{(1+\alpha|x''|)^{(k-1)/2}}\|\nabla u\|^q_{r,\omega}\, dx''$$

proves (4.5). ■

We note that the estimates (4.5) and (4.6) can be improved concerning the denominator $(1+\alpha|x''|)^{(k-1)/2}$ in the L^2–case for small α and for $q \ne 2$ by using e.g. interpolation theory.

References

[1] T. Abe, Y. Shibata: On a generalized resolvent estimate of the Stokes equation on an infinite layer. Part 1: $|\lambda| > 0$ Case. Part 2: $|\lambda| = 0$ Case. Preprint Waseda University 2000

[2] H. Amann: Linear and Quasilinear Parabolic Problems, Vol. I. Abstract Linear Theory. Monographs in Mathematics 89, Birkhäuser, Basel 1995

[3] H. Amann: Operator–valued Fourier multipliers, vector—valued Besov spaces and applications. Math. Nachr. 186, 5–56 (1997)

[4] S.-K. Chua: Extension theorems for weighted Sobolev spaces. Indiana Univ. Math. J. 41, 1027–1076 (1992)

[5] Ph. Clément, B. dePagter, F.A. Sukochev, H. Witvliet: Schauder decompositions and multiplier theorems. Studia Math. 138, 135–163 (2000)

[6] J. Diestel, H. Jarchow, A. Tonge: Absolutely Summing Operators. Cambridge University Press 1995

[7] R. Farwig: The weak Neumann problem and the Helmholtz decomposition in general aperture domains. Progress in partial differential equations: the Metz surveys 2, Ed. M. Chipot. Pitman Research Notes Math. Seriies 296, 86–96 (1993)

[8] R. Farwig, H. Sohr: Helmholtz decomposition and Stokes resolvent system for aperture domains in L^q–spaces. Analysis 16, 1–26 (1996)

[9] R. Farwig, H. Sohr: Weighted L^q–theory for the Stokes resolvent in exterior domains. J. Math. Soc. Japan 49, 251–288 (1997)

[10] A. Fröhlich: The Helmholtz decomposition of weighted L^q–spaces for Muckenhoupt weights. Ann. Univ. Ferrara – Sez. VII–Sc. Mat. 46, 11–19 (2000)

[11] A. Fröhlich: The Stokes operator in weighted L^q–spaces I: Weighted estimates for the Stokes resolvent problem in a half space. Dept. of Math., Darmstadt Univ. of Technology, Preprint no. 2172 (2001). To appear in: J. Math. Fluid Mech.

[12] A. Fröhlich: The Stokes operator in weighted L^q–spaces II: Weighted resolvent estimates and maximal L^p–regularity. Dept. of Math., Darmstadt Univ. of Technology, Preprint no. 2173 (2001). To appear in: Math. Ann.

[13] D. Fujiwara, H. Morimoto: An L_r–theorem of the Helmholtz decomposition of vector fields. J. Fac. Sci. Univ. Tokyo, Sec. I, 24, 685–700 (1977)

[14] J. Garcia-Cuerva, J.L. Rubio de Francia: Weighted norm inequalities and related topics. North Holland, Amsterdam 1985

[15] R. Haller, H. Heck, A. Noll: Miklin's theorem for operator–valued Fourier multipliers in n variables. Dept. of Math., Darmstadt Univ. of Technology, Preprint no. 2123 (2000). To appear in: Math. Nachr.

[16] T. Miyakawa: On nonstationary solutions of the Navier–Stokes equations in an exterior domain. Hiroshima Math. J. 12, 115–140 (1982)

[17] T. Miyakawa: The Helmholtz decomposition of vector fields in some unbounded domains. Math. J. Toyama Univ. 17, 115–149 (1994)

[18] A.S. Nazarov: Asymptotics of the solution to a boundary value problem in a thin cylinder with nonsmooth lateral surface. Izv. Ross. Akad. Nauk. Ser. Mat. 57, 202–239 (1993). English Transl.: Math. Izvestiya 42, 183–217 (1994)

[19] S.A. Nazarov, K.I. Pileckas: The Fredholm property of the Neumann problem operator in the domains that are layer–like at infinity. Algebra i Analiz 8, 57–104 (1996). English Transl.: St. Petersburg Math. J. 8, 951–983 (1997)

[20] S.A. Nazarov, B.A. Plamenevskii: Asymptotics of the spectrum of the Neumann problem in singularly perturbed thin domains. Algebra i Analiz 2, 85–111 (1990). English Transl.: Leningrad Math. J. 2, 287–311 (1991)

[21] C. Simander, H. Sohr: A new approach to the Helmholtz decomposition and the Neumann problem in L^q–spaces for bounded and unbounded domains. Mathematical Problems Relating to Navier-Stokes Equations, Ed. G.P. Galdi. Singapore World Scientific Publ. 1992

[22] C. Simader, T. Ziegler: The weak Dirichlet and Neumann problems in L^q for the Laplacian and the Helmholtz decomposition in infinite cylinders and layers. Recent Developments in Partial Differential Equations, Ed. V.A. Solonnikov. quaderni di matematica 2, 37–162 (1998)

[23] H. Sohr: The Navier-Stokes Equations. An Elementary Functional Analytic Approach. Birkhäuser Advanced Texts, Basel 2001

[24] H. Sohr, G. Thäter: Imaginary powers of second order differential operators and L^q–Helmholtz decomposition in the infinite cylinder. Math. Ann. 311, 577–602 (1998)

[25] V.A. Solonnikov: Estimates for solutions of nonstationary Navier–Stokes equations. J. Soviet Math. 8, 467–529 (1977)

[26] Z. Štrkalj, L. Weis: On operator–valued Fourier multiplier theorems. Fakultät für Mathematik, Univ. Karlsruhe, Preprint no. 36 (2000)

[27] G. Thäter: Neumann problem in domains with outlets of bounded diameter. Univ. Bonn, SFB 256, Preprint no. 686 (2000). To appear in: Acta Appl. Math. 2002

[28] G. Thäter: Neumann problem in cylinders and Sobolev spaces. Univ. Bonn, SFB 256, Prprint no. 687 (2000). To appear in: Math. Meth. Appl. Sci. 2002

[29] W. von Wahl: The Equations of Navier-Stokes and Abstract Parabolic Equations. Vieweg Braunschweig 1985

[30] W. von Wahl: Abschätzungen für das Neumann–Problem und die Helmholtz–Zerlegung von L^p. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, (1990) Nr. 2

[31] L. Weis: Operator-valued Fourier multiplier theorems and maximal L_p–regularity. Math. Ann. 319, 735–758 (2001)

[32] M. Wiegner: Resolvent estimates for the Stokes operator on an infinite layer. Unpublished manuscript 1995