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Infinite-dimensional groups and their representations

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Abstract. These lecture notes provide an introduction to the representation theory of Banach–Lie groups of operators on Hilbert spaces, where our main focus lies on highest weight representations and their geometric realization as spaces of holomorphic sections of a complex line bundle. After discussing the finite-dimensional case in Section I, we describe the algebraic side of the theory in Sections II and III. Then we turn in Sections IV and V to Banach–Lie groups and holomorphic representations of complex classical ones. The geometry of the coadjoint action is discussed in Section VI, and in the concluding Section VII all threads lead to a full discussion of the theory for the group $U_2(H)$ of unitary operators u on a Hilbert space H for which $u - \mathbf{1}$ is Hilbert–Schmidt.

Introduction

As in finite dimensions, Lie theory is an exciting combination of algebraic and analytic methods. In the finite-dimensional situation one studies a connected Lie group G by the exponential function $\exp\colon \mathfrak{g} \to G$ which is a local diffeomorphism. Therefore the Lie algebra structure of $\mathfrak{g}$ carries essentially all the local information on G. This means that all groups with the same Lie algebra $\mathfrak{g}$ are quotients of an essentially unique simply connected group $\widetilde{G}$ modulo discrete central subgroups. Viewing $\mathfrak{g}$ as a "linearization" of G, the heart of the Lie theoretic methods is a dictionary translating analytic and global properties of G into algebraic properties of its Lie algebra $\mathfrak{g}$, which are then studied by algebraic methods.

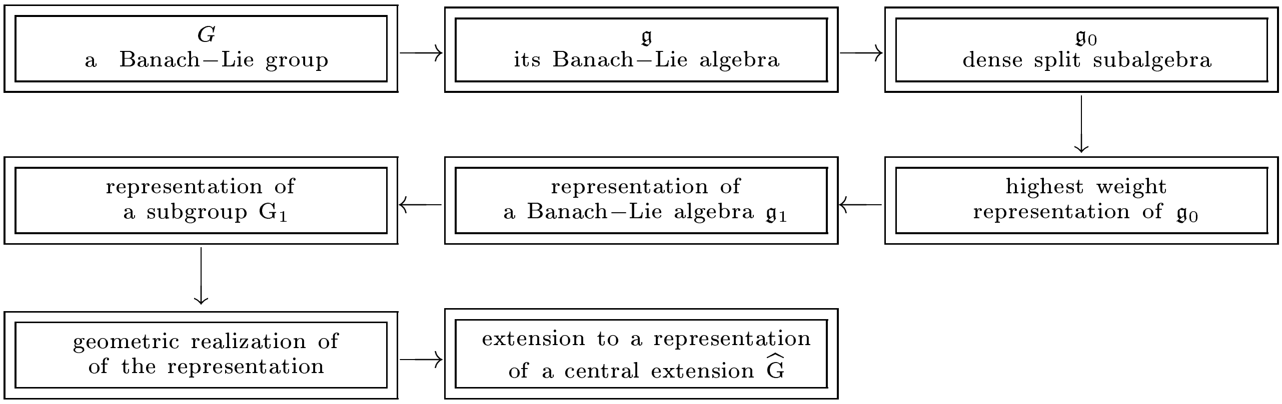
This picture is different for infinite-dimensional groups, and how bad it becomes depends on the setting one is working in. The central objects of these lectures will be groups of operators on Hilbert spaces. These groups will always have a natural topology for which they are Banach–Lie groups, i.e., manifolds modeled over a Banach space endowed with a smooth group structure (multiplication and inversion). In this setting one still has an exponential function

$\exp: \mathfrak{g} \to G$ which is a local diffeomorphism, hence a good translation mechanism from G to $\mathfrak{g}$ and back to G. A new feature is that the Lie algebra $\mathfrak{g}$ now is a Banach space with a continuous Lie bracket, a Banach–Lie algebra, so that we need functional analytic as well as algebraic concepts to study the Lie algebra and the group.

As we will see below, one often finds many incarnations of such a Lie group in the sense that there is a great variety of dense subgroups $G_1 \subseteq G$ which are Lie groups in their own right, but which are better suited for several constructions than G itself. Sometimes G is simply too big or has to be replaced by a suitable central extension. On the Lie algebraic side these groups correspond to dense subalgebras $\mathfrak{g}_1$ of $\mathfrak{g}$ which are much smaller, and one often has certain "minimal" subalgebras which are purely algebraic objects. It is this phenomenon that makes infinite-dimensional Lie theory more difficult and also more interesting than the finite-dimensional theory. One first has to find the right "version" of the group which is best suited for the setting one has in mind, and then one has to analyze this group which might differ from the original one.

The following diagram shows schematically which way one has to go to obtain a thorough understanding of the class of (unitary) highest weight representations of Banach–Lie groups. Starting with a Banach–Lie group (in these notes this will essentially be a group of operators on a Hilbert space), we specify a certain dense subalgebra $\mathfrak{g}_0$ of its Lie algebra which has a root decomposition. For this Lie algebra we are then able to classify all unitary highest weight representations in a completely algebraic context. The next step consists in extending these representations under natural boundedness conditions to a continuous representation of a Banach–Lie algebra completion $\mathfrak{g}_1$ of $\mathfrak{g}_0$ and then integrating this representation to a holomorphic representation of some complex Banach–Lie group G_1. In many cases it turns out that the group G_1 is far from being the maximal group to which this representation integrates, and to understand the subtleties involved in this integration process, we will have to obtain a natural geometric realization of the representation under consideration by a space of holomorphic sections of a complex line bundle. In this geometric context we will then determine the natural groups acting in the representations. This involves in particular a discussion of central extensions of these groups.

Below we will see several examples where such translations procedures become crucial. We think that the quite accessible class of operator groups displays these techniques quite well. They also lead to a good understanding of many phenomena in the physical literature concerning central extensions and the implementability of symmetries. For the sake of simplicity, we will mainly discuss the group $\mathrm{GL}_2(H)$ of a complex Hilbert space H which consists of all those invertible operators g on H for which $g - \mathbf{1}$ is a Hilbert–Schmidt operator.


G
a Banach–Lie group
$\mathfrak{g}$
its Banach–Lie algebra
$\mathfrak{g}_0$
dense split subalgebra
highest weight representation of $\mathfrak{g}_0$
representation of a Banach–Lie algebra $\mathfrak{g}_1$
representation of a subgroup G_1
geometric realization of of the representation
extension to a representation of a central extension $\widehat{\mathrm{G}}$

I. The finite-dimensional case

Before we turn to infinite-dimensional groups, it is worthwhile to recall the picture for finite-dimensional groups to clarify which kind of representations and type of geometry we will be looking for in the infinite-dimensional context.

There are several paths along which one can approach the picture which presents itself as a circle of ideas with several entry points. One possibility is to start with compact groups. Here the problem is to classify all irreducible unitary representations of a compact connected Lie group U and to find natural geometric realizations of these representations which then can in turn be used to get more information on the representations. Functional analytic arguments imply that all irreducible representations of a compact group are finite-dimensional, so that we may limit our considerations to finite-dimensional representations. To be able to obtain a classification, it turns out to be very fruitful to use a certain analytic extension process to translate the problem as follows. First one shows that there exists a complex connected Lie group $G = U_{\mathbb{C}}$ containing U as a subgroup for which the polar map

$$U \times \mathfrak{u} \to G, \quad (u, X) \mapsto u \exp iX$$

is a diffeomorphism. Here $\mathfrak{u} = \mathbf{L}(U)$ denotes the Lie algebra of U. We call the resulting decomposition $G = U \exp(i\mathfrak{u})$ the *polar decomposition of* G. The simplest example is the circle group

$$U = \mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\} \quad \text{with} \quad U_{\mathbb{C}} = \mathbb{C}^{\times},$$

where the polar map corresponds to polar coordinates in the complex plane. Groups of the form $U_{\mathbb{C}}$ are called *complex reductive groups*. This terminology comes the theory of algebraic groups. One should be aware of the fact that the Lie algebras of complex reductive groups are reductive, but that the converse is not true. In particular the group $\mathbb{C}^n$ is not "complex reductive" in the sense above.

An important consequence of the polar decomposition is that every homomorphism $\varphi \colon U \to H$ to a finite-dimensional complex Lie group H extends to a holomorphic homomorphism

$$\varphi_{\mathbb{C}} \colon U_{\mathbb{C}} \to H \quad \text{by} \quad \varphi_{\mathbb{C}}(u \exp(iX)) := \varphi(u) \exp(i\,\mathbf{L}(\varphi)(X)),$$

where

$$\mathbf{L}(\varphi) = d\varphi(\mathbf{1}) \colon \mathbf{L}(U) \to \mathbf{L}(H)$$

is the corresponding Lie algebra homomorphism. We thus obtain a one-to-one correspondence between irreducible unitary representations of U and irreducible (finite-dimensional) holomorphic representations of $G = U_{\mathbb{C}}$ (cf. Exercise I.2), so that we are left with the problem of describing the irreducible finite-dimensional holomorphic representations of a complex reductive group G. For simplicity we assume in the following that G is simply connected. A particular example is the group $G = \mathrm{SL}(n, \mathbb{C})$ which arises as $U_{\mathbb{C}}$ for $U = \mathrm{U}(n, \mathbb{C})$.

The algebraic approach to the classification

We are interested in the geometry and the structure of the irreducible representations of U, resp., G. The best accessible picture is the algebraic one, dealing with simple finite-dimensional modules of the reductive Lie algebra $\mathfrak{g}$. To see the connection between group and Lie algebra representations requires some translation mechanism, a method which is characteristic for Lie theory as a whole.

First we have to get hold of the algebraic structure of the Lie algebra $\mathfrak{g}$ of G. The crucial tool is the root decomposition of $\mathfrak{g}$: There exists a maximal abelian subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with the property that all operators $\operatorname{ad} h$, $h \in \mathfrak{h}$, are diagonalizable, so that one obtains a decomposition of $\mathfrak{g}$ into simultaneous eigenspaces

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \colon (\forall h \in \mathfrak{h})\, [h, x] = \alpha(h)x\}$$

for the action of $\mathfrak{h}$ on $\mathfrak{g}$, where $\alpha \colon \mathfrak{h} \to \mathbb{C}$ is a linear functional. A non-zero functional $\alpha \in \mathfrak{h}^*$ is called a *root of* $\mathfrak{g}$ if $\mathfrak{g}^\alpha \neq \{0\}$. We write $\Delta \subseteq \mathfrak{h}^*$ for the set of roots. It turns out that $\mathfrak{g}^0 = \mathfrak{h}$, so that we obtain the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha.$$

It is an important fact that for every root $\alpha \in \Delta$ the subspace

$$\mathfrak{g}(\alpha) := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$$

is a three-dimensional simple subalgebra, hence isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. From that one derives the existence of a unique element $\check{\alpha} \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \subseteq \mathfrak{h}$ with $\alpha(\check{\alpha}) = 2$. This element is called the *coroot* corresponding to α.

To use the information on the structure of $\mathfrak{g}$ to classify irreducible representations, we consider a maximal solvable subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ containing $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian, hence solvable, the existence of such a subalgebra follows from the fact that $\mathfrak{g}$ is finite-dimensional. One simply chooses a solvable subalgebra containing $\mathfrak{h}$ which is of maximal dimension. One can show that, in terms of the root decomposition, $\mathfrak{b}$ can be described as

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha,$$

where $\Delta^+ \subseteq \Delta$ is a *positive system*, i.e.,

$$\Delta^+ \dot{\cup} - \Delta^+ = \Delta \quad \text{and} \quad (\Delta^+ + \Delta^+) \cap \Delta \subseteq \Delta^+.$$

The next step is to apply Lie's Theorem on the finite-dimensional representations of solvable Lie algebras to see that every simple $\mathfrak{g}$-module V contains a (unique) one-dimensional $\mathfrak{b}$-eigenspace $\mathbb{C}v$. Since the linear functional

$$\lambda \colon \mathfrak{b} \to \mathbb{C}$$

given by $b.v = \lambda(b)v$, $b \in \mathfrak{b}$, is a Lie algebra homomorphism, it vanishes on each $\mathfrak{g}^\alpha$, $\alpha \in \Delta^+$. Moreover, the representation theory of $\mathfrak{sl}(2,\mathbb{C})$, applied to the subalgebras $\mathfrak{g}(\alpha)$, implies that λ is *dominant integral*:

$$\lambda(\check{\alpha}) \in \mathbb{N}_0 \quad \text{for all} \quad \alpha \in \Delta^+.$$

This sets the stage for the classification, and now one shows that if $V_1 \cong V_2$, then $\lambda_1 = \lambda_2$, and that for each dominant integral λ, there exists a simple $\mathfrak{g}$-module which is called $L(\lambda)$.

Theorem I.1. *The finite-dimensional simple $\mathfrak{g}$-modules of are in one-to-one correspondence with the dominant integral weights λ with respect to Δ^+.* ■

A detailed proof of the preceding result can be found in [Hum72]. It is remarkable that the choice of $\mathfrak{b}$, resp., Δ^+ is irrelevant. A different choice only leads to a parametrization of the simple modules by a different set of dominant integral weights. This will be drastically different in the infinite-dimensional setting.

