Central extensions of current groups

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Abstract. In this paper we study central extensions of the identity component G of the Lie group $C^\infty(M,K)$ of smooth maps from a compact manifold M into a Lie group K which might be infinite-dimensional. We restrict our attention to Lie algebra cocycles of the form $\omega(\xi,\eta)=[\kappa(\xi,d\eta)]$, where $\kappa\colon\mathfrak{k}\times\mathfrak{k}\to Y$ is a symmetric invariant bilinear map on the Lie algebra $\mathfrak{k}$ of K and the values of ω lie in $\Omega^1(M,Y)/dC^\infty(M,Y)$. For such cocycles we show that a corresponding central Lie group extension exists if and only if this is the case for $M=\mathbb{S}^1$. If K is finite-dimensional semisimple, this implies the existence of a universal central Lie group extension $\widehat{G}$ of G. The groups $\mathrm{Diff}(M)$ and $C^\infty(M,K)$ act naturally on G by automorphisms. We also show that these smooth actions can be lifted to smooth actions on the central extension $\widehat{G}$ if it also is a central extension of the universal covering group $\widetilde{G}$ of G.

Introduction

Let M be a compact manifold and K a Lie group (which may be infinite-dimensional). Then the so called current groups $C^\infty(M,K)$ with pointwise multiplication are interesting infinite-dimensional Lie groups arising in many circumstances. The most studied class of such groups are the loop groups ($M=\mathbb{S}^1$ and K compact) which is completely covered by Pressley and Segal's monograph [PS86]. The goal of this paper is a systematic understanding of a certain class of central extensions of the identity components of these groups, namely those whose Lie algebra cocycle is of product type, which is defined in more detail below. Here the main point is to see which Lie algebra cocycle can be integrated to a central Lie group extension. These central extensions occur naturally in mathematical physics, where the problem to integrate projective representations of groups to representations of central extensions is at the heart of quantum mechanics ([Mi87], [LMNS98], [Wu01]). The central extensions of current groups are often constructed via representatations by pulling back central extensions of certain operator groups ([Mi89]). It is our philosophy that one should try to understand the central extensions of a Lie group G first, and then try to construct representations of these central extensions. In this context certain discreteness conditions for Lie algebra cocycles appear naturally because they ensure that the corresponding central Lie algebra extensions integrate to group representations ([Ne02b]). We think of these discreteness conditions as an abstract version of the phenomenon of discreteness of quantum numbers in quantum physics. As an outcome of our analysis, we will see that we do not have to impose any conditions on the group K for our general results.

We now describe our results in some more detail. Let M be a compact manifold, Y a sequentially complete locally convex space, $\Omega^p(M,Y)$ the space of smooth Y-valued p-forms on M, and $\mathfrak{z}_M(Y)=\Omega^1(M,Y)/dC^\infty(M,Y)$. Then $\mathfrak{z}_M(Y)$ carries a natural locally convex topology and if Y is Fréchet, then the same holds for $\mathfrak{z}_M(Y)$. Now let K be a possibly infinite-dimensional connected Lie group and $\mathfrak{k}$ its Lie algebra. We associate to each invariant continuous bilinear form $\kappa\colon\mathfrak{k}\times\mathfrak{k}\to Y$ a continuous Lie algebra cocycle on $\mathfrak{g}:=C^\infty(M,\mathfrak{k})$ by $\omega(\xi,\eta):=[\kappa(\xi,d\eta)]\in\mathfrak{z}_M(Y)$. We call such cocycles *of product type*. The main objective of this paper is to understand central Lie group extensions of the identity component $G:=C^\infty(M,K)_e$ of the Lie group $C^\infty(M,K)$ corresponding to the Lie algebra cocycle ω. According to the results in [Ne02b, Sect. VII], there are two obstructions for the existence of a central Lie group extension $\widehat{G}$ of G corresponding to ω. First the image of the associated period map $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}_M(Y)$ may not be discrete, and second, the adjoint action of $\mathfrak{g}$ on the Lie algebra $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z}_M(Y)$ does not integrate to a smooth representation of G. The main point in the choice of this general setting is that it permits us to use arbitrary infinite-dimensional Lie groups K, hence in particular groups of the type $K = C^\infty(N,H)$, H a finite-dimensional Lie group. Then $C^\infty(M,K) \cong C^\infty(M \times N, H)$, so that we may use product decompositions of manifolds to study current groups on manifolds.

In the first section we investigate the discreteness of the period group $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega)$. Our main result states that Π_ω is discrete for all compact manifolds M if and only if it is discrete for the manifold $M = \mathbb{S}^1$. This is remarkable because the group $\pi_2(G)$ is not well accessible for $\dim M > 2$. In Section II we turn to the case where K is finite-dimensional and $\kappa\colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is the universal invariant symmetric bilinear form on $\mathfrak{k}$. In this case we show that the period group is discrete for $M = \mathbb{S}^1$, hence also for arbitrary M by the results of Section I.

In Section III we turn to the central Lie group extensions. Here we show in particular that for any Lie algebra cocycle ω of product type the adjoint representation of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}$ integrates to a smooth Lie group representation of the generally non-connected group $C^\infty(M,K)$. Therefore the second obstruction to the existence of a central Lie group extension is always trivial, and we obtain for each κ for which the period group Π_ω is discrete a central Lie group extension of the identity component $G = C^\infty(M,K)_e$. In Section IV we show that if K is finite-dimensional and semisimple, then we even obtain a universal central Lie group extension of G by the abelian group $\pi_1(G) \times (\mathfrak{z}_M(V(\mathfrak{k}))/\Pi_\omega)$.

Because of its relevance for the construction of representations of $\mathrm{Diff}(M)$ and abelian extensions of this group, it is interesting to know to which extent the Lie group $\mathrm{Diff}(M)$ acts on the central extensions of G. It obviously acts on G itself by composition $\varphi.f := f \circ \varphi^{-1}$ for $f \in G$, $\varphi \in \mathrm{Diff}(M)$. Suppose that $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is a central Lie group extension corresponding to a cocycle of product type and that $\widehat{G}$ also is a central extension of the universal covering group $\widetilde{G}$ of G, which means that the connecting homomorphism $\pi_1(G) \to \pi_0(Z)$ is an isomorphism. Then we show in Section VI that the action of $\mathrm{Diff}(M)$ has a unique lift to an action on $\widehat{G}$. This result is based on general results in Section V which are concerned with lifting automorphic Lie group actions $R \times G \to G$ to actions of R on central extensions $\widehat{G}$ of G by Z. We show that if G is simply connected, a pair of smooth actions of R on G and Z can be lifted to a smooth action of R on $\widehat{G}$ whenever there is a smooth action of R on the Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ extending the actions on $\mathfrak{g}$ and $\mathfrak{z}$.

The universal central extension $\widehat{G}$ of the universal covering group $\widetilde{G}$ of $G = C^\infty(M,K)_e$, K a simple compact Lie group, appears in [PS86] for the first time, although no rigorous argument for its existence is given there. As we will see in Section III, the group $\pi_2(\widehat{G})$ is not always trivial, contradicting a corresponding statement in [PS86]. The construction of a central extension of the group G, instead of its universal covering group, seems to be new (see [LMNS98] for a construction for which it is not clear to the authors that it produces a Lie group). It is clear that this point of view has the advantage that the group G itself has a concrete realization, which need not be the case for its universal covering group.

As is well known from the case of loop groups ([PS86]) and the case of current groups over Riemann surfaces ([EF94]), the coadjoint orbits of the central extensions $\widehat{G}$ have fascinating geometric interpretations. In [VN02] we address various aspects of the geometry of coadjoint orbits for general M.

It is also interesting to study "algebraic" relatives of the central extensions of current groups arising in this paper. In [Shi92] Shi constructs so-called toroidal groups associated to the universal central extension $\widehat{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g} := \mathbb{C}[t^\pm, s^\pm] \otimes \mathfrak{k}$, where $\mathfrak{k}$ is a simple complex Lie algebra. These groups are defined as groups generated by root groups in such a way that they act in all integrable representations of $\widehat{\mathfrak{g}}$. He also makes a connection to Steinberg groups of the algebra $\mathbb{C}[t^\pm, s^\pm]$ of Laurent polynomials. It would be interesting to understand the precise relationship between these groups and the universal central Lie group extension of $C^\infty(\mathbb{T}^2, K)_e$. For $M = \mathbb{T}^d$, the d-dimensional torus, we think of our central extensions $\widehat{G}$, or the corresponding semidirect product groups $\widehat{G} \rtimes \mathbb{T}^d$, as natural Lie group versions of toroidal groups. The Lie algebras of these groups and their representations have been studied intensively in recent years (see f.i. [CF01], [Tan99], [Pi00], [BB99]). In [Ta98] Takebayashi approaches the problem to find groups for the Lie algebra $\widehat{\mathfrak{g}}$, or rather for $\mathfrak{g}$ in his context, by using a Chevalley basis of $\mathfrak{k}$ to construct a group corresponding to $\mathfrak{g}$ as an algebraic group over the algebra $\mathbb{C}[t^{\pm}, s^{\pm}]$ via the Chevalley-Demazure construction. He also examines the structure of the "elementary subgroup" generated by all root groups, which is a quotient of the group constructed by Shi.

This paper contributes to a larger program dealing with Lie groups G whose Lie algebras $\mathfrak{g}$ are *root graded* in the sense that there exists a finite irreducible root system Δ such that $\mathfrak{g}$ has a Δ-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, it contains the split simple Lie algebra $\mathfrak{k}$ corresponding to Δ as a graded subalgebra, and is generated, topologically, by the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Delta$. All Lie groups of the type $C^\infty(M,K)$, M compact and K simple complex, are of this type, and the same holds for their central extension. A different but related class of groups arising in this context are the Lie groups $\mathrm{SL}_n(A)$ and their central extensions, where A is a continuous inverse algebra, i.e., a locally convex unital associative algebra with open unit group and continuous inversion ([Gl01c]). In [Ne01c] we discuss the universal central extensions of the groups $\mathrm{SL}_n(A)$, which are Lie group versions of the Steinberg groups $\mathrm{St}_n(A)$. In the end of Section II we show that for $K = \mathrm{SL}_n(A)$, A a commutative continuous inverse algebra, we have $V(\mathfrak{k}) \cong A$ with $\kappa(x,y) = tr(xy)$ and that the image of the corresponding period map is discrete for the corresponding product type cocycle on the Lie algebra $C^\infty(M,\mathfrak{k})$ of the group $C^\infty(M,K)$. For non-commutative algebras the image of the period map is not always discrete.

The present results on current groups over compact manifolds are extended in [Ne02c] to current groups over non-compact manifolds in several settings. It is remarkable that the difficulties arising in this context are of a completely different nature as those in this paper because they are mainly caused by the possible infinite dimension of the first cohomology of M.

I. The period map

Definition I.1. For a finite-dimensional manifold M (for this definition we do not have to assume that M is compact) and a sequentially complete locally convex (s.c.l.c.) space Y we define

$$\mathfrak{z}_M(Y) := \Omega^1(M,Y)/d\Omega^0(M,Y)$$

and observe that the image of the space of closed forms in $\mathfrak{z}_M(Y)$ is the subspace $H^1_{\mathrm{dR}}(M,Y)$.

We endow $\Omega^1(M,Y)$ with the natural topology given by locally uniform convergence of all derivatives. Then we obtain for each $\alpha \in C^\infty(\mathbb{S}^1,M)$ a continuous linear map $\Omega^1(M,Y) \to Y$ by integration over α. Since the space $d\Omega^0(M,Y)$ of all exact 1-forms coincides with the annihilator of these functionals, it is a closed subspace, and we thus obtain on $\mathfrak{z}_M(Y)$ a natural locally convex Hausdorff topology and continuous linear maps given by

$$\alpha_{\mathfrak{z}}\colon \mathfrak{z}_M(Y) \to Y, \quad [\beta] \mapsto \int_\alpha \beta. \qquad \blacksquare$$

In the following we write $\mathrm{Lin}(E,F)$ for the space of continuous linear maps between topological vector spaces E and F.

Remark I.2. (a) Since an element $\beta \in \Omega^1(M,Y)$ is an exact form if and only if all integrals $\int_\alpha \beta$, $\alpha \in C^\infty(\mathbb{S}^1,M)$, vanish, the linear functions $\alpha_{\mathfrak{z}} \in \mathrm{Lin}(\mathfrak{z}_M(Y),Y)$ separate the points of $\mathfrak{z}_M(Y)$.
(b) A 1-form $\beta \in \Omega^1(M,Y)$ is closed if and only if for all pairs of homotopic paths α_1, α_2 the integrals of β over α_1 and α_2 coincide. Therefore the subspace $H^1_{\mathrm{dR}}(M,Y) \subseteq \mathfrak{z}_M(Y)$ is the annihilator of the functionals $\alpha_{1,\mathfrak{z}} - \alpha_{2,\mathfrak{z}}$, $[\alpha_1] = [\alpha_2]$ in $\pi_1(M)$, which implies in particular that it is closed. Moreover, for $[\beta] \in \mathfrak{z}_M(Y)$ the condition $[\beta] \in H^1_{\mathrm{dR}}(M,Y)$ is equivalent to the independence of $\alpha_{\mathfrak{z}}([\beta])$ from the homotopy class of α.
(c) For $M = \mathbb{S}^1$ we have $\mathfrak{z}_{\mathbb{S}^1}(Y) \cong Y$ because the map $\Omega^1(M,Y) \to Y, \beta \mapsto \int_{\mathbb{S}^1} \beta$ is surjective with kernel $d\Omega^0(M,Y)$. We identify the class of $\beta \in \Omega^1(\mathbb{S}^1,Y)$ in $\mathfrak{z}_{\mathbb{S}^1}(Y)$ with the integral $\int_{\mathbb{S}^1} \beta$.

(d) On the subspace $H^1_{\rm dR}(M,Y)$ we can define continuous linear maps by integration over continuous loops because we may use the isomorphism

$$H^1_{\rm dR}(M,Y) \cong H^1_{\rm sing}(M,Y) \cong \mathrm{Hom}(\pi_1(M),Y). \qquad \blacksquare$$

From now on we assume M to be compact.

The following remark will be helpful for the calculation of period groups.

Remark I.3. For every compact connected smooth manifold M the group $\pi_1(M)$ is finitely generated (M can be triangulated), which is inherited by the singular homology group $H_1(M) \cong \pi_1(M)/(\pi_1(M),\pi_1(M))$ (Hurewicz). Let $k := b_1(M) := \mathrm{rank}\, H_1(M)$ and fix $\alpha_1,\ldots,\alpha_k \in C(\mathbb{S}^1,M)$ such that the corresponding 1-cycles $[\alpha_j]$ form a basis of the free abelian group $H_1(M)/\mathrm{tor}(H_1(M))$.

Since $H_0(M)$ is a free abelian group, the Universal Coefficient Theorem implies that

$$H^1_{\rm sing}(M,\mathbb{Z}) \cong \mathrm{Hom}(H_1(M),\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{Z})).$$

Moreover, in view of Huber's Theorem ([Hu61]) and the local contractibility of M, this group is isomorphic to

$$\check{H}^1(M,\mathbb{Z}) \cong [M,\mathbb{S}^1].$$

In particular there exist continuous functions $f_1,\ldots,f_k\colon M \to \mathbb{S}^1$ such that $[f_j \circ \alpha_i] = \delta_{ij} \in \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. Since every homotopy class in $[M,\mathbb{S}^1]$ contains a smooth function ([Ne02b, Th. A.3.7]), we will assume in the following that the functions f_j are smooth. This implies in particular that its logarithmic derivative $\delta(f_j) := f_j^{-1}.df_j$ can be viewed as a closed 1-form on M, which is not exact because $\int_{\alpha_j} \delta(f_j) = 1$.

With the basis $[\alpha_j]$ of the group $H_1(M)/\mathrm{tor}\, H_1(M)$, we immediately obtain an isomorphism

$$\Phi\colon H^1_{\rm dR}(M,Y) \cong \mathrm{Hom}(H_1(M),Y) \cong \mathrm{Hom}(H_1(M)/\mathrm{tor}\, H_1(M),Y) \to Y^k, \quad [\beta] \mapsto \Big(\int_{\alpha_j} \beta\Big)_{j=1,\ldots,k}$$

whose continuous inverse is given by

$$\Phi^{-1}(y_1,\ldots,y_k) = \Big[\sum_{j=1}^k \delta(f_j)\cdot y_j\Big]. \qquad \blacksquare$$

Definition I.4. (The topology on $C^\infty(M,K)$) (a) If K is a Lie group and X is a compact space, then $C(X,K)$, endowed with the topology of uniform convergence is a Lie group with Lie algebra $C(X,\mathfrak{k})$ ([Ne02b, App. A.3]).

(b) If K is a Lie group with Lie algebra $\mathfrak{k}$, then the tangent bundle of K is a Lie group isomorphic to $\mathfrak{k} \rtimes K$, where K acts by the adjoint representation on $\mathfrak{k}$ (cf. [Ne01b]). Iterating this procedure, we obtain a Lie group structure on all higher tangent bundles T^nK which are diffeomorphic to $\mathfrak{k}^{2^n-1} \times K$.

For each $n \in \mathbb{N}_0$ we obtain topological groups $C(T^nM,T^nK)$ by using the topology of uniform convergence on compact subsets. Therefore the inclusion

$$C^\infty(M,K) \hookrightarrow \prod_{n\in\mathbb{N}_0} C(T^nM,T^nK)$$

leads to a natural topology on $C^\infty(M,K)$ turning it into a topological group. For compact manifolds M these groups can even be turned into Lie groups with Lie algebra $C^\infty(M,\mathfrak{k})$. Here $C^\infty(M,\mathfrak{k})$ is endowed with the topology defined above if we consider $\mathfrak{k}$ as an additive Lie group. For details we refer to [Gl01b]. $\blacksquare$

Definition I.5. (a) Let $\mathfrak{z}$ be a topological vector space and $\mathfrak{g}$ a topological Lie algebra. A *continuous* $\mathfrak{z}$*-valued* 2*-cocycle* is a continuous skew-symmetric bilinear function $\omega\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{z}$ satisfying

$$\omega([x,y],z)+\omega([y,z],x)+\omega([z,x],y)=0.$$

It is called a *coboundary* if there exists a continuous linear map $\alpha\in\mathrm{Lin}(\mathfrak{g},\mathfrak{z})$ with $\omega(x,y)=\alpha([x,y])$ for all $x,y\in\mathfrak{g}$. We write $Z^2_c(\mathfrak{g},\mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles and $B^2_c(\mathfrak{g},\mathfrak{z})$ for the subspace of coboundaries defined by continuous linear maps. We define the *second continuous Lie algebra cohomology space* to be

$$H^2_c(\mathfrak{g},\mathfrak{z}):=Z^2_c(\mathfrak{g},\mathfrak{z})/B^2_c(\mathfrak{g},\mathfrak{z}).$$

(b) If ω is a continuous $\mathfrak{z}$-valued cocycle on $\mathfrak{g}$, then we write $\mathfrak{g}\oplus_\omega\mathfrak{z}$ for the topological Lie algebra whose underlying topological vector space is the product space $\mathfrak{g}\times\mathfrak{z}$, and the bracket is defined by

$$[(x,z),(x',z')]=\big([x,x'],\omega(x,x')\big).$$

Then $q\colon\mathfrak{g}\oplus_\omega\mathfrak{z}\to\mathfrak{g},(x,z)\mapsto x$ is a central extension and $\sigma\colon\mathfrak{g}\to\mathfrak{g}\oplus_\omega\mathfrak{z}, x\mapsto(x,0)$ is a continuous linear section of q. ■

Let K be a Lie group and $\mathfrak{k}$ its Lie algebra. Further let $G:=C^\infty(M,K)_e$ denote the identity component of the Lie group $C^\infty(M,K)$ with Lie algebra $\mathfrak{g}=C^\infty(M,\mathfrak{k})$. We consider a continuous invariant symmetric bilinear map $\kappa\colon\mathfrak{k}\times\mathfrak{k}\to Y$. We thus obtain a continuous $\mathfrak{z}_M(Y)$-valued cocycle on $\mathfrak{g}$ by

$$\omega_M(\xi,\eta):=\omega_{M,\kappa}(\xi,\eta):=[\kappa(\xi,d\eta)]\in\mathfrak{z}_M(Y), \tag{1.1}$$

where we view $\kappa(\xi,d\eta)$ as the element of $\Omega^1(M,Y)$ whose value in a tangent vector $v\in T_p(M)$ is given by $\kappa(\xi(p),d\eta(p)(v))$. We write Ω_M for the left invariant $\mathfrak{z}_M(Y)$-valued 2-form on G with $\Omega_M(e)=\omega_M$.

In this first section we will discuss the image of the period homomorphism

$$\mathrm{per}_{\omega_M}\colon\pi_2(G)\to\mathfrak{z}_M(Y)$$

which is defined on piecewise smooth maps (with respect to a triangulation) $\sigma\colon\mathbb{S}^2\to G$ by

$$\mathrm{per}_{\omega_M}([\sigma]):=\int_\sigma\Omega_M$$

([Ne02b, Sect. V]). We also recall from [Ne02b, Th. A.3.7] that each homotopy class in $\pi_2(G)$ has smooth representatives and that the integration formula defines a group homomorphism per_{ω_M}. In particular we are interested in whether or not the *period group*

$$\Pi_{M,\kappa}:=\mathrm{im}(\mathrm{per}_{\omega_{M,\kappa}})$$

is a discrete subgroup of $\mathfrak{z}_M(Y)$.

The following theorem is the key result of this section.

Theorem I.6. (Reduction Theorem) *The period group* $\Pi_{M,\kappa}$ *is contained in the subspace* $H^1_{\mathrm{dR}}(M,Y)$ *of* $\mathfrak{z}_M(Y)$. *Identifying* $H^1_{\mathrm{dR}}(M,Y)$ *with* Y^k *via the map* Φ, *where* $k:=b_1(M):=\dim H^1_{\mathrm{dR}}(M,\mathbb{R})$ *is the first Betti number of* M, *we have*

$$\Pi_{M,\kappa}\cong\Pi^k_{\mathbb{S}^1,\kappa}\subseteq Y^k\cong H^1_{\mathrm{dR}}(M,Y)\subseteq\mathfrak{z}_M(Y).$$

In particular $\Pi_{M,\kappa}$ *is discrete if and only if* $\Pi_{\mathbb{S}^1,\kappa}$ *is discrete.* ■

For the proof we need several lemmas. Since the linear maps $\alpha_\mathfrak{z}$ on $\mathfrak{z}_M$ separate points (Remark I.2), it is crucial to get a better description of the compositions $\alpha_\mathfrak{z}\circ\mathrm{per}_{\omega_M}$.

Lemma I.7. *For each $\alpha \in C^\infty(\mathbb{S}^1, M)$ we have*

$$\alpha_{\mathfrak{z}} \circ \mathrm{per}_{\omega_M} = \mathrm{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_K), \tag{1.2}$$

where $\pi_2(\alpha_K)\colon \pi_2(G) \to \pi_2(C^\infty(\mathbb{S}^1, K))$ is the group homomorphism induced by the Lie group homomorphism $\alpha_K\colon G \to C^\infty(\mathbb{S}^1, K), f \mapsto f \circ \alpha$.

Proof. First we observe that $\alpha_{\mathfrak{z}} \circ \Omega_M$ is a Y-valued left invariant 2-form on G whose value in e is $\alpha_{\mathfrak{z}} \circ \omega_M$. Further $\alpha_K^* \Omega_{\mathbb{S}^1}$ is a left invariant 2-form on G whose value in e is given by

$$\begin{aligned}(\xi, \eta) &\mapsto \omega_{\mathbb{S}^1}(\xi \circ \alpha, \eta \circ \alpha) = [\kappa(\xi \circ \alpha, d(\eta \circ \alpha))] \\ &= [\kappa(\alpha^*\xi, \alpha^*(d\eta))] = \int_{\mathbb{S}^1} \kappa(\alpha^*\xi, \alpha^*(d\eta)) = \int_\alpha \kappa(\xi, d\eta) = \alpha_{\mathfrak{z}}\big(\omega_M(\xi, \eta)\big).\end{aligned}$$

This implies $\alpha_{\mathfrak{z}} \circ \Omega_M = \alpha_K^* \Omega_{\mathbb{S}^1}$, which in turn leads to (1.2). ■

Lemma I.8. *Let M_i, $i = 1, 2$, be two compact manifolds with base points x_{M_i} and $\alpha_{1,2}\colon M_1 \to M_2$ two smooth homotopic maps with $\alpha_j(x_{M_1}) = x_{M_2}$. Then the Lie group homomorphisms*

$$\alpha_{j,K}\colon C^\infty(M_2, K) \to C^\infty(M_1, K), \quad f \mapsto f \circ \alpha_j$$

satisfy $\pi_m(\alpha_{1,K}) = \pi_m(\alpha_{2,K})$ for each $m \in \mathbb{N}_0$.

Proof. Let $F\colon [1,2] \times M_1 \to M_2$ be a homotopy with $F_1 = \alpha_1$ and $F_2 = \alpha_2$. Then the map

$$\Phi\colon [1,2] \times C(M_2, K) \to C(M_1, K), \quad \Phi(t, f)(s) := f(F(t, s))$$

is continuous because the map

$$\widetilde{\Phi}\colon [1,2] \times C(M_2, K) \times M_1 \to K, \quad \widetilde{\Phi}(t, f, s) := f(F(t, s)) = \mathrm{ev}(f, F(t, s))$$

is continuous, which in turn follows from the continuity of the evaluation map

$$\mathrm{ev}\colon C(M_2, K) \times M_2 \to K.$$

We conclude that the two maps $\Phi_1, \Phi_2\colon C(M_2, K) \to C(M_1, K)$ are homotopic, hence induce the same homomorphisms $\pi_m(C(M_2, K)) \to \pi_m(C(M_1, K))$ for each $m \in \mathbb{N}_0$.