Now we come back to the group level. Since G is assumed to be simply connected, the irreducible representations of G are in one-to-one correspondence with the irreducible representations of $\mathfrak{g}$, so that the classification described above also yields a classification for G and hence for the corresponding compact group U. Here we refer to the general theorem that every Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ integrates to a homomorphism $G \to \mathrm{GL}(V)$, which is quite inexplicit and does not lead to any kind of geometric information about the representation. It is much more desirable to have a more geometric realization of the G-representation on the space $L(\lambda)$ which will be described in the remainder of this section.

Holomorphic vector bundles

In this subsection we explain some of the geometry which is involved in the geometric realization of the irreducible representations of a complex reductive Lie group as the space of holomorphic sections of a complex line bundle. Still all Lie groups are assumed to be finite-dimensional.

Let G be a complex Lie group and $P \subseteq G$ a closed complex subgroup. Then the quotient space $M := G/P$ carries the structure of a complex manifold. To each holomorphic representation (ρ, V) of P, i.e., to each holomorphic homomorphism $\rho\colon P \to \mathrm{GL}(V)$, we will associate a holomorphic vector bundle over M.

Definition I.2. Let M be a complex manifold. A *holomorphic vector bundle* with fiber V is a holomorphic map $\pi\colon \mathcal{V} \to M$ of complex manifolds for which there exists a complex vector space V, an open covering $(U_j)_{j\in J}$ of M, and biholomorphic maps

$$\varphi_j\colon \pi^{-1}(U_j) \to U_j \times V$$

with

$$\pi(\varphi_j^{-1}(x,v)) = x \quad \text{for} \quad (x,v) \in U_j \times V,$$

and such that for each pair $i,j \in J$ there exists a holomorphic map

$$g_{ij}: U_i \cap U_j \to \mathrm{GL}(V)$$

with

$$\varphi_i\varphi_j^{-1}: (U_i \cap U_j) \times V \to (U_i \cap U_j) \times V, \quad (x,v) \mapsto (x, g_{ij}(x).v).$$

The spaces $\pi^{-1}(x)$ are called the *fibers of the bundle* $\mathcal{V}$. Since the maps $\varphi_i\varphi_j^{-1}$ are fiberwise linear, each fiber $\pi^{-1}(x)$ carries a natural complex vector space structure such that $\varphi_j\,|_{\pi^{-1}(x)}: \pi^{-1}(x) \to \{x\} \times V \cong V$ is an isomorphism of complex vector spaces.

A *holomorphic section* of $\mathcal{V}$ is a holomorphic map $\sigma: M \to \mathcal{V}$ with $\pi \circ \sigma = \mathrm{id}_M$. Using the vector space structure on the fibers, we obtain on the space $\Gamma(\mathcal{V})$ of holomorphic sections of $\mathcal{V}$ the structure of a complex vector space via

$$(\lambda\sigma)(x) := \lambda\sigma(x) \quad \text{and} \quad (\sigma_1 + \sigma_2)(x) := \sigma_1(x) + \sigma_2(x). \qquad \blacksquare$$

Homogeneous vector bundles

The only type of bundles we will deal with in these notes are of a rather simple nature because they are so called *homogeneous bundles*. Such bundles are constructed as follows. We return to the setting where G is a complex Lie group, $P \subseteq G$ is a closed complex subgroup, and $M = G/P$. We write $q: G \to M$ for the quotient map. Let (ρ, V) be a holomorphic representation of P on V and write $h.v := \rho(h)(v)$.

On the product manifold $G \times V$ we consider the action of P given by $h.(g,v) := (gh^{-1}, h.v)$. Let

$$\mathcal{V} := G \times_P V := (G \times V)/P$$

denote the space of all P-orbits in $G \times V$. We write $[g,v] := P.(x,v)$ for the orbit of (x,v) and observe that we have a well-defined map

$$\pi: \mathcal{V} \to M, \quad [x,v] \mapsto q(x) = xP.$$

Let $U \subseteq G/P$ be an open subset for which there exists a holomorphic map $\sigma_U: U \to G$ with $x = \sigma_U(x)P$ for all $x \in U$. Then the map

$$q^{-1}(U) \to U \times P, \quad g \mapsto (q(g), \sigma_U(q(g))^{-1}g)$$

is biholomorphic and its inverse is given by the multiplication map

$$U \times P \to q^{-1}(U), (u,h) \mapsto \sigma_U(u)h.$$

We further have a bijection

$$\varphi_U\colon \pi^{-1}(U) \to U \times V, \quad [x,v] \to \Big(q(x), \big(\sigma_U(q(x))^{-1}x\big).v\Big).$$

If, in addition, U is chosen such that it is the domain of a chart of M, then the maps φ_U can be used to obtain a chart of $\mathcal{V}$ as a complex manifold. Moreover, we have

$$\pi \circ \varphi_U^{-1}(x,v) = x \quad \text{for} \quad (x,v) \in U \times V,$$

and for two open subsets $U, W \subseteq M$ with sections σ_U and σ_W, we obtain the map

$$\varphi_U\varphi_W^{-1}\colon (U \cap W) \times V \to (U \cap W) \times V, \quad (x,v) \mapsto (x, g_{U,W}(x).v)$$

with

$$g_{U,W}(x) = \rho\Big(\sigma_U(xP)^{-1}\sigma_W(q(x))\Big).$$

We conclude that $\pi\colon \mathcal{V} \to M$ is a holomorphic vector bundle over M. It is homogeneous in the sense that the natural action of the group G on $\mathcal{V}$ given by $g.[x,v] := [g.x, v]$ is a holomorphic action $G \times \mathcal{V} \to \mathcal{V}$ which is fiberwise linear, i.e., an action by automorphisms of the holomorphic vector bundle.

On the space $\Gamma(\mathcal{V})$ of holomorphic sections we now obtain a natural representation of G by

$$(g.s)(x) := g.s(g^{-1}.x)$$

(Exercise).

It often is convenient to have a more accessible description of the space $\Gamma(\mathcal{V})$ as holomorphic functions $G \to V$. This description is obtained as follows. Let $s\colon M \to \mathcal{V}$ be a holomorphic section of $\mathcal{V}$. Then we can write $s(q(x)) = [x, f(x)]$, where $f\colon G \to V$ is a function. In fact, for each $x \in G$ each element of the fiber $\pi^{-1}(q(x))$ has a unique representative of the form (x,v) and all other representatives are given by $(xp^{-1}, p.v)$, $p \in P$. This leads to

$$f(xp^{-1}) = p.f(x) \quad \text{for} \quad x \in G, p \in P. \tag{1.1}$$

In local coordinates we then have

$$\varphi_U(s(q(x))) = \Big(q(x), \big(\sigma_U(q(x))^{-1}x\big).f(x)\Big),$$

showing that the function $f\colon G \to V$ is holomorphic because $q^{-1}(U) \to P, x \mapsto \sigma_U(q(x))^{-1}x$ is a holomorphic map. If, conversely, $f \in \mathrm{Hol}(G,V)$ is a holomorphic map satisfying (1.1), then the holomorphic map $G \to \mathcal{V}, (g,v) \mapsto [g, f(g)]$ is constant on the P-orbits and therefore factors through a holomorphic map $s\colon M = G/P \to \mathcal{V}$ which is a holomorphic section of $\mathcal{V}$.

We summarize the results of the preceding discussion in the following lemma.

Lemma I.3. *If $\mathcal{V} = G \times_P V$ is a homogeneous holomorphic vector bundle over $M = G/P$, then the space $\Gamma(\mathcal{V})$ of holomorphic sections is in one-to-one correspondence with the space*

$$\Gamma_G(\mathcal{V}) := \{f \in \mathrm{Hol}(G, V) \colon (\forall x \in G, p \in P)\, f(xp^{-1}) = p.f(x)\}.$$

The corresponding map is given by

$$\Phi \colon \Gamma(\mathcal{V}) \to \Gamma_G(\mathcal{V}), \quad s(xP) = [x, \Phi(s)(x)].$$

On $\Gamma_G(\mathcal{V}) \subseteq \mathrm{Hol}(G, V)$ the representation of G is given by

$$(g.f)(x) = f(g^{-1}.x).$$

Proof. In view of the preceding discussion, it only remains to verify the formula for the action of G on $\Gamma_G(\mathcal{V})$: For $s(q(x)) = [x, f(x)]$, $q(x) = xP$, we have

$$\begin{aligned}(g.s)(q(x)) &= g.\big(s(g^{-1}.q(x))\big) = g.\big(s(q(g^{-1}x))\big) \\ &= g.[g^{-1}x, f(g^{-1}x)] = [x, f(g^{-1}x)].\end{aligned}$$

■

A key example: $\mathrm{SL}(2, \mathbb{C})$

We consider the special case where $G = \mathrm{SL}(2, \mathbb{C})$ and

$$P := \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times, c \in \mathbb{C} \right\}.$$

Then P is the stabilizer of the line $\mathbb{C}e_2 \subseteq \mathbb{C}^2$, so that the quotient space G/P can be identified with the complex projective line $\mathbb{P}_1(\mathbb{C}) := \mathbb{P}(\mathbb{C}^2)$, i.e., the set of all one-dimensional subspaces of $\mathbb{C}^2$, via the map $gP \mapsto \mathbb{C}g.e_2$. We write $[z, w] := \mathbb{C}(z, w)$ for the one-dimensional space represented by $(z, w) = ze_1 + we_2 \in \mathbb{C}^2 \setminus \{0\}$. There are two natural open subsets of $\mathbb{P}_1(\mathbb{C})$ given by

$$U_1 := \{[z, w] \colon w \neq 0\}\} = \{[z, 1] \colon z \in \mathbb{C}\}, \quad U_2 := \{[z, w] \colon z \neq 0\} = \{[1, w] \colon w \in \mathbb{C}\}$$

with $U_1 \cup U_2 = \mathbb{P}_1(\mathbb{C})$. On U_1 we define a section

$$\sigma_1 \colon U_1 \to G, \quad \sigma_1([z, 1]) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

and on U_2 we put

$$\sigma_2 \colon U_2 \to G, \quad \sigma_2([1, w]) := \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

One immediately verifies that these two maps are indeed sections of $q\colon G \to G/P \cong \mathbb{P}_1(\mathbb{C})$, i.e., $q(\sigma_1([z,1])) = [z,1]$ and $q(\sigma_2([1,w])) = [1,w]$.

For $n \in \mathbb{Z}$ we consider the one-dimensional holomorphic representation (a holomorphic character)

$$\rho_n\colon P \to \mathrm{GL}(1,\mathbb{C}) \cong \mathbb{C}^\times, \quad \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \mapsto a^n$$

and consider the corresponding holomorphic line bundle $\mathcal{L}_n \to \mathbb{P}_1(\mathbb{C})$. We are interested in the question whether this bundle has a non-zero holomorphic section.

Let $s\colon \mathbb{P}_1(\mathbb{C}) \to \mathcal{L}_n$ be a holomorphic section and $f\colon G \to \mathbb{C}$ the corresponding holomorphic function satisfying $f(xp^{-1}) = \rho_n(p)f(x)$ for $x \in G, p \in P$. We define an entire function $h\colon \mathbb{C} \to \mathbb{C}$ by

$$h(z) := f\Big(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\Big).$$

From

$$\sigma_2([1,w]) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & w^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w^{-1} & 0 \\ -1 & w \end{pmatrix}, \quad w \in \mathbb{C}^\times,$$

in $G = \mathrm{SL}(2,\mathbb{C})$ we obtain

$$f(\sigma_2([1,w])) = \rho_n\begin{pmatrix} w^{-1} & 0 \\ -1 & w \end{pmatrix}^{-1}.h(w^{-1}) = w^n h(w^{-1}) \quad \text{for} \quad w \in \mathbb{C}^\times.$$

The fact that this function extends holomorphically to 0 leads in particular to

$$\limsup_{|z|\to\infty} |h(z)| \cdot |z|^{-n} < \infty. \tag{1.2}$$

For $n < 0$ this implies that h is bounded so that Liouville's Theorem shows that h is constant. For $h \neq 0$ we then obtain a contradiction to the holomorphic extendability of the function $w \mapsto w^n h(w^{-1})$ to 0. This implies that

$$\Gamma(\mathcal{L}_n) = \{0\} \quad \text{for} \quad n < 0.$$

For $n \geq 0$ the condition (1.2) means that h is a polynomial of degree at most n (this follows from the Cauchy estimates for Laurent series). Conversely, for every such polynomial the function $w \mapsto w^n h(w^{-1})$ extends holomorphically to 0, so that it corresponds to a holomorphic section of $\mathcal{L}_n$.

Next we ask which representation of $\mathrm{SL}(2,\mathbb{C})$ we find in the space $\Gamma(\mathcal{L}_n)$ for $n \geq 0$. We know already that the dimension is $n+1$. To determine the representation, we consider the realization in the space $\Gamma_G(\mathcal{L}_n) \subseteq \mathrm{Hol}(G)$. Let

$$N := \Big\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \Big\}$$

and observe that NP is dense in $\mathrm{SL}(2,\mathbb{C})$, so that the restriction map

$$\Gamma_G(\mathcal{L}_n) \to \mathrm{Hol}(N) \cong \mathrm{Hol}(\mathbb{C})$$

is injective. Let $f \in \Gamma_G(\mathcal{L}_n)$ denote the element corresponding to a non-zero constant function in $\mathrm{Hol}(N)$. For $x \in N$ and $p \in P$ we then have $f(xp) = \rho_n(p)^{-1}f(\mathbf{1})$. This implies that for $x \in N$ we have $x.f = f$, and for a diagonal matrix h we get for $x \in N$:

$$(1.3) \qquad (h.f)(x) = f(h^{-1}x) = f(h^{-1}xhh^{-1}) = \rho_n(h).f(\underbrace{h^{-1}xh}_{\in N}) = \rho_n(h).f(x).$$

On the Lie algebra level we have $\mathfrak{g} = \mathfrak{g}^{-\alpha} + \mathfrak{h} + \mathfrak{g}^{\alpha}$ with $\mathfrak{h} + \mathfrak{g}^{-\alpha} = \mathbf{L}(P)$ and $N = \exp(\mathfrak{g}^{\alpha})$, where $\check{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The derived representation of $\mathfrak{g}$ on $\Gamma_G(\mathcal{L}_n)$ is a representation containing a vector f with

$$\check{\alpha}.f = nf \quad \text{and} \quad \mathfrak{g}^{\alpha}.f = \mathbf{L}(N).f = \{0\},$$

Now the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ implies that the submodule generated by f is an $(n+1)$-dimensional simple module and therefore that $\Gamma(\mathcal{L}_n)$ is a simple module of $\mathrm{SL}(2,\mathbb{C})$.

With these elementary considerations we have proved the Borel–Weil Theorem for the group $\mathrm{SL}(2,\mathbb{C})$:

Theorem I.4. (Borel–Weil Theorem for $\mathrm{SL}(2,\mathbb{C})$) *Consider the closed subgroup*

$$P := \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times, c \in \mathbb{C} \right\}$$

of $G := \mathrm{SL}(2,\mathbb{C})$ and its holomorphic characters

$$\chi_n : P \to \mathrm{GL}(1,\mathbb{C}) \cong \mathbb{C}^\times, \quad \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \mapsto a^n, \quad n \in \mathbb{Z}.$$

For the associated holomorphic line bundles $\mathcal{L}_n := G \times_P \mathbb{C}$ we then have

$$\dim \Gamma(\mathcal{L}_n) = \begin{cases} 0 & \text{for } n < 0 \\ n+1 & \text{for } n \geq 0. \end{cases}$$

For $n \geq 0$ the natural representation of G on the space $\Gamma(\mathcal{L}_n)$ of holomorphic sections is the irreducible representation of dimension $n+1$. ■

For $n = 0$ the bundle $\mathcal{L}_0 \to \mathbb{P}_1(\mathbb{C})$ is trivial. Therefore $\Gamma(\mathcal{L}_0) = \mathrm{Hol}(\mathbb{P}_1(\mathbb{C}))$ is the space of holomorphic functions on the Riemann sphere $\mathbb{P}_1(\mathbb{C})$ which consists only of the constant functions.

The Borel–Weil Theorem for complex reductive groups

The general Borel–Weil Theorem can be stated as follows:

Theorem I.5. (Borel–Weil Theorem for complex reductive groups) *Let G be a complex reductive group, $\mathfrak{p} \subseteq \mathfrak{g}$ a subalgebra of the form*

$$\mathfrak{p} = \mathfrak{h} + \sum_{\alpha\in\Delta_P} \mathfrak{g}^\alpha \quad \text{with} \quad \Delta_P \cup -\Delta_P = \Delta,$$

and $\Delta^+ \supseteq \Delta \setminus \Delta_P$ a positive system. We consider the closed connected subgroup $P \subseteq G$ with Lie algebra $\mathfrak{p}$ and a holomorphic character $\chi\colon P \to \mathbb{C}^\times$. Let $\lambda := d\chi\mid_{\mathfrak{h}}$. For the associated holomorphic line bundles $\mathcal{L}_\lambda := G \times_P \mathbb{C}$ we then have

$$\Gamma(\mathcal{L}_\lambda) \neq \{0\} \qquad \Longleftrightarrow \qquad (\forall \alpha \in \Delta_P)\ \lambda(\check{\alpha}) \in -\mathbb{N}_0.$$

In this case the representation of G on $\Gamma(\mathcal{L}_\lambda)$ is the irreducible holomorphic representation of highest weight λ with respect to Δ^+.

The Borel-Weil Theorem shows in particular that if λ is dominant integral and we choose the parabolic subalgebra $\mathfrak{p}$ such that

$$\Delta_P := \{\alpha \in \Delta\colon \lambda(\check{\alpha}) \leq 0\},$$

then $L(\lambda)$ is isomorphic to the space of holomorphic sections of $\mathcal{L}_\lambda$, whenever λ integrates to a holomorphic character of P. If G is semisimple and simply connected, this is always the case if λ is dominant integral (Exercise I.4). In general we need that $\lambda(x) \in 2\pi i\mathbb{Z}$ for each $x \in \mathfrak{h}$ with $\exp x = \mathbf{1}$ (λ is then called *analytically integral*).

We can also take $\Delta_P := -\Delta^+$ if λ is dominant integral with respect to Δ^+.

Proof. Idea of the proof (for a detailed proof of the more general Bott–Borel–Weil Theorem we refer to [KV95]): First we observe that the group P is a semidirect product

$$P \cong N_P \rtimes L_P,$$

where

$$N_P = \exp\big(\sum_{\alpha\in\Delta_P\setminus-\Delta_P} \mathfrak{g}^\alpha\big) \quad \text{and} \quad L_P = \big\langle \exp(\mathfrak{h} + \sum_{\alpha\in\Delta_P\cap-\Delta_P} \mathfrak{g}^\alpha)\big\rangle,$$

Let $0 \neq s \in \Gamma(\mathcal{L}_\lambda)$ be a non-zero section. Pick $p \in M = G/P$ with $s(p) \neq \{0\}$ and write $x_0 := q(\mathbf{1}) \in M$ for the base point. Then there exists a $g \in G$ with $g.p = x_0$, and the section $g.s$ does not vanish in x_0, so that we may assume that $s(x_0) \neq \{0\}$. Let $f\colon G \to \mathbb{C}$ denote the corresponding holomorphic function.

For $\alpha \in -\Delta_P$ and the corresponding subalgebra

$$\mathfrak{g}(\alpha) := \mathbb{C}\check{\alpha} + \mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha} \cong \mathfrak{sl}(2,\mathbb{C})$$

we have a corresponding holomorphic homomorphism

$$\eta_\alpha : \mathrm{SL}(2,\mathbb{C}) \to G,$$

where $\check{\alpha} = \mathbf{L}(\eta_\alpha).h$ for $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Put $\mathfrak{p}(\alpha) := \mathfrak{g}^{-\alpha} + \mathbb{C}\check{\alpha}$ and write $P_\alpha \subseteq \mathrm{SL}(2,\mathbb{C})$ for the corresponding analytic subgroup. Since $f(\mathbf{1}) \neq 0$, the function $f_\alpha := f \circ \eta_\alpha$ is a non-zero holomorphic function on $\mathrm{SL}(2,\mathbb{C})$ satisfying

$$f_\alpha(xp^{-1}) = \chi(\eta_\alpha(p)) f_\alpha(x) \quad \text{for} \quad x \in \mathrm{SL}(2,\mathbb{C}), p \in P_\alpha.$$

The character $\chi \circ \eta_\alpha |_{P_\alpha} : P_\alpha \to \mathbb{C}^\times$ is determined by the integer

$$n := \lambda\big(\mathbf{L}(\eta_\alpha).h\big) = \lambda(\check{\alpha}),$$

so that Theorem I.4 implies that $n \in \mathbb{N}_0$ because the bundle $\mathcal{L}_n \to \mathbb{P}_1(\mathbb{C})$ has a non-zero holomorphic section given by the function f_α on $\mathrm{SL}(2,\mathbb{C})$. This proves that $\lambda(\check{\alpha}) \in \mathbb{N}_0$ for $\alpha \in -\Delta_P$ is necessary for the existence of non-zero holomorphic sections of $\mathcal{L}_\lambda$.

Next we assume that this condition is satisfied. Then there are several ways to show that $\Gamma(\mathcal{L}_\lambda)$ is non-trivial. One possibility is to use the Bruhat decomposition of the group G to construct directly a holomorphic section $f \in \Gamma(\mathcal{L}_\lambda)$ with $f(\mathbf{1}) = 1$ (cf. [CSM95, Sect. II.14] and also [PS86]). Since this method will not work in the infinite-dimensional cases, we use the representation theory of the Lie algebra $\mathfrak{g}$ to obtain a simple highest weight module $L(\lambda)$ of highest weight λ (Theorem I.1). Then the representation of $\mathfrak{g}$ on $L(\lambda)$ integrates to a representation of the simply connected covering group $\widetilde{G}$ on $L(\lambda)$, but since λ integrates to a character of P and therefore in particular to a character of the subgroup $H := \exp \mathfrak{h}$, it factors through a holomorphic representation $(\pi_\lambda, L(\lambda))$ of G (see Exercise I.4).

To realize this representation by holomorphic sections of $\mathcal{L}_\lambda$, we first consider the dual space $L(\lambda)^*$. This space is a $\mathfrak{g}$-module with respect to the action given by

$$(x.\beta)(v) := -\beta(x.v), \quad x \in \mathfrak{g}, \beta \in L(\lambda)^*, v \in L(\lambda).$$

Since the λ-weight space V^λ of V with respect to $\mathfrak{h}$ is one-dimensional, there exists a linear functional $\delta \in L(\lambda)^*$ and a basis element $v_\lambda \in V^\lambda$ with $\delta(v_\lambda) = 1$ and $\ker \delta = \sum_{\mu \neq \lambda} V^\mu$.

For the parabolic subalgebra

$$\mathfrak{p} := \mathfrak{h} + \sum_{\lambda(\check{\alpha}) \leq 0} \mathfrak{g}^\alpha$$

we then have $\mathfrak{p}.\ker\delta \subseteq \ker\delta$ (Exercise I.3), which easily implies that δ is a $\mathfrak{p}$-eigenfunctional of weight $-\lambda$. The group G acts on $L(\lambda)^*$ by

$$(g.\beta)(v) := \beta(g^{-1}.v), \quad g \in G, \beta \in L(\lambda)^*, v \in L(\lambda),$$

and for the connected subgroup $P := \langle \exp \mathfrak{p} \rangle \subseteq G$ corresponding to $\mathfrak{p}$, we obtain that

$$p.\delta = \chi(p)^{-1}\delta,$$

where $\chi: P \to \mathbb{C}^\times$ is the unique holomorphic character whose differential is λ, viewed as a linear function on $\mathfrak{p}$ vanishing on all the root spaces. Let

$$\Psi: L(\lambda) \to \mathrm{Hol}(G), \quad \Psi(v)(g) := \langle \delta, g^{-1}.v \rangle = (g.\delta)(v).$$

Then Ψ is a G-equivariant linear map with respect to the natural representation of G on $\mathrm{Hol}(G)$ given by $(g.f)(x) := f(g^{-1}x)$, and each function f in the range of Ψ satisfies

$$f(gp) = \chi(p)^{-1} f(g), \quad g \in G, p \in P.$$

This means that

$$\Psi(L(\lambda)) \subseteq \Gamma_G(\mathcal{L}_\lambda) = \{f \in \mathrm{Hol}(G): (\forall g \in G)(\forall p \in P)\, f(gp) = \chi(p)^{-1} f(g)\},$$

showing that $\Gamma(\mathcal{L}_\lambda)$ contains a subspace isomorphic to the highest weight module $L(\lambda)$. It remains to show that this subspace exhausts $\Gamma(\mathcal{L}_\lambda)$.

To analyze the representation on the non-zero space $\Gamma(\mathcal{L}_\lambda)$ of holomorphic sections, we need a fact whose proof we do not want to reproduce:

$$\dim \Gamma(\mathcal{L}_\lambda) < \infty.$$

This is a special case of a more general theorem on spaces of holomorphic sections of vector bundles over compact complex manifolds,[1] and G/P is compact because the compact real form U acts transitively on G/P. A more direct proof is outlined in Remark I.6 below.