The restriction, resp., corestriction of these two maps to the subgroup $C^\infty(M_2, K)$ of smooth functions are the maps $\alpha_{1,K}$ and $\alpha_{2,K}$. Since the inclusions $C^\infty(M_j, K) \hookrightarrow C(M_j, K)$ is a homotopy equivalence ([Ne02b, Th. A.3.7]), the commutativity of the diagram

$$\begin{array}{ccc}\pi_m(C^\infty(M_2, K)) & \xrightarrow{\cong} & \pi_m(C(M_2, K)) \\ \big\downarrow{\scriptstyle \pi_m(\alpha_{j,K})} & & \big\downarrow{\scriptstyle \pi_m(\Phi_j)} \\ \pi_m(C^\infty(M_1, K)) & \xrightarrow{\cong} & \pi_m(C(M_1, K))\end{array}$$

implies $\pi_m(\alpha_{1,K}) = \pi_m(\alpha_{2,K})$ because of $\pi_m(\Phi_1) = \pi_m(\Phi_2)$. ■

Corollary I.9. $\Pi_{M,\kappa} \subseteq H^1_{\mathrm{dR}}(M, Y)$.

Proof. From (1.2) and Lemma I.8 we derive that for each $\alpha \in C^\infty(\mathbb{S}^1, M)$ the map $\alpha_{\mathfrak{z}} \circ \mathrm{per}_{\omega_M}$ only depends on the homotopy class of α, and therefore that $\mathrm{im}(\mathrm{per}_{\omega_M}) \subseteq H^1_{\mathrm{dR}}(M, Y)$ (Remark I.2(b)). ■

Lemma I.10. *Let $C^\infty_*(\mathbb{S}^1, K) := \{f \in C^\infty(\mathbb{S}^1, K) : f(1) = e\}$ denote the Lie group of based loops. For $h \in C^\infty(\mathbb{S}^1, \mathbb{S}^1)$ and $m \in \mathbb{N}_0$ the map*

$$\pi_m(h_K) : \pi_m(C^\infty_*(\mathbb{S}^1, K)) \to \pi_m(C^\infty_*(\mathbb{S}^1, K))$$

is given by

$$\pi_m(h_K)([\sigma]) = \deg(h) \cdot [\sigma],$$

where $\deg(h) = [h] \in \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ is the mapping degree of h.

Proof. We realize $\mathbb{S}^1$ as $\mathbb{R}/\mathbb{Z}$, so that continuous functions $\mathbb{S}^1 \to K$ correspond to continuous 1-periodic functions $\mathbb{R} \to K$. In view of Lemma I.8, $\pi_m(h_K)$ only depends on the homotopy class of h, so that we may assume that $h(z) = nz$ for some $n \in \mathbb{Z}$. In this case $n = \deg(h)$.

Since the inclusion $C^\infty_*(\mathbb{S}^1, K) \hookrightarrow C_*(\mathbb{S}^1, K)$ is a homotopy equivalence ([Ne02b, Th. A.3.7]), it suffices to consider the maps

$$\varphi_n : C_*(\mathbb{S}^1, K) \to C^\infty_*(\mathbb{S}^1, K), \quad \varphi_n(f)(t) = f(nt).$$

We claim that φ_n is homotopy equivalent to the map $\psi_n(f) := f^n$.

We assume that $n > 0$. The case $n = 0$ is trivial and the case $n < 0$ is treated similarly. For each interval $[\frac{i}{n}, \frac{i+1}{n}]$, $i = 0, \ldots, n-1$, we define a continuous map

$$\alpha_i : C_*(\mathbb{S}^1, K) \to C_*(\mathbb{S}^1, K), \quad \alpha_i(f)(t) := f(\widetilde{\alpha}_i(t)), \qquad 0 \leq t \leq 1,$$

where

$$\widetilde{\alpha}_i : [0,1] \to [0,1], \quad t \mapsto \begin{cases} 0 & \text{for } t \leq \frac{i}{n} \\ nt - i & \text{for } \frac{i}{n} \leq t \leq \frac{i+1}{n} \\ 1 & \text{for } \frac{i+1}{n} \leq t \leq 1. \end{cases}$$

This means that the functions $\alpha_i(f)$ are "supported" by the $\mathbb{Z}$-translates of the interval $[\frac{i}{n}, \frac{i+1}{n}]$. Then each map $\widetilde{\alpha}_i$ is homotopic to the identity of $[0,1]$ with fixed endpoints, and the same carries over to α_i. Now

$$\varphi_n(f) = \alpha_1(f) \cdot \alpha_2(f) \cdots \alpha_n(f)$$

is a pointwise product because the supports of the factors are disjoint. As each map α_i is homotopic to $\mathrm{id}_{C_*(\mathbb{S}^1, K)}$, the map φ_n is homotopic to the nth power map.

The nth power map on $C_*(\mathbb{S}^1, K)$ induces the nth power map on the corresponding homotopy groups, where the multiplication is induced by pointwise multiplication in K, and we conclude that

$$\pi_m(\varphi_n) : \pi_m(C_*(\mathbb{S}^1, K)) \to \pi_m(C_*(\mathbb{S}^1, K))$$

is the nth power map in the abelian group $\pi_m(C_*(\mathbb{S}^1, K))$. ■

In the appendix we give an alternative proof of Lemma I.10 using the homotopy cogroup structure of $\mathbb{S}^1$.

Proof. (of Theorem I.6) We already know from Corollary I.9 that

$$\Pi_{M,\kappa} \subseteq H^1_{\mathrm{dR}}(M, Y) = \bigoplus_{j=1}^{k} [\delta(f_j)] \cdot Y \cong Y^k,$$

and the linear maps $\alpha_{j,\mathfrak{z}}$ correspond to the projections onto the components in Y^k. We have to evaluate these maps on Π_M. To approach Π_M from below, we associate to each $f \in C^\infty(M, \mathbb{S}^1)$ the map

$$f_K : C^\infty(\mathbb{S}^1, K) \to G = C^\infty(M, K), \quad \eta \mapsto \eta \circ f,$$

which in turn induces a map $\pi_2(f_K) : \pi_2(C^\infty(\mathbb{S}^1, K)) \to \pi_2(G)$. For $\alpha \in C^\infty(\mathbb{S}^1, M)$ we obtain with Lemma I.10

$$\begin{aligned} \alpha_{\mathfrak{z}} \circ \mathrm{per}_{\omega_M} \circ \pi_2(f_K) &= \mathrm{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_K) \circ \pi_2(f_K) = \mathrm{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_K \circ f_K) \\ &= \mathrm{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2((f \circ \alpha)_K) = \deg(f \circ \alpha) \cdot \mathrm{per}_{\omega_{\mathbb{S}^1}}. \end{aligned}$$

For $f = f_i$ and $\alpha = \alpha_j$ it follows in particular that

$$\alpha_{i,\mathfrak{z}} \circ \mathrm{per}_{\omega_M} \circ \pi_2(f_{j,K}) = \delta_{ij}\,\mathrm{per}_{\omega_{\mathbb{S}^1}}.$$

Hence

$$\mathrm{per}_{\omega_M}\big(\mathrm{im}\,\pi_2(f_{j,K})\big) = [\delta(f_j)]\cdot \Pi_{\mathbb{S}^1,\kappa},$$

and further

$$\Pi_M \supseteq \sum_{j=1}^k [\delta(f_j)]\cdot \Pi_{\mathbb{S}^1} \cong \Pi_{\mathbb{S}^1}^k.$$

For the converse inclusion, we observe that

$$\alpha_{j,\mathfrak{z}} \circ \mathrm{per}_{\omega_M} = \mathrm{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_K)$$

implies that for each j we have $\alpha_{j,\mathfrak{z}} \circ \mathrm{per}_{\omega_M} \subseteq \Pi_{\mathbb{S}^1,\kappa}$ and therefore $\Pi_M \subseteq \Pi_{\mathbb{S}^1,\kappa}^k$. ■

In view of Theorem I.6, the discreteness of the group $\Pi_{M,\kappa}$ does not depend on M (if $b_1(M) > 0$), so that as far as the discreteness of the period group is concerned, it suffices to consider the simplest non-trivial compact manifold $M = \mathbb{S}^1$. In this first section we did not use any specific information on κ, but for the discreteness of $\Pi_{\mathbb{S}^1,\kappa}$ the specific choice of κ plays a crucial role.

Remark I.11. (a) In this section we have analyzed the period map

$$\pi_2(C^\infty(M,K)) \to \mathfrak{z}_M(Y)$$

by indirect methods based on smooth homomorphisms of loop groups into $C^\infty(M,K)$ and on homomorphisms into loop groups. It is remarkable that this method provides a complete description of the period group.

Let $x_M \in M$ be a base point and $C_*(M,K) \subseteq C(M,K)$ denote the kernel of the evaluation homomorphism $C(M,K) \to K, f \mapsto f(x_M)$. For general groups K and general compact manifolds the Approximation Theorem ([Ne02b, Th. A.3.7] implies that

$$\begin{aligned}\pi_2(C^\infty(M,K)) &\cong \pi_2(C(M,K)) \cong \pi_2(K)\times\pi_2(C_*(M,K)) \cong \pi_2(K)\times[\mathbb{S}^2, C(M,K)]_*\\ &\cong \pi_2(K)\times[\mathbb{S}^2\wedge M, K]_* \cong \pi_2(K)\times\pi_0(C_*(\mathbb{S}^2\wedge M,K)).\end{aligned}$$

In general the group of homotopy classes $[M,K]$ for a CW-complex M may be quite hard to access if $\dim M \geq 3$. For 2-dimensional manifolds one can use the classification of compact surfaces to obtain good descriptions of $\pi_2(C(M,K))$.
(b) We consider the case where $M = \mathbb{T}^d$ is a d-dimensional torus. Then

$$C(\mathbb{T}^d,K) \cong C(\mathbb{T}, C(\mathbb{T}^{d-1},K)) \cong C_*\big(\mathbb{T}, C(\mathbb{T}^{d-1},K)\big) \rtimes C(\mathbb{T}^{d-1},K)$$

implies that

$$\pi_k(C(\mathbb{T}^d,K)) \cong \pi_{k+1}(C(\mathbb{T}^{d-1},K)) \oplus \pi_k(C(\mathbb{T}^{d-1},K))$$

and by induction we obtain

$$\pi_k(C(\mathbb{T}^d,K)) \cong \sum_{j=0}^d \pi_{k+j}(K)^{\binom{d}{j}}.$$ ■

II. The case of loop groups

We keep the notation of Section I. In addition, we assume in this section that K is finite-dimensional. In this case we show that if κ is the universal invariant symmetric bilinear form on $\mathfrak{k}$, then the period group $\Pi_{\mathbb{S}^1,\kappa}$ is discrete.

Definition II.1. For a finite-dimensional Lie algebra $\mathfrak{k}$ we write $V(\mathfrak{k}) := S^2(\mathfrak{k})/\mathfrak{k}.S^2(\mathfrak{k})$, where the action of $\mathfrak{k}$ on $S^2(\mathfrak{k})$ is the natural action inherited by the one on the tensor product $\mathfrak{k}\otimes\mathfrak{k}$ by $x.(y\otimes z) = [x,y]\otimes z + y\otimes[x,z]$. There exists a natural invariant symmetric bilinear form

$$\kappa\colon \mathfrak{k}\times\mathfrak{k}\to V(\mathfrak{k}), \quad x\otimes y\mapsto [x\vee y]$$

such that for each invariant symmetric bilinear form $\beta\colon\mathfrak{k}\times\mathfrak{k}\to W$ there exists a unique linear map $\varphi\colon V(\mathfrak{k})\to W$ with $\varphi\circ\kappa=\beta$. We call the natural map $\kappa\colon\mathfrak{k}\times\mathfrak{k}\to V(\mathfrak{k})$ the *universal invariant symmetric bilinear form on* $\mathfrak{k}$. ■

We start with some observations that will be needed later on.

Remark II.2. (1) The assignment $\mathfrak{g}\to V(\mathfrak{g})$ is a covariant functor from Lie algebras to vector spaces.
(2) If $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{b}$ with $\mathfrak{a}$ perfect, then $V(\mathfrak{g})\cong V(\mathfrak{a})\oplus V(\mathfrak{b})$ because for every symmetric invariant bilinear map $\kappa\colon\mathfrak{g}\times\mathfrak{g}\to V$ we have for $x,y\in\mathfrak{a}$, $z\in\mathfrak{b}$ the relation $\kappa([x,y],z)=\kappa(x,[y,z])=\kappa(x,0)=0$.
(3) If $\mathfrak{h}\trianglelefteq\mathfrak{g}$ is an ideal and the quotient morphism $q\colon\mathfrak{g}\to\mathfrak{q}:=\mathfrak{g}/\mathfrak{h}$ splits, then $\mathfrak{g}\cong\mathfrak{h}\rtimes\mathfrak{q}$, and the natural map $V(\mathfrak{q})\to V(\mathfrak{g})$ is an embedding. In fact, let $\eta\colon\mathfrak{q}\to\mathfrak{g}$ be the inclusion map. Then $q\circ\eta=\mathrm{id}_{\mathfrak{q}}$ and this leads to $V(q)\circ V(\eta)=\mathrm{id}_{V(\mathfrak{q})}$, showing that $V(\eta)$ is injective.
(4) If $\mathfrak{s}$ is reductive with the simple ideals $\mathfrak{s}_1,\ldots,\mathfrak{s}_n$, then (2) implies that

$$V(\mathfrak{s})\cong V(\mathfrak{z}(\mathfrak{s}))\oplus\bigoplus_{j=1}^n V(\mathfrak{s}_j)\cong V(\mathfrak{z}(\mathfrak{s}))\oplus\mathbb{R}^n.$$

(5) If $\mathfrak{k}=\mathfrak{r}\rtimes\mathfrak{s}$ is a Levi decomposition, then (3) implies that the natural map $V(\mathfrak{s})\to V(\mathfrak{k})$ is an embedding.
(6) If $\mathfrak{k}=\mathfrak{gl}(n,\mathbb{R})$, then $V(\mathfrak{k})\cong\mathbb{R}^2$ follows from (4). ■

Remark II.3. We recall some results on the homotopy groups of finite-dimensional Lie groups K. First we recall E. Cartan's Theorem

$$\pi_2(K)=\mathbf{1}$$

([Mi95, Th. 3.7]), and further Bott's Theorem that for a compact connected simple Lie group C we have

$$\pi_3(C)\cong\mathbb{Z}$$

([Mi95, Th. 3.9]).

In [Mi95, pp. 969/970] one also finds a table with $\pi_k(G)$ up to $k=15$, showing that

$$\pi_4(G)\cong\begin{cases}\mathbb{Z}_2\oplus\mathbb{Z}_2 & \text{for } G=\mathrm{SO}(4)\\ \mathbb{Z}_2 & \text{for } G=\mathrm{Sp}(n),\mathrm{SU}(2),\mathrm{SO}(3),\mathrm{SO}(5)\\ \mathbf{1} & \text{for } G=\mathrm{SU}(n),\ n\geq 3 \text{ and } \mathrm{SO}(n),\ n\geq 6\\ \mathbf{1} & \text{for } G=G_2,F_4,E_6,E_7,E_8.\end{cases}$$

$$\pi_5(G)\cong\begin{cases}\mathbb{Z}_2\oplus\mathbb{Z}_2 & \text{for } G=\mathrm{SO}(4)\\ \mathbb{Z}_2 & \text{for } G=\mathrm{Sp}(n),\mathrm{SU}(2),\mathrm{SO}(3),\mathrm{SO}(5)\\ \mathbb{Z} & \text{for } G=\mathrm{SU}(n),\ n\geq 3 \text{ and } \mathrm{SO}(6)\\ \mathbf{1} & \text{for } G=\mathrm{SO}(n),\ n\geq 7,\ G_2,F_4,E_6,E_7,E_8.\end{cases}$$

■

Remark II.4. (a) Let $C \subseteq K$ be a maximal compact subgroup, C_0 the identity component of the center of C and $C_1, \ldots, C_m$ the connected simple normal subgroups of C. Then the multiplication map

$$C_0 \times C_1 \times \ldots \times C_m \to C$$

has finite kernel, hence is a covering map. Now the existence of a manifold factor in K implies that

$$\pi_3(K) \cong \pi_3(C) \cong \prod_{j=1}^m \pi_3(C_j) \cong \mathbb{Z}^m$$

(Remark II.3) because C_0 is a torus, so that $\pi_3(C_0)$ is trivial.
(b) If C is compact and simple, then a generator of $\pi_3(C)$ can be obtained from a homomorphism $\eta\colon \mathrm{SU}(2) \to C$. More precisely, let α be a long root in the root system $\Delta_{\mathfrak{c}}$ of $\mathfrak{c}$ and $\mathfrak{c}(\alpha) \subseteq \mathfrak{c}$ the corresponding $\mathfrak{su}(2)$-subalgebra. Then the corresponding homomorphism $\mathrm{SU}(2) \cong \mathbb{S}^3 \to C$ represents a generator of $\pi_3(C)$ ([Bo58]). ■

Remark II.5. If E and F are locally convex vector spaces, then we write $E \otimes_\pi F$ for the tensor product space endowed with the projective tensor product topology (cf. [Tr67]) and $E\widehat{\otimes}F$ for the completion of this space.

If M is a finite-dimensional σ-compact manifold and E a complete locally convex space, then

$$C^\infty(M,E) \cong C^\infty(M,\mathbb{R})\widehat{\otimes}E$$

follows from [Gr55, Ch. 2, p.81]. In particular, the subspace

$$C^\infty(M,\mathbb{R}) \otimes E \cong \mathrm{span}\{\varphi \cdot y \colon \varphi \in C^\infty(M,\mathbb{R}), y \in E\}$$

is dense in $C^\infty(M,E)$. ■

Lemma II.6. *Let Y be a s.c.l.c. space and $\mathfrak{z}_M(Y)$ as in* Definition I.1. *Then the subspace $\mathfrak{z}_M(\mathbb{R}) \cdot Y$ spanned by the elements of the form $[\beta \cdot y]$, $\beta \in \Omega^1(M,\mathbb{R})$, $y \in Y$, is dense in $\mathfrak{z}_M(Y)$.*

Proof. It suffices to show that $\Omega^1(M,\mathbb{R}) \cdot Y$ spans a dense subspace of $\Omega^1(M,Y)$.

Let $(\varphi_j)_{j\in J}$ be a finite partition of unity in $C^\infty(M,\mathbb{R})$ such that the support of each function φ_j is contained in an open set U_j diffeomorphic to an open subset of $\mathbb{R}^d$ for $d := \dim M$. For each U_j we then have

$$\Omega^1(U_j,Y) \cong C^\infty(U_j,Y)^d,$$

and Remark II.5 implies that for the completion $\overline{Y}$ of Y we have

$$C^\infty(U_j,\overline{Y}) \cong C^\infty(U_j,\mathbb{R})\widehat{\otimes}\overline{Y}.$$

Since $C^\infty(U_j,\mathbb{R}) \cdot Y$ is dense in $C^\infty(U_j,\mathbb{R})\widehat{\otimes}\overline{Y}$, it is also dense in $C^\infty(U_j,Y)$.

Writing $\beta \in \Omega^1(M,Y)$ as a sum $\beta = \sum_j \varphi_j\beta$, the preceding argument implies that each $\varphi_j\beta$ is contained in the closure of $\Omega^1(M,\mathbb{R}) \cdot Y$, and this proves that $\Omega^1(M,\mathbb{R}) \cdot Y$ is dense in $\Omega^1(M,Y)$. ■

Lemma II.7. *Let $\mathfrak{k}$ be a locally convex Lie algebra, M a smooth manifold, $\mathfrak{g} := C^\infty(M,\mathfrak{k})$, $\kappa\colon \mathfrak{k} \times \mathfrak{k} \to Y$ a continuous invariant symmetric bilinear form, and $\omega_{M,\kappa} \in Z^2_c(\mathfrak{g},\mathfrak{z}_M(Y))$ defined by*

$$\omega_{M,\kappa}(\eta,\xi) := [\kappa(\eta,d\xi)],$$

so that in particular $\omega_{M,\kappa}(f\otimes x, g\otimes y) := [fdg]\kappa(x,y) \in \mathfrak{z}_M(Y)$. If $\mathrm{im}(\kappa)$ spans Y, then the central extension $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_{\omega_{M,\kappa}} \mathfrak{z}_M(Y)$ is a covering, i.e., $\mathfrak{z}_M(Y)$ is contained in the closure of the commutator algebra of $\widehat{\mathfrak{g}}$.

Proof. For $x,y \in \mathfrak{k}$ and $f,g \in C^\infty(M,\mathbb{R})$ we have in $\widehat{\mathfrak{g}}$ the relation

$$[f\otimes x, g\otimes y] - [g\otimes x, f\otimes y] = \big(fg\otimes[x,y] - gf\otimes[x,y], 2[fdg]\cdot\kappa(x,y)\big) = \big(0, 2[fdg]\cdot\kappa(x,y)\big).$$

This implies that the dense subspace $\mathfrak{z}_M(\mathbb{R}) \cdot Y$ of $\mathfrak{z}_M(Y)$ (Lemma II.6) is contained in $[\widehat{\mathfrak{g}},\widehat{\mathfrak{g}}]$ and therefore that $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is a covering. ■

We now return to our assumption that K is finite-dimensional and consider the loop group $G := C^\infty(\mathbb{S}^1,K)$. Let $\kappa\colon \mathfrak{k}\times\mathfrak{k} \to V(\mathfrak{k})$ denote the universal invariant symmetric bilinear form and define a cocycle on $\mathfrak{g} = C^\infty(\mathbb{S}^1,\mathfrak{k})$ as in Section I by $\omega(f,g) := \omega_{\mathbb{S}^1,\kappa}(f,g) := [\kappa(f,dg)]$.

Remark II.8. (a) If K is a finite-dimensional Lie group, then $\pi_2(K) = \mathbf{1}$ implies that $\pi_3(K) \cong \pi_2(C_*(\mathbb{S}^1, K)) \cong \pi_2(G)$, and we can view the period map of ω as a homomorphism

$$\mathrm{per}_K \colon \pi_3(K) \to V(\mathfrak{k}).$$

(b) For any infinite-dimensional Lie group K we can also define a homomorphism $\pi_3(K) \to V(\mathfrak{k})$ as follows. To define $V(\mathfrak{k})$ for an infinite-dimensional Lie algebra $\mathfrak{k}$, we first endow $\mathfrak{k} \otimes \mathfrak{k}$ with the projective tensor product topology and define $V(\mathfrak{k})$ as the quotient of this space by the closure of the subspace spanned by all elements of the form

$$x \otimes y - y \otimes x \quad \text{and} \quad [x, y] \otimes z + y \otimes [x, z], \quad x, y, z \in \mathfrak{k}.$$

If $[z]$ denotes the image of $z \in \mathfrak{k} \otimes \mathfrak{k}$ in $V(\mathfrak{k})$, we obtain a continuous invariant bilinear map

$$\kappa \colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k}), \quad \kappa(x, y) := [x \otimes y]$$

which leads to the cocycle $\omega \in Z^2_c(\mathfrak{g}, V(\mathfrak{k}))$ on $\mathfrak{g} := C^\infty(\mathbb{S}^1, \mathfrak{k})$ given by $\omega(\xi, \eta) := [\kappa(\xi, d\eta)]$.

Let $G := C^\infty(\mathbb{S}^1, K)_e$. Since the restriction of ω to the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ consisting of constant $\mathfrak{k}$-valued functions vanishes, the period map $\mathrm{per}_\omega \colon \pi_2(G) \cong \pi_3(K) \times \pi_2(K) \to V(\mathfrak{k})$ vanishes on $\pi_2(K)$ and defines group homomorphism

$$\mathrm{per}_K \colon \pi_3(K) \to V(\mathfrak{k})$$

with the same image. ■

The following theorem shows that for each finite-dimensional Lie group K the homomorphism per_K has discrete image, and it is not so easy to find infinite-dimensional Lie groups where this is not the case. Below we discuss some related examples and special classes.

Theorem II.9. *For every finite-dimensional connected Lie group K and the $V(\mathfrak{k})$-valued cocycle $\omega(f, g) = [\kappa(f, dg)]$ on $C^\infty(\mathbb{S}^1, \mathfrak{k})$, the image of per_ω in $V(\mathfrak{k})$ is discrete.*

Proof. If $\varphi \colon K_1 \to K_2$ is a Lie group morphism and $\mathbf{L}(\varphi) \colon \mathfrak{k}_1 \to \mathfrak{k}_2$ the corresponding Lie algebra morphism, then we have

$$\kappa_{\mathfrak{k}_2} \circ \mathbf{L}(\varphi \times \varphi) = V(\mathbf{L}(\varphi)) \circ \kappa_{\mathfrak{k}_1}, \quad \text{and} \quad \mathrm{per}_{\omega_{M, \mathfrak{k}_2}} \circ \pi_3(\varphi) = V(\mathbf{L}(\varphi)) \circ \mathrm{per}_{\omega_M, \mathfrak{k}_1}.$$

In view of Remark II.4, this reduces the problem to the determination of $V(\mathbf{L}(\eta_j))$ for the generators $\eta_j \colon \mathrm{SU}(2) \to K$, $j = 1, \ldots, m$, of $\pi_3(K)$.

For $K = \mathrm{SU}(2)$ pick $x \in \mathfrak{k}$ with $\mathrm{Spec}(\mathrm{ad}\, x) = \{0, \pm 2i\}$. All these elements are conjugate under inner automorphisms. Therefore $v_{\mathfrak{k}} := \frac{1}{2}\kappa(x, x) \in V(\mathfrak{k})$ is well defined (κ can be viewed as a multiple of the Cartan-Killing form; see also Remark II.2(4)). Then the calculations in Appendix IIa to Section II in [Ne01a] imply that $\mathrm{per}_\omega([\mathrm{id}_K]) = v_{\mathfrak{k}}$.