Now we can argue as follows. First we use Lie's Theorem for the solvable Lie algebra $\mathfrak{b} := \mathfrak{h} + \sum_{\alpha\in\Delta^+} \mathfrak{g}^\alpha$ to see that there exists an $f \in \Gamma_G(\mathcal{L}_\lambda)$ which is a $\mathfrak{b}$-eigenvector. Then f is fixed by the group $N := \exp(\sum_{\alpha\in\Delta^+} \mathfrak{g}^\alpha)$ for which $NP \subseteq G$ is an open subset of G (Exercise I.5). Therefore each element of $\Gamma_G(\mathcal{L}_\lambda)$ is uniquely determined by its restriction to N, which in particular implies that the space

$$\Gamma_G(\mathcal{L}_\lambda)^N = \{f \in \Gamma_G(\mathcal{L}_\lambda): (\forall u \in N)\, u.f = f\}$$

[1] We refer to [GR65, Th. VIII.19] for the Theorem of Cartan–Serre asserting that the cohomology of any coherent sheaf on a compact analytic space is finite-dimensional. Since compact complex manifolds are in particular compact analytic spaces, and holomorphic vector bundles define coherent sheaves, this implies the finite-dimensionality of the space of holomorphic sections for every holomorphic vector bundle over a compact complex manifold.

is one-dimensional. We may assume that $f(\mathbf{1}) = 1$, so that $f(u) = 1$ for all $u \in N$. Then we obtain for $h \in H$:

$$(h.f)(u) = f(h^{-1}u) = f(h^{-1}uhh^{-1}) = \chi(h)f(\underbrace{h^{-1}uh}_{\in N}) = \chi(h) = \chi(h)f(u).$$

This shows that for $h \in \mathfrak{h}$ we have $h.f = \lambda(h)f$, and therefore that f is a $\mathfrak{b}$-eigenvector of weight λ. As in the proof of Theorem I.4, the finite-dimensional representation theory of $\mathfrak{g}$ implies that the submodule generated by f is isomorphic to $L(\lambda)$. It remains to see that this subspace exhausts $\Gamma_G(\mathcal{L}_\lambda)$. If this is not the case, then Weyl's Theorem implies that there exists a complementary submodule W. Repeating the argument above, we find a non-zero function $\widetilde{f} \in W$ which is N-invariant, but this contradicts the fact that $\Gamma_G(\mathcal{L}_\lambda)^N$ is one-dimensional. ∎

Remark I.6. (a) Let $T \cong \mathbb{T}^n$ be a torus group and (ρ, V) a continuous representation of T on the finite-dimensional vector space V. Then T also acts on the space $\mathrm{Hol}(V)$ of complex-valued holomorphic functions on V by

$$(t.f)(x) := f(t^{-1}.x).$$

Since T is abelian, V decomposes into a finite sum of weight spaces of the Lie algebra $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$:

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V^\mu.$$

Let v_j, $j = 1, \ldots, m$, be a basis of V with $v_j \in V^{\mu_j}$. We then have a Taylor expansion

$$f\Big(\sum_j z_j v_j\Big) = \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ and

$$z^\alpha := z_1^{\alpha_1} \cdots z_m^{\alpha_m}.$$

In these terms we obtain

$$((\exp x).f)\Big(\sum_j z_j v_j\Big) = \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha e^{-\sum_j \alpha_j \mu_j(x)} z^\alpha,$$

showing that the weight spaces $\mathrm{Hol}(V)^\mu$ are given by

$$\mathrm{Hol}(V)^\mu = \Big\{ f \in \mathrm{Hol}(V) \colon f\big(\sum_j z_j v_j\big) = \sum_{\sum_j \alpha_j \mu_j = -\mu} c_\alpha z^\alpha \Big\}.$$

If there exists an element $x_0 \in i\mathfrak{t}$ with $\mu_j(x_0) > 0$ for all j, then we may w.l.o.g. assume that $\mu_j(x_0) > 1$ for all j. This condition means that the set

$\{\mu_1, \ldots, \mu_m\}$ is contained in an open half space. Then $\sum_j \alpha_j \mu_j = -\mu$ implies that $-\mu(x_0) = \sum_j \alpha_j \mu_j(x_0) \geq \sum_j \alpha_j$, so that there are only finitely many $\alpha \in \mathbb{N}_0^m$ with $\sum_j \alpha_j \mu_j = -\mu$, which implies that

$$\dim \mathrm{Hol}(V)^\mu = \Big|\Big\{\alpha \in \mathbb{N}_0^m : \sum_j \alpha_j \mu_j = -\mu\Big\}\Big| < \infty.$$

If $F \subseteq \mathrm{Hol}(V)$ is a closed subspace invariant under T, then for each $f \in F$ and each character $\chi_\mu : T \to \mathbb{T}$ with $\chi_\mu(\exp x) = e^{\mu(x)}$ the holomorphic function

$$f_\mu(x) := \int_T f(t.x)\chi_\mu(t)\, d\nu_T(t)$$

is also contained in F, where ν_T is the normalized Haar measure on T. For $\mu = d\chi_\mu$ we have $f_\mu \in \mathrm{Hol}(V)^\mu$, and this implies that each $f \in F$ has a convergent expansion

$$f = \sum_{\mu \in \mathcal{P}_F} f_\mu, \quad \mathcal{P}_F := \{\mu \in \mathfrak{t}_{\mathbb{C}}^* : F^\mu \neq \{0\}\}.$$

This leads to the following observation: If all weight spaces $\mathrm{Hol}(V)^\mu$ are finite-dimensional and $F \subseteq \mathrm{Hol}(V)$ is a closed T-invariant subspace for which $\mathcal{P}_F$ is a finite set, then

$$\dim F < \infty.$$

(b) Now we explain how the preceding discussion can be applied to show that in the proof of the Borel–Weil Theorem we have $\dim \Gamma(\mathcal{L}_\lambda) < \infty$. First we have to specify the torus group to which (a) will be applied. We consider the torus $T = \exp(\mathfrak{t})$, where

$$\mathfrak{t} = \{x \in \mathfrak{h} : \overline{\exp(\mathbb{R}x)} \text{ is compact}\}$$

(Exercise I.7). If G is semisimple, this means that $\mathfrak{t} = \mathrm{span}_{\mathbb{R}}\{i\check{\alpha} : \alpha \in \Delta\}$. In this situation we put

$$V := \sum_{\alpha \notin \Delta_P} \mathfrak{g}^\alpha \quad \text{with} \quad \Delta \setminus \Delta_P \subseteq \Delta^+.$$

Since $\exp(V)P \subseteq G$ is an open subset,

$$\Gamma_G(\mathcal{L}_\lambda) \to \mathrm{Hol}(V), \quad f \mapsto (x \mapsto f(\exp x))$$

is a T-equivariant injective map, where the action of T on $\mathrm{Hol}(V)$ is given by

$$(t.f)(z) = \chi(t) f(\mathrm{Ad}(t)^{-1}.z).$$

The discussion above shows that the set of $\mathfrak{t}$-weights, resp., $\mathfrak{h}$-weights in $\mathrm{Hol}(V)$ is given by

$$\lambda - \sum_{\alpha \in \Delta \setminus \Delta_P} \mathbb{N}_0 \alpha \subseteq \lambda - \sum_{\alpha \in \Delta^+} \mathbb{N}_0 \alpha,$$

and that all multiplicities are finite.

On the other hand the set of $\mathfrak{h}$-weights is invariant under the Weyl group $\mathcal{W}$ (Exercise!), so that the set of weights is contained in

$$\bigcap_{w\in\mathcal{W}} (\lambda - \sum_{\alpha\in\Delta^+} \mathbb{N}_0\alpha) \subseteq \mathrm{conv}(\mathcal{W}.\lambda) \cap (\lambda + \mathcal{R}),$$

where $\mathcal{R}$ is the root lattice. Here the inclusion "$\subseteq$" is not obvious (see [Bou90, Ch. VIII]; see also Section V.2 in [Ne99a]). Since the latter set is finite, and all multiplicities are finite, we derive that the space $\Gamma(\mathcal{L}_\lambda)$ is finite-dimensional. ■

Remark I.7. A third possibility to get hold of the representation on the space $\Gamma(\mathcal{L}_\lambda)$ is to view it as a representation of the compact real form $U \subseteq G$ on a Fréchet space. Then the Big Peter–Weyl Theorem (cf. [HoMo98, Th. 3.51]) implies that it contains a dense subspace of finite-dimensional submodules. Now the argument given in the proof of Theorem I.5 shows that this subspace is an irreducible module, and therefore that the representation of G on the space $\Gamma(\mathcal{L}_\lambda)$ is finite-dimensional and irreducible. ■

Exercises for Section I

Exercise I.1. Let V be a finite-dimensional real vector space and $W \subseteq V$ a subspace. For $A \in \mathrm{End}(V)$ the following are equivalent:
(a) $A(W) \subseteq W$.
(b) For all $t \in \mathbb{R}$ we have $e^{tA}(W) \subseteq W$.
(c) There exists an $\varepsilon > 0$ such that for all $t \in [-\varepsilon, \varepsilon]$ we have $e^{tA}(W) \subseteq W$. ■

Exercise I.2. Let $\rho\colon G \to \mathrm{GL}(V)$ be a holomorphic representation of a finite-dimensional connected complex Lie group G, $d\rho\colon \mathbf{L}(G) \to \mathfrak{gl}(V)$ the derived representation, and $U \subseteq G$ a subgroup such that $\mathbf{L}(G) = \mathbf{L}(U) + i\,\mathbf{L}(U)$. Then for a subspace $W \subseteq V$ the following are equivalent:
(a) W is invariant under G.
(b) W is invariant under U.
(c) W is invariant under $d\rho(\mathbf{L}(U))$.
(d) W is invariant under $d\rho(\mathbf{L}(G))$. ■

Exercise I.3. Let $L(\lambda)$ be a simple highest weight module of a complex reductive Lie algebra $\mathfrak{g}$ of highest weight λ and write

$$L(\lambda) = W \oplus L(\lambda)^\lambda,$$

where $W = \sum_{\mu\neq\lambda} L(\lambda)^\mu$ is the sum of all other weight spaces. Then W is invariant under the subalgebra

$$\mathfrak{p} := \mathfrak{h} + \sum_{\lambda(\check{\alpha})\leq 0} \mathfrak{g}^\alpha.$$

Hint: If v_μ is a weight vector of weight μ and $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}^\alpha$, with $x_\alpha.v_\mu = v_\lambda$, then $\lambda(\check{\alpha}) > 0$ follows from the representation theory of $\mathfrak{sl}(2,\mathbb{C})$. ■

Exercise I.4. Let G be a connected complex reductive Lie group and $q\colon \widetilde{G} \to G$ its universal covering group. Further let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. We consider the complex abelian subgroups $H := \exp_G \mathfrak{h}$, resp., $\widetilde{H} := \exp_{\widetilde{G}} \mathfrak{h}$ of G, resp., $\widetilde{G}$. For a dominant integral λ let $\widetilde{\rho}_\lambda\colon \widetilde{G} \to \mathrm{GL}(L(\lambda))$ denote the corresponding holomorphic representation of $\widetilde{G}$. Show that the following are equivalent:

(a) $\widetilde{\rho}_\lambda$ factors through a holomorphic representation $\rho_\lambda\colon G \to \mathrm{GL}(L(\lambda))$.

(b) $\pi_1(G) := \ker q \subseteq \ker \widetilde{\rho}_\lambda$.

(c) There exists a holomorphic character $\chi\colon H \to \mathbb{C}^\times$ with $d\chi = \lambda$.

(d) λ satisfies $\lambda(\Gamma) \subseteq 2\pi i\mathbb{Z}$ for the subgroup $\Gamma := \{x \in \mathfrak{h}\colon \exp x = \mathbf{1}\}$.

Hint: Use that $\pi_1(G) \subseteq Z(\widetilde{G}) \subseteq \widetilde{H}$. ■

Exercise I.5. Let G be a Lie group and $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$ subalgebras with $\mathfrak{a} + \mathfrak{b} = \mathfrak{g}$. Let $A, B \subseteq G$ be the corresponding analytic subgroups endowed with their intrinsic Lie group topology. Show that the map

$$m\colon A \times B \to G$$

has an open image and that m is a diffeomorphism onto the open subset AB if $A \cap B = \{\mathbf{1}\}$. Hint: Consider the action of the direct product group on G given by $(a, b).g := agb^{-1}$. ■

Exercise I.6. (Integrating representations of $\mathfrak{sl}(2, \mathbb{C})$)

(1) Let V_n be the $n+1$-dimensional simple module of $\mathfrak{sl}(2, \mathbb{C})$. We consider the space

$$P_n := \operatorname{span}\{z_1^j z_2^k\colon j + k = n\} \subseteq \mathbb{C}[z_1, z_2] \cong \operatorname{Pol}(\mathbb{C}^2)$$

of homogeneous polynomials of degree k on $\mathbb{C}^2$. Then the group $\mathrm{SL}(2, \mathbb{C})$ acts on P_n by $(g.f)(x) := f(g^{-1}.x)$. Show that the corresponding derived $\mathfrak{sl}(2, \mathbb{C})$-module is isomorphic to V_n and hence that the Lie algebra action on V_n can be integrated to a representation of $\mathrm{SL}(2, \mathbb{C})$ on V_n.