Therefore, in the general case, $\mathrm{im}(\mathrm{per}\,\omega) \subseteq V(\mathfrak{k})$ is the subgroup generated by the elements $v_1, \ldots, v_m$ corresponding to the homomorphisms $\eta_j \colon \mathrm{SU}(2) \to C_j$ mentioned above. If $\mathfrak{s} \subseteq \mathfrak{k}$ is a Levi complement, then we may assume that $\mathrm{im}(\mathbf{L}(\eta_j)) \subseteq \mathfrak{s}$ for each j, so that it suffices to determine the image of per_ω in the case where $\mathfrak{k} = \mathfrak{s}$ is semisimple (Remark II.2(5)). This problem immediately reduces to the case where $\mathfrak{s}$ is simple. Let $\mathfrak{s}_c \subseteq \mathfrak{s}$ be a maximal compact semisimple subalgebra. Then $\mathfrak{s}_c$ need not be simple and we write $\mathfrak{s}_c^j$, $j = 1, \ldots, l$, for its simple ideals. (For $\mathfrak{s} = \mathfrak{su}(p, q)$ we have $\mathfrak{s}_c \cong \mathfrak{su}(p) \times \mathfrak{su}(q)$, so that $l = 2$ for $p, q \geq 2$.)

We are interested in the subgroup of $V(\mathfrak{s}) \cong \mathbb{R}$ generated by the elements v_j coming from the basis elements $v_{\mathfrak{s}_c^j} = \frac{1}{2}\kappa(x_j, x_j) \in V(\mathfrak{s}_c^j)$, where x_j denotes an element in a suitable $\mathfrak{su}_2$-subalgebra of the simple ideal $\mathfrak{s}_c^j$ of $\mathfrak{s}_c$ which is normalized in such a way that $\mathrm{Spec}(\mathrm{ad}\, x_j) = \{\pm 2i, 0\}$ holds on the $\mathfrak{su}_2$-subalgebra. The choice of the elements $x_j \in \mathfrak{s}_c^j$ and the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ imply that all eigenvalues of $\mathrm{ad}\, x_j$ are contained in $i\mathbb{Z}$,so that $\mathrm{tr}((\mathrm{ad}\, x_j)^2) \in -\mathbb{N}_0$. Therefore the values of the Cartan–Killing form on the v_j are integral, so that they generate a discrete subgroup of $V(\mathfrak{s}) \cong \mathbb{R}$. We finally conclude that in the general situation the image of per_ω in $V(\mathfrak{k})$ is discrete. ■

Remark II.10. Let $\gamma \in V(\mathfrak{k})^*$, so that $\kappa_\gamma := \gamma \circ \kappa$ defines a real-valued symmetric bilinear form on $\mathfrak{k}$. Then the image of the corresponding period map in $\mathbb{R}$ is determined by the values of γ on the image of the period map $\pi_3(K) \to V(\mathfrak{k})$ in Theorem II.9 which is generated by the elements $v_1, \ldots, v_m \in V(\mathfrak{k})$ obtained as follows. Let $\mathfrak{c}_j$ denote the simple ideals in the Lie algebra $\mathfrak{c}$ of a maximal compact subgroup $C \subseteq K$. Further let $\mathfrak{su}(2)_j \subseteq \mathfrak{c}_j$ be a subalgebra corresponding to a long root and $x_j \in \mathfrak{su}(2)_j$ with $\mathrm{Spec}(\mathrm{ad}\, x_j|_{\mathfrak{su}(2)_j}) = \{0, \pm 2i\}$. Then $v_j = \frac{1}{2}\kappa(x_j, x_j) \in V(\mathfrak{k})$, and we have

$$\mathrm{im}(\mathrm{per}_\omega) = \sum_{j=1}^{k} \mathbb{Z}\gamma(v_j) = \sum_{j=1}^{k} \frac{1}{2}\mathbb{Z}\gamma(\kappa(x_j, x_j)). \qquad \blacksquare$$

Lemma II.11. *Let $\mathfrak{k}$ be a finite-dimensional simple Lie algebra, and $\kappa_{\mathfrak{k}}$ its Cartan-Killing form of $\mathfrak{k}$. Further let A be a locally convex unital commutative associative algebra and consider the locally convex Lie algebra $\mathfrak{g} := A \otimes_\pi \mathfrak{k}$ with the bracket given by $[a \otimes x, b \otimes y] := ab \otimes [x, y]$. Then the map*

$$\kappa\colon \mathfrak{g} \times \mathfrak{g} \to A, \quad (a \otimes x, b \otimes y) \mapsto \kappa_{\mathfrak{k}}(x, y)ab$$

has the universal property of the universal invariant symmetric bilinear form. In particular we have $V(\mathfrak{g}) \cong A$.

Proof. From

$$\kappa([a \otimes x, b \otimes y], c \otimes z) = \kappa_{\mathfrak{k}}([x, y], z)abc = \kappa_{\mathfrak{k}}(x, [y, z])abc = \kappa(a \otimes x, [b \otimes y, c \otimes z])$$

we see that κ is an invariant symmetric bilinear form on $\mathfrak{g}$. Its construction implies the continuity.

To verify the universal property, let $\beta\colon \mathfrak{g} \times \mathfrak{g} \to Y$ be a continuous invariant symmetric bilinear form. For each pair $a, b \in A$ we then obtain an invariant bilinear form

$$\beta_{a,b}\colon \mathfrak{k} \times \mathfrak{k} \to Y, \quad (x, y) \mapsto \beta(a \otimes x, b \otimes y).$$

Now $V(\mathfrak{k}) = \mathbb{R}\kappa_{\mathfrak{k}}$ implies the existence of a unique element $\eta(a, b) \in Y$ with $\beta_{a,b} = \kappa_{\mathfrak{k}} \cdot \eta(a, b)$. Pick $x, y \in \mathfrak{k}$ with $\kappa_{\mathfrak{k}}(x, y) \neq 0$. Then the continuity of the map

$$A \times A \to Y, (a, b) \mapsto \beta(a \otimes x, b \otimes y) = \kappa_{\mathfrak{k}}(x, y)\eta(a, b)$$

implies the continuity of $\eta\colon A \times A \to Y$.

Since $\mathfrak{k}$ is a perfect Lie algebra, we also find three elements $x, y, z \in \mathfrak{k}$ with $\kappa_{\mathfrak{k}}([x, y], z) \neq 0$. Then the invariance of β further leads to

$$\begin{aligned}\kappa_{\mathfrak{k}}([x, y], z)\eta(ab, c) &= \beta([a \otimes x, b \otimes y], c \otimes z) = \beta(a \otimes x, [b \otimes y, c \otimes z]) \\ &= \kappa_{\mathfrak{k}}(x, [y, z])\eta(a, bc) = \kappa_{\mathfrak{k}}([x, y], z)\eta(a, bc),\end{aligned}$$

so that

$$\eta(ab, c) = \eta(a, bc), \quad a, b, c \in A.$$

Let $\mathbf{1} \in A$ denote the unit element and define the continuous linear map $\gamma\colon A \to Y, a \mapsto \eta(a, \mathbf{1})$. Then

$$\beta(a \otimes x, b \otimes y) = \kappa_{\mathfrak{k}}(x, y)\eta(a, b) = \kappa_{\mathfrak{k}}(x, y)\eta(ab, \mathbf{1}) = \kappa_{\mathfrak{k}}(x, y)\gamma(ab) = (\gamma \circ \kappa)(a \otimes x, b \otimes y)$$

shows that β factors through κ, which implies the universal property of κ. Here the uniqueness of γ follows from $A = \mathbf{1} \cdot A = A \cdot A$. $\blacksquare$

Remark II.12. (a) We call an associative unital locally convex algebra A a *continuous inverse algebra* if its group of units $A^\times$ is open and the inversion $A^\times \to A^\times$ is a continuous map. Such algebras have been studied in [Gl01c]. In particular the following results have been obtained:

(1) If A is a sequentially complete continuous inverse algebra, then all matrix algebras $M_n(A)$, $n \in \mathbb{N}$, also have this property ([Gl01c, Prop. 4.5]).

(2) If A is a continuous inverse algebra, then $A^\times$ is a Baker–Campbell–Hausdorff–Lie group (BCH-Lie group), i.e., it has an exponential map $\exp\colon \mathfrak{a} \to A$ (given by holomorphic functional calculus) which restricts to a diffeomorphism of some open 0-neighborhood U in A to some open $\mathbf{1}$-neighborhood in $A^\times$ and on some 0-neighborhood $W \subseteq U$ with $\exp W \exp W \subseteq \exp U$ the multiplication

$$x * y := \exp|_U^{-1}(\exp x \exp y)$$

is given by the BCH-series.

By combining (1) and (2), we can use the theory of analytic subgroups of BCH-Lie groups ([Gl01b]) to derive for each closed Lie subalgebra $\mathfrak{g} \subseteq M_n(A)$ the existence of a global Lie group G with an exponential function obtained by restricting the one of $M_n(A)$ ([Gl01b, Prop. 2.13]).

(b) Let A be a unital locally convex algebra and $HC_0(A) := A/\overline{[A,A]}$. We write $[a]$ for the class of $a \in A$ in $HC_0(A)$. Then the map

$$\mathrm{Tr}\colon M_r(A) \to HC_0(A), \quad x \mapsto [\sum_j x_{jj}]$$

is a continuous Lie algebra homomorphism and we define $\mathfrak{sl}_r(A) := \ker \mathrm{Tr}$. Inspecting the arguments in [BGK96, Lemma 2.8] in the algebraic setting, it is easy to see that $V(\mathfrak{sl}_r(A)) \cong HC_0(A)$ and that a universal invariant symmetric bilinear form is given by $\kappa(x,y) := \mathrm{Tr}(xy)$.

Suppose that A is a complete complex commutative continuous inverse algebra. According to [Bos95, Prop. A.1.5], A satisfies

$$K_0(A) \cong K_2(A) := \mathrm{indlim}_{n\to\infty} \pi_3(\mathrm{GL}_n(A)).$$

One can show that the period map

$$\mathrm{per}_{\mathrm{SL}_r(K)}\colon \pi_3(\mathrm{SL}_r(A)) \to HC_0(A)$$

is the composition of the natural maps $\pi_3(\mathrm{SL}_r(A)) \to \pi_3(\mathrm{GL}_r(A)) \to K_0(A)$ and the trace map

$$T_A\colon K_0(A) \to HC_0(A), \quad [p] \mapsto \mathrm{Tr}(p),$$

where $p = p^2 \in M_n(A)$ is an idempotent representing an element of $K_0(A)$ (see [Ne02a] for details).

If A is commutative, then $HC_0(A) = A$ and the image of the trace map T_A is contained in the kernel of the exponential function $\exp_A\colon A \to A^\times, x \mapsto e^{2\pi i x}$, hence discrete. This implies that $\mathrm{im}(\mathrm{per}_{\mathrm{SL}_r(A)})$ is discrete. The smallest examples of non-commutative algebras for which $\mathrm{im}(T_A)$ is not discrete are the irrational rotation algebras, certain 2-dimensional quantum tori. In this case $HC_0(A) \cong \mathbb{C}$ and $\mathrm{im}(T_A) = \mathbb{Z} + \theta\mathbb{Z}$ for some irrational real number.

(c) In the context of (b), we can use (a) to obtain for each simple complex Lie algebra $\mathfrak{k}$ the existence of a Lie group G with Lie algebra $\mathfrak{g} := A \otimes \mathfrak{k}$ because we can embed $\mathfrak{k}$ into some $M_n(\mathbb{C})$ and then extend scalars to obtain an embedding $\mathfrak{g} \hookrightarrow M_n(A)$. We then have $\mathfrak{g} \subseteq \mathfrak{sl}_n(A)$, and the natural map $V(\mathfrak{g}) \to V(\mathfrak{sl}_n(A))$ is an isomorphism (Lemma II.11). Therefore (b) implies that $\mathrm{im}(\mathrm{per}_G)$ is discrete if $\mathrm{im}(\mathrm{per}_{\mathrm{SL}_r(A)})$ is discrete, which holds whenever A is a complete commutative continuous inverse algebra. ■

III. Existence of corresponding central Lie group extensions

In the following we will use the concept of an infinite-dimensional Lie group described in detail in [Mi83] (see also [Gl01a] and [Ne01b]). This means that a Lie group G is a smooth manifold modeled on a locally convex space $\mathfrak{g}$ for which the group multiplication and the inversion are smooth maps. We write $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G. Then each $X \in T_e(G)$ corresponds to a unique left invariant vector field X_l with $X_l(g) := d\lambda_g(\mathbf{1}).X, g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_e(G)$ a continuous Lie bracket which is uniquely determined by $[X,Y]_l = [X_l, Y_l]$.

In this context central extensions of Lie groups are always assumed to have a smooth local section. Let $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ be a central extension of the connected Lie group G by the abelian group Z. We assume that the identity component Z_e of Z can be written as $Z_e = \mathfrak{z}/\pi_1(Z)$, where the Lie algebra $\mathfrak{z}$ of Z is a s.c.l.c. space. This means that the additive group of $\mathfrak{z}$ can be identified in a natural way with the universal covering group of Z_e, and that Z_e is a quotient $\mathfrak{z}$ modulo a discrete subgroup which can be identified with $\pi_1(Z)$. Since the quotient map $q: \widehat{G} \to G$ has a smooth local section, the corresponding Lie algebra homomorphism $\widehat{\mathfrak{g}} \to \mathfrak{g}$ has a continuous linear section, hence can be described by a continuous Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ as

$$\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z} \quad \text{with the bracket} \quad [(x,z),(x',z')] = ([x,x'], \omega(x,x')).$$

Let $Z^2_s(G,Z)$ denote the abelian group of 2-cocycles $f: G \times G \to Z$ which are smooth in a neighborhood of (e,e) and $B^2_s(G,Z)$ the subgroup of all functions of the form $(g,g') \mapsto h(gg')h(g)^{-1}h(g')^{-1}$, where $h: G \to Z$ is smooth in an identity neighborhood. We recall from [Ne02b, Prop. IV.2] that central Lie group extensions as above can always be written as

$$\widehat{G} \cong G \times_f Z \quad \text{with} \quad (g,z)(g',z') = \big(gg', zz'f(g,g')\big),$$

with $f \in Z^2_s(G,Z)$. Two cocycles f_1, f_2 define equivalent Lie group extensions if and only if $f_1 \cdot f_2^{-1} \in B^2_s(G,Z)$ (for $f_2^{-1}(x,y) := f_2(x,y)^{-1}$), and the quotient group $H^2_s(G,Z) := Z^2_s(G,Z)/B^2_s(G,Z)$ parametrizes the equivalence classes of central Z-extensions of G with smooth local sections ([Ne02b, Remark IV.4]). On the Lie algebra level the space $H^2_c(\mathfrak{g},\mathfrak{z}) = Z^2_c(\mathfrak{g},\mathfrak{z})/B^2_c(\mathfrak{g},\mathfrak{z})$ classifies the central $\mathfrak{z}$-extensions of $\mathfrak{g}$ with continuous linear sections. There is a natural map $H^2_s(G,Z) \to H^2_c(\mathfrak{g},\mathfrak{z})$ induced by the map

$$D: Z^2_s(G,Z) \to Z^2_c(\mathfrak{g},\mathfrak{z}), \quad D(f)(x,y) = d^2f(e,e)(x,y) - d^2f(e,e)(y,x) \tag{3.1}$$

([Ne02b, Lemma IV.6]), where $d^2f(e,e)$ has to be understood as

$$d^2f(e,e)(x,y) := \frac{\partial^2}{\partial s \partial t}\,|_{t,s=0}\, f(\gamma(t),\eta(s))$$

where $\gamma:]-\varepsilon,\varepsilon[\to G$ and $\eta:]-\varepsilon,\varepsilon[\to G$ are smooth curves with $\gamma(0) = e$, $\gamma'(0) = x$, $\eta(0) = e$, and $\eta'(0) = y$. For more details we refer to [Ne02b].

In this section we discuss the existence of a central Lie group extension for the Lie algebra cocycles $\omega_{M,\kappa}$ of product type (see (1.1)), where K may be an infinite-dimensional Lie group.

The group $G := C^\infty(M,K)$ acts on $\mathfrak{g}$ by the adjoint action which is given by

$$(\mathrm{Ad}(f).\xi)(m) := \mathrm{Ad}(f(m)).\xi(m) \quad \text{for} \quad m \in M.$$

We also define an action of G on $\mathfrak{k}$-valued 1-forms on M by

$$(\mathrm{Ad}(f).\alpha)(m) := \mathrm{Ad}(f(m)) \circ \alpha(m) \quad \text{for} \quad m \in M.$$

Definition III.1. For an element $f \in C^\infty(M,K)$ we write

$$\delta^l(f)(m) := d\lambda_{f(m)^{-1}}(f(m))df(m): T_m(M) \to \mathfrak{k} \cong T_e(K)$$

for the *left logarithmic derivative of* f. This derivative can be viewed as a $\mathfrak{k}$-valued 1-form on M. We also write simply $\delta^l(f) = f^{-1}.df$ and observe the following product rule

$$\delta^l(f_1 f_2) = \mathrm{Ad}(f_2)^{-1}.\delta^l(f_1) + \delta^l(f_2). \tag{3.2}$$

The *right logarithmic derivative* $\delta^r(f) = df.f^{-1}$ satisfies the product rule

$$\delta^r(f_1 f_2) = \delta^r(f_1) + \mathrm{Ad}(f_1).\delta^r(f_2) \tag{3.3}$$

([KM97, 38.1]).

The form $\theta^l_K := \delta^l(\mathrm{id}_K) \in \Omega^1(K,\mathfrak{k})$ is called the *left Maurer–Cartan form on* K and $\theta^r_K := \delta^r(\mathrm{id}_K)$ the *right Maurer–Cartan form.* Using the Maurer–Cartan forms, we have

$$\delta^l(f) = f^*\theta^l_K \quad \text{and} \quad \delta^r(f) = f^*\theta^r_K. \qquad \blacksquare$$

Lemma III.2. *The smooth maps* $\delta^l, \delta^r: C^\infty(M,K) \to \Omega^1(M,\mathfrak{k})$ *satisfy*

$$(d\delta^l)(e)(\eta) = (d\delta^r)(e)(\eta) = d\eta \quad \textit{for} \quad \eta \in C^\infty(M,\mathfrak{k}) \cong T_e(C^\infty(M,K)).$$

Proof. Let $V \subseteq \mathfrak{k}$ be an open convex 0-neighborhood and $\varphi: V \to U := \varphi(V)$ a chart of K with $\varphi(0) = e$ and $d\varphi(0) = \mathrm{id}_{\mathfrak{k}}$. Let $\eta \in \mathfrak{g} = C^\infty(M,\mathfrak{k})$. Then there exists an $\varepsilon > 0$ such that for each $t \in [0,\varepsilon]$ we have $t\eta(M) \subseteq V$. Then

$$\gamma: [0,\varepsilon] \to C^\infty(M,K), \quad \gamma_t(m) := \varphi(t\eta(m))$$

is a smooth curve on $C^\infty(M,K)$ with $\gamma(0) = e$ and $\gamma'(0) = \eta$. We now have for $v \in T_m(M)$

$$d\gamma_t(m).v = d\varphi(t\eta(m))td\eta(m)v \in T_{\gamma(m)}(K)$$

and therefore

$$\delta^l(\gamma_t)(m).v = \gamma_t(m)^{-1}.(d\gamma_t(m).v) = \varphi(t\eta(m))^{-1}.d\varphi(t\eta(m)) \cdot t \cdot d\eta(m)v \in \mathfrak{k}.$$

In view of $d\gamma_0 = 0$, it follows that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \gamma_t(m)^{-1}.(d\gamma_t(m).v) &= \lim_{t\to 0} \varphi(t\eta(m))^{-1}.d\varphi(t\eta(m))d\eta(m)v \\ &= \varphi(0)^{-1}.d\varphi(0)d\eta(m)v = d\eta(m)v. \end{aligned}$$

A similar argument works for the right logarithmic derivatives. $\blacksquare$

Proposition III.3. *Let* $\mathfrak{g} := C^\infty(M,\mathfrak{k})$, $\kappa: \mathfrak{k} \times \mathfrak{k} \to Y$ *be a continuous invariant symmetric bilinear form, and define*

$$\Theta: C^\infty(M,K) \to \mathrm{Lin}(\mathfrak{g}, \mathfrak{z}_M(Y)), \quad \Theta(f)(\xi) := [\kappa(\delta^l(f),\xi)].$$

Then we obtain for the cocycle $\omega(\xi,\eta) := [\kappa(\xi,d\eta)]$ *an automorphic action of* $C^\infty(M,K)$ *on* $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z}_M(Y)$ *by*

$$f.(\xi,z) := (\mathrm{Ad}(f).\xi, z - \Theta(f)(\xi)) = (\mathrm{Ad}(f).\xi, z - [\kappa(\delta^l(f),\xi)]). \tag{3.4}$$

The corresponding derived action is given by

$$\eta.(\xi,z) = [(\eta,0),(\xi,z)] = ([\eta,\xi],\omega(\eta,\xi)). \tag{3.5}$$

Proof. Using (3.2), we first verify the cocycle condition for Θ:

$$\begin{aligned}\Theta(f_1f_2)(\xi) &= [\kappa(\delta^l(f_1f_2),\xi)] = [\kappa(\delta^l(f_2)+\mathrm{Ad}(f_2)^{-1}.\delta^l(f_1),\xi)]\\ &= \Theta(f_2)(\xi)+[\kappa(\delta^l(f_1),\mathrm{Ad}(f_2).\xi)] = \Theta(f_2)(\xi)+\Theta(f_1)(\mathrm{Ad}(f_2).\xi).\end{aligned}$$

This relation implies that

$$\begin{aligned}f_1.(f_2.(\xi,z)) = f_1.(\mathrm{Ad}(f_2).\xi, z-\Theta(f_2)(\xi)) &= \big(\mathrm{Ad}(f_1f_2).\xi, z-\Theta(f_2)(\xi)-\Theta(f_1)(\mathrm{Ad}(f_2).\xi)\big)\\ &= \big(\mathrm{Ad}(f_1f_2).\xi, z-\Theta(f_1f_2)(\xi)\big).\end{aligned}$$

To see that $C^\infty(M,K)$ acts by automorphisms of $\widehat{\mathfrak g}$, we note that

$$\begin{aligned}d(\mathrm{Ad}(f).\eta)(m) &= \big((d\,\mathrm{Ad})(f(m))df(m)\big).\eta(m)+\mathrm{Ad}(f(m))\circ d\eta(m)\\ &= \big(\mathrm{Ad}(f(m))d\,\mathrm{Ad}(e)d\lambda_{f(m)^{-1}}(f(m))df(m)\big).\eta(m)+\mathrm{Ad}(f(m))\circ d\eta(m)\\ &= \big(\mathrm{Ad}(f(m))\circ \mathrm{ad}\,\delta^l(f)(m)\big).\eta(m)+\mathrm{Ad}(f(m))\circ d\eta(m),\end{aligned}$$

which means that

$$d(\mathrm{Ad}(f).\eta) = \mathrm{Ad}(f).[\delta^l(f),\eta]+\mathrm{Ad}(f).d\eta. \tag{3.6}$$

Therefore

$$\begin{aligned}\omega(\mathrm{Ad}(f).\xi,\mathrm{Ad}(f).\eta) &= [\kappa(\mathrm{Ad}(f).\xi, d(\mathrm{Ad}(f).\eta))] = [\kappa(\mathrm{Ad}(f).\xi,\mathrm{Ad}(f).d\eta+\mathrm{Ad}(f).[\delta^l(f),\eta])]\\ &= [\kappa(\xi,d\eta)]+[\kappa(\xi,[\delta^l(f),\eta])] = [\kappa(\xi,d\eta)]-[\kappa(\delta^l(f),[\xi,\eta])]\\ &= \omega(\xi,\eta)-\Theta(f)([\xi,\eta]).\end{aligned}$$

That $C^\infty(M,K)$ acts by automorphisms on $\widehat{\mathfrak g}$ now follows from

$$\begin{aligned}f.[(\xi_1,z_1),(\xi_2,z_2)] &= (\mathrm{Ad}(f).[\xi_1,\xi_2],\omega(\xi_1,\xi_2)-\Theta(f)([\xi_1,\xi_2]))\\ &= \big([\mathrm{Ad}(f).\xi_1,\mathrm{Ad}(f).\xi_2],\omega(\mathrm{Ad}(f).\xi_1,\mathrm{Ad}(f).\xi_2)\big) = [f.(\xi_1,z_1),f.(\xi_2,z_2)].\end{aligned}$$

To verify (3.5), we have to show that the differential of Θ in e is given by

$$d\Theta(e)(\eta)(\xi) = \omega(\xi,\eta).$$

Using Lemma III.2, we obtain

$$d\Theta(e)(\eta)(\xi) = [\kappa\big((d\delta^l)(e)(\eta),\xi\big)] = [\kappa(d\eta,\xi)] = [\kappa(\xi,d\eta)] = \omega(\xi,\eta).$$ ∎

Definition III.4. Let G be a connected Lie group with Lie algebra $\mathfrak g$ and $\omega\in Z^2_c(\mathfrak g,\mathfrak z)$ a continuous Lie algebra cocycle with values in the s.c.l.c. space $\mathfrak z$. Let $\Gamma\subseteq\mathfrak z$ be a discrete subgroup and $Z := \mathfrak z/\Gamma$ the corresponding quotient Lie group. Further let Ω be the corresponding left invariant closed $\mathfrak z$-valued 2-form on G. Then we define a homomorphism

$$P\colon H^2_c(\mathfrak g,\mathfrak z)\to \mathrm{Hom}(\pi_2(G),Z)\times\mathrm{Hom}(\pi_1(G),\mathrm{Lin}(\mathfrak g,\mathfrak z))$$

as follows. For the first component we take

$$P_1([\omega]) := q_Z\circ \mathrm{per}_\omega,$$

where $q_Z\colon\mathfrak z\to Z$ is the quotient map and $\mathrm{per}_\omega\colon\pi_2(G)\to\mathfrak z$ is the period map of ω. To define the second component, for each $X\in\mathfrak g$ we write X_r for the corresponding right invariant vector field on G. Then $i_{X_r}\Omega$ is a closed $\mathfrak z$-valued 1-form ([Ne02b, Lemma III.11]) to which we associate a homomorphism $\pi_1(G)\to\mathfrak z$ via

$$P_2([\omega])([\gamma])(X) := \int_\gamma i_{X_r}\Omega.$$

We refer to [Ne02b, Sect. VII] for arguments showing that P is well defined, i.e., that the right hand sides only depend on the Lie algebra cohomology class of ω. ∎

Theorem III.5. *Let $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ be a continuous Lie algebra cocycle. Then the central Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z} \twoheadrightarrow \mathfrak{g}$ integrates to a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ if and only if $P([\omega]) = 0$.*

Proof. [Ne02b, Th. VII.12]. ■

Theorem III.6. *Let K be a connected Lie group, M a compact manifold, $G := C^\infty(M,K)_e$ and $\omega_{M,\kappa} \in Z^2_c(\mathfrak{g},\mathfrak{z}_M(Y))$ as above. Suppose that the period group $\Pi_{M,\kappa} \subseteq \mathfrak{z}_M(Y)$ is discrete. For $Z := \mathfrak{z}_M(Y)/\Pi_{\omega_{M,\kappa}}$ we then obtain a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ corresponding to the cocycle $\omega_{M,\kappa}$.*

Proof. In view of Theorem III.5, we only have to see that $P_2([\omega_{M,\kappa}]) = 0$, but this follows from Proposition III.3 and [Ne02b, Prop. VII.6]. ■

Corollary III.7. *If $\dim K < \infty$, $Y = V(\mathfrak{k})$, and $\kappa\colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is the universal symmetric invariant bilinear map, then there exists for $Z := V(\mathfrak{k})/\Pi_{M,\kappa}$ a central Lie group extension*

$$Z \hookrightarrow \widehat{G} \twoheadrightarrow G = C^\infty(M,K)_e.$$

Proof. This is a consequence of Theorem II.9 and Theorem III.6. ■

Remark III.8. (a) (cf. [Ne02b, Rem. V.12]) Let $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ be a central extension of Lie groups, where G and $\widehat{G}$ are connected. In view of [Ne02b, Prop. V.11], the long exact homotopy sequence of the principal Z-bundle $\widehat{G}$ over G leads to an exact sequence

$$\pi_2(Z) \to \pi_2(\widehat{G}) \to \pi_2(G) \xrightarrow{\mathrm{per}_\omega} \pi_1(Z) \to \pi_1(\widehat{G}) \to \pi_1(G) \to \pi_0(Z) \to \pi_0(\widehat{G}) = \mathbf{1},$$

so that $\pi_2(Z) \cong \pi_2(\mathfrak{z}) = \mathbf{1}$ leads to

$$\pi_2(\widehat{G}) \hookrightarrow \pi_2(G) \xrightarrow{\mathrm{per}_\omega} \pi_1(Z) \to \pi_1(\widehat{G}) \twoheadrightarrow \pi_1(G) \to \pi_0(Z).$$

If the connecting map $\pi_1(G) \to \pi_0(Z)$ is injective, then the map $\pi_1(Z) \to \pi_1(\widehat{G})$ is surjective, and we obtain

$$\pi_2(\widehat{G}) \cong \ker \mathrm{per}_\omega \subseteq \pi_2(G) \quad \text{ and } \quad \pi_1(G) \cong \pi_1(\widehat{G})/\operatorname{coker} \mathrm{per}_\omega.$$

These relations show how the period homomorphism controls how the first two homotopy groups of G and $\widehat{G}$ are related.

(b) We consider the special case where K is a simple compact Lie group and $G = C^\infty(\mathbb{T}^d,K)_e$, where $M = \mathbb{T}^d$ is a d-dimensional torus. Then $Y = V(\mathfrak{k}) \cong \mathbb{R}$, where the Cartan-Killing form $\kappa_\mathfrak{k}$ of $\mathfrak{k}$ is universal, and $\pi_1(\mathbb{T}^d) \cong \mathbb{Z}^d$ implies $\mathfrak{z}_{\mathbb{T}^d}(\mathbb{R}) \cong \mathbb{R}^d$, where the projection onto the components is given by integrating over the coordinate loops $\alpha_j\colon \mathbb{T} \hookrightarrow \mathbb{T}^d$, $j = 1,\ldots,d$.

According to Remark I.11(b), we have

$$\pi_2(G) \cong \pi_2(K) \oplus \pi_3(K)^d \oplus \pi_4(K)^{\binom{d}{2}} \oplus \ldots.$$

Since $\pi_2(K)$ is trivial and $\pi_3(K) \cong \mathbb{Z}$ (Remark II.3), we have

$$\pi_2(G) \cong \mathbb{Z}^d \oplus E,$$

where $E \cong \pi_4(K)^{\binom{d}{2}} \oplus \ldots$. The natural homomorphism $\mathbb{Z}^d \hookrightarrow \pi_2(G)$ is obtained from the map

$$C^\infty(\mathbb{T},K)^d \to G, \quad (g_j)_{j=1,\ldots,d} \mapsto (g_1 \circ p_1) \cdots (g_d \circ p_d),$$

where $p_j\colon \mathbb{T}^d \to \mathbb{T}$ is the projection onto the j-component. As we have seen above, the period map $\mathrm{per}_{\omega_{M,\kappa}}$ maps the subgroup $\mathbb{Z}^d$ bijectively onto the full period group

$$\Pi_{\mathbb{T}^d,\kappa} \cong \Pi^d_{\mathbb{S}^1,\kappa} \cong \mathbb{Z}^d \subseteq \mathfrak{z}_{\mathbb{T}^d}(\mathbb{R}) \cong \mathbb{R}^d.$$

We conclude in particular with (a) that

$$\pi_2(\widehat{G}) \cong \ker(\mathrm{per}_{\omega_{\mathbb{T}^d,\kappa}}) \cong \pi_2(G)/\pi_3(K)^d \cong \pi_4(K)^{\binom{d}{2}} \oplus \ldots.$$

As we have seen in Remark II.3, this group is not always trivial, showing that $\pi_2(\widehat{G})$ is not always trivial. This contradicts a statement in [PS86, Prop. 4.10.1] saying that $\pi_2(\widehat{G})$ is trivial. ■

For the following theorem we recall that we can use the continuous bilinear form $\kappa\colon \mathfrak{k}\times\mathfrak{k} \to Y$ to define a wedge product

$$\wedge_\kappa\colon \Omega^1(M,\mathfrak{k}) \times \Omega^1(M,\mathfrak{k}) \to \Omega^2(M,Y)$$

by

$$(\alpha \wedge_\kappa \beta)(v,w) := \kappa(\alpha_p(v),\beta_p(w)) - \kappa(\beta_p(v),\alpha_p(w)), \quad v,w \in T_p(M).$$

We also define for $\xi \in C^\infty(M,\mathfrak{k})$ and $\alpha \in \Omega^1(M,\mathfrak{k})$ the wedge product

$$\xi \wedge_\kappa \alpha := -\alpha \wedge_\kappa \xi := \kappa(\xi,\alpha)$$

and observe that

$$d(\xi \wedge_\kappa \alpha) = d\xi \wedge_\kappa \alpha + \kappa(\xi, d\alpha).$$

For each smooth map $f\colon M \to G$ we then have

$$\big(\operatorname{Ad}(f).\alpha\big) \wedge_\kappa \beta = \alpha \wedge_\kappa \big(\operatorname{Ad}(f)^{-1}.\beta\big), \tag{3.7}$$

where $(\operatorname{Ad}(f).\alpha)(v) = \operatorname{Ad}(f(p)).\alpha(v)$ for $v \in T_p(M)$, because the bilinear map κ is invariant under $\operatorname{Ad}(K)$. We likewise get

$$[\xi,\alpha] \wedge_\kappa \beta = -\alpha \wedge_\kappa [\xi,\beta] \tag{3.8}$$

for $\xi \in C^\infty(M,\mathfrak{k})$, where $[\beta,\xi]_p(v) := -[\xi,\beta]_p(v) := [\beta_p(v),\xi(p)]$. We also have a wedge product

$$[\cdot,\cdot]_\wedge\colon \Omega^1(M,\mathfrak{k}) \times \Omega^1(M,\mathfrak{k}) \to \Omega^2(M,\mathfrak{k})$$

defined by

$$[\alpha,\beta]_\wedge(v,w) := [\alpha_p(v),\beta_p(w)] - [\alpha_p(w),\beta_p(v)], \quad v,w \in T_p(M).$$

Note that $[\alpha,\beta]_\wedge = [\beta,\alpha]_\wedge$. The two wedge products are related by the formula

$$\kappa([\alpha,\beta]_\wedge,\xi) = \alpha \wedge_\kappa [\beta,\xi], \quad \xi \in C^\infty(M,\mathfrak{k}). \tag{3.9}$$

Theorem III.9. *Let $G^+ := C^\infty(M,K)$. Then the map*

$$c\colon G^+ \times G^+ \to \Omega^2(M,Y), \quad c(f,g) := \delta^l(f) \wedge_\kappa \delta^r(g)$$

defines a a smooth $\Omega^2(M,Y)$-valued group 2-cocycle on G^+, so that we obtain a central Lie group extension $\widehat{G}^+ := G^+ \times_c \Omega^2(M,Y)$. The corresponding Lie algebra cocycle Dc from (3.1) *is given by*

$$Dc(\xi,\eta) = 2d\xi \wedge_\kappa d\eta \quad \text{for} \quad \xi,\eta \in C^\infty(M,\mathfrak{k}).$$

The map $\gamma\colon \mathfrak{z}_M(Y) \to \Omega^2(M,Y), [\beta] \mapsto 2d\beta$ satisfies $\gamma \circ \omega_{M,\kappa} = Dc$ and induces a Lie algebra homomorphism

$$\gamma_{\mathfrak{g}}\colon \widehat{\mathfrak{g}} = \mathfrak{g} \oplus_{\omega_{M,\kappa}} \mathfrak{z}_M(Y) \to \widehat{\mathfrak{g}}^+ := \mathfrak{g} \oplus_{Dc} \Omega^2(M,Y), \quad (X,[\beta]) \mapsto (X, 2d\beta).$$

This homomorphism is G^+-equivariant with respect to the action on $\widehat{\mathfrak{g}}^+$ induced by the adjoint action of $\widehat{G}^+$, given by

$$\operatorname{Ad}_{\widehat{\mathfrak{g}}^+}(g).(\xi,z) = \big(\operatorname{Ad}(g).\xi, z - d(\kappa(\delta^l(g),\xi))\big).$$

Proof. The smoothness of the cocycle follows from the smoothness of the maps

$$\delta^l, \delta^r\colon C^\infty(M,K) \to \Omega^1(M,\mathfrak{k})$$

and the continuity of κ.

For the constant function $f = e$ we have $\delta^l(f) = \delta^r(f) = 0$, so that $c(g,e) = c(e,g) = 0$. Moreover, we obtain with (3.2), (3.3) and (3.7):

$$\begin{aligned} c(f,gh) - c(fg,h) &= \delta^l(f) \wedge_\kappa \delta^r(gh) - \delta^l(fg) \wedge_\kappa \delta^r(h) \\ &= \delta^l(f) \wedge_\kappa \big(\delta^r(g) + \mathrm{Ad}(g).\delta^r(h)\big) - \big(\delta^l(g) + \mathrm{Ad}(g)^{-1}.\delta^l(f)\big) \wedge_\kappa \delta^r(h) \\ &= c(f,g) - c(g,h) + \delta^l(f) \wedge_\kappa \big(\mathrm{Ad}(g).\delta^r(h)\big) - \big(\mathrm{Ad}(g)^{-1}.\delta^l(f)\big) \wedge_\kappa \delta^r(h) \\ &= c(f,g) - c(g,h) \end{aligned}$$

Therefore c is a group cocycle.

According to [Ne02, Lemma IV.6] and Lemma III.2, the corresponding Lie algebra cocycle $Dc \in Z^2_c(C^\infty(M,\mathfrak{k}), Y)$, is given by

$$\begin{aligned} Dc(\xi,\eta) &= d^2c(e,e)(\xi,\eta) - d^2c(e,e)(\eta,\xi) \\ &= d\delta^l(e)(\xi) \wedge_\kappa d\delta^r(e)(\eta) - d\delta^l(e)(\eta) \wedge_\kappa d\delta^r(e)(\xi) \\ &= d\xi \wedge_\kappa d\eta - d\eta \wedge_\kappa d\xi = 2 d\xi \wedge_\kappa d\eta. \end{aligned}$$

To relate the Lie algebra cocycles $\omega_{M,\kappa}$ and Dc, we first observe that the differential $d\colon \Omega^1(M,Y) \to \Omega^2(M,Y)$ leads to a linear map $\gamma\colon \mathfrak{z}_M(Y) \to \Omega^2(M,Y), [\beta] \mapsto 2d\beta$. This map satisfies

$$\gamma \circ \omega_{M,\kappa}(\xi,\eta) = 2d(\kappa(\xi, d\eta)) = 2d(\xi \wedge_\kappa d\eta) = 2d\xi \wedge_\kappa d\eta = Dc(\xi,\eta).$$

This implies that $\gamma_\mathfrak{g}$ is a Lie algebra homomorphism.

Next we derive an explicit formula for the action of G^+ on the Lie algebra

$$\widehat{\mathfrak{g}}^+ := \mathfrak{g} \oplus_{Dc} \Omega^2(M,Y)$$

from which it will follow that $\gamma_\mathfrak{g}$ is G^+-equivariant. The conjugation action of G^+ on the group $\widehat{G}^+$ is given by

$$g.(h,0) := (g,0)(h,0)(g,0)^{-1} = \big(ghg^{-1}, c(g,h) - c(ghg^{-1}, g)\big)$$

([Ne02b, Rem. I.2]) which implies that the derived action is given by

$$\mathrm{Ad}_{\widehat{\mathfrak{g}}^+}(g).(\xi,0) = \big(\mathrm{Ad}(g).\xi, dc(g,e)(0,\xi) - dc(e,g)(\mathrm{Ad}(g).\xi,0)\big).$$

We have seen in Lemma III.2 that

$$dc(g,e)(0,\xi) = \delta^l(g) \wedge_\kappa d\xi,$$

and with (3.6) we further get

$$\begin{aligned} dc(e,g)(\mathrm{Ad}(g).\xi,0) &= d(\mathrm{Ad}(g).\xi) \wedge_\kappa \delta^r(g) = \mathrm{Ad}(g).[\delta^l(g),\xi] \wedge_\kappa \delta^r(g) + \big(\mathrm{Ad}(g).d\xi\big) \wedge_\kappa \delta^r(g) \\ &= \mathrm{Ad}(g).[\delta^l(g),\xi] \wedge_\kappa \delta^r(g) + d\xi \wedge_\kappa \big(\mathrm{Ad}(g)^{-1}.\delta^r(g)\big) \\ &= [\delta^l(g),\xi] \wedge_\kappa \delta^l(g) + d\xi \wedge_\kappa \delta^l(g). \end{aligned}$$

This leads to

$$\mathrm{Ad}_{\widehat{\mathfrak{g}}^+}(g).(\xi,0) = \big(\mathrm{Ad}(g).\xi, 2\delta^l(g) \wedge_\kappa d\xi + \delta^l(g) \wedge_\kappa [\delta^l(g),\xi]\big).$$

To show that $\gamma_\mathfrak{g}$ is G^+-equivariant, we have to verify that

$$\mathrm{Ad}_{\widehat{\mathfrak{g}}^+}(g).(\xi,0) := \big(\mathrm{Ad}(g).\xi, -2d\big(\kappa(\delta^l(g),\xi)\big)\big) \tag{3.10}$$

(see (3.4)). The Maurer–Cartan Equation

$$d\delta^l(f) = -\frac{1}{2}[\delta^l(f),\delta^l(f)]_\wedge, \quad f \in C^\infty(M,K)$$

([KM97, p.405]) implies

$$\begin{aligned} d\big(\kappa(\delta^l(f),\xi)\big) &= d(\delta^l(f) \wedge_\kappa \xi) = d\delta^l(f) \wedge_\kappa \xi - \delta^l(f) \wedge_\kappa d\xi \\ &= -\frac{1}{2}[\delta^l(f),\delta^l(f)]_\wedge \wedge_\kappa \xi - \delta^l(f) \wedge_\kappa d\xi = -\frac{1}{2}\delta^l(f) \wedge_\kappa [\delta^l(f),\xi] - \delta^l(f) \wedge_\kappa d\xi. \end{aligned}$$

This relation immediately gives the desired formula for $\mathrm{Ad}_{\widehat{\mathfrak{g}}^+}(f)$. ∎

Remark III.10. Since the central extension $\widehat{G}^+$ of G^+ has a smooth global section, its period group $\Pi_{Dc} = \gamma(\Pi_{M,\kappa}) \subseteq \Omega^2(M,Y)$ is trivial ([Ne02b, Prop. VIII.5]). This is another argument for the inclusion $\Pi_{M,\kappa} \subseteq H^1_{\rm dR}(M,Y)$ (Corollary I.9). It is remarkable that we obtain a central extension of the whole group G^+ and not only of its identity component G. ■

Remark III.11. (a) Since M is compact, its fundamental group $\pi_1(M)$ is finitely generated. Let $k := b_1(M) := \operatorname{rk} H_1(M)$ and choose $\alpha_1, \ldots, \alpha_k \in C^\infty(\mathbb{S}^1, M)$ as in Remark I.3. Then the integration map

$$\Phi: \mathfrak{z}_M(Y) \to Y^k, \quad [\beta] \mapsto \Big(\int_{\alpha_j} \beta\Big)_{j=1,\ldots,k}$$

maps the subspace $H^1_{\rm dR}(M,Y)$ bijectively onto Y^k, so that we obtain a topological splitting

$$\mathfrak{z}_M(Y) \cong H^1_{\rm dR}(M,Y) \oplus \ker \Phi.$$

Then the differential $d: \mathfrak{z}_M(Y) \to \Omega^2(M,Y), [\beta] \mapsto d\beta$ maps $\ker\Phi$ continuously onto the closed subspace $B^2_{\rm dR}(M,Y)$ of exact 2-forms in $\Omega^2(M,Y)$.

Suppose that $\Pi_{M,\kappa}$ is discrete. Then the group Z from Theorem III.6 has a product decomposition

$$Z \cong \Big(H^1_{\rm dR}(M,Y)/\Pi_{M,\kappa}\Big) \times \ker\Phi \cong \Big(Y/\Pi_{\mathbb{S}^1,\kappa}\Big)^k \times \ker\Phi$$

(cf. Theorem I.6).
(b) The differential $d: \mathfrak{z}_M(Y) \to \Omega^2(M,Y)$ induces a Lie algebra homomorphism

$$\gamma_{\mathfrak{g}}: \widehat{\mathfrak{g}} = \mathfrak{g} \oplus_{\omega_{M,\kappa}} \mathfrak{z}_M(Y) \to \widehat{\mathfrak{g}}^+ = \mathfrak{g} \oplus_{Dc} \Omega^2(M,Y), \quad (\xi,[\beta]) \mapsto (\xi, 2d\beta).$$

The construction of a corresponding Lie group homomorphism $\widehat{G} \to \widehat{G}^+$, where $\widehat{G}$ is a central extension of G by $Z = \mathfrak{z}_M(Y)/\Pi_{M,\kappa}$ (Theorem III.6) is not so obvious because the values of the cocycle c in Theorem III.9 are in general not exact forms (Remark III.13 below), hence do not lie in the range of the map d. Nevertheless, the range of the Lie algebra cocycle Dc is contained in the space of exact forms. Suppose that Y is a Fréchet space. Then the quotient map $p: \Omega^2(M,Y) \to E := \Omega^2(M,Y)/B^2_{\rm dR}(M,Y)$ is an open morphism of Fréchet spaces. We obtain a smooth group cocycle $c_- := p \circ c \in Z^2_s(G^+, E)$ whose corresponding Lie algebra cocycle is trivial. According to [Ne02b, Th. VIII.8], there exists a homomorphism $\alpha: \pi_1(G) \to E$ such that $G \times_{c_-} E \cong (\widetilde{G} \times E)/\Gamma(\alpha)$, where $\Gamma(\alpha) \subseteq \pi_1(G) \times E$ is the graph of α. Is this extension trivial? Since G is smoothly paracompact, there exists a smooth function $f: \widetilde{G} \to E$ with $f(gd) = f(g) + \alpha(d)$, $g \in \widetilde{G}$, $d \in \pi_1(G)$ ([Ne02b, Prop. III.8]).
(c) If Y is Fréchet, the same holds for the space $\Omega^2(M,Y)$. Therefore $\widehat{G}^+$ is a central extension of the regular Fréchet–Lie group G^+ by the regular Fréchet–Lie group $\Omega^2(M,Y)$, hence regular ([KM97, Th. 38.6]). Therefore the Lie algebra homomorphism $\gamma_{\mathfrak{g}}: \widehat{\mathfrak{g}} \to \mathfrak{g} \oplus_{Dc} B^2_{\rm dR}(M,Y)$ integrates to a unique Lie group homomorphism $\widetilde{\gamma}_G: G^\sharp \to \widetilde{G} \times_{\widetilde{c}} \Omega^2(M,Y)$, where $G^\sharp$ is the central Lie group extension of the universal covering group $\widetilde{G}$ of G by $Z = \mathfrak{z}_M(Y)/\Pi_{M,\kappa}$ (Theorem III.6). Then the surjectivity of the period homomorphism $\pi_2(G) \cong \pi_2(\widetilde{G}) \to \pi_1(Z)$ implies that $G^\sharp$ is simply connected (Remark III.8). Since the natural map $\pi_1(\widehat{G}) \to \pi_1(G)$ is an isomorphism (Remark III.8), it follows that $\widetilde{\gamma}_G(\pi_1(\widehat{G})) \subseteq \pi_1(G)$, and hence that $\widetilde{\gamma}_G$ factors through a Lie group homomorphism $\gamma_G: \widehat{G} \to \widehat{G}^+$ with $\mathbf{L}(\gamma_G) = \gamma_{\mathfrak{g}}$. ■

Remark III.12. (The abelian case) We assume that K is a connected abelian Lie group with universal covering group $\widetilde{K} = (\mathfrak{k}, +)$. Then $K \cong \mathfrak{k}/\Gamma$, where $\Gamma \cong \pi_1(K)$ is a discrete subgroup of $\mathfrak{k}$. Let $q_K: \mathfrak{k} \to K$ denote the quotient map.

Let M be a compact connected manifold. Then the group $G^+ = C^\infty(M,K)$ is abelian and its identity component $G = C^\infty(M,K)_e$ is the image of the exponential map

$$\exp_G: \mathfrak{g} = C^\infty(M,\mathfrak{k}) \to G, \quad \xi \mapsto q_K \circ \xi.$$

Therefore $\widetilde{G} = \mathfrak{g} = C^\infty(M, \mathfrak{k})$ is contractible, and $\pi_k(G) = \mathbf{1}$ for $k \geq 2$. We further have

$$\pi_1(G) \cong \ker \exp_G \cong C^\infty(M, \Gamma) \cong \Gamma \quad \text{ and } \quad \pi_0(G) \cong \mathrm{Hom}(\pi_1(M), \Gamma) \cong \Gamma^k$$

for $k = b_1(M)$. Here we use [Ne02b, Prop. III.9] to see that each homomorphism $\pi_1(M) \to \Gamma$ is obtained from a smooth map $M \to K$ and that a smooth map $f: M \to K$ lifts to a smooth map $M \to \mathfrak{k}$ if and only if $\pi_1(f): \pi_1(M) \to \pi_1(K) \cong \Gamma$ is trivial. Let $\kappa: \mathfrak{k} \times \mathfrak{k} \to Y$ be a continuous bilinear form and $\omega(\xi, \eta) := [\kappa(\xi, d\eta)]$ the corresponding Lie algebra cocycle.

(a) Since each element of $\pi_1(G) \subseteq \mathfrak{g}$ corresponds to a constant function, we have $\omega(\pi_1(G), \mathfrak{g}) = \{0\}$, so that

$$c_G(\exp_G \xi, \exp_G \eta) := \tfrac{1}{2}\omega(\xi, \eta) = \tfrac{1}{2}[\kappa(\xi, d\eta)]$$

defines a global $\mathfrak{z}_M(Y)$-valued group cocycle on G, and we obtain a central extension

$$\widehat{G} = G \times_{c_G} \mathfrak{z}_M(Y)$$

which can be lifted to a central Lie group extension

$$\widetilde{G} \times_{\widetilde{c}_G} \mathfrak{z}_M(Y) \quad \text{ with } \quad \widetilde{c}_G := c_G \circ (\exp_G \times \exp_G),$$

i.e., $\widetilde{c}_G(\xi, \eta) = [\kappa(\xi, d\eta)]$.

On the other hand we have the central extension $G^+ \times_c \Omega^2(M, Y)$ given by the cocycle

$$c(g, h) = \delta^l(g) \wedge_\kappa \delta^r(h) = \delta^l(g) \wedge_\kappa \delta^l(h)$$

(Theorem III.9). Note that $\delta^r = \delta^l$ follows from K being abelian. Since each left invariant 1-form on an abelian Lie group is closed, the Maurer–Cartan form θ_K is closed, hence $\delta^l(f) = f^*\theta_K$ is closed for each smooth function $f: M \to K$, so that all 2-forms $c(g, h)$ are closed.

As we will see below, they are not always exact. For elements $g = \exp_G \xi$ and $h = \exp_G \eta$ in the identity component G of G^+ we have

$$c(g, h) = d\xi \wedge_\kappa d\eta = d\big(\kappa(\xi, d\eta)\big) = 2d\big(c_G(g, h)\big),$$

so that

$$G \times_{c_G} \mathfrak{z}_M(Y) \to G^+ \times_c \Omega^2(M, Y), \quad (g, [\beta]) \mapsto (g, 2d\beta)$$

is a Lie group homomorphism.