(2) We call a module (ρ, V) of $\mathfrak{sl}(2, \mathbb{C})$ *integrable* if the operators $\rho(e)$ and $\rho(f)$ are locally nilpotent and $\rho(h)$ is diagonalizable. Using the PBW-Theorem, show that V is a locally finite module, i.e., every element generates a finite-dimensional submodule.

(3) If (ρ, V) is a locally finite $\mathfrak{sl}(2, \mathbb{C})$-module, then there exists a representation $\widetilde{\rho}\colon \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(V)$ such that

$$\rho(X).v = \frac{d}{dt}\bigg|_{t=0} \widetilde{\rho}(e^{tX}).v \qquad \text{for all} \quad v \in V.$$

Hint: Use Weyl's Theorem to see that V is a sum of simple modules, hence semisimple and therefore a direct sum of simple finite-dimensional modules. Then use (1).

(4) Justify the terminology "integrable module." ■

Exercise I.7. Let A be a connected real abelian Lie group and $\mathfrak{a}$ its Lie algebra. Then

$$\mathfrak{t} := \{x \in \mathfrak{a} \colon \overline{\exp \mathbb{R}x} \text{ is compact}\}$$

is a subspace of $\mathfrak{a}$ and $T := \exp \mathfrak{t}$ is a torus, which is the unique maximal compact subgroup of A. ■

II. Split Lie algebras

In this section we describe, in a completely algebraic context, several types of Lie algebras that occur as the algebraic skeleton of operator Lie algebras. These are always locally finite Lie algebras with a root decomposition. For this class of Lie algebras the root decomposition leads to an effective structure theory which is almost comparable to the results one has for finite-dimensional Lie algebras. In particular one has a classification of the simple Lie algebras. The structure of these Lie algebras will be refined by endowing them with an involution, a structural feature that will be crucial in the study of unitary highest weight representations in the next section. The algebraic analysis of these representations is the first step in our approach to the highest weight representations of operator groups.

Throughout this section all Lie algebras are complex if not stated otherwise.

II. Root decompositions

Definition II.1. (a) We call an abelian subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ a *splitting Cartan subalgebra* if $\mathfrak{h}$ is maximal abelian and the derivations $\operatorname{ad} h$ for $h \in \mathfrak{h}$ are simultaneously diagonalizable. If $\mathfrak{g}$ contains a splitting Cartan subalgebra $\mathfrak{h}$, then $\mathfrak{g}$, respectively the pair $(\mathfrak{g}, \mathfrak{h})$, is called a *split Lie algebra* and $\mathfrak{h}$ a *splitting Cartan subalgebra.* This means that we have a *root decomposition*

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h})\, [h, x] = \alpha(h)x\}$ for a linear functional $\alpha \in \mathfrak{h}^*$, and

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \backslash \{0\} : \mathfrak{g}^\alpha \neq \{0\}\}$$

is the corresponding *root system.* The subspaces $\mathfrak{g}^\alpha$ for $\alpha \in \Delta$ are called *root spaces* and its elements are called *root vectors.*

(b) A root $\alpha \in \Delta$ is called *integrable* if $\mathfrak{g}(\alpha) := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \cong \mathfrak{sl}(2, \mathbb{C})$ and there exist non-zero elements $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that $\operatorname{ad} x_{\pm\alpha}$ are locally nilpotent. (An endomorphism A of a vector space V is called *locally nilpotent* if $V = \bigcup_{n \in \mathbb{N}} \ker A^n$.) If $\mathfrak{g}$ is *locally finite*, i.e., every finite subset generates a finite-dimensional subalgebra, then the latter condition is redundant (Exercise II.1).

We write Δ_i for the set of integrable roots. For $\alpha \in \Delta_i$ the space $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ is one-dimensional and α does not vanish on it. Hence there exists a unique element $\check{\alpha} \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ with $\alpha(\check{\alpha}) = 2$ which is called the associated *coroot.* To each coroot we associate the reflection $r_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ given by

$$r_\alpha(\beta) = \beta - \beta(\check{\alpha})\alpha$$

and write $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h}^*)$ for the subgroup generated by these reflections. It is called the *Weyl group of* $\mathfrak{g}$. ■

It is well known that every finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$ has a root decomposition and that all roots are integrable ([Hum72]). One can show that the integrability of a root is equivalent to the existence of a representation $\rho_\alpha\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g})$ for which the derived representation is given by restricting the adjoint representation to $\mathfrak{g}$ (cf. [MP95]; see also Exercise I.6). This justifies the terminology.

Example II.2. Since the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ plays an important role in root decompositions, we first have a look at its standard root decomposition. It is given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The brackets of these basis elements are given by

$$[h,e] = 2e, \quad [h,f] = -2f \quad \text{and} \quad [e,f] = h.$$

Therefore we have the root decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}, \qquad \mathfrak{h} = \mathbb{C}h, \quad \mathfrak{g}^\alpha = \mathbb{C}e, \quad \mathfrak{g}^{-\alpha} = \mathbb{C}f,$$

with $\alpha(h) = 2$, so that $\check{\alpha} = 2$ and $r_\alpha.\alpha = -\alpha$. ■

Example II.3. Let J be a set and $\mathbb{C}^{(J)}$ the vector space with the basis $(e_j)_{j\in J}$. One may also think of this space as the space of all functions $J \to \mathbb{C}$ with finite support. We write $\mathfrak{g} := \mathfrak{gl}(J,\mathbb{C}) \subseteq \mathrm{End}(\mathbb{C}^{(J)})$ for the Lie algebra consisting of all those endomorphisms whose corresponding $J \times J$-matrices have only finitely many non-zero entries. Then the elementary matrices E_{ij} with a single non-zero entry in the (i,j)-position form a basis of the vector space $\mathfrak{g}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be the subalgebra of diagonal matrices and define $\varepsilon_j \in \mathfrak{h}^*$ by $\varepsilon_j(\mathrm{diag}(x_{ii})) := x_{jj}$. Then the set of of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is given by

$$\Delta := \{\varepsilon_j - \varepsilon_k\colon j \neq k, j,k \in J\}$$

where

$$\mathfrak{g}^{\varepsilon_j - \varepsilon_k} = \mathbb{C}E_{jk} \quad \text{and} \quad (\varepsilon_j - \varepsilon_k)\check{} = E_{jj} - E_{kk}.$$

For every pair $i \neq j$ the subalgebra $\mathfrak{g}(\varepsilon_i - \varepsilon_j)$ spanned by $h := E_{ii} - E_{jj}$, $e = E_{ij}$ and $f := E_{ji}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Since, moreover, $(\mathrm{ad}\, E_{ij})^3 = 0$, every root is integrable.

We define

$$\mathfrak{sl}(J,\mathbb{C}) := \Big\{X \in \mathfrak{gl}(J,\mathbb{C})\colon \mathrm{tr}\, X = \sum_{j\in J} x_{jj} = 0\Big\}$$

and note that this subalgebra also has a root decomposition with respect to the Cartan subalgebra $\mathfrak{h} \cap \mathfrak{sl}(J,\mathbb{C})$. ■

For infinite-dimensional Lie algebras there are some subtleties involving the notion of a "reductive" Lie algebra which come from the fact that for many simple Lie algebras not every derivation is inner (cf. Exercise II.10).

Definition II.4. We call a Lie algebra $\mathfrak{g}$ *semisimple* if it is a direct sum of simple ideals. It is said to be *almost reductive* if $[\mathfrak{g},\mathfrak{g}]$ is *semisimple*. It is called *reductive* if $\mathfrak{g} \cong \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$. ■

Example II.5. For every $n \in \mathbb{N}$ the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ is reductive. We have

$$\mathfrak{gl}(n,\mathbb{C}) \cong \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}\mathbf{1},$$

where $\mathfrak{sl}(n,\mathbb{C})$ is simple.

For an infinite set J the identity matrix $\mathbf{1}$ is not contained in $\mathfrak{gl}(J,\mathbb{C})$, which implies that

$$\mathfrak{z}(\mathfrak{gl}(J,\mathbb{C})) = \{x \in \mathfrak{gl}(J,\mathbb{C}) \colon [x, \mathfrak{gl}(J,\mathbb{C})] = \{0\}\} = \{0\}.$$

What survives is the Lie algebra homomorphism

$$\mathrm{tr} \colon \mathfrak{gl}(J,\mathbb{C}) \to \mathbb{C}$$

with $\ker \mathrm{tr} = \mathfrak{sl}(J,\mathbb{C})$. Since $\mathfrak{sl}(J,\mathbb{C})$ is simple (Exercise!), we obtain

$$[\mathfrak{gl}(J,\mathbb{C}), \mathfrak{gl}(J,\mathbb{C})] = \mathfrak{sl}(J,\mathbb{C}),$$

showing that $\mathfrak{gl}(J,\mathbb{C})$ is almost reductive but not reductive. ■

The following theorem shows that the abundance of integrable roots in a split Lie algebra has strong consequences for its structure.

Theorem II.6. *A split Lie algebra $\mathfrak{g}$ is almost reductive and locally finite if and only if all roots are integrable, i.e., $\Delta = \Delta_i$.*

Proof. If $\Delta = \Delta_i$, then Theorem VI.3 in [Ne00a] implies that $\mathfrak{g}$ is locally finite. Now Theorem III.12 in [St99a] shows that $\mathfrak{g}$ is almost reductive. The converse follows from Lemma IV.8 and Theorem III.19 in [St99a]. ■

Positive systems

Definition II.7. A subset $\Delta^+ \subseteq \Delta$ is called a *positive system* if $\Delta = \Delta^+ \cup -\Delta^+$ and no non-trivial linear combination $\sum_{j=1}^n \lambda_j \alpha_j$ with $\alpha_j \in \Delta^+$ and $\lambda_j \geq 0$ vanishes. Geometrically this condition means that

$$\mathrm{cone}(\Delta^+) := \mathbb{R}^+[\Delta^+] := \Big\{ \sum_{j=1}^n \lambda_j \alpha_j \colon \lambda_j \in \mathbb{R}^+, \alpha_j \in \Delta^+ \Big\}$$

is a *pointed convex cone* in the sense that

$$\mathrm{cone}(\Delta^+) \cap -\mathrm{cone}(\Delta^+) = \{0\}.$$

This requirement implies in particular that each positive system contains exactly one root of each set $\{\alpha, -\alpha\}$ and that $\Delta^+ \cap -\Delta^+ = \emptyset$. ■

Proposition II.8. *The positive systems in the root system*

$$\Delta = A_J := \{\varepsilon_j - \varepsilon_k : j \neq k \in J\}$$

of $\mathfrak{gl}(J,\mathbb{C})$ *are in one-to-one correspondence with the linear orderings* $\preceq$ *of the set* J. *This correspondence is established by assigning to* $\preceq$ *the positive system*

$$\Delta^+_{\preceq} := \{\varepsilon_j - \varepsilon_k : j \prec k\}.$$

Proof. First we show that for a linear order $\preceq$ on J the set $\Delta^+_{\preceq}$ is a positive system. So let $\alpha_i = \varepsilon_{j_i} - \varepsilon_{k_i}$, $i = 1,\ldots,n$, be positive roots and $J_0 := \{j_i, k_i : i = 1,\ldots,n\}$. Let $f: J_0 \to \mathbb{R}$ be an injective decreasing function. Then there exists a linear functional F on $\operatorname{span}\{\varepsilon_j : j \in J_0\}$ with $F(\varepsilon_j) = f(j)$ and we have for each positive linear combination

$$F\Big(\sum_{i=1}^n \lambda_i \alpha_i\Big) = \sum_{i=1}^n \lambda_i\big(f(j_i) - f(k_i)\big) > 0$$

if at least one λ_i is positive. This shows that the set $\Delta^+_{\preceq}$ is a positive system in Δ.

If, conversely, Δ^+ is a positive system, then we define $j \preceq k$ by $j = k$ or $\varepsilon_j - \varepsilon_k \in \Delta^+$. It is clear that we thus obtain a reflexive, transitive relation which defines a linear order on J. ■

Remark II.9. (a) The Weyl group $\mathcal{W}$ of $\Delta = A_J$ is isomorphic to the group $S_{(J)}$ of finite permutations of the set J (the subgroup generated by all transpositions). It acts on the diagonal matrices by permuting the entries. Since $S_{(J)}$ acts transitively on the set of all pairs of elements of J, we see that $\mathcal{W}$ acts transitively on Δ.