(b) Let $q_M: \widetilde{M} \to M$ denote the universal covering map and $g \in G^+$. Then the map $\widetilde{g} := g \circ q_M$ can be written as $\exp_K \circ \widetilde{\xi}$, where $\widetilde{\xi} \in C^\infty(\widetilde{M}, \mathfrak{k})$. We likewise write $\widetilde{h} = \exp_K \circ \widetilde{\eta}$ for a second element $h \in G^+$. Then

$$q_M^* c(g, h) = q_M^*(\delta^l(g) \wedge_\kappa \delta^l(h)) = d\widetilde{\xi} \wedge_\kappa d\widetilde{\eta} = d(\widetilde{\xi} \wedge_\kappa d\widetilde{\eta})$$

is an exact 2-form on $\widetilde{M}$. This means that $[c(g, h)] \in H^2(M, Y) \cong \mathrm{Hom}(H_2(M), Y)$ vanishes on the image of $\pi_2(M) \cong H_2(\widetilde{M})$ in $H_2(M)$.

(c) For $M = \mathbb{T}^2$, $K = \mathbb{T}$, $Y = \mathbb{R}$, $\kappa(x, y) = xy$, $g(t_1, t_2) = t_1$ and $h(t_1, t_2) = t_2$ we obtain on $\widetilde{M} \cong \mathbb{R}^2$:

$$q_M^* c(g, h) = dx \wedge dy$$

and therefore $\int_M c(g, h) \neq 0$. In particular $c(g, h)$ is not exact.

(d) Since K is abelian, the group $\pi_0(G^+)$ acts trivially on $\widetilde{G}$ and hence on $\pi_1(G)$. The actions of G^+ on $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}^+$ are given by

$$\mathrm{Ad}_{\widehat{\mathfrak{g}}}(g).(\xi, z) = (\xi, z - [\kappa(\delta^l(g), \xi)])$$

and

$$\mathrm{Ad}_{\widehat{\mathfrak{g}}^+}(g).(\xi, z) = (\xi, z - d\kappa(\delta^l(g), \xi)) = (\xi, z - \delta^l(g) \wedge_\kappa d\xi).$$

For each constant map $\xi \in \Gamma \subseteq \widetilde{G} \cong \mathfrak{g}$ we therefore obtain $\mathrm{Ad}_{\widehat{\mathfrak{g}}^+}(g).(\xi, z) = (\xi, z)$, but for each $g \in G$ the 1-form $\kappa(\delta^l(g), \xi) = \delta^l(g) \wedge_\kappa \xi$ is closed, and for $\alpha \in C^\infty(\mathbb{S}^1, M)$ we have

$$\int_\alpha \kappa(\delta^l(g), \xi) = \kappa\Big(\int_\alpha \delta^l(g), \xi\Big)$$

with $\int_\alpha \delta^l(g) \in \Gamma$. Therefore the action of $\pi_0(G^+)$ on $\pi_1(G) \times \mathfrak{z}_M(Y) \cong \Gamma \times \mathfrak{z}_M(Y)$ is given by

$$g.(\gamma, z) = (\gamma, z - [\kappa(\delta^l(g), \gamma)]),$$

where $[\kappa(\delta^l(g), \gamma)] \in H^1_{\mathrm{dR}}(M, Y) \cong H^1(M, Y) \cong \mathrm{Hom}(\pi_1(M), Y)$ corresponds to the homomorphism $\kappa(\pi_1(g), \gamma): \pi_1(M) \to Y$. This action is non-trivial if and only if $\kappa(\Gamma, \Gamma) \neq \{0\}$. ■

Remark III.13. Let K be a compact Lie group and $M := K \times K$. We consider the smooth maps

$$f: M \to K, \quad (k_1, k_2) \mapsto k_1 \quad \text{and} \quad g: M \to K, \quad (k_1, k_2) \mapsto k_2^{-1}.$$

Let $p_1, p_2: M \to K$ denote the projections onto the factors. Then $\delta^l(f) = p_1^*\theta_K^l$ and $\delta^r(g) = -p_2^*\theta_K^l$ holds for the left Maurer–Cartan form θ_K^l on K. Hence $c(f,g) = -p_1^*\theta_K^l \wedge_\kappa p_2^*\theta_K^l$ is a left invariant 2-form on the compact Lie group $M = K \times K$. Let $\beta := c(f,g)_e$. Then

$$\beta((x,y),(x',y')) = -\kappa(x,y') + \kappa(x',y).$$

Since K is a compact connected Lie group, the form $c(f,g)$ is closed/exact if and only if β is closed/exact as a Lie algebra cochain. For every continuous linear map $\alpha: \mathfrak{k} \times \mathfrak{k} \to Y$ we have

$$\alpha([(x,y),(x',y')]) = \alpha([x,x'],0) + \alpha(0,[y,y']).$$

Therefore $c(f,g)$ is exact if and only if $\kappa = 0$.

The closedness of $c(f,g)$ is equivalent to the vanishing of

$$\kappa([x',x''],y) - \kappa([y',y''],x) + \kappa([x'',x],y') - \kappa([y'',y],x') + \kappa([x,x'],y'') - \kappa([y,y'],x'').$$

Using this identity for $y' = y'' = 0$, we see that $c(f,g)$ is closed if and only if $\kappa(\mathfrak{k}, [\mathfrak{k}, \mathfrak{k}]) = \{0\}$. ■

IV. Universal central extensions

In this section we turn to the question whether the central extension from Corollary III.7 is universal. This question will be answered affirmatively if $\mathfrak{k}$ is finite-dimensional and semisimple. First we recall some concepts and a result from [Ne01c] on weakly universal central extensions of Lie groups and Lie algebras.

Definition IV.1. (cf. [Ne01c]) Let $\mathfrak{g}$ be a topological Lie algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathfrak{a}$ be a topological vector space considered as a trivial $\mathfrak{g}$-module. We call a central extension $q: \widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$ with $\mathfrak{z} = \ker q$ (or simply the Lie algebra $\widehat{\mathfrak{g}}$) *weakly universal for* $\mathfrak{a}$ if the corresponding map $\delta_{\mathfrak{a}}: \mathrm{Lin}(\mathfrak{z}, \mathfrak{a}) \to H^2_c(\mathfrak{g}, \mathfrak{a}), \gamma \mapsto [\gamma \circ \omega]$ is bijective.

We call $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ *universal for* $\mathfrak{a}$ if for every linearly split central extension $q_1: \widehat{\mathfrak{g}}_1 \to \mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{a}$ there exists a unique homomorphism $\varphi: \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}_1$ with $q_1 \circ \varphi = q$. Note that this universal property immediately implies that two central extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by $\mathfrak{a}_1$ and $\mathfrak{a}_2$ which are both universal for both spaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are isomorphic. A central extension is said to be *(weakly) universal* if it is (weakly) universal for all locally convex spaces $\mathfrak{a}$. ■

Definition IV.2. We call a central extension $\widehat{G} = G \times_f Z$ of the connected Lie group G by the abelian Lie group Z *weakly universal for the abelian Lie group* A if the map

$$\delta_A \colon \mathrm{Hom}(Z, A) \to H^2_s(G, A), \quad \gamma \mapsto [\gamma \circ f]$$

is bijective. It is called *universal for the abelian group* A if for every central extension

$$q_1 \colon G \times_\varphi A \to G, \quad \varphi \in Z^2_c(G, A),$$

there exists a unique Lie group homomorphism $\psi \colon G \times_f Z \to G \times_\varphi A$ with $q_1 \circ \psi = q$. A central extensional is said to be *(weakly) universal* if it is (weakly) universal for all Lie groups A with $A_e \cong \mathfrak{a}/\pi_1(A)$ and $\mathfrak{a}$ s.c.l.c. ■

Definition IV.3. If $\mathfrak{g}$ is a Fréchet–Lie algebra, then we write $H_1(\mathfrak{g}) := \mathfrak{g}/\mathfrak{g}'$, where $\mathfrak{g}' := \overline{[\mathfrak{g}, \mathfrak{g}]}$ is the closed commutator algebra. The space $H_1(\mathfrak{g})$ is a Fréchet space because $\mathfrak{g}'$ is closed. If G is a connected Lie group with Lie algebra $\mathfrak{g}$ and $\widetilde{G}$ its universal covering group, then we have a natural homomorphism $d_G \colon \widetilde{G} \to H_1(\mathfrak{g})$. Its kernel is denoted by $(\widetilde{G}, \widetilde{G})$. If G is finite-dimensional, then $(\widetilde{G}, \widetilde{G})$ is the commutator group of $\widetilde{G}$. ■

The following theorem is [Ne01c, Th. IV.13].

Theorem IV.4. (Recognition Theorem) *Assume that $q \colon \widehat{G} \to G$ is a central Z-extension of Fréchet–Lie groups over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ for which*

(1) *the corresponding Lie algebra extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is weakly $\mathbb{K}$-universal,*

(2) *$\widehat{G}$ is simply connected, and*

(3) $\pi_1(G) \subseteq (\widetilde{G}, \widetilde{G})$.

If $\widehat{\mathfrak{g}}$ is weakly universal for a Fréchet space $\mathfrak{a}$, then $\widehat{G}$ is weakly universal for each abelian Fréchet–Lie group A with Lie algebra $\mathfrak{a}$ and $A_e \cong \mathfrak{a}/\pi_1(A)$. ■

Lemma IV.5. *If $q \colon Y \to Z$ is a surjective morphism of Fréchet spaces and X a Fréchet space, then the natural map $\mathrm{id}_X \otimes q \colon X \widehat{\otimes} Y \to X \widehat{\otimes} Z$ is a quotient map.*

Proof. Let $f \in X \widehat{\otimes} Z$. Then f can be written as $f = \sum_{n \in \mathbb{N}} \lambda_n x_n \otimes z_n$ with $\lambda \in l^1(\mathbb{N}, \mathbb{R})$, $x_n \to 0$ and $z_n \to 0$ ([Tr67, Th. 45.1]). Let p_X, p_Y, resp., p_Z, be a continuous seminorm on X, Y, resp., Z. We further assume that

$$p_Z(q(y)) = \inf p_Y(y + \ker q),$$

because the seminorms on the right hand side define the topology on Z.

Using the quotient metric on Z, we find $y_n \in Y$ with $q(y_n) = z_n$ and $y_n \to 0$. Then

$$\widetilde{f} := \sum_n \lambda_n x_n \otimes y_n$$

converges absolutely in $X \widehat{\otimes} Y$ because

$$\sum_n |\lambda_n| (p_X \otimes p_Y)(x_n \otimes y_n) \leq \sum_n |\lambda_n| p_X(x_n) p_Y(y_n)$$

and the right hand side converges since $p_X(x_n) \to 0$ and $p_Y(y_n) \to 0$. Moreover, we have $q(\widetilde{f}) = f$. This implies that $\mathrm{id}_X \otimes q$ is surjective, hence a quotient map by the Open Mapping Theorem. ■

Proposition IV.6. *Let M be a compact manifold, K a connected Fréchet–Lie group and $G := C^\infty(M,K)_e$. Then*

$$\pi_1(G) \subseteq (\widetilde G, \widetilde G) \qquad \iff \qquad \pi_1(K) \subseteq (\widetilde K, \widetilde K).$$

Moreover, we have

$$\mathfrak g' = C^\infty(M, \mathfrak k') \quad \text{and} \quad H_1(\mathfrak g) \cong C^\infty(M, H_1(\mathfrak k)).$$

Proof. Let $x_M \in M$ be any point and consider the evaluation map $\delta: G \to K, f \mapsto f(x_M)$. We write $\delta_{\mathfrak g}: \mathfrak g \to \mathfrak k$ for the corresponding Lie algebra homomorphism. With $G_* := C^\infty_*(M,K)_e := \ker\delta$ we obtain $G \cong G_* \rtimes K$, i.e., the Lie group homomorphism δ is split by the natural inclusion $\iota: K \to G$ by viewing elements of K as constant maps on M. We thus obtain a commutative diagram, where the vertical homomorphisms split:

$$\begin{array}{ccc} \widetilde G & \xrightarrow{d_G} & H_1(\mathfrak g) \\ \big\downarrow{\scriptstyle \widetilde\delta} & & \big\downarrow{\scriptstyle H_1(\delta_{\mathfrak g})} \\ \widetilde K & \xrightarrow{d_K} & H_1(\mathfrak k). \end{array}$$

The splitting of the vertical arrows implies that we may view the lower horizontal homomorphism as the restriction of d_G to $\widetilde K$. As $\pi_1(K)$ can be viewed as a subgroup of $\pi_1(G)$, this shows that $\pi_1(G) \subseteq (\widetilde G, \widetilde G) = \ker d_G$ implies $\pi_1(K) \subseteq \ker d_K = (\widetilde K, \widetilde K)$.

Suppose, conversely, that $\pi_1(K) \subseteq \ker d_K$, i.e., that d_K factors through a Lie group homomorphism $d_K': K \to H_1(\mathfrak k)$ which in turn induces a Lie group homomorphism

$$\eta: C^\infty(M,K) \to C^\infty(M, H_1(\mathfrak k)), \quad \eta(f)(m) := d_K'(f(m)).$$

On the other hand, the natural Lie algebra homomorphism $\beta: C^\infty(M,\mathfrak k) \to C^\infty(M, H_1(\mathfrak k))$ is a quotient map (Remark II.5, Lemma IV.5) with kernel $C^\infty(M, \mathfrak k')$.

We claim that $\mathfrak g' = C^\infty(M,\mathfrak k')$. It is obvious that the commutator algebra $\mathfrak g'$ is contained in $C^\infty(M,\mathfrak k')$. We further have for each $f \in C^\infty(M,\mathfrak k)$ the relation

$$f \otimes \sum_j [x_j, y_j] = \sum_j [1 \otimes x_j, f \otimes y_j] \in \mathfrak g',$$

which leads to $C^\infty(M,\mathbb R)\widehat\otimes \mathfrak k' \cong C^\infty(M,\mathfrak k') \subseteq \mathfrak g'$ because $\mathfrak k'$ is a complete locally convex space (Remark II.5), and hence to $C^\infty(M,\mathfrak k') = \mathfrak g'$. Putting the information together, we conclude that

$$H_1(\mathfrak g) = \mathfrak g/\mathfrak g' = C^\infty(M,\mathfrak k)/C^\infty(M,\mathfrak k') \cong C^\infty(M, H_1(\mathfrak k)).$$

Therefore the Lie group homomorphism $G \subseteq C^\infty(M,K) \to C^\infty(M, H_1(\mathfrak k))$ integrates the Lie algebra homomorphism $\mathfrak g \to H_1(\mathfrak g)$, which implies that $\pi_1(G) \subseteq \ker d_G = (\widetilde G, \widetilde G)$. ■

Theorem IV.7. *Suppose that K is finite-dimensional semisimple and let $G := C^\infty(M,K)_e$. Let $\mathfrak z := \mathfrak z_M(V(\mathfrak k))$ and $\omega \in Z^2_c(\mathfrak g, \mathfrak z)$ the cocycle given by $\omega(\eta,\xi) = [\kappa(\eta, d\xi)]$. Then the corresponding central Lie algebra extension $\widehat{\mathfrak g} := \mathfrak g \oplus_\omega \mathfrak z$ is universal and there exists a corresponding central Lie group extension $Z \hookrightarrow \widehat G \twoheadrightarrow G$ with $Z \cong \pi_1(G) \times (\mathfrak z/\Pi_\omega)$ which is universal for all abelian Fréchet–Lie groups A with $A_e \cong \mathfrak a/\pi_1(A)$.*

Proof. First we note that $\widehat{\mathfrak g} \to \mathfrak g$ is a covering (Lemma II.7), so that for each locally convex space $\mathfrak a$ the natural map

$$\delta: \mathrm{Lin}(\mathfrak z, \mathfrak a) \to H^2_c(\mathfrak g, \mathfrak a), \quad \gamma \mapsto [\gamma \circ \omega]$$

is injective ([Ne01c, Rem. I.6]).

It has been shown in [Ma02, Thm. 16] that δ is also surjective, so that $\widehat{\mathfrak g}$ is weakly universal for all locally convex spaces $\mathfrak a$. Since $\mathfrak g$ is perfect by Proposition IV.6, the Lie algebra $\widehat{\mathfrak g}$ is a universal central extension of $\mathfrak g$.

Furthermore, the period map $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ has discrete image Π_ω (Theorem II.9). In view of Theorem III.6, [Ne02b, Prop. VII.13] now implies the existence of a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ with $Z \cong (\mathfrak{z}/\Pi_\omega) \times \pi_1(G)$ corresponding to the Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \to \mathfrak{g}$ and such that the connecting homomorphism $\pi_1(G) \to \pi_0(Z)$ is an isomorphism.

To prove the universality of $\widehat{G}$, we use the Recognition Theorem IV.4. For that we have to verify that

(1) $\widehat{\mathfrak{g}}$ is weakly universal,

(2) $\mathfrak{k}$ is Fréchet,

(3) $\pi_1(\widehat{G}) = \mathbf{1}$,

(4) $\pi_1(G) \subseteq (\widetilde{G}, \widetilde{G})$.

Since $\mathfrak{k}$ is finite-dimensional semisimple, (2) is trivially satisfied, and (1) has been verified above. Further (4) follows from $\pi_1(K) \subseteq \widetilde{K} = (\widetilde{K}, \widetilde{K})$ (Proposition IV.6). It therefore remains to verify (3). For that we consider a part of the long exact homotopy sequence of the Z-principal bundle $q\colon \widehat{G} \to G$ (cf. Remark III.8):

$$\pi_2(G) \xrightarrow{\delta} \pi_1(Z) \to \pi_1(\widehat{G}) \to \pi_1(G) \to \pi_0(Z). \tag{3.11}$$

According to [Ne02b, Prop. V.11], we have $\delta = -\mathrm{per}_\omega$, so that $\pi_1(Z) = \Pi_\omega$ (as subsets of $\mathfrak{z}$) implies that δ is surjective. Moreover, the natural homomorphism $\pi_1(G) \to \pi_0(Z)$ is an isomorphism by the construction of $\widehat{G}$, so that the exactness of (3.11) implies that $\widehat{G}$ is simply connected. ■

Remark IV.8. (a) If K is finite-dimensional and reductive, then $\widetilde{K} \cong \mathfrak{z}(\mathfrak{k}) \times (\widetilde{K}, \widetilde{K})$. Therefore $\pi_1(K)$ is contained in $(\widetilde{K}, \widetilde{K})$ if and only if $K \cong \mathfrak{z}(\mathfrak{k}) \times (K, K)$. In this case we have

$$C^\infty(M, K) \cong C^\infty(M, \mathfrak{z}(\mathfrak{k})) \times C^\infty(M, (K, K))$$

and hence we have for $G = C^\infty(M, K)_e$ the direct product decomposition

$$G = G_D \times G_Z \quad \text{with} \quad G_D := C^\infty(M, (K, K))_e \quad \text{and} \quad G_Z := C^\infty(M, \mathfrak{z}(\mathfrak{k})).$$

In this case the Lie algebra $\mathfrak{g} = C^\infty(M, \mathfrak{k})$ has the direct decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$ with $\mathfrak{g}' = C^\infty(M, \mathfrak{k}')$ (Proposition IV.6) and $\mathfrak{z}(\mathfrak{g}) = C^\infty(M, \mathfrak{z}(\mathfrak{k}))$. It is easy to see that every Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g}, Y)$ vanishes on $\mathfrak{g}' \times \mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g} \times \mathfrak{g}$ because $\mathfrak{g}'$ is perfect. From that one further derives that a weakly universal central extension of $\mathfrak{g}$ can be obtained with

$$\mathfrak{z} := \mathfrak{z}_M(V(\mathfrak{k}')) \oplus \Lambda^2(\mathfrak{z}(\mathfrak{g})),$$

where for a locally convex space E the space $\Lambda^2(E)$ is defined as the quotient of $E \otimes_\pi E$ modulo the closure of the subspace spanned by the elements $e \otimes e$, $e \in E$. To describe the corresponding cocycle, we write $\xi \in \mathfrak{g}$ as $\xi = (\xi', \xi_\mathfrak{z})$ with $\xi' \in \mathfrak{g}'$ and $\xi_\mathfrak{z} \in \mathfrak{z}(\mathfrak{g})$. Then a weakly universal cocycle is given by

$$\omega(\xi, \eta) = ([\kappa_{\mathfrak{k}'}(\xi', d\eta')], \xi_\mathfrak{z} \wedge \eta_\mathfrak{z}).$$

Let $\widehat{G}_D$ be the universal central extension of G_D from Theorem IV.7 and define $\widehat{G} := \widehat{G}_D \times \widehat{G}_Z$, where $\widehat{G}_Z$ is the 2-step nilpotent Lie algebra

$$\mathfrak{z}(\mathfrak{g}) \times_{\omega_Z} \Lambda^2(\mathfrak{z}(\mathfrak{g})) \quad \text{with} \quad \omega_Z(\xi, \eta) = \xi \wedge \eta.$$

Using Theorem IV.4, we see that $\widehat{G}_Z$ is a weakly universal central extension of $G_Z \cong \mathfrak{g}_Z$. Theorems IV.4 and IV.7 now imply that $\widehat{G}$ is a weakly universal central extension of G.

(b) As we have seen in Proposition IV.6, the Lie algebra $\mathfrak{g} = C^\infty(M, \mathfrak{k})$ has the commutator algebra $\mathfrak{g}' = C^\infty(M, \mathfrak{k}')$. On the other hand $\mathfrak{g} = \mathfrak{g}_* \rtimes \mathfrak{k}$, where $\mathfrak{k}$ corresponds to the constant functions in $\mathfrak{g}$, and $\mathfrak{g}_* := \{\xi \in \mathfrak{g} \colon \xi(x_M) = 0\}$, where $x_M \in M$ is any point. For two elements $\xi, \eta \in \mathfrak{g}_*$ we then have $d[\xi, \eta](x_M) = 0$, showing that $[\mathfrak{g}_*, \mathfrak{g}_*]$ is in general not dense in $C^\infty_*(M, \mathfrak{k}')$. This defect comes from the observation that in the algebra $C^\infty_*(M, \mathbb{R})$ the ideal $C^\infty_*(M, \mathbb{R})^2$ is contained in $\{f \in C^\infty_*(M, \mathbb{R}) \colon df(x_M) = 0\}$, and it is easy to see that we actually have equality. ■

V. Lifting automorphisms to central extensions

In this section we discuss the problem to associate to a pair (γ_G, γ_Z) of an automorphism γ_G of G and γ_Z of Z an automorphism $\widehat{\gamma}$ of a central extension $\widehat{G}$ of G by Z restricting to γ_Z on Z and inducing γ_G on G. This section is independent of the others. Its results apply to general infinite-dimensional Lie groups. The key results of this section are Proposition V.4 which gives for a simply connected G a necessary and sufficient condition for the existence of $\widehat{\gamma}$, and Theorem V.9, saying that for smooth actions of a Lie group R on G and Z which lead to a smooth action on the Lie algebra $\widehat{\mathfrak{g}}$, there exists a smooth action on the group $\widehat{G}$. In Section VI we will apply these results to the actions of the groups $\mathrm{Diff}(M)$ and $C^\infty(M, K)$ on $C^\infty(M, K)_e$.

For a Lie group G we write $\mathrm{Aut}(G)$ for the group of Lie group automorphisms of G and $\mathrm{Hom}(G_1, G_2)$ for the set of Lie group morphisms from G_1 to G_2. For a homomorphism $\varphi: G_1 \to G_2$ of Lie groups we write $\mathbf{L}(\varphi): \mathfrak{g}_1 \to \mathfrak{g}_2$ for the corresponding homomorphism of Lie algebras. In particular we then obtain a group homomorphism $\mathbf{L}: \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$. As above, let $Z \hookrightarrow \widehat{G} \xrightarrow{q} G$ be a central extension of connected Lie groups, where $Z_e \cong \mathfrak{z}/\pi_1(Z)$.

In the following we write $\gamma = (\gamma_G, \gamma_Z)$ for elements $\gamma \in \mathrm{Aut}(G) \times \mathrm{Aut}(Z)$. The group $\mathrm{Aut}(G) \times \mathrm{Aut}(Z)$ acts on the group $Z^2_s(G, Z)$ by

$$\gamma.f := \gamma_Z \circ f \circ (\gamma_G^{-1}, \gamma_G^{-1}).$$

It likewise acts on $Z^2_c(\mathfrak{g}, \mathfrak{z})$ by

$$f.\omega := \mathbf{L}(\gamma_Z) \circ f \circ (\mathbf{L}(\gamma_G)^{-1} \times \mathbf{L}(\gamma_G)^{-1}).$$

The following purely algebraic lemma will be quite useful in the following.

Lemma V.1. (a) *For $i = 1, 2$ let $\widehat{G}_i = G_i \times_{f_i} Z_i$ be a central Lie group extension of G_i by the abelian Lie group Z_i defined by $f_i \in Z^2_s(G_i, Z_i)$. For $\gamma = (\gamma_G, \gamma_Z) \in \mathrm{Hom}(G_1, G_2) \times \mathrm{Hom}(Z_1, Z_2)$ and a function $h: G_1 \to Z_2$ which is smooth in an identity neighborhood, the formula*

$$\widehat{\gamma}(g, z) := (\gamma_G(g), \gamma_Z(z)h(g)), \quad g \in G_1, z \in Z_1$$

defines a Lie group morphism $\widehat{G}_1 \to \widehat{G}_2$ if and only if the relation

$$\gamma_Z(f_1(g, g'))h(gg') = f_2\big(\gamma_G(g), \gamma_G(g')\big)h(g)h(g') \tag{5.1}$$

holds. Every Lie group homomorphism $\widehat{\gamma}: \widehat{G}_1 \to \widehat{G}_2$ mapping Z_1 into Z_2 is of this form.

For $G = G_1 = G_2$, $Z = Z_1 = Z_2$ and $(\gamma_G, \gamma_Z) \in \mathrm{Aut}(G) \times \mathrm{Aut}(Z)$, formula (5.1) *is equivalent to*

$$(\gamma.f)(g, g')f(g, g')^{-1} = h_0(gg')h_0(g)^{-1}h_0(g')^{-1}, \quad g, g' \in G \tag{5.2}$$

for the function $h_0 := \mathrm{inv}(h) \circ \gamma_G^{-1}$, where $\mathrm{inv}(h)(x) := h(x)^{-1}$.
(b) *For $i = 1, 2$ let $\widehat{\mathfrak{g}}_i = \mathfrak{g}_i \times_{\omega_i} \mathfrak{z}_i$ be a central extension of the topological Lie algebra $\mathfrak{g}_i$ by the abelian Lie algebra $\mathfrak{z}_i$ defined by $\omega_i \in Z^2_c(\mathfrak{g}_i, \mathfrak{z}_i)$. If $\gamma = (\gamma_{\mathfrak{g}}, \gamma_{\mathfrak{z}}) \in \mathrm{Lin}(\mathfrak{g}_1, \mathfrak{g}_2) \times \mathrm{Lin}(\mathfrak{z}_1, \mathfrak{z}_2)$, then for $\alpha \in \mathrm{Lin}(\mathfrak{g}_1, \mathfrak{z}_2)$ the formula*

$$\widehat{\gamma}(x, z) := (\gamma_{\mathfrak{g}}(x), \gamma_{\mathfrak{z}}(z) + \alpha(x)), \quad x \in \mathfrak{g}_1, z \in \mathfrak{z}_1,$$

defines a continuous Lie algebra morphism $\widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ if and only if the relation

$$\omega_2(\gamma_{\mathfrak{g}}(x), \gamma_{\mathfrak{g}}(x')) = \gamma_{\mathfrak{z}}(\omega_1(x, x')) + \alpha([x, x']) \tag{5.3}$$

holds. Every morphism $\widehat{\mathfrak g}_1 \to \widehat{\mathfrak g}_2$ *mapping* $\mathfrak z_1 \hookrightarrow \mathfrak z_2$ *is of this form.*

For $\mathfrak g = \mathfrak g_1 = \mathfrak g_2$, $\mathfrak z = \mathfrak z_1 = \mathfrak z_2$, $(\gamma_{\mathfrak g}, \gamma_{\mathfrak z}) \in \mathrm{Aut}(\mathfrak g) \times \mathrm{Aut}(\mathfrak z)$, *and* $\alpha_0 := \alpha \circ \gamma_{\mathfrak g}^{-1}$, *formula* (5.3) *is equivalent to* $\gamma.\omega - \omega = d\alpha_0$.

(c) *Let* R *be a Lie group and* $\gamma\colon R \to \mathrm{Aut}(\mathfrak g) \times \mathrm{Aut}(\mathfrak z), r \mapsto (r_{\mathfrak g}, r_{\mathfrak z})$ *a homomorphism such that the corresponding actions on* $\mathfrak g$ *and* $\mathfrak z$ *are smooth. Let* $\alpha\colon R \times \mathfrak g \to \mathfrak z$ *be a smooth map which is linear in the second argument. Then*

$$\widehat\gamma(r).(x,z) := (r_{\mathfrak g}(x), r_{\mathfrak z}(z) + \alpha(r,x)), \quad r \in R, x \in \mathfrak g, z \in \mathfrak z,$$

defines a smooth action of R *by automorphisms of* $\widehat{\mathfrak g}$ *if and only if for each* $r \in R$ *the function* $\alpha_r := \alpha(r,\cdot)$ *satisfies* (5.3) *for* $\gamma(r)$, *and* α *satisfies the cocycle condition*

$$\alpha(r\widetilde r, x) = r_{\mathfrak z}.\alpha(\widetilde r, x) + \alpha(r, \widetilde r_{\mathfrak g}.x), \quad r, \widetilde r \in R, x \in \mathfrak g. \tag{5.4}$$

Proof. (a) If (5.1) is satisfied for some function h which is smooth in an identity neighborhood, then $\widehat\gamma$ is a group homomorphism which is smooth in an identity neighborhood, hence a morphism of Lie groups.

Assume, conversely, that $\widehat\gamma\colon \widehat G_1 \to \widehat G_2$ is a Lie group homomorphism mapping Z_1 into Z_2. Then $\widehat\gamma$ has the form

$$\widehat\gamma(g,z) = \big(\gamma_G(g), \gamma_Z(z)h(g)\big),$$

where $h\colon G_1 \to Z_2$ is a function which is smooth in an identity neighborhood, and an easy calculation leads to (5.1).

(b) The proof is a straightforward verification.

(c) According to (b), the requirement $\widehat\gamma(r) \in \mathrm{Aut}(\widehat{\mathfrak g})$ is equivalent to (5.3) for $\gamma(r)$ and α_r. Suppose that these conditions are satisfied. It is clear that $\widehat\gamma$ defines a smooth function $R\times\widehat{\mathfrak g} \to \widehat{\mathfrak g}$, so that we only have to see which condition on α means that $\widehat\gamma$ defines a representation of R on $\widehat{\mathfrak g}$. That this is equivalent to (5.4) follows from

$$r.(\widetilde r.(x,z)) = (r_{\mathfrak g}\widetilde r_{\mathfrak g}.x, r_{\mathfrak z}\widetilde r_{\mathfrak z}.z + r_{\mathfrak z}.\alpha(\widetilde r, x) + \alpha(r, \widetilde r_{\mathfrak g}.x))$$

and

$$(r\widetilde r).(x,z) = (r_{\mathfrak g}\widetilde r_{\mathfrak g}.x, r_{\mathfrak z}\widetilde r_{\mathfrak z}.z + \alpha(r\widetilde r, x)). \qquad \blacksquare$$

Lemma V.2. *If* $\gamma \in \mathrm{Aut}(\widehat G)$ *preserves the subgroup* Z, *then* $\gamma_Z := \gamma \mid_Z$ *is a smooth endomorphism of* Z.

Proof. This follows from the fact that Z is a submanifold of $\widehat G$ in the sense that each point in Z has a neighborhood which is diffeomorphic to a product of an open subset of Z and a transversal manifold. ■

If $Z \hookrightarrow \widehat G \twoheadrightarrow G$ is a central extension as discussed above, then we define

$$\mathrm{Aut}(\widehat G, Z) := \{\gamma \in \mathrm{Aut}(\widehat G)\colon \gamma(Z) = Z\}.$$

In view of Lemma V.2, we then have a natural homomorphism

$$\eta\colon \mathrm{Aut}(\widehat G, Z) \to \mathrm{Aut}(G) \times \mathrm{Aut}(Z), \quad \eta(\gamma)(q(g), z) = \big(q(\gamma(g)), \gamma(z)\big).$$

To each $f \in \mathrm{Hom}(G,Z)$ we assign the element of $\mathrm{Aut}(\widehat G, Z)$ given by $\widehat f(g) := gf(q(g))$. Then

$$\ker\eta = \{\widehat f\colon f \in \mathrm{Hom}(G,Z)\} \cong \mathrm{Hom}(G,Z).$$

([Ne01a, Lemma II.9]).

Lemma V.3. *If $\gamma = (\gamma_G, \gamma_Z) \in \mathrm{Aut}(G) \times \mathrm{Aut}(Z)$ is contained in the range of η, then there exists $\alpha \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})$ satisfying* (5.3). *If, conversely, G is simply connected and $\alpha \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})$ satisfies* (5.3), *then there exists a unique automorphism $\widehat{\gamma} \in \mathrm{Aut}(\widehat{G}, Z)$ with $\eta(\widehat{\gamma}) = \gamma$ and*

$$\mathbf{L}(\widehat{\gamma})(x,z) = \big(\mathbf{L}(\gamma_G).x, \mathbf{L}(\gamma_Z)(z) + \alpha(x)\big), \quad x \in \mathfrak{g}, z \in \mathfrak{z}.$$

Proof. If $\gamma = \eta(\widehat{\gamma})$, then $\mathbf{L}(\widehat{\gamma}) \in \mathrm{Aut}(\widehat{\mathfrak{g}})$ preserves $\mathfrak{z}$ and induces an automorphism of $\mathfrak{z}$ (Lemma V.2). Hence it is of the form

$$\mathbf{L}(\widehat{\gamma}).(x,z) = (\mathbf{L}(\gamma_G).x, \mathbf{L}(\gamma_Z).z + \alpha(x)),$$

where $\alpha \colon \mathfrak{g} \to \mathfrak{z}$ is a continuous linear map (Lemma V.1(b)). This implies the first part of the assertion.

Suppose, conversely, that (5.3) is satisfied by $\alpha \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})$ for $\gamma_{\mathfrak{g}} := \mathbf{L}(\gamma_G)$ and $\gamma_{\mathfrak{z}} := \mathbf{L}(\gamma_Z)$. Since G is simply connected, the exact sequence for central Lie group extensions ([Ne02b, Th. VII.12]) implies that the natural map $H^2_s(G,Z) \to H^2_c(\mathfrak{g}, \mathfrak{z})$ is injective.

Now it easily follows that it is equivariant with respect to the action of $\mathrm{Aut}(G) \times \mathrm{Aut}(Z)$ on both sides. Our assumption implies that $[\gamma.\omega] = [\omega]$ in $H^2_c(\mathfrak{g}, \mathfrak{z})$, so that the equivariance of D together with the injectivity of the corresponding map on the cohomology groups implies that $[\gamma.f] = [f]$ in $H^2_s(G,Z)$. Now the existence of the automorphism $\widehat{\gamma}$ follows from Lemma V.1(a). The uniquenss of the automorphism $\widehat{\gamma}$ follows from the fact that any automorphism of the connected Lie group $\widehat{G}$ is uniquely determined by the corresponding automorphism of the Lie algebra ([Mi83, Lemma 7.1]). ■

Proposition V.4. *If G is simply connected and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ is a Lie algebra cocycle corresponding to the Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$, and $\widehat{G}$ a corresponding Lie group extension of G by Z, then $\gamma = (\gamma_G, \gamma_Z) \in \mathrm{Aut}(G) \times \mathrm{Aut}(Z)$ lifts to an automorphism $\widehat{\gamma} \in \mathrm{Aut}(\widehat{G}, Z)$ if and only if $[\gamma.\omega] = [\omega]$, i.e., if the corresponding automorphism of $\mathfrak{g}$ lifts to an automorphism of $\widehat{\mathfrak{g}}$.*

Proof. This is a direct consequence of Lemma V.3. ■

Lemma V.5. *Suppose that $\sigma \colon R \times G \to G$ is a smooth action of the Lie group R by automorphisms of the connected Lie group G. Then the action of R on G lifts to a smooth action $\widetilde{\sigma} \colon R \times \widetilde{G} \to \widetilde{G}$ by automorphisms of the simply connected covering group $\widetilde{G}$ of G.*

Proof. [Ne01a, Lemma II.17] ■

If G is not simply connected, then it might have non-trivial central Z-extensions corresponding to trivial Lie algebra extension. These are discussed in the following lemma.

Lemma V.6. *If $\widehat{G}$ is of the form $\widehat{G} = (\widetilde{G} \times Z)/\Gamma(\varphi)$, where $q_G \colon \widetilde{G} \to G$ is the universal covering morphism of G, $\pi_1(G) \cong \ker q_G$ is identified with a subgroup of $\widetilde{G}$, $\varphi \colon \pi_1(G) \to Z$ is a homomorphism, and*

$$\Gamma(\varphi) := \{(d, \varphi(d)) \colon d \in \pi_1(G)\}$$

the graph of φ, then $\gamma = (\gamma_G, \gamma_Z) \in \mathrm{Aut}(G) \times \mathrm{Aut}(Z)$ is in the range of η if and only if $(\gamma_Z^{-1} \circ \varphi \circ \pi_1(\gamma_G)) \cdot \varphi^{-1}$ extends to a smooth homomorphism $\widetilde{G} \to Z$.

Proof. Let $\widetilde{\gamma}_G$ be the natural lift of γ_G to $\widetilde{G}$ (Lemma V.5). The canonical map $\widetilde{G} \times Z \to \widehat{G}$ is a covering, and $\widetilde{G} \times \mathfrak{z}$ is the universal covering group of $\widehat{G}$. Therefore, if $\gamma = \eta(\widehat{\gamma})$, the automorphism $\widehat{\gamma}$ also lifts to some automorphism $\widetilde{\gamma}$ of $\widetilde{G} \times Z$ preserving the subgroup $\Gamma(\varphi)$. Then $\widetilde{\gamma}$ is of the form

$$\widetilde{\gamma}(g,z) = (\widetilde{\gamma}_G(g), \gamma_Z(z) f(g)),$$

with $f \in \mathrm{Hom}(\widetilde{G}, Z)$. The condition that $\widetilde{\gamma}$ preserves $\Gamma(\varphi)$ means that

$$f\,|_{\pi_1(G)} = (\gamma_Z \circ \varphi)^{-1} \cdot \varphi \circ \pi_1(\gamma_G),$$

where $\pi_1(\gamma_G) = \widetilde{\gamma}_G|_{\pi_1(G)}$. If, conversely, $(\gamma_Z \circ \varphi)^{-1} \cdot \varphi \circ \pi_1(\gamma_G)$ extends to a morphism $\widetilde{G} \to Z$, then the above formula yields an automorphism $\widetilde{\gamma}$ on $\widetilde{G} \times Z$ preserving $\Gamma(\varphi)$ which then factors to the quotient group $\widehat{G}$. ■

If $\pi_1(G) \subseteq (\widetilde{G}, \widetilde{G})$, then $\gamma \in \mathrm{im}(\eta)$ is equivalent to $\gamma_Z \circ \varphi = \varphi \circ \pi_1(\gamma_G)$ because for every homomorphism of $\widetilde{G}$ to an abelian Lie group the restriction to $\pi_1(G)$ is trivial.

Lifting automorphic group actions to central extensions

In the preceding subsection we have lifted automorphisms of G to automorphisms of $\widehat{G}$. Now we assume that we have a smooth automorphic action of the Lie group R on G, which leads to a semidirect product Lie group $G \rtimes R$. We are looking for sufficient conditions to lift the smooth action of R on G to a smooth action on $\widehat{G}$ which apply in particular to the action of $\mathrm{Diff}(M)$ and $C^\infty(M,K)$ on $C^\infty(M,K)_e$, where K is a Lie group and M a compact manifold.

The following lemma will be used to reduce the problem to the case where the group $\widehat{G}$ is simply connected.

Lemma V.7. *Let $Z^\sharp := \mathfrak{z}/\mathrm{im}(\mathrm{per}_\omega)$. Then there exists a central Lie group extension*

$$Z^\sharp \hookrightarrow G^\sharp \xrightarrow{q^\sharp} \widetilde{G}$$

corresponding to the cocycle ω, and $G^\sharp$ is the universal covering group of $\widehat{G}$.

Proof. [Ne01a, Lemma II.16] ■

The following remark will be relevant for the argument in the proof of the Lifting Theorem IV.9 below.

Remark V.8. (Local description of central Lie group extensions) Let $q\colon \widehat{G} \to G$ be a central Lie group extension with kernel Z.

Let Ω be the left invariant 2-form on G with $\Omega_e = \omega$, where $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$. Further let $p_{\mathfrak{z}}\colon \widehat{\mathfrak{g}} \to \mathfrak{z}$ denote the projection onto $\mathfrak{z}$ defined by this identification. We write α for the left invariant $\mathfrak{z}$-valued 1-form on $\widehat{G}$ with $\alpha_e = p_{\mathfrak{z}}$. Then the 2-form $q^*\Omega$ is exact with $q^*\Omega = -d\alpha$ because $-dp_{\mathfrak{z}}((x,z),(x',z')) = p_{\mathfrak{z}}([(x,z),(x',z')]) = \omega(x,x')$.

In $\widehat{G}$ we have an open e-neighborhood of the form $U \times Z \subseteq \widehat{G}$, where the multiplication is given for $x, x', xx' \in U$ by

$$(x,z)(x',z') = (xx', zz' f^Z(x,x'))$$

for a smooth function $f^Z\colon U \times U \to Z$. This means that the left multiplication map $\lambda_{(x,e)}$ is given by $(x',z') \mapsto (xx', z' f^Z_x(x'))$ for a smooth function $f^Z_x\colon U \to Z$. Let $\sigma\colon U \to \widehat{G}$ denote the smooth section given by $\sigma(g) = (g,e)$. Then $\theta := -\sigma^*\alpha$ is a $\mathfrak{z}$-valued 1-form on G with

$$d\theta = -d\sigma^*\alpha = -\sigma^* d\alpha = \sigma^* q^*\Omega = \Omega \quad \text{and} \quad \theta_e = -p_{\mathfrak{z}} \circ d\sigma(e) = 0.$$

In view of the left invariance of α, we have on $U \times Z$ the relation

$$\alpha = q^*\theta + p_Z^*\theta_Z,$$

where $\theta_Z = \delta^l(\mathrm{id}_Z)$ is the Maurer–Cartan form on Z with $\theta_Z(e) = \mathrm{id}_{\mathfrak{z}}$ and $p_Z\colon U \times Z \to Z$ is the projection onto Z. Therefore

$$\alpha = \lambda_{(x,e)}^*\alpha = q^*\lambda_x^*\theta + p_Z^*\theta_Z + q^*\delta^l(f_x^Z),$$

which leads to

$$\theta = \lambda_x^*\theta - \delta^l(f_x^Z)$$

and hence to

$$\lambda_x^*\theta - \theta = \delta^l(f_x^Z), \quad f_x^Z(e) = e.$$

We assume that W is an open identity neighborhood in G diffeomorphic to an open convex subset of $\mathfrak{g}$ with $WW \subseteq U$. Then the Poincaré Lemma ([Ne02b, Lemma III.3]) implies for each $x \in W$ the existence of a smooth function

$$f_x^{\mathfrak{z}}: W \to \mathfrak{z} \quad \text{with} \quad f_x^{\mathfrak{z}}(e) = 0 \quad \text{and} \quad df_x^{\mathfrak{z}} = (\lambda_x^* \theta - \theta)|_W.$$

Moreover, this function depends smoothly on x, so that the function

$$f^{\mathfrak{z}}: W \times W \to \mathfrak{z}, \quad f^{\mathfrak{z}}(x,y) := f_x^{\mathfrak{z}}(y)$$

is smooth. From the uniqueness we now conclude that on W we have for each $x \in W$ the relation $f_x^Z = q_Z \circ f_x^{\mathfrak{z}}$. This construction of the functions f_x^Z will become crucial, when we lift automorphic group actions on G to group actions on $\widehat{G}$ in Theorem V.9. ■

Theorem V.9. (Lifting Theorem) *Let $\sigma_G: R \times G \to G$, resp., $\sigma_Z: R \times Z \to Z$ be smooth automorphic actions of the Lie group R on the connected Lie groups G, resp., Z. Assume further that G is simply connected and that there exists a smooth function $\alpha: R \times \mathfrak{g} \to \mathfrak{z}$ such that*

$$\sigma_{\widehat{\mathfrak{g}}}(r)(x,z) := (r.x, r.z + \alpha(r,x)), \quad r \in R, x \in \mathfrak{g}, z \in \mathfrak{z}$$

is an action of R on $\widehat{\mathfrak{g}}$ by automorphisms. Then there exists a unique smooth action

$$\sigma_{\widehat{G}}: R \times \widehat{G} \to \widehat{G}$$

by automorphisms such that the corresponding derived action is $\sigma_{\widehat{\mathfrak{g}}}$.

Proof. In view of Lemma V.3, each automorphism $\sigma_{\widehat{\mathfrak{g}}}(r)$ of $\widehat{\mathfrak{g}}$ integrates to a unique automorphism of $\widehat{G}$. It is clear that the uniqueness implies that we obtain an action $\sigma_{\widehat{G}}$ of R on $\widehat{G}$ by smooth automorphisms. It remains to show that this action is smooth.

The action $\sigma_{\widehat{G}}$ lifts uniquely to an action $\sigma_{G^\sharp}$ on the universal covering group $G^\sharp$ of $\widehat{G}$ by Lie group automorphisms which can also be viewed as a central extension of the simply connected group G by a group $Z^\sharp \cong \mathfrak{z}/\pi_1(Z^\sharp)$ (Lemma V.7). If the action $\sigma_{G^\sharp}$ is smooth, then the induced action $\sigma_{\widehat{G}}$ is also smooth. Hence it suffices to show that $\sigma_{G^\sharp}$ is smooth. Therefore we may w.l.o.g. assume that $\widehat{G}$ is simply connected, i.e., $\widehat{G} = G^\sharp$.

First we consider the local situation in a suitable small neighborhood of the identity in $\widehat{G}$. For $r \in R$ we write $r_G := \sigma_G(r,\cdot)$ and $r_Z := \sigma_Z(r,\cdot)$. In $\widehat{G}$ we have an open e-neighborhood of the form $U \times Z \subseteq \widehat{G}$, where the multiplication is given for $x, x', xx' \in U$ by

$$(x,z)(x',z') = (xx', zz' f^Z(x,x'))$$

for a smooth function $f^Z: U \times U \to Z$. Let W and $f := f^{\mathfrak{z}}: W \times W \to \mathfrak{z}$ with $f^Z = q_Z \circ f$ be as in Remark V.8 determined by

$$df_x = (\lambda_x^* \theta - \theta)|_W \quad \text{for} \quad f_x := f(x,\cdot).$$

Now let $r \in R$ and $W_1 \subseteq W$ be an open e-neighborhood diffeomorphic to a convex set such that $r.W_1 \subseteq W$. Let α_r be the left invariant $\mathfrak{z}$-valued 1-form on G with $\alpha_r(e) = \alpha(r,\cdot)$. Then (5.3) implies that

$$r_G^* \Omega - \mathbf{L}(r_Z) \circ \Omega = -d\alpha_r$$

because both sides are left invariant 2-forms which coincide in e because

$$\omega(\mathbf{L}(r_G).x, \mathbf{L}(r_G).y) - \mathbf{L}(r_Z).\omega(x,y) = \alpha([x,y]), \quad x, y \in \mathfrak{g}.$$

On W_1 we therefore have $d(r_G^*\theta - \mathbf{L}(r_Z) \circ \theta + \alpha_r) = 0$, so that there exists a unique function $h_r: W_1 \to \mathfrak{z}$ with $h_r(e) = 0$ and $dh_r = r_G^*\theta - \mathbf{L}(r_Z) \circ \theta + \alpha_r$.

On $W_1 \times W_1$ we consider the function $(r^\sharp.f)(x,y) := \mathbf{L}(r_Z)^{-1}.f(r_G.x, r_G.y)$. Then $(r^\sharp.f)_x = \mathbf{L}(r_Z)^{-1} r_G^* f_{r_G.x}$, so that on W_1 we have

$$\begin{aligned} d\big((r^\sharp.f)_x\big) &= \mathbf{L}(r_Z)^{-1} r_G^* df_{r_G.x} = \mathbf{L}(r_Z)^{-1} r_G^*(\lambda_{r_G.x}^*\theta - \theta) \\ &= \mathbf{L}(r_Z)^{-1}\big((\lambda_{r_G.x}\circ r_G)^*\theta - r_G^*\theta\big) = \mathbf{L}(r_Z)^{-1}\big((r_G\circ\lambda_x)^*\theta - r_G^*\theta\big) \\ &= \mathbf{L}(r_Z)^{-1}\big(\lambda_x^* r_G^*\theta - r_G^*\theta\big). \end{aligned}$$

Now the left invariance of α_r leads to

$$\begin{aligned} d((r^\sharp.f)_x - f_x) &= \mathbf{L}(r_Z)^{-1}\big(\lambda_x^* r_G^*\theta - r_G^*\theta\big) - \lambda_x^*\theta + \theta \\ &= \mathbf{L}(r_Z)^{-1}\Big(\lambda_x^*\big(r_G^*\theta - \mathbf{L}(r_Z)\circ\theta\big) - \big(r_G^*\theta - \mathbf{L}(r_Z)\circ\theta\big)\Big) \\ &= \mathbf{L}(r_Z)^{-1}\Big(\lambda_x^*\big(r_G^*\theta - \mathbf{L}(r_Z)\circ\theta + \alpha_r\big) - \big(r_G^*\theta - \mathbf{L}(r_Z)\circ\theta + \alpha_r\big)\Big) \\ &= \mathbf{L}(r_Z)^{-1}(\lambda_x^* dh_r - dh_r) = d\big(\mathbf{L}(r_Z)^{-1}(\lambda_x^* h_r - h_r)\big). \end{aligned}$$

In view of the normalizations $f_x(e) = f(x,e) = 0 = h_r(e)$, we have

$$((r^\sharp.f)_x - f_x)(e) = \mathbf{L}(r_Z)^{-1}.f(r_G.x, e) = 0$$

and

$$\mathbf{L}(r_Z)^{-1}(\lambda_x^* h_r - h_r)(e) = \mathbf{L}(r_Z)^{-1} h_r(x).$$

Therefore

$$(r^\sharp.f)_x - f_x = \mathbf{L}(r_Z)^{-1}(\lambda_x^* h_r - h_r) - \mathbf{L}(r_Z)^{-1} h_r(x),$$

which leads to

$$f(r_G.x, r_G.y) - \mathbf{L}(r_Z).f(x,y) = h_r(xy) - h_r(y) - h_r(x) \tag{5.5}$$

for x,y sufficiently close to e.

Let $q_Z\colon \mathfrak{z} \to Z$ be the quotient map, $f^Z := q_Z \circ f$ and $h_r^Z := q_Z \circ h_r$. Then we have an e-neighborhood of the form $W_2 \times Z$ in $\widehat{G}$, where $W_2 \subseteq W_1$, and the multiplication on $W_2 \times Z$ is given by

$$(g,z)(g',z') = (gg', zz' f^Z(g,g')).$$

Pick an open symmetric connected e-neighborhood $W_3 \subseteq W_2$ with $r.W_3 \subseteq W_2$ such that (5.5) is satisfied for $x,y \in W_3$. Then a similar argument as in Lemma V.1 shows that the map

$$\sigma_0(r)\colon W_3 \times Z \to W_2 \times Z \subseteq \widehat{G}, \quad (g,z) \mapsto (r_G.g, r_Z(z) h_r^Z(g))$$

is a smooth homomorphism of local groups. Using Lemma II.3 in [Ne02b] and the simple connectedness of $\widehat{G}$, we see that $\sigma_0(r)$ extends to a smooth homomorphism $\sigma_0(r)\colon \widehat{G} \to \widehat{G}$. The derivative of this automorphism in $e \in \widehat{G}$ is given by

$$\begin{aligned} d\sigma_0(r)(e)(x,z) &= (r_G.x, r_Z.z + dh_r^Z(e)(x)) = (r_G.x, r_Z.z + dh_r(e)(x)) \\ &= (r_G.x, r_Z.z + \alpha(r,x) + \theta(e)(r_G.x) - r_Z.\theta(e)(x)) \\ &= (r_G.x, r_Z.z + \alpha(r,x)) = \sigma_{\widehat{\mathfrak{g}}}(r)(x,z). \end{aligned}$$

Since both automorphisms induce the same Lie algebra automorphism, $\sigma_0(r) = \sigma_{\widehat{G}}(r)$ for each $r \in R$, so that we obtain an explicit description of $\sigma_{\widehat{G}}$ near to the identity in $\widehat{G}$.