As the preceding proposition shows, the $\mathcal{W}$-orbits on the set of all positive systems in Δ correspond to the $S_{(J)}$-orbits on the set of all linear orders on J. If J is finite, then $\mathcal{W}$ acts transitively on the set of all linear orders, hence on the set of all positive systems. This does not make it necessary to consider different positive systems for $\mathfrak{gl}(n,\mathbb{C})$ because every finite linearly ordered set $(J,\preceq)$ is isomorphic to $(\{1,\ldots,n\},\leq)$.
(b) If $J = \mathbb{N}$, then it is clear that the natural order $\leq$ on $\mathbb{N}$ corresponds to the standard positive system. A linear order is $\mathcal{W}$-conjugate to this one if there are only finitely many pairs (j,k) with $j < k$ and $k \prec j$ (Exercise!). Interesting other orders are the following: $2 \prec 3 \prec 4 \prec \ldots \prec 1$ or $3 \prec 4 \prec 5 \prec \ldots \prec 1 \prec 2$ etc. Another class of interesting orders arises from bijections with $\mathbb{Z}$:

$$\ldots 7 \prec 5 \prec 3 \prec 1 \prec 2 \prec 4 \prec 6 \ldots$$

One could even define a linear order on $\mathbb{N}$ by using a bijection to $\mathbb{Q}$.
(c) Note that for orders like those described above on $J = \mathbb{N}$ one can think of the elements of $\mathfrak{gl}(\mathbb{N},\mathbb{C})$ as $\mathbb{N}\times\mathbb{N}$-matrices with finitely many non-zero entries, where the basis is ordered according to the linear order $\preceq$. For the order coming from the bijection with $\mathbb{Z}$, this leads to the representation by $\mathbb{Z}\times\mathbb{Z}$-matrices with finitely many non-zero entries. ■

Involutions on split Lie algebras

Definition II.10. (a) An *involutive Lie algebra* is a complex Lie algebra $\mathfrak{g}$ endowed with an involutive antilinear antiautomorphism $z \mapsto z^*$. This means in particular that

$$(x^*)^* = x \quad \text{and} \quad [x,y]^* = [y^*, x^*].$$

Such an involution determines the real form

$$\mathfrak{g}_{\mathbb{R}} := \{x \in \mathfrak{g} : x^* = -x\}$$

of $\mathfrak{g}$. This is a real subalgebra with $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus i\mathfrak{g}_{\mathbb{R}}$ (direct sum of vector spaces). If, conversely, $\mathfrak{g}_{\mathbb{R}}$ is a real form of $\mathfrak{g}$, then there exists a unique involution $*$ defining $\mathfrak{g}_{\mathbb{R}}$ which is given by $(x+iy)^* := -x + iy$ for $x, y \in \mathfrak{g}_{\mathbb{R}}$.
(b) Let $(\mathfrak{g}, \mathfrak{h})$ be a complex split Lie algebra and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ the corresponding root decomposition. An involution $*$ of $\mathfrak{g}$ is said to be *compatible* with the root decomposition if $x^* \in \mathfrak{g}^{-\alpha}$ for $x \in \mathfrak{g}^\alpha$ and $\alpha \in \Delta \cup \{0\}$. In this case the triple $(\mathfrak{g}, \mathfrak{h}, *)$ is called an *involutive split Lie algebra.*
(c) Let $(\mathfrak{g}, \mathfrak{h}, *)$ be an involutive split Lie algebra and Δ the corresponding root system. For $\alpha \in \Delta_i$ the space $\mathfrak{g}(\alpha)_{\mathbb{R}} := \mathfrak{g}(\alpha) \cap \mathfrak{g}_{\mathbb{R}}$ is a real form of the test algebra $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{C})$, so that $\mathfrak{g}(\alpha)_{\mathbb{R}} \cong \mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{su}(1,1)$ or $\mathfrak{g}(\alpha)_{\mathbb{R}} \cong \mathfrak{su}(2)$. We call α *compact* if $\mathfrak{g}(\alpha)_{\mathbb{R}} \cong \mathfrak{su}(2)$ and write Δ_k for the set of compact roots. The roots in $\Delta_p := \Delta \setminus \Delta_k$ are called *non-compact.* We write $\mathcal{W}_{\mathfrak{k}}$ for the subgroup of $\mathcal{W}$ generated by the reflections r_α, $\alpha \in \Delta_k$. This group is called the *compact Weyl group*, which of course does not mean that $\mathcal{W}_{\mathfrak{k}}$ is a compact topological group.■

Examples II.11. (a) If H is a Hilbert space and $\mathfrak{gl}(H) := B(H)$ the space of all bounded linear operators on H, then $\mathfrak{gl}(H)$ is an involutive Lie algebra with respect to the operator adjoint which is defined by

$$\langle X^*.v, w \rangle = \langle v, X.w \rangle \quad \text{for all} \quad v, w \in H.$$

The corresponding real form is the subalgebra

$$\mathfrak{u}(H) := \mathfrak{gl}(H)_{\mathbb{R}} = \{X \in B(H) \colon X^* = -X\}$$

of skew-hermitian operators. We will see later how the notation $\mathfrak{u}(H)$ and $\mathfrak{gl}(H)$ will be justified by the corresponding Lie groups.
(b) For $H = \mathbb{C}^n$ with $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ we also write

$$\mathfrak{gl}(n, \mathbb{C}) := \mathfrak{gl}(H) \quad \text{and} \quad \mathfrak{u}(n, \mathbb{C}) := \mathfrak{u}(H).$$

The subalgebra

$$\mathfrak{sl}(n, \mathbb{C}) := \{X \in \mathfrak{gl}(n, \mathbb{C}) \colon \operatorname{tr} X = 0\}$$

is invariant under the involution, and we thus obtain the real form

$$\mathfrak{su}(n, \mathbb{C}) := \mathfrak{u}(n, \mathbb{C}) \cap \mathfrak{sl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{C})_{\mathbb{R}}.$$

(c) For every set J the Lie algebra $\mathfrak{gl}(J, \mathbb{C})$ can be viewed as operators on the Hilbert space

$$H = l^2(J, \mathbb{C}) = \Big\{(x_j)_{j \in J} \in \mathbb{C}^J : \sum_j |x_j|^2 < \infty\Big\}.$$

The corresponding involution is given by $X^* = \overline{X}^\top$.
(d) The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ has another natural involution given by $X^\sharp = -\overline{X}$, where $\overline{X} = (\overline{x_{ij}})_{i,j \in J}$ for $X = (x_{ij})_{i,j \in J}$. In this case the corresponding real form is

$$\mathfrak{gl}(n, \mathbb{C})_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{R}),$$

the Lie algebra of real $(n \times n)$-matrices.
(e) For $\tau := \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \in \mathfrak{gl}(p+q, \mathbb{C})$ we obtain on $\mathfrak{gl}(p+q, \mathbb{C})$ an involution by $X^\sharp := \tau X^* \tau$. The corresponding real form is called $\mathfrak{u}(p, q, \mathbb{C})$. We likewise have the real form $\mathfrak{su}(p, q, \mathbb{C})$ of $\mathfrak{sl}(p+q, \mathbb{C})$. ■

In the first part of these lectures we will mainly be concerned with the case where all roots are compact. In this case we call $\mathfrak{g}_{\mathbb{R}}$ a *compact real form* of $\mathfrak{g}$. Note that the standard involution $X^* = \overline{X}^\top$ on $\mathfrak{gl}(J, \mathbb{C})$ has this property.

Exercises for Section II

Exercise II.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with root decomposition.
(a) Show that for each root α and $x_\alpha \in \mathfrak{g}^\alpha$ the endomorphism $\operatorname{ad} x_\alpha : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Hint: The set of roots is finite.
(b) If $\mathfrak{g}$ is finite-dimensional, then $\alpha \in \Delta$ is integrable if and only if $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{C})$.
(c) If $\mathfrak{g}$ is locally finite (every finite subset generates a finite-dimensional subalgebra), then $\alpha \in \Delta$ is integrable if and only if $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{C})$.
(d)* If $\mathfrak{g}$ is locally finite, then $\alpha \in \Delta$ is integrable if and only if

$$\alpha([\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]) \neq \{0\}.$$

Hint: Use the representation theory of $\mathfrak{sl}(2, \mathbb{C})$. ■

* Exercises marked with $*$ require more work than the others.

Exercise II.2. In the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ of $(n\times n)$-matrices, we consider the subalgebra $\mathfrak g$ of upper triangular matrices.
(a) Show that the diagonal matrices $\mathfrak h$ are a splitting Cartan subalgebra of $\mathfrak g$.
(b) $\Delta\cap -\Delta=\{0\}$ holds for the corresponding root system.
(c) Are there any integrable roots? ∎

Exercise II.3. (a) Find on the vector space $\mathbb{C}[Z]$ (the algebra of all polynomials in one indeterminate Z over $\mathbb{C}$) two linear maps P and Q with $[P,Q]=\mathbf{1}$.
(b) Let V be a vector space and $\mathfrak{gl}(V):=\mathrm{End}(V)$ the Lie algebra of all linear maps $V\to V$. Show that

$$\mathbf{1}\in[\mathfrak{gl}(V),\mathfrak{gl}(V)] \quad\Longleftrightarrow\quad \dim V=\infty.$$

Hint: If V is infinite, then $V\cong V\otimes\mathbb{C}[Z]$ (why?).
(c)* Show that if $\dim V=\infty$, then

$$\mathfrak{gl}(V)=[\mathfrak{gl}(V),\mathfrak{gl}(V)].$$

Hint: Write $V\cong V\otimes\mathbb{C}[Z]$ and write a given $A\in\mathfrak{gl}(V)$ as $A=[S,B]$ with $S(v\otimes f(Z))=v\otimes f'(Z)$. ∎

Exercise II.4. (Block structure of classical Lie algebras) Let J be a set and consider the disjoint union $2J:=J\dot\cup -J$, where $-J$ means a set whose elements are formally written as $-j$, $j\in J$. We write $\mathbb{C}^{(2J)}=\mathbb{C}^{(J)}\oplus\mathbb{C}^{(J)}$ and accordingly elements of $\mathfrak{gl}(2J,\mathbb{C})$ as block (2×2)-matrices with entries in $\mathfrak{gl}(J,\mathbb{C})$.
(a) Show that

$$\mathfrak{o}(2J,\mathbb{C}):=\Big\{\begin{pmatrix} a & b\\ c & -a^\top\end{pmatrix}\in\mathfrak{gl}(2J,\mathbb{C})\colon b=-b^\top, c=-c^\top\Big\}$$

is a Lie algebra and that

$$\mathfrak h:=\mathrm{span}\{E_{jj}-E_{-j,-j}\colon j\in J\}$$

is a splitting Cartan subalgebra of $\mathfrak g$ with the root system

$$\Delta=D_J:=\{\pm(\varepsilon_j\pm\varepsilon_k)\colon j\neq k, j,k\in J\},$$

where we define $\varepsilon_j\in\mathfrak h^*$ by $\varepsilon_j\big(\mathrm{diag}(x_{ii})\big):=x_{jj}$ for $j\in J$. Hint: Show that the symmetric bilinear form $\beta(v,w):=\sum_{j\in 2J}v_jw_{-j}$ satisfies

$$\mathfrak{o}(2J,\mathbb{C})=\{X\in\mathfrak{gl}(2J,\mathbb{C})\colon(\forall v,w\in V)\,\beta(X.v,w)+\beta(v,X.w)=0\}.$$

(b) Show that

$$\mathfrak{sp}(2J,\mathbb{C})=\Big\{\begin{pmatrix} a & b\\ c & -a^\top\end{pmatrix}\in\mathfrak{gl}(2J,\mathbb{C})\colon b=b^\top, c=c^\top\Big\}$$

is a Lie algebra and that

$$\mathfrak{h} := \operatorname{span}\{E_{jj} - E_{-j,-j} : j \in J\}$$

is a splitting Cartan subalgebra of $\mathfrak{g}$ with the root system

$$\Delta = C_J := \{\pm 2\varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k) : j \neq k, j, k \in J\}.$$

Hint: Show that the skew-symmetric bilinear form

$$\beta(v, w) := \sum_{j \in J} x_j y_{-j} - x_{-j} y_j$$

satisfies

$$\mathfrak{sp}(J, \mathbb{C}) = \{X \in \mathfrak{gl}(2J, \mathbb{C}) : (\forall v, w \in V)\, \beta(X.v, w) + \beta(v, X.w) = 0\}.$$

(c) We write $\mathbb{C}^{(2J+1)} = \mathbb{C}^{(J)} \oplus \mathbb{C} \oplus \mathbb{C}^{(J)}$ and accordingly elements of $\mathfrak{gl}(2J+1, \mathbb{C})$ as block (3×3)-matrices. Show that

$$\mathfrak{o}(2J+1, \mathbb{C}) := \left\{ \begin{pmatrix} a & b & c \\ -b^\top & 0 & d \\ e & -d^\top & -a^\top \end{pmatrix} \in \mathfrak{gl}(2J+1, \mathbb{C}) : c = -c^\top, e = -e^\top \right\}$$

is a Lie algebra and that

$$\mathfrak{h} := \operatorname{span}\{E_{jj} - E_{-j,-j} : j \in J\}$$

is a splitting Cartan subalgebra of $\mathfrak{g}$ with the root system

$$\Delta = B_J := \{\pm \varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k) : j \neq k, j, k \in J\}.$$

Hint: Show that the symmetric bilinear form $\beta(v, w) := \sum_{j \in 2J+1} v_j w_{-j}$ satisfies

$$\mathfrak{o}(2J+1, \mathbb{C}) = \{X \in \mathfrak{gl}(2J+1, \mathbb{C}) : (\forall v, w \in V)\, \beta(X.v, w) + \beta(v, X.w) = 0\}. \blacksquare$$

The preceding exercise shows that there are split Lie algebras with $\mathfrak{h} \cong \mathbb{C}^{(J)}$, where $\Delta = \Delta_i$ is one of the following root systems:

$$\begin{aligned}
A_J &= \{\varepsilon_j - \varepsilon_k : j, k \in J, j \neq k\}, && \text{for} \quad \mathfrak{sl}(J, \mathbb{C}), \mathfrak{gl}(J, \mathbb{C}), \\
B_J &= \{\pm\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\} && \text{for} \quad \mathfrak{o}(2J+1, \mathbb{C}) \\
C_J &= \{\pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\} && \text{for} \quad \mathfrak{sp}(J, \mathbb{C}), \quad \text{and} \\
D_J &= \{\pm\varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\} && \text{for} \quad \mathfrak{o}(2J, \mathbb{C}).
\end{aligned}$$

One can show that these are precisely the infinite root systems of simple split locally finite Lie algebras which then leads to the classification of this class of simple Lie algebras ([NeSt00]): Every infinite-dimensional locally finite split simple complex Lie algebra $\mathfrak{g}$ is isomorphic to one of the following three types:

$$\mathfrak{sl}(J, \mathbb{C}), \quad \mathfrak{sp}(J, \mathbb{C}) \quad \text{or} \quad \mathfrak{o}(2J, \mathbb{C}) \cong \mathfrak{o}(2J+1, \mathbb{C}).$$

The latter isomorphism is specific for the infinite-dimensional situation. It is discussed in Exercise II.6 below.