It remains to show that this action is smooth. Since R acts by smooth automorphism on $\widehat{G}$, it suffices to show that the action is smooth in a neighborhood of (e,e) and that all orbit maps $R \to \widehat{G}$ are smooth in a neighborhood of e. Since the latter property can be derived from the first one ($\widehat{G}$ is connected), it remains to see that the action is smooth in a neighborhood of (e,e). To this end, we slightly adjust the choices of W_1 and W_3 above. First we choose an open e-neighborhood V in R and W_1 such that, in addition, $V.W_1 \subseteq W$. Likewise we choose $V_1 \subseteq V$ and $W_3 \subseteq W_2$ with $V_1.W_3 \subseteq W_2$. Then the function $(r,x) \mapsto h_r(x)$ is defined on $V \times W_1$, and the construction of h_r with the Poincaré Lemma implies that this function is smooth in a neighborhood of (e,e) (cf. [Ne02b, Lemma III.3]). This implies that the action map $\sigma_{\widehat{G}}$ is smooth on a neighborhood of (e,e) contained in $V_1 \times W_3$, and this completes the proof. ∎

Corollary V.10. *Let $\sigma_G: R \times G \to G$ be a smooth automorphic action of the Lie group R on the connected Lie group G. Assume that G is simply connected and that $r_G^*\omega = \mathbf{L}(r_Z) \circ \omega$ holds for all $r \in R$. Then the action of R on G lifts uniquely to a smooth automorphic action of R on $\widehat{G}$ such that the corresponding action of R on $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$ is given by*

$$r.(x, z) = (\mathbf{L}(r_G).x, \mathbf{L}(r_Z).z), \quad r \in R, x \in \mathfrak{g}, z \in \mathfrak{z}.$$

Proof. We apply Theorem V.9 with $\alpha = 0$. ■

Remark V.11. Suppose that $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is a central Lie group extension and that the R-action on the group $G^\sharp$ from Lemma V.7 exists. If this action preserves the discrete subgroup $\pi_1(\widehat{G})$, then it factors through an action on $\widehat{G} \cong G^\sharp/\pi_1(\widehat{G})$, but this condition has to be checked directly in concrete cases because there is no general reason for it to be satisfied. If G is simply connected, then the natural mal $H_s^2(G, Z) \to H_c^2(\mathfrak{g}, \mathfrak{z})$ is injective, which permits us to lift every $\gamma \in \mathrm{Aut}(G) \times \mathrm{Aut}(Z)$ fixing the cohomology class $[\omega]$ in $H_c^2(\mathfrak{g}, \mathfrak{z})$ to an automorphism of $\widehat{G}$. If G is not simply connected, then we only have an exact sequence

$$\ldots \to \mathrm{Hom}(\pi_1(G), Z) \to H_s^2(G, Z) \to H_c^2(\mathfrak{g}, \mathfrak{z}) \to \ldots$$

([Ne02b, Th. VII.12]) which shows that in general there are inequivalent central Z-extensions $\widehat{G}$ of G with the same Lie algebra, so that there is no reason for a $\gamma \in \mathrm{Aut}(G) \times \mathrm{Aut}(Z)$ to lift to a particular one. ■

Remark V.12. (a) If $\mathfrak{g}$ is *topologically perfect*, i.e., the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is dense in $\mathfrak{g}$, then in (5.5) the continuous linear map $\alpha_r := \alpha(r, \cdot): \mathfrak{g} \to \mathfrak{z}$ is uniquely determined by $r^*\omega - \omega = -d\alpha_r$. Therefore

$$\begin{aligned} -d\alpha_{r\widetilde{r}} &= (r\widetilde{r})^*\omega - \mathbf{L}(r_Z)\,\mathbf{L}(\widetilde{r}_Z)\omega \\ &= \widetilde{r}^*(r^*\omega - \mathbf{L}(r_Z)\omega) + \widetilde{r}^*\,\mathbf{L}(r_Z)\omega - \mathbf{L}(r_Z)\,\mathbf{L}(\widetilde{r}_Z)\omega = -\widetilde{r}^*d\alpha_r - \mathbf{L}(r_Z)d\alpha_{\widetilde{r}} \end{aligned}$$

implies the relation (5.4). In view of this, (5.4) is only needed if $\mathfrak{g}$ is not topologically perfect.
(b) If $\widehat{G}$ is a regular Lie group in the sense of [Mi83], then every automorphism of $\widehat{\mathfrak{g}}$ integrates uniquely to an automorphism of $\widehat{G}$ ([Mi83, Th. 8.1]). In our context it does not make sense to work with this additional assumption because we anyway need the more explicit information obtained in the proof of Theorem V.9 to show that the action is smooth. ■

Problem V.1. Let G be Lie group and $\sigma_G: R \times G \to G$ an action of the Lie group R on G by Lie automorphisms such that the corresponding action $\sigma_{\mathfrak{g}}: R \times \mathfrak{g} \to \mathfrak{g}$ is smooth. Does this imply that σ_G is a smooth action? ■

VI. Diffeomorphism groups acting on current groups

If M is a compact manifold, then the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M has a natural Lie group structure and the action of this group on M induces a natural smooth action on each group $C^\infty(M, K)$ of smooth maps into some Lie group K. In this section we apply the Lifting Theorem of the preceding section to see how the action of $\mathrm{Diff}(M)$ on $G = C^\infty(M, K)_e$ can be lifted to a smooth action of $\mathrm{Diff}(M)$ on a central extension $\widehat{G}$ whenever this central extension of G is such that the connecting homomorphism $\pi_1(G) \to \pi_0(Z)$ is an isomorphism. The latter means that $\widehat{G}$ is weakly universal for discrete abelian groups. This condition is in particular satisfied for the universal central extension of G if K is finite-dimensional and simple (Theorem IV.7). We also lift the conjugation action of $C^\infty(M, K)$ on G to $\widehat{G}$.

The manifold structure on $\mathrm{Diff}(M)$ is obtained by the observation that this group is an open subset of the mapping space $C^\infty(M, M)$ which is a smooth manifold ([KM97, Th. 43.1]). Let $E: \mathrm{Diff}(M) \times M \to M$ be the natural action of $\mathrm{Diff}(M)$ on M given by the evaluation. To see that E is a smooth map, it suffices to observe that the corresponding map

$$E: C^\infty(M, M) \times M \to M, \ (\varphi, m) \mapsto \varphi(m)$$

is smooth ([KM97, Th. 42.13]).

Lemma VI.1. *If M is a compact manifold and K a Fréchet–Lie group, then the natural action*

$$\mathrm{Diff}(M) \times C^\infty(M,K) \to C^\infty(M,K), \quad (\varphi, f) \mapsto f \circ \varphi^{-1}$$

is smooth.

Proof. Let $U \subseteq K$ be an open identity neighborhood diffeomorphic to an open subset of $\mathfrak{k}$. Then [Ne01b, Th. III.5] implies that the action of $\mathrm{Diff}(M)$ on the open subset $C^\infty(M,U) \subseteq C^\infty(M,K)$ is smooth.

For a smooth function $f\colon M \to K$ the orbit map

$$\mathrm{Diff}(M) \to C^\infty(M,K), \quad \varphi \mapsto f \circ \varphi^{-1}$$

is smooth because the map $\mathrm{Diff}(M) \times M \to K, (\varphi, m) \mapsto f(\varphi^{-1}(m))$ is smooth, which in turn follows from the smoothness of the action of $\mathrm{Diff}(M)$ on M.

Now the smoothness of the action of $\mathrm{Diff}(M)$ on $C^\infty(M,K)$ follows from the observation that for each $f \in C^\infty(M,K)$ the map

$$\mathrm{Diff}(M) \times C^\infty(M,U) \to C^\infty(M,K), \quad (\varphi, h) \mapsto \varphi.(fh) = \varphi.f \cdot \varphi.h$$

is smooth because the orbit map of f is smooth and the action on $C^\infty(M,U)$ is smooth. ■

The general argument behind the proof of Lemma VI.1 is that an automorphic action of a Lie group R on the Lie group G is smooth if

(1) there exists an open invariant identity neighborhood on which the action is smooth, and

(2) all orbit maps are smooth.

Remark VI.2. (a) Let $G := C^\infty(M,G)_e$. On the Lie algebra $\mathfrak{g} = C^\infty(M,\mathfrak{k})$ of G we consider the continuous cocycle

$$\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}_M(Y) = \Omega^1(M,Y)/d\Omega^0(M,Y), \quad \omega(\xi,\eta) = [\kappa(\xi, d\eta)],$$

where κ is a continuous invariant symmetric bilinear form $\mathfrak{k} \times \mathfrak{k} \to Y$ and Y is a s.c.l.c. space. For $\varphi \in \mathrm{Diff}(M)$ we have

$$\begin{aligned}\omega(\varphi^{-1}.\xi, \varphi^{-1}.\eta) &= \omega(\varphi^*\xi, \varphi^*\eta) = [\kappa(\varphi^*\xi, d\varphi^*\eta)]\\ &= [\kappa(\varphi^*\xi, \varphi^* d\eta)] = [\varphi^*\kappa(\xi, d\eta)] = \varphi^{-1}.[\kappa(\xi, d\eta)] = \varphi^{-1}.\omega(\xi,\eta).\end{aligned}$$

Here the last expression refers to the natural action of $\mathrm{Diff}(M)$ on $\mathfrak{z}_M(Y)$ which exists because the natural action on $\Omega^1(M,Y)$ preserves the closed subspace $d\Omega^0(M,Y)$ because $\varphi^*(df) = d\varphi^* f$ for $f \in \Omega^0(M,Y)$. Lemma V.1(b) now implies that

$$\varphi.(\xi, z) := (\xi \circ \varphi^{-1}, (\varphi^{-1})^*.z)$$

defines a smooth action of R on the Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$ by Lie algebra automorphisms.

(b) The cocycle ω is fixed by $\mathrm{Diff}(M)$ if and only if this group acts trivial on $\mathfrak{z}_M(Y)$, which (for $Y \neq \mathbf{0}$) is equivalent to the triviality of the action on $\mathfrak{z}_M(\mathbb{R})$. If this is the case, then we have in particular that for each vector field X on M and each 1-form α the 1-form

$$\mathcal{L}_X.\alpha = i_X d\alpha + d(i_X.\alpha)$$

is exact, which implies $d\alpha = 0$. That all 1-forms are closed means that $\dim M \leq 1$, so that $M = \mathbb{S}^1$ is the only non-trivial compact manifold for which the Lie algebra of vector fields acts trivially on $\mathfrak{z}_M(\mathbb{R})$. For a 1-form α on M and $\varphi \in \mathrm{Diff}(M)$ we have

$$\int_{\mathbb{S}^1} \varphi^*\alpha = \deg(\varphi) \int_{\mathbb{S}^1} \alpha.$$

Therefore the identity component $\mathrm{Diff}(\mathbb{S}^1)_e$ of orientation preserving diffeomorphisms acts trivially on $\mathfrak{z}_{\mathbb{S}^1}(\mathbb{R}) \cong \mathbb{R}$, and if a diffeomorphism changes orientation, it acts by multiplication by -1 on $\mathfrak{z}_{\mathbb{S}^1}(\mathbb{R})$. ■

Theorem VI.3. *Let K be a connected Fréchet–Lie group, M a compact manifold, $G := C^\infty(M,K)_e$, $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z}_M(Y))$ a cocycle of product type with discrete period group. Further let $\widehat{G} \to G$ be a corresponding central extension of G by a Lie group Z with Lie algebra $\mathfrak{z}_M(Y)$ for which the connecting homomorphism $\pi_1(G) \to \pi_0(Z)$ is an isomorphism. Then the following assertions hold:*

(1) *The automorphic action of* $\mathrm{Diff}(M)$ *on* $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}_M(Y)$ *by* $\varphi.(\xi, z) := (\xi \circ \varphi^{-1}, (\varphi^{-1})^*.z)$ *integrates to a smooth action of* $\mathrm{Diff}(M)$ *on* $\widehat{G}$.

(2) *The automorphic action of* $C^\infty(M,K)$ *on* $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}_M(Y)$ *by*

$$f.(\xi, z) := (\mathrm{Ad}(f).\xi, z - [\kappa(\delta^l(f), \xi)])$$

integrates to a smooth action on $\widehat{G}$.

Proof. First we use [Ne01c, Lemma 4.6] to see that the condition that the connecting homomorphism $\pi_1(G) \to \pi_0(Z)$ is an isomorphism implies that the central extension $q\colon \widehat{G} \to G$ is weakly universal for all discrete abelian groups A. Now [Ne01c, Prop. 4.7] further implies that $\widehat{G}/Z_e \cong \widetilde{G}$, showing that $\widehat{G}$ can be viewed as a central extension of the simply connected group $\widetilde{G}$ by Z_e.

(1) Using Lemma V.5, we lift the smooth action of $\mathrm{Diff}(M)$ on G to a smooth action on $\widetilde{G}$. Now the Lifting Theorem V.9 implies that this action can be lifted to a smooth action of $\mathrm{Diff}(M)$ on $\widehat{G}$, integrating the given action on the Lie algebra $\widehat{\mathfrak{g}}$.

(2) follows as in (1) from Proposition III.3 and the Lifting Theorem V.9. ■

For the case of loop groups, part (2) of Theorem VI.3 has already been observed in [PS86]. Theorem VI.3 is a good starting point for a systematic investigation of the action of subgroups of $\mathrm{Diff}(M)$ on coadjoint orbits of the central extension $\widehat{G}$. Although $\mathrm{Diff}(M)$ acts on the group $\widehat{G}$ and its Lie algebra $\widehat{\mathfrak{g}}$, the corresponding action on the topological dual $\widehat{\mathfrak{g}}'$ mixes the coadjoint orbits of $\widehat{G}$. Here the interesting point is that specific coadjoint orbits of $\widehat{G}$ can be assigned to geometric structures on the manifold M and one can only expect the corresponding subgroups of $\mathrm{Diff}(M)$ to act on these orbits. This point of view will be explored in [NV02] (see also [PS86] for the case of loop groups which is somehow trivial, and [EF94] for the case of complex Riemann surfaces).

VII. Problems arising for non-connected groups

In this section we discuss some of the additional difficulties arising for non-connected groups. One such difficulty is that for a non-connected group the conjugation action of G on G might induce a non-trivial action on the fundamental group $\pi_1(G)$. A related problem is that the surjective homomorphism $G \to \pi_0(G)$ does in general not split. Another problem is that we find for connected groups K certain natural maps

$$\chi_{m,n}\colon \pi_m(K) \times \pi_n(K) \to \pi_{n+m}(K)$$

given by commutators, and we do not know whether they are always trivial or not. This is of particular interest for $m = n = 1$. If K is finite-dimensional, then $\pi_2(K)$ is trivial, so that $\chi_{1,1}$ is trivial. For infinite-dimensional groups, the maps $\chi_{1,1}$ are harder to study because $\pi_1(K)$ need not be generated by $\mathrm{Hom}(\mathbb{T}, K)$.

Central extensions of non-connected groups

Remark VII.1. Let G be the identity component of the Lie group G^+ and assume that we have a central extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ as above. When can we extend this central extension to a central extension $Z \hookrightarrow \widehat{G}^+ \twoheadrightarrow G^+$ of the full group G^+?

Since $Z \subseteq \widehat{G}^+$ is central, the subgroup $\widehat{G} \subseteq \widehat{G}^+$ acts trivially by conjugation on Z, so that we obtain an action of $\widehat{G}^+/\widehat{G} \cong G^+/G = \pi_0(G^+)$ by Lie automorphisms on the group Z. Let σ_Z denote the corresponding action of G^+, resp., $\pi_0(G^+)$, on Z. A necessary condition for the existence of a central extension $\widehat{G}^+$ of G^+ is that the adjoint action of G^+ on $\mathfrak{g}$ can be extended to an action of $G^+ \cong \widehat{G}^+/Z$ on $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$ of the form

$$c(g).(x,z) := \big(\mathrm{Ad}(g).x, \sigma_Z(g).z + \alpha(g,x)\big),$$

where $\alpha\colon G^+ \times \mathfrak{g} \to \mathfrak{z}$ is a cocycle, so that $c\colon G^+ \to \mathrm{Aut}(\widehat{\mathfrak{g}})$ defines a representation of G^+ on $\widehat{\mathfrak{g}}$. The existence of this action implies in particular that

$$\sigma_Z(g) \circ \omega - \omega \circ (\mathrm{Ad}(g) \times \mathrm{Ad}(g)) \in B^2_c(\mathfrak{g}, \mathfrak{z})$$

for all $g \in G^+$ (Lemma V.1). For $g \in G$ this follows automatically from the existence of the conjugation action of G on $\widehat{G}$. ■

In the preceding section we have constructed central extensions of the identity component $C^\infty(M,K)_e$ of the group $C^\infty(M,K)$ which in general is not connected. In this subsection we briefly discuss the difficulties involved in extending central Lie group extensions from the identity component of a Lie group to the whole group.

Remark VII.2. We resume the situation of Theorem VI.3. As we have seen in Proposition III.3, the condition under (a) is satisfied for the group $G^+ = C^\infty(M,K)$ and the cocycle $\omega(\xi,\eta) = [\kappa(\xi, d\eta)]$ for $\sigma_Z(g) = \mathrm{id}_{\mathfrak{z}}$. We recall that $\pi_0(Z) \cong \pi_1(G)$, so that the divisibility of $Z_e \cong \mathfrak{z}/\Pi_{M,\kappa}$ implies that $Z \cong Z_e \times \pi_1(G)$. Since the action of G^+ on $\widehat{\mathfrak{g}}$ fixes $\mathfrak{z}$ pointwise, the corresponding action on $\widehat{G}$ fixes Z_e pointwise. Therefore the action is given by an action of $\pi_0(G^+) \cong [M,K]$ on $\pi_0(Z) \cong \pi_1(G) \cong \pi_1(G^+)$ and a map

$$\zeta\colon \pi_0(G^+) \times \pi_1(G) \to Z_e \quad \text{defined by} \quad \alpha.(z,\beta) = (z\zeta(\alpha,\beta), \alpha.\beta).$$

The map ζ satisfies the cocycle identity

$$\zeta(\alpha_1\alpha_2, \beta) = \zeta(\alpha_1, \alpha_2.\beta)\zeta(\alpha_2,\beta),$$

so that ζ is a bihomomorphism if the action of $\pi_0(G^+)$ on $\pi_1(G^+) = \pi_1(G)$ is trivial. Since the splitting of Z_e in Z is not natural, we cannot expect to find a complement which is invariant under the action of $\pi_0(G^+)$. Nevertheless, if $q\colon \widehat{G} \to G$ is the quotient map of the central extension and we consider K as a subgroup of G, then $q^{-1}(K) \cong \widetilde{K} \times Z_1$, where Z_1 is an open subgroup of Z. To see this, we first construct the central extension $\widehat{G}_*$ of the subgroup $G_* := C^\infty_*(M,K)_e$ of $G \cong G_* \rtimes K$, and then observe that $\widehat{G} \cong \widehat{G}_* \rtimes \widetilde{K}$ because this group is simply connected with the Lie algebra $\widehat{\mathfrak{g}} \cong \mathfrak{g}_* \rtimes \mathfrak{k}$. As the cocycle ω on $\mathfrak{g}$ is invariant under $\mathrm{Ad}(K)$, there is no obstruction to lifting the action of K on G_* to $\widehat{G}_*$ (Theorem V.9). In this picture $\pi_1(K)$, realized as a subgroup of $\widetilde{K}$, arises naturally as a subgroup of Z, but the action of G^+ does not leave the subgroup $\widetilde{K}$ of $\widehat{G}$ invariant. ■

In Proposition A.3 below we will see that the action of $\pi_0(C_*(M,K))$ on $\pi_1(C_*(M,K))$ is trivial for $M = \mathbb{S}^d$, $d \geq 1$, and more generally if M is homotopic to a space of the form $\mathbb{S}^1 \wedge N$. In this case the action of $\pi_0(G^+)$ on Z is completely encoded in the map ζ. Passing from G^+ to the open subgroup $C^\infty(M,\widetilde{K})$, where $\widetilde{K}$ is the universal covering group of K, reduces the number of connected components, so that in this context it is more probable that G^+ acts trivially on Z.

Remark VII.3. In this remark we discuss the problem of finding a formula for ζ which is as explicit as possible. For that we have to understand how an element $\gamma \in G^+ = C^\infty(M,K)$ acts on the group $\widehat{G}$ (Theorem VI.3), where the action on the Lie algebra $\widehat{\mathfrak g}$ is given by

$$\mathrm{Ad}_{\widehat{\mathfrak g}}(\gamma).(\xi,z) = \big(\mathrm{Ad}(\gamma).\xi, z - [\kappa(\delta^l(\gamma),\xi)]\big).$$

Let $\Omega \in \Omega^2(G,\mathfrak z_M(Y))$ be the left invariant 2-form with $\Omega_e = \omega_{\mathbb S^1,\kappa}$. Then the calculations in the proof of Proposition III.3 show that $\mathrm{Ad}(\gamma)^*\omega - \omega = d\theta(\gamma)$ with $\theta(\gamma) = [\kappa(\delta^l(\gamma),\cdot)] \in \mathrm{Lin}(\mathfrak g,\mathfrak z_M(Y))$. Let $\Theta(\gamma) \in \Omega^1(G,\mathfrak z_M(Y))$ denote the corresponding left invariant 1-form on G. Then the conjugation automorphism $c_\gamma(f) := \gamma f\gamma^{-1}$ of G satisfies $c_f^*\Omega - \Omega = d\Theta(\gamma)$. For a smooth map $\eta \in C^\infty_*(\mathbb S^1,G)$ we then obtain

$$\int_\eta \Theta(\gamma) = \int_{\mathbb S^1} [\kappa(\underbrace{\delta^l(\gamma)}_{\in\Omega^1(M,\mathfrak k)}, \underbrace{\delta^l(\eta)(t)}_{\in C^\infty(M,\mathfrak k)})]\,dt \in \mathfrak z_M(Y).$$

Let $\mathbb S^1 \cong \mathbb R/2\pi\mathbb Z$, and $z\colon[0,2\pi]\to Z$ a smooth curve with

$$z(0)=0 \quad\text{and}\quad \delta^l(z)(t) = -\Theta(\gamma)(\eta'(t)) = -\theta(\gamma)(\delta^l(\eta)(t)).$$

Further let $\widehat\eta\colon[0,2\pi]\to\widehat G$ denote the *horizontal lift of the curve* $c_\gamma.\eta$ defined by $\widehat\eta(0)=e$ and $\delta^l(\widehat\eta)(t) = (\mathrm{Ad}(\gamma).\delta^l(\eta)(t),0)$, $t\in[0,2\pi]$. Then the pointwise product $\widehat\eta\cdot z\colon[0,2\pi]\to\widehat G$ is a smooth curve with

$$\delta^l(\widehat\eta\cdot z) = \delta^l(\widehat\eta) + \delta^l(z) = \mathrm{Ad}_{\widehat{\mathfrak g}}(\gamma).(\delta^l(\eta)(t),0) = \big(\mathrm{Ad}(\gamma).\delta^l(\eta)(t), -[\kappa(\delta^l(\gamma),\delta^l(\eta)(t))]\big)$$

because z is a curve with central values. The endpoint $\widehat\eta(2\pi)z(2\pi)$ lies over $\widetilde\eta(2\pi)$ for the lift $\widetilde\eta$ of η to $\widetilde G$, hence corresponds to $\gamma.([\eta],0) = (\gamma.[\eta],\zeta(\gamma,\eta)) \in \pi_1(G)\times Z$.

Let us assume, in addition, that $\eta(\mathbb S^1)\subseteq K$, i.e., that each map $\eta(t)\colon M\to K$ is constant, so that we can think of $\widetilde\eta$ as a curve in $\widetilde K \subseteq \widehat G$ from e to the element $[\eta]\in\pi_1(K)\hookrightarrow\pi_1(G)$. This curve is mapped by $\widehat c_f \in \mathrm{Aut}(\widehat G)$ to $\widehat\eta\cdot z$ ending in $\widehat\eta(2\pi)z(2\pi)$. If, in addition, $\mathrm{Ad}(\gamma).\delta^l(\eta)(t) = \delta^l(\eta)(t)$ holds for each $t\in[0,2\pi]$, then $\widehat\eta(t)=\eta(t)$, and therefore

$$\zeta([\gamma],[\eta]) = z(2\pi) = q_Z\Big(-\int_0^{2\pi}[\kappa\big(\delta^l(\gamma),\delta^l(\eta)(t)\big)]\,dt\Big). \tag{7.1}$$

Since each $\delta^l(\eta)(t)$ is a constant function, we identify it with an element of $\mathfrak k$, and write $[\delta^l(\gamma)]$ for the class of $\delta^l(\gamma)\in\Omega^1(M,\mathfrak k)$ in $\mathfrak z_M(\mathfrak k)$. Then we have for each t the relation

$$[\kappa\big(\delta^l(\gamma),\delta^l(\eta)(t)\big)] = \kappa\big([\delta^l(\gamma)],\delta^l(\eta)(t)\big) \in \mathfrak z_M(Y),$$

via the map

$$\mathfrak z_M(\mathfrak k)\times\mathfrak k \to \mathfrak z_M(Y), \quad ([\beta],x)\mapsto[\kappa(\beta,x)],$$

which is well defined because $d\kappa(\xi,x) = \kappa(d\xi,x)$ for $\xi\in C^\infty(M,\mathfrak k)$. In this sense we also have

$$\zeta([\gamma],[\eta]) = q_Z\Big(-\kappa\Big([\delta^l(\gamma)],\int_0^{2\pi}\delta^l(\eta)(t)\,dt\Big)\Big). \tag{7.2}$$

■

Example VII.4. (a) In [PS86] one finds an explicit description of the action of $\pi_0(G^+)$ on Z for the loop group case $M = \mathbb{S}^1$ and K compact and simple. We now consider the situation, where $M = \mathbb{S}^1$ for a general connected groups K satisfying $\pi_2(K) = \mathbf{1}$. This holds in particular for finite-dimensional Lie groups K. In this case $\pi_0(G^+) \cong \pi_1(K)$ and $\pi_1(G^+) \cong \pi_2(K) \times \pi_1(K) \cong \pi_1(K)$. As the conjugation action of $\pi_1(K)$ on itself is trivial, the action of $\pi_1(K)$ on Z is completely determined by the bihomomorphism

$$\zeta\colon \pi_1(K) \times \pi_1(K) \to Z_e.$$

We think of $\mathbb{S}^1$ as $\mathbb{R}/2\pi\mathbb{Z}$, so that we think of functions on $\mathbb{S}^1$ as 2π-periodic functions on $\mathbb{R}$. Further $\mathfrak{z}_{\mathbb{S}^1}(Y) \cong Y$ via the integration isomorphism $[\beta] \mapsto \frac{1}{2\pi}\int_{\mathbb{S}^1} \beta$, and $Z_e \cong Y/\Pi_{\mathbb{S}^1,\kappa}$.

Let $\gamma \in C^\infty_*(\mathbb{S}^1, K)$ be a smooth loop. Then we identify $[\delta^l(\gamma)] \in \mathfrak{z}_M(\mathfrak{k})$ with $\frac{1}{2\pi}\int_{\mathbb{S}^1} \delta^l(\gamma)$ and obtain with (7.2) for $\eta \in C^\infty_*(\mathbb{S}^1, K)$:

$$\zeta([\gamma],[\eta]) = q_Z\Big(-\kappa\Big(\frac{1}{2\pi}\int_{\mathbb{S}^1} \delta^l(\gamma), \int_{\mathbb{S}^1} \delta^l(\eta)\Big)\Big).$$

If K is finite-dimensional and $T \subseteq K$ a maximal torus, then the natural map $\mathrm{Hom}(\mathbb{T}, T) \to \pi_1(K)$ is surjective, so that $[\gamma]$ and $[\eta]$ have representatives for which $\delta^l(\gamma) = x$ and $\delta^l(\eta) = y$ are constant functions. As $[x,y] = 0$, the assumptions leading to (7.2) are satisfied, and we obtain the simple formula

$$\zeta([\gamma],[\eta]) = q_Z\big(-2\pi\kappa(x,y)\big).$$

We conclude that ζ is trivial if and only if $2\pi x, 2\pi y \in \ker \exp_T$ for the exponential function $\exp_T\colon \mathfrak{t} \to T$ of the maximal torus $T \subseteq K$ implies $\kappa(x,y) \in \frac{1}{2\pi}\Pi_{\mathbb{S}^1,\kappa}$.
(b) To understand this condition, let us assume that K is compact and simple. Then $V(\mathfrak{k})$ is one-dimensional, so that we may w.l.o.g. assume that $Y = \mathbb{R}$. Further $\pi_2(G) \cong \pi_3(K) \cong \mathbb{Z}$, and we may therefore assume that $\Pi_{\mathbb{S}^1,\kappa} = 2\pi\mathbb{Z}$, where

$$\omega(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\xi(\theta), \eta'(\theta))\, d\theta.$$

Let $\mathfrak{t} \subseteq \mathfrak{k}$ be the Lie algebra of a maximal torus of K. For the coroots $\check\alpha$ of the long roots $\alpha \in \Delta_{\mathfrak{k}} \subseteq i\mathfrak{t}^*$ we then have

$$-\kappa(\check\alpha,\check\alpha) = \kappa(i\check\alpha, i\check\alpha) = 2$$

for the complex bilinear extension of κ to $\mathfrak{k}_{\mathbb{C}}$ (see Appendix IIa to Section II in [Ne01a]). We claim that for $x \in \mathfrak{t}_{\mathbb{C}}$ we then have

$$\kappa(\check\Delta, x) \subseteq \mathbb{Z}\Delta(x).$$

In fact, let $\alpha \in \Delta$ and $r_\alpha(x) := x - \alpha(x)\check\alpha$ the corresponding reflection in $\mathfrak{t}_{\mathbb{C}}$. Since the restriction of κ to $\mathfrak{t}_{\mathbb{C}}$ is invariant under all these reflections, we have

$$\kappa(\check\alpha, x) = -\kappa(r_\alpha.\check\alpha, x) = -\kappa(\check\alpha, r_\alpha.x) = -\kappa(\check\alpha, x) + \alpha(x)\kappa(\check\alpha,\check\alpha),$$

so that

$$\kappa(\check\alpha, x) = \frac{1}{2}\alpha(x)\kappa(\check\alpha,\check\alpha) \in \mathbb{Z}\alpha(x)$$

follows from $\kappa(\check\alpha,\check\alpha) \in 2\mathbb{Z}$ for all roots (including the short ones) (see [Ne01a, loc.cit.]). From $\Delta(\check\alpha) \subseteq \mathbb{Z}$ for each coroot, we obtain in particular

$$\kappa(\check\Delta, \check\Delta) \subseteq \mathbb{Z}.$$

If $Z(K)$ is trivial, then for $x \in \mathfrak{t}$ the condition $\exp 2\pi x = e$ is equivalent to $e^{2\pi\,\mathrm{ad}\, x} = \mathrm{id}_{\mathfrak{t}}$, which means that $\Delta(x) \subseteq i\mathbb{Z}$. This is satisfied in particular for $x \in i\check\Delta$. We have

$$\kappa(x, i\check\Delta) \subseteq i\mathbb{Z}\Delta(x) \subseteq \mathbb{Z}$$

whenever $\Delta(x) \subseteq i\mathbb{Z}$. Nevertheless, it may happen that there are two elements $x, y \in \mathfrak{t}$ with $2\pi x, 2\pi y \in \ker \exp_T$ but $\kappa(x,y) \not\in \mathbb{Z}$.
(c) Finally we consider an example where ζ is non-trivial. For $\mathfrak{k} = \mathfrak{su}(2)$ and $K = \mathrm{SO}(3,\mathbb{R}) \cong \mathrm{SU}(2,\mathbb{C})/\{\pm e\}$ we have $\ker \exp_T = \mathbb{Z}\pi i \check\alpha$, where $\Delta = \{\pm\alpha\}$. For $x = y = \frac{i}{2}\check\alpha$ we therefore get

$$\kappa(x,y) = -\frac{1}{4}\kappa(\check\alpha,\check\alpha) = \frac{1}{2} \not\in \mathbb{Z}.$$

We conclude that for $K = \mathrm{SO}(3,\mathbb{R})$ the group $\pi_0(G^+) \cong \pi_1(K) \cong \mathbb{Z}_2 = \{\pm 1\}$ acts non-trivially on $Z \cong \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{Z}_2$ by $s.(x,t) = (c(s,t)x, t)$, where $c\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ is the unique non-trivial bicharacter satisfying $c(-1,-1) = -1$. ■

Remark VII.5. (a) Let $x_o \in M$ be a base point, and assume that M is connected of positive dimension and K is a Banach–Lie group. We consider the group $G_* := C^\infty_*(M,K)_e$. If $\varphi \in \mathrm{Hom}(\mathbb{T}, G_*)$, then the map

$$\Phi\colon M \to \mathrm{Hom}(\mathbb{T}, K), \quad x \mapsto (t \mapsto \varphi(t)(x))$$

is a continuous map with $\Phi(x_o) = e$ (the constant homomorphism). Since K has no small subgroups, the constant homomorphism e is isolated in the set $\mathrm{Hom}(\mathbb{T},K) \subseteq C(\mathbb{T},K)$. Therefore the continuity of Φ implies that it is constant, and thus $\mathrm{Hom}(\mathbb{T}, G_*) = \{e\}$. On the other hand $\pi_1(G_*) \cong [M \wedge \mathbb{S}^1, K]$ may be non-trivial. A typical example is $K = \mathrm{SU}(2,\mathbb{C})$ and $M = \mathbb{S}^2$, where $\pi_1(G_*) \cong \pi_3(K) \cong \mathbb{Z}$. Hence G_* is an example of an infinite-dimensional Lie group for which $\pi_1(G_*)$ is not generated by the homotopy classes of homomorphisms $\mathbb{T} \to G_*$.
(b) According to [ASS71], the unit groups $G := A^\times$ of von Neumann algebras on separable Hilbert spaces all have the property that $\mathrm{Hom}(\mathbb{T}, A^\times)$ generates $\pi_1(A^\times)$. ■

Problems VII. (1) Find a good characterization of those groups G for which a "universal covering group" exists even if G is not connected.
(2) Generalize (7.1) to a general formula for ζ without any additional assumption. ■

The following two examples show that in general the universal covering group $q\colon \widetilde G \to G$ cannot be extended to a central/abelian extension of the full group G^+. If the homomorphism $G^+ \twoheadrightarrow \pi_0(G^+)$ splits, then we can simply form $\widetilde G \rtimes \pi_0(G^+)$ by lifting the natural conjugation action of $\pi_0(G^+)$ on G to an action on $\widetilde G$.

Example VII.6. We describe an example of a non-connected Lie group for which G_e does not split. Let

$$G := \Big\{ \begin{pmatrix} 1 & p & z \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : p, q \in \mathbb{Z}, z \in \mathbb{R} \Big\}.$$

Then $G_e \cong \mathbb{R}$ and $\pi_0(G) \cong \mathbb{Z}^2$. The group G is a central extension of $\mathbb{Z}^2$ by $\mathbb{R}$. An easy calculation shows that the commutator group (G,G) of G is

$$(G,G) = \Big\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \Big\}.$$

As the commutator group is non-trivial, G is not a semidirect product of G_e and $\pi_0(G)$. ■

Example VII.7. In the group G of Example VII.6, we consider the normal subgroup

$$N := \Big\{ \begin{pmatrix} 1 & p & z \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : p, q, z \in 2\mathbb{Z} \Big\}.$$

Then G/N is a central extension of $\pi_0(G/N) \cong \mathbb{Z}_2^2$ by $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. The commutator group of G/N is given by $(G,G)/((G,G) \cap N) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$. Therefore

$$\mathbb{T} = (G/N)_e \hookrightarrow G/N \twoheadrightarrow \mathbb{Z}_2^2 \cong \pi_0(G/N)$$

is a non-trivial central extension.

Suppose that we have an extension

$$\widetilde{\mathbb{T}} \cong \mathbb{R} \hookrightarrow \widehat{G} \twoheadrightarrow \mathbb{Z}_2^2,$$

where $\widehat{G}$ is a covering group of G. Then $\widetilde{\mathbb{T}}$ is central in $\widehat{G}$ because $\mathbb{Z}_2^2 \cong \pi_0(G/N)$ acts trivially on the Lie algebra of G/N, hence on $(G/N)_e$. Therefore $\widehat{G}$ is a central extension of $\mathbb{Z}_2^2$ by $\mathbb{R}$. Since the commutator map $(x,y) \mapsto xyx^{-1}y^{-1}$ factors through a bihomomorphism $\mathbb{Z}_2^2 \to \mathbb{R}$. Since $\mathbb{R}$ has no non-trivial finite subgroups, the commutator group of $\widehat{G}$ is trivial. Therefore $\widehat{G}$ is abelian, contradicting the assumption that $\widehat{G}$ is a covering of the non-abelian group G.

We have thus shown that the group G has no universal covering group. ■

Lemma VII.8. *Let $G := C_*(M,K)$, where K is a Banach–Lie group and M a connected topological space. Then the constant map e is the only element of G of finite order.*

Proof. Assume that $f^k = e$ holds for some continuous base point preserving map $f: M \to K$. Further let $U \subseteq K$ be an identity neighborhood containing no small subgroups and $V \subseteq U$ an open identity neighborhood with $V^k \subseteq U$. Then the only element of order k in V is e because otherwise U would contain a non-trivial subgroup of K. Therefore $f^{-1}(V)$ is an open subset of M which coincides with $f^{-1}(\{e\})$, hence is also closed. As f preserves base points, this set is non-empty, and the connectedness of M implies that f is constant e. ■

Example VII.9. Let $M = \mathbb{S}^1$, K be a compact connected semisimple Lie group, and $G := C_*(M,K)$. Then $\pi_0(G) \cong \pi_1(K)$ is a finite group and Lemma VII.8 implies that the exact sequence $G_e \hookrightarrow G \twoheadrightarrow \pi_0(G)$ does not split. ■

Appendix: Some homotopy theoretic background

A version of Hilton's Lemma

This section grew out of an attempt to obtain a more conceptual proof of Lemma I.10.

Lemma A.1. *Let G be a set with two group structures $m_1(a,b) = ab$ and $m_2(a,b) = a * b$ with identity elements e_1 and e_2. We assume that m_2 is a group homomorphism $(G,m_1)^2 \to (G,m_1)$, i.e., that*

$$(ab) * (cd) = (a * c)(b * d), \quad a,b,c,d \in G.$$

Then $e_1 = e_2$, $m_1 = m_2$, and the multiplications are abelian.

Proof. First we obtain

$$e_1 * e_1 = (e_1e_1) * (e_1e_1) = (e_1 * e_1)(e_1 * e_1),$$

showing that $e_1 * e_1$ is an idempotent for m_1, and hence that $e_1 = e_1 * e_1$. Now e_1 is an idempotent for m_2, so that we also get $e_2 = e_1$. Therefore $e := e_1 = e_2$ is a unit for both group structures. We now obtain for $a,b,c,d \in G$:

$$b * c = (eb) * (ce) = (e * c)(b * e) = cb \quad \text{and} \quad a * d = (ae) * (ed) = (a * e)(e * d) = ad.$$

We conclude that both group structures coincide and turn G into an abelian group. ■

The above argument works with somewhat weaker assumptions. It also shows that if we have two monoid structures with the same identity on a set M, then both coincide and are abelian.

If G is a topological group and X an H-cogroup, such as a sphere $\mathbb{S}^n$, then we have on the set $[X, G]$ of homotopy classes of continuous maps $X \to G$ two natural group structures. The first one is given by pointwise multiplication:

$$[f][g] := [f \cdot g],$$

and the second one is obtained by the comultiplication $\gamma\colon X \to X \vee X$ by

$$[f] * [g] := [f * g] := [(f \vee g) \circ \gamma].$$

According to [Br93, Lemma VII.3.4], both group structures satisfy the assumptions of Lemma A.1, hence coincide. A similar argument applies to the set $[X, G]_*$ of homotopy classes of base point preserving maps, but if G is arcwise connected, we trivially have $[X, G] = [X, G]_*$.

Remark A.2. If X itself is a topological group, for example if $X = \mathbb{S}^1 = \mathbb{T}$, then we consider the element $[\mathrm{id}_X]$ in the abelian group $[X, X]$. Its powers form a cyclic subgroup $[\mathrm{id}_X]^{\mathbb{Z}}$ of $[X, X]$.

Let

$$[X, X] \times [X, G] \to [X, G], \quad ([f], [g]) \mapsto [g \circ f]$$

denote the natural composition map. We claim that in the group $[X, G]$ we have

$$[f][\mathrm{id}_X]^n = [f^n], \quad n \in \mathbb{Z}.$$

In fact, for a fixed $[f] \in [X, G]$ the map

$$[X, X] \to [X, G], \quad [h] \mapsto [f \circ h]$$

is a group homomorphism because

$$[h_1][h_2] = [h_1 * h_2] \mapsto [f \circ (h_1 \vee h_2) \circ \gamma] = [f \circ h_1] * [f \circ h_2].$$

Therefore

$$[f] \circ [\mathrm{id}_X]^n = [f \circ \mathrm{id}_X]^n = [f]^n = [f^n], \quad n \in \mathbb{Z},$$

and in particular $[\mathrm{id}_X]^n \cong [\mathrm{id}_X^n]$. For $X = \mathbb{S}^1 = \mathbb{T}$ this implies that for each $n \in \mathbb{Z}$ and $f \in C^\infty(\mathbb{S}^1, G)$ the maps

$$t \mapsto f(t^n) \quad \text{and} \quad t \mapsto f(t)^n$$

are homotopic. ■

Whitehead products and some homotopy theory

Let G be a topological group. Then G acts by inner automorphism $c_g(x) := gxg^{-1}$ on itself. For each $k \in \mathbb{N}_0$ the automorphism c_g induces an automorphism $\pi_k(c_g)$ of the homotopy group $\pi_k(G)$ which is trivial if $g \in G_e$, the arc-component of the identity of G. We thus obtain an action of $\pi_0(G)$ on the groups $\pi_k(G)$, which for $k = 0$ is the conjugation action of $\pi_0(G)$ on itself.

There are circumstance under which these actions are trivial.

Proposition A.3. *If $G = C_*(\mathbb{S}^1, K)$ holds for a topological group K, then $\pi_0(G)$ acts trivially on all homotopy groups $\pi_k(G)$, $k \in \mathbb{N}_0$.*

Proof. If $G = C_*(\mathbb{S}^1, K)$ for some topological group K, then we have to consider the action of $\pi_0(G) \cong \pi_1(K)$ on the groups $\pi_k(G) \cong \pi_{k+1}(K)$. To see that this action is trivial, we view the pointed spaces $\mathbb{S}^k$ as $I^k/\partial I^k$, where $I = [0,1]$. Then we view elements of G as continuous function $f\colon I \to K$, and an element of $\pi_k(G)$ is represented by a continuous function $h\colon I^{k+1} \to K$. Then the action of $[f] \in \pi_0(G)$ on $[h] \in \pi_k(G)$ is given by

$$(f.h)(x_1, \ldots, x_k, x_{k+1}) = f(x_1)h(x_1, \ldots, x_k, x_{k+1})f(x_1)^{-1}.$$

The map f is homotopic to f_1 with support contained in $[0, \frac{1}{2}] \subseteq I$, and h_1 is homotopic to a maps whose support is contained in $[\frac{1}{2}, 1] \times I^k$. Therefore $[f.h] = [f_1.h_1] = [h_1]$ implies that $\pi_0(G)$ acts trivially on $\pi_k(G)$. ■

Corollary A.4. *For $G = C_*(\mathbb{S}^d, K)$, $d > 0$ and K a topological group, the action of $\pi_0(G)$ on all homotopy groups $\pi_k(G)$ is trivial.*

Proof. In view of $d \geq 1$, we have

$$C_*(\mathbb{S}^d, K) \cong C_*(\mathbb{S}^1 \wedge \mathbb{S}^{d-1}, K) \cong C_*(\mathbb{S}^1, C_*(\mathbb{S}^{d-1}, K)).$$

Therefore Proposition A.3 applies. ■

Now let $G = C(\mathbb{S}^d, K)$ with $d > 0$ and assume that K is connected. Then the evaluation map $\mathrm{ev}\colon G \to K, f \mapsto f(x_o)$ in the base point $x_o \in \mathbb{S}^d$ is split by the map $K \hookrightarrow G$ mapping $k \in K$ to the corresponding constant map. Let $G_* := C_*(\mathbb{S}^d, K) := \ker \mathrm{ev}$. We then obtain an isomorphism $G \cong G_* \rtimes K$, and therefore

$$\pi_0(G) \cong \pi_0(G_*) \cong \pi_d(K),$$

and for $k > 0$:

$$\pi_k(G) \cong \pi_k(G_*) \times \pi_k(K) \cong \pi_{k+d}(K) \times \pi_k(K).$$

From Corollary A.4 we derive that the action of $\pi_0(G)$ on $\pi_k(G_*)$ is trivial. Since the projection $\pi_k(G) \to \pi_k(K)$ corresponds to the evaluation in x_o, and all the elements of G_* are trivial in x_o, we see that the action of $\pi_0(G)$ on $\pi_k(G)$ can be written as

$$[f].([v],[w]) = ([v] + \chi_{d,k}([f],[w]),[w]),$$

where the map

$$\chi_{d,k}\colon \pi_d(K) \times \pi_k(K) \to \pi_{k+d}(K)$$

is a bihomomorphism. This map is given on the level of functions by $\chi_{d,k}([f],[v]) = [f * v]$ with

$$(f * v)(x,y) = f(x)v(y)f(x)^{-1}v(y)^{-1},$$

where we observe that this defines a map $I^{k+d} = I^k \times I^d \to K$ vanishing on the boundary of I^{k+d}. A particular interesting case is $d = k = 1$, where we have a bihomomorphism

$$\chi_{1,1}\colon \pi_1(K) \times \pi_1(K) \to \pi_2(K).$$

Proposition A.5. *If $K = C_*(\mathbb{S}^1, H)$ holds for a topological group H, then the maps $\chi_{d,k}\colon \pi_d(K) \times \pi_k(K) \to \pi_{k+d}(K)$ are trivial.*

Proof. Let $f\colon I^d \to K$ and $v\colon I^k \to K$ be continuous maps representing the elements $[f] \in \pi_d(K)$ and $[v] \in \pi_k(K)$. In view of $C_*(\mathbb{S}^d, K) = C_*(\mathbb{S}^d, C_*(\mathbb{S}^1, H)) \cong C_*(\mathbb{S}^{d+1}, H)$, we may think of f as a function $\widetilde{f}\colon I \times I^d \to H$, and likewise interprete v as a function $\widetilde{v}\colon I \times I^k \to H$. As these functions vanish on $\partial I \times I^d$, resp., $\partial I \times I^k$, they can be represented by functions $\overline{f}$ supported by $[0, \frac{1}{2}] \times I^d$, resp., $\overline{v}$ supported by $[\frac{1}{2}, 1] \times I^k$. Then the function

$$I \times I^d \times I^k \to H, \quad (t,x,y) \mapsto \overline{f}(t,x)\overline{v}(t,y)\overline{f}(t,x)^{-1}\overline{v}(t,y)^{-1}$$

is constant e, which implies that $\chi_{d,k}([f],[v]) \in \pi_{d+k}(K)$ is trivial. ■

Remark A.6. For each pointed topological space X one has the Whitehead products

$$\pi_k(X) \times \pi_m(X) \to \pi_{k+m-1}(X), \quad k, m \in \mathbb{N}.$$

These maps are defined as follows. We use the natural map

$$g \colon \mathbb{S}^{k+m-1} \cong \partial I^{k+m} = \partial I^k \times I^m \cup I^k \times \partial I^m \to \mathbb{S}^k \vee \mathbb{S}^m$$

to define for two maps $\varphi \in C_*(\mathbb{S}^k, X)$ and $\psi \in C_*(\mathbb{S}^m, X)$ the map

$$\varphi * \psi := (\varphi \vee \psi) \circ g \colon \mathbb{S}^{k+m-1} \to X,$$

and by passing to the homotopy classes a map

$$\pi_k(X) \times \pi_m(X) \to \pi_{k+m-1}(X), \quad [\varphi] * [\psi] \mapsto [\varphi * \psi].$$

With a similar argument as in the proof of Proposition A.5, one can see that if X is a topological group, then all Whitehead products are trivial ([Br93, Probl. VII.7.5]). For $k = m = 1$ this implies in particular that $\pi_1(X)$ is abelian because in this case

$$[\varphi] * [\psi] = [\varphi][\psi][\varphi]^{-1}[\psi]^{-1}.$$

The Whitehead products are somehow close, but different from the maps $\chi_{d,k}$ defined above. If $X = H$ is a topological group and $K := C_*(\mathbb{S}^1, H)$, then the Whitehead products are maps

$$\pi_k(K) \times \pi_m(K) \to \pi_{k+m}(K), \quad k, m \in \mathbb{N}_0.$$

We do not know whether the Whitehead products correspond in this setting to the maps $\chi_{d,k}$ from above, but in this context we also have the direct argument in Proposition A.5 showing that each $\chi_{d,k}$ is trivial. ■

Remark A.7. (Some information on $[M, K]_*$) Let $\widetilde{K}$ denote the simply connected covering group of K. Then a base point preserving smooth map $f \colon M \to K$ lifts to $\widetilde{K}$ if and only if the corresponding homomorphism $\pi_1(f) \colon \pi_1(M) \to \pi_1(K)$ is trivial. We therefore obtain an exact sequence

$$C_*(M, \widetilde{K}) \hookrightarrow C_*(M, K) \to \operatorname{Hom}(\pi_1(M), \pi_1(K))$$

which immediately leads to an exact sequence

$$[M, \widetilde{K}]_* \cong \pi_0(C_*(M, \widetilde{K})) \hookrightarrow [M, K]_* \cong \pi_0(C_*(M, K)) \to \operatorname{Hom}(\pi_1(M), \pi_1(K)).$$

Here we use that the image of $C_*(M, \widetilde{K})$ is open because maps with image contained in a sufficiently small identity neighborhood can be lifted to the universal covering group of K. ■

Remark A.8. One may also try to get some information on the $\chi_{d,k}$'s for the groups $G_d := C(\mathbb{T}^d, K)$ for $d \geq 2$, but this becomes quite involved for the following reasons. In this case

$$G_d \cong C(\mathbb{T}, G_{d-1}) \cong C_*(\mathbb{T}, G_{d-1}) \rtimes G_{d-1}$$

(cf. Remark I.11(b)). If we ask for the action of $\pi_0(G_d)$ on $\pi_1(G_d)$, then we can use the same argument as after Corollary A.4 to see that this action is trivial if and only if the commutator map

$$\chi_{1,1} \colon \pi_1(G_{d-1}) \times \pi_1(G_{d-1}) \to \pi_2(G_{d-1})$$

is trivial. Even for $d = 2$, this does not follow from Proposition A.5 because $C(\mathbb{T}, K)$ is larger than the group $C_*(\mathbb{T}, K)$. ■

Problem A.1. Let K be a connected topological group. Is the mapping

$$\chi_{d,k} \colon \pi_d(K) \times \pi_k(K) \to \pi_{k+d}(K)$$

defined above trivial for $d, k \geq 1$? The case $k = d = 1$ is of particular interest. ■

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