Exercise II.5. (a) Show that $\mathfrak{gl}(J,\mathbb{C})$ and therefore every subalgebra of $\mathfrak{gl}(J,\mathbb{C})$ is a locally finite Lie algebra.
(b) Show that all roots of the Lie algebras $\mathfrak{o}(2J,\mathbb{C})$, $\mathfrak{o}(2J+1,\mathbb{C})$ and $\mathfrak{sp}(J,\mathbb{C})$ are integrable.
(c) Determine the structure of the Weyl groups for $\mathfrak{sl}(J,\mathbb{C})$, $\mathfrak{o}(2J,\mathbb{C})$, $\mathfrak{o}(2J+1,\mathbb{C})$ and $\mathfrak{sp}(J,\mathbb{C})$. ■

Exercise II.6. (Isomorphisms of orthogonal Lie algebras) Let $J = J_1 \dot\cup J_2$ be a set and $S \in \mathrm{End}(\mathbb{C}^{(J)})$ the symmetric block (2×2)-matrix

$$S := \begin{pmatrix} \mathbf{1}_{J_1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{J_2} \end{pmatrix}$$

with respect to the decomposition $\mathbb{C}^{(J)} = \mathbb{C}^{(J_1)} \oplus \mathbb{C}^{(J_2)}$. We define the Lie algebra

$$\mathfrak{o}(J_1, J_2, \mathbb{C}) := \{X \in \mathfrak{gl}(2J,\mathbb{C}) \colon X^\top S + SX = 0\}.$$

(a) Show that $\mathfrak{o}(2J,\mathbb{C}) \cong \mathfrak{o}(J,J,\mathbb{C})$.
(b) Show that $\mathfrak{o}(2J+1,\mathbb{C}) \cong \mathfrak{o}(J+1,J,\mathbb{C})$.
(c) Show that $\mathfrak{o}(J_1,J_2,\mathbb{C}) \cong \mathfrak{o}(J_1\dot\cup J_2, 0,\mathbb{C})$.
(d) Deduce that $\mathfrak{o}(2J,\mathbb{C}) \cong \mathfrak{o}(2J+1,\mathbb{C})$ for infinite sets J.
(e) Which of the arguments in (a)–(d) work over arbitrary fields $\mathbb{K}$ of characteristic zero? When is $\mathfrak{o}(J_1,J_2,\mathbb{K}) \cong \mathfrak{o}(J_1',J_2',\mathbb{K})$? What happens over $\mathbb{R}$? ■

Exercise II.7. (a) Describe the endomorphisms of the vector space $\mathbb{C}^{(J)}$ in terms of $(J\times J)$-matrices. Which matrices occur?
(b) Show that every $(J\times J)$-matrix A for which every column contains at most finitely many non-zero entries and which is invertible in the sense that there exists another matrix A^{-1} of this type with $AA^{-1} = A^{-1}A = \mathbf{1}$ defines an isomorphism φ_A of $\mathfrak{gl}(J,\mathbb{C})$ by $\varphi_A(x) = AxA^{-1}$.
(c) Show that the group S_J of *all* bijections of J acts naturally on the Lie algebra $\mathfrak{gl}(J,\mathbb{C})$ by automorphisms. ■

Exercise II.8. (a) Show that the real forms $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(1,1)$ of $\mathfrak{sl}(2,\mathbb{C})$ are isomorphic.
(b) Describe the isomorphisms $\mathfrak{so}(3,\mathbb{R}) \cong \mathfrak{su}(2,\mathbb{C})$ and $\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$.
(c) Is every real form of $\mathfrak{sl}(2,\mathbb{C})$ isomorphic to $\mathfrak{su}(2,\mathbb{C})$ or $\mathfrak{su}(1,1,\mathbb{C})$?
(d) Describe the corresponding real form in terms of (2×2)-block matrices.
(e) How can Example II.10(e) be generalized to Lie algebras of operators on Hilbert spaces? ■

Exercise II.9. We call a module V of the split Lie algebra $\mathfrak{g}$ *integrable* if for each integrable root $\alpha \in \Delta_i$ the module V is locally finite for the subalgebra $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2,\mathbb{C})$. Show that if V is an integrable module of the finite-dimensional split Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ a splitting Cartan subalgebra, then the set $\mathcal{P}_V \subseteq \mathfrak{h}^*$ of $\mathfrak{h}$-weights of V is invariant under the Weyl group. Hint: For each weight $\beta \in \mathcal{P}_V$ and each integrable root α consider the $\mathfrak{g}(\alpha)$-module $\sum_{k\in\mathbb{Z}} V^{\beta+k\alpha}$. ■

Exercise II.10. (a) Let $\mathfrak{g} = \mathfrak{sl}(J, \mathbb{C})$, where J is an infinite set. Show that for each element $x \in \mathfrak{gl}(J, \mathbb{C}) \setminus \mathfrak{sl}(J, \mathbb{C})$ the derivation $D := \operatorname{ad} x|_{\mathfrak{g}}$ of $\mathfrak{sl}(J, \mathbb{C})$ is not inner.

(b) Let d be a diagonal matrix $d = \operatorname{diag}(d_j)_{j \in J}$. Then $D(x) := dx - xd$ defines a derivation of $\mathfrak{gl}(J, \mathbb{C})$ and $\mathfrak{sl}(J, \mathbb{C})$ which is diagonalizable as an operator on both Lie algebras.

(c) Let A be a complex $J \times J$-matrix such that each row and each column of A contains at most finitely many entries. Then $D_A(x) := Ax - xA$ maps $\mathfrak{gl}(J, \mathbb{C})$ and $\mathfrak{sl}(J, \mathbb{C})$ into itself and defines a derivation of these algebras. ∎

Exercise II.11. A subset Π of a positive system Δ^+ is called a *basis* if $\Delta^+ \subseteq \mathbb{N}_0[\Pi]$, i.e., every positive root is a sum of elements of Π. We assume that $\Delta = A_J$.

(a) A positive system $\Delta^+_{\preceq}$ has a basis if and only if for each pair $j, k \in J$ with $j \prec k$ the order interval $[j, k] := \{i \in J : j \preceq i \preceq k\}$ is finite.

(b) If Δ^+ has a basis, then J is countable.

(c) If J is infinite and countable, then there are three types of positive systems $\Delta^+_{\preceq}$ which have a basis. They correspond to the linearly ordered sets $(\mathbb{N}, \leq)$, $(\mathbb{N}, \geq)$ (the reversed order) and $(\mathbb{Z}, \leq)$. ∎

III. Unitary highest weight modules

In the representation theory of infinite-dimensional Lie groups, the unitary highest weight representations are a very prominent class of representations. This has several reasons. First of all they arise most naturally in physical models because boundedness properties of spectra (such as the lower boundedness of the energy) often imply that representations are highest weight representations (see the discussion in Chapter X of [Ne99a]). On the other hand, highest weight representations enjoy some very close connections to complex geometry and Kähler manifolds. We have already seen part of this connection in our discussion of the Borel–Weil Theorem in Section I.

In this section we study unitary highest weight representations from a purely algebraic point of view. After describing the general setting, we will explain some specific classification results for locally finite Lie algebras.

Unitary highest weight modules of locally finite Lie algebras

Definition III.1. Let $\mathfrak{g}$ be a split Lie algebra.
(a) For a $\mathfrak{g}$-module V and $\beta \in \mathfrak{h}^*$ we write

$$V^\beta := \{v \in V : (\forall X \in \mathfrak{h}) X.v = \beta(X)v\}$$

for the *weight space of weight* β and

$$\mathcal{P}_V = \{\beta \in \mathfrak{h}^* : V^\beta \neq \{0\}\}$$

for the set of $\mathfrak{h}$*-weights* of V.
(b) A non-zero element $v \in V^\lambda$, $\lambda \in \mathcal{P}_V$, is called *primitive* (with respect to the positive system Δ^+) if $\mathfrak{g}^\alpha.v = \{0\}$ holds for all $\alpha \in \Delta^+$. A $\mathfrak{g}$-module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.
(c) Suppose, in addition, that $\mathfrak{g}$ is an involutive Lie algebra. Then we call a hermitian form $\langle \cdot, \cdot \rangle$ on a $\mathfrak{g}$-module V *contravariant* if

$$\langle X.v, w \rangle = \langle v, X^*.w \rangle \quad \text{for all} \quad v, w \in V, X \in \mathfrak{g}.$$

A $\mathfrak{g}$-module V is said to be *unitary* if it carries a contravariant positive definite hermitian form. Note that this property depends on the involution $*$ on the Lie algebra $\mathfrak{g}$. ■

In the following we will define $\alpha^*(x) := \overline{\alpha(x^*)}$ for $\alpha \in \mathfrak{h}^*$.

Proposition III.2. *Let $\mathfrak{g}$ be an involutive split complex Lie algebra and Δ^+ a positive system. Then the following assertions hold:*

(i) *Each module V of highest weight λ satisfies*

$$\mathcal{P}_V \subseteq \lambda - \mathbb{N}_0[\Delta^+].$$

Moreover, it has a unique maximal submodule and satisfies $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}\mathbf{1}$.

(ii) *For each $\lambda \in \mathfrak{h}^*$ there exists a unique irreducible highest weight module $L(\lambda, \Delta^+)$.*

(iii) *If $L(\lambda, \Delta^+)$ is unitary, then $\lambda = \lambda^*$ and moreover $\mu = \mu^*$ for each $\mu \in \mathcal{P}_V$.*

(iv) *Each unitary highest weight module is irreducible.*

(v) *If $\lambda = \lambda^*$ and $v_\lambda \in L(\lambda, \Delta^+)$ is a primitive element, then $L(\lambda, \Delta^+)$ carries a unique contravariant hermitian form $\langle \cdot, \cdot \rangle$ with $\langle v_\lambda, v_\lambda \rangle = 1$. This form is non-degenerate.*

Proof. (i) Let $\mathfrak{n}^\pm := \sum_{\alpha \in \Delta^+} \mathfrak{g}^{\pm\alpha}$ and $v_\lambda \in V^\lambda$ be a primitive element. From $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}^+)$ (which follows from the Poincaré–Birkhoff–Witt Theorem) we obtain $V = U(\mathfrak{n}^-).v_\lambda$, showing that V has an $\mathfrak{h}$-weight decomposition. Since the set of $\mathfrak{h}$-weights on $U(\mathfrak{n}^-)$ is $-\mathbb{N}_0[\Delta^+]$, the set $\mathcal{P}_V$ of $\mathfrak{h}$-weights of V is contained in $\lambda - \mathbb{N}_0[\Delta^+]$.

To see that V^λ is one-dimensional, we observe that

$$V = U(\mathfrak{n}^-).v_\lambda \subseteq \mathbb{C}v_\lambda + \mathfrak{n}^-.V,$$

where all $\mathfrak{h}$-weights in $\mathfrak{n}^-.V$ are contained in $\bigcup_{\alpha \in \Delta^+}(\lambda - \alpha - \mathbb{N}_0[\Delta^+])$. The relation $(-\Delta^+) \cap \mathbb{N}_0[\Delta^+] = \emptyset$ further implies that λ is not a weight of $\mathfrak{n}^-.V$, and therefore $\dim V^\lambda = 1$.

In view of $\dim V^\lambda = 1$, each $A \in \mathrm{End}_{\mathfrak{g}}(V)$ maps the primitive element v_λ to a multiple cv_λ of v_λ. Then $A = c\mathbf{1}$ is a consequence of the fact that v_λ generates V.

If $N \subseteq V$ is a proper submodule, then it does not contain v_λ. Further the fact that it is invariant under $\mathfrak{h}$ implies that it decomposes according to the $\mathfrak{h}$-weight decomposition (Exercise III.1). Hence it is contained in the proper subspace

$$\sum_{0 \neq \alpha \in \mathbb{N}_0[\Delta^+]} V^{\lambda - \alpha}.$$

This implies that the sum of all proper submodules is a proper submodule and therefore a maximal submodule.

(ii) Let $\mathbb{C}_\lambda$ denote the one-dimensional module of the Lie algebra $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}^+$ on which $\mathfrak{n}^+$ acts trivially and $\mathfrak{h}$ acts by $X.v = \lambda(X)v$. We consider the induced $\mathfrak{g}$-module

$$M(\lambda, \Delta^+) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n}^+)} \mathbb{C}_\lambda$$

which is called the *Verma module of highest weight* λ. We think of $M(\lambda, \Delta^+)$ as a $\mathfrak{g}$-module quotient of the tensor product $U(\mathfrak{g}) \otimes \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is considered as a trivial $\mathfrak{g}$-module, modulo the subspace spanned by the elements of the form

$DX \otimes 1 - D \otimes \lambda(X)$, $D \in U(\mathfrak{g})$, $X \in \mathfrak{b}$. In this sense, we write $[v \otimes z] \in M(\lambda, \mathfrak{b})$ for the image of the element $v \otimes z \in U(\mathfrak{g}) \otimes \mathbb{C}_\lambda$ under the natural surjection to $M(\lambda, \mathfrak{b})$. We also write $\lambda: U(\mathfrak{b}) \to \mathbb{C}$ for the algebra homomorphism obtained by the homomorphic extension of λ.

Claim 1: The module $M(\lambda, \mathfrak{b})$ is a highest weight module of highest weight λ, and $[\mathbf{1} \otimes 1]$ is a primitive element: This is immediate from the definitions because for $X \in \mathfrak{b}$ we have

$$X.[\mathbf{1} \otimes 1] = [X \otimes 1] = [\mathbf{1} \otimes \lambda(X)] = \lambda(X)[\mathbf{1} \otimes 1],$$

and $U(\mathfrak{g}).[\mathbf{1} \otimes 1] = [U(\mathfrak{g}) \otimes 1] = M(\lambda, \Delta^+)$.

Claim 2: Each $\mathfrak{g}$-module V of highest weight λ is a quotient of $M(\lambda, \Delta^+)$: Let $v_\lambda \in V$ be a primitive element of weight λ. Then we have a unique surjective $\mathfrak{g}$-equivariant map

$$U(\mathfrak{g}) \otimes \mathbb{C}_\lambda \to V \quad \text{with} \quad D \otimes 1 \mapsto D.v_\lambda.$$

Since v_λ is a $\mathfrak{b}$-weight vector of weight λ, this map factors through a surjective map

$$M(\lambda, \Delta^+) \to V \quad \text{with} \quad [D \otimes 1] \mapsto D.v_\lambda.$$

Now Claim 2 and (i) show that every irreducible module of highest weight λ is isomorphic to the quotient of $M(\lambda, \Delta^+)$ modulo its maximal submodule.

(iii) The first part follows directly from

$$\lambda(X)\langle v_\lambda, v_\lambda \rangle = \langle X.v_\lambda, v_\lambda \rangle = \langle v_\lambda, X^*.v_\lambda \rangle = \overline{\lambda(X^*)}\langle v_\lambda, v_\lambda \rangle$$

for all $X \in \mathfrak{h}$ and a primitive element v_λ. The second part now follows from (i) and $\alpha^* = \alpha$ for all roots $\alpha \in \Delta$.

(iv) First we observe that for unitary modules the $\mathfrak{h}$-weight decomposition is orthogonal (Exercise III.1). Let $N \subseteq V$ be a proper submodule. As we have seen in (i),

$$N \subseteq \sum_{0 \neq \alpha \in \mathbb{N}_0[\Delta^+]} V^{\lambda - \alpha} \subseteq v_\lambda^\perp.$$

Hence

$$\langle N, V \rangle = \langle N, U(\mathfrak{g}).v_\lambda \rangle = \langle U(\mathfrak{g}).N, v_\lambda \rangle \subseteq \langle N, v_\lambda \rangle = \{0\}.$$

Since the $\langle \cdot, \cdot \rangle$ is non-degenerate, it follows that $N = \{0\}$ and therefore that V is irreducible.

(v) Uniqueness of the form: We define a linear functional on $U(\mathfrak{g})$ by

$$\varphi(D) := \langle D.v_\lambda, v_\lambda \rangle.$$

In view of

$$\langle D_1.v_\lambda, D_2.v_\lambda \rangle = \langle D_2^* D_1.v_\lambda, v_\lambda \rangle = \varphi(D_2^* D_1),$$

it suffices to show that φ is uniquely determined by λ and does not depend on $\langle\cdot,\cdot\rangle$. For $D \in U(\mathfrak{h})$ we have $\varphi(D) = \lambda(D)$, and

$$U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g}) \subseteq \ker\varphi.$$

Since the Poincaré–Birkhoff–Witt Theorem implies that $U(\mathfrak{g})$ is a direct vector space sum

$$U(\mathfrak{g}) = \mathfrak{n}^- U(\mathfrak{g}) \oplus U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}^+, \tag{3.1}$$

it follows that φ is uniquely determined by λ.

Existence: We use the decomposition in (3.1) to define a linear function φ on $U(\mathfrak{g})$ with $U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g}) \subseteq \ker\varphi$ and $\varphi|_{U(\mathfrak{h})} = \lambda$. First we observe that the form

$$\big(U(\mathfrak{g}) \otimes \mathbb{C}_\lambda\big) \times \big(U(\mathfrak{g}) \otimes \mathbb{C}_\lambda\big) \to \mathbb{C}, \quad (C \otimes 1, D \otimes 1) \mapsto \varphi(D^* C)$$

is sesquilinear and factors through a form on $M(\lambda, \Delta^+)$ with

$$\langle [C \otimes 1], [D \otimes 1] \rangle := \varphi(D^* C), \quad C, D \in U(\mathfrak{g}).$$

The assumption $\lambda = \lambda^*$ first implies that $\varphi(x^*) = \varphi(x)^*$ holds for all $x \in U(\mathfrak{h})$, and further $\big(U(\mathfrak{g})\mathfrak{n}^+\big)^* = \mathfrak{n}^- U(\mathfrak{g})$ implies that the preceding relation holds for all $x \in U(\mathfrak{g})$. Therefore $\langle\cdot,\cdot\rangle$ is a hermitian form, and the contravariance follows immediately from the definition. ∎

The following proposition is quite useful to prove that highest weight modules of locally finite Lie algebras are unitary because it permits to use information on finite-dimensional Lie algebras.

In the following we call a family $(\mathfrak{g}_j)_{j \in J}$ of subalgebras of $\mathfrak{g}$ *directed* if for $j_1, j_2 \in J$ there exists a $j_3 \in J$ with $\mathfrak{g}_{j_1} \cup \mathfrak{g}_{j_3} \subseteq \mathfrak{g}_{j_3}$.

Proposition III.3. *Let $(\mathfrak{g}_j)_{j \in J}$ be a directed family of involutive subalgebras of $\mathfrak{g}$ with the following properties:*

(1) $\mathfrak{g} = \bigcup_j \mathfrak{g}_j$.

(2) *Each $\mathfrak{g}_j$ is invariant under $\mathfrak{h}$ such that $\mathfrak{h}_j := \mathfrak{h} \cap \mathfrak{g}_j$ is a splitting Cartan subalgebra of $\mathfrak{g}_j$.*

(3) *$\mathfrak{h}_j$ separates the points in the vector space spanned by*

$$\Delta_j := \{\alpha \in \Delta : \mathfrak{g}^\alpha \cap \mathfrak{g}_j \neq \{0\}\},$$

so that we may identify Δ_j with the roots of $\mathfrak{g}_j$ with respect to $\mathfrak{h}_j$.

For a positive system $\Delta^+ \subseteq \Delta$ we consider the positive system $\Delta_j^+ := \Delta^+ \cap \Delta_j$ in Δ_j. Then the highest weight module $L(\lambda, \Delta^+)$ of $\mathfrak{g}$ is unitary if and only if all the highest weight modules $L(\lambda|_{\mathfrak{h}_j}, \Delta_j^+)$ for the subalgebras $\mathfrak{g}_j$, $j \in J$, are unitary.

Proof. If $L(\lambda, \Delta^+)$ is unitary and $v_\lambda \in L(\lambda, \Delta^+)$ is a primitive element, then $V_j := U(\mathfrak{g}_j).v_\lambda$ is a unitary highest weight module of highest weight $\lambda_j := \lambda|_{\mathfrak{h}_j}$ of $\mathfrak{g}_j$, hence irreducible (Proposition III.2(iv)). We conclude that $V_j \cong L(\lambda_j, \Delta_j^+)$ is unitary.

If, conversely, all the modules $L(\lambda_j, \Delta_j^+)$ are unitary, and $\langle \cdot, \cdot \rangle$ denotes the unique contravariant hermitian form on $L(\lambda, \Delta^+)$ with $\langle v_\lambda, v_\lambda \rangle = 1$ (cf. Proposition III.2(v)), then the uniqueness of the contravariant form on V_j implies that it is positive semidefinite on the submodules V_j whose union coincides with $L(\lambda, \Delta^+)$. Therefore it is positive semidefinite on $L(\lambda, \Delta^+)$ and hence positive definite because $L(\lambda, \Delta^+)$ is irreducible. ∎

Some necessary conditions for unitarity

In this subsection $\mathfrak{g}$ denotes an involutive split Lie algebra. We use explicit calculations involving root vectors to derive some necessary conditions for the unitarity of a highest weight module $L(\lambda, \Delta^+)$, which for the particular case of Lie algebras with $\Delta^+ = \{\alpha\}$ and $\dim \mathfrak{g}^\alpha = 1$ turn out to be sufficient.

Lemma III.4. *For $\alpha \in \Delta$, $Z \in \mathfrak{g}^\alpha$ and $Y \in \mathfrak{g}^{-\alpha}$, the following assertions hold:*

(i) $[Z, Y^n] = nY^{n-1}\big([Z, Y] - \frac{n-1}{2}\alpha([Z, Y])\mathbf{1}\big)$ *in* $U(\mathfrak{g})$.

(ii) *If $v_\lambda \in L(\lambda, \Delta^+)$ is a primitive element, then*

$$Z^n (Z^*)^n.v_\lambda = n! \prod_{j=0}^{n-1} \big(\lambda - \tfrac{j}{2}\alpha\big)([Z, Z^*])v_\lambda.$$

Proof. (i) Repeated application of the Leibniz rule leads to

$$[Z, Y^n] = \sum_{i+j=n-1} Y^i [Z, Y] Y^j = \sum_{i+j=n-1} Y^{i+j}[Z, Y] + Y^i\big[[Z, Y], Y^j\big]$$

$$= \sum_{i+j=n-1} Y^{i+j}[Z, Y] - j\alpha([Z, Y])Y^{i+j} = nY^{n-1}[Z, Y] - \tfrac{n(n-1)}{2}\alpha([Z, Y])Y^{n-1}$$

$$= nY^{n-1}\Big([Z, Y] - \tfrac{(n-1)}{2}\alpha([Z, Y])\mathbf{1}\Big).$$

(ii) Again repeated application of the Leibniz rule yields

$$[Z^n, (Z^*)^n] = \sum_{i+j=n-1} Z^i [Z, (Z^*)^n] Z^j,$$

so that $Z^n(Z^*)^n.v_\lambda = [Z^n, (Z^*)^n].v_\lambda = Z^{n-1}[Z, (Z^*)^n].v_\lambda$. Hence the formula under (i) gives

$$Z^n (Z^*)^n.v_\lambda = n(\lambda - \tfrac{n-1}{2}\alpha)([Z, Z^*])\ Z^{n-1}(Z^*)^{n-1}.v_\lambda.$$

Now the assertion follows from an easy induction. ∎

Proposition III.5. *Suppose that* $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha}$ *is an involutive Lie algebra with* $\mathfrak{g}^{\alpha} = \mathbb{C}Z$, *and let* $\lambda = \lambda^* \in \mathfrak{h}^*$.

(i) *If* $\alpha([Z, Z^*]) > 0$, *then* $L(\lambda, \Delta^+)$ *is unitary if and only if there exists an* $n \in \mathbb{N}_0$ *with*

$$\lambda([Z, Z^*]) = n\frac{\alpha([Z, Z^*])}{2}.$$

For $\alpha \in \Delta_k$ *this means that* $\lambda(\check{\alpha}) \in \mathbb{N}_0$. *In this case,* $\dim L(\lambda, \Delta^+) = n+1$.

(ii) *If* $\alpha([Z, Z^*]) \leq 0$, *then* $L(\lambda, \Delta^+)$ *is unitary if and only if* $\lambda([Z, Z^*]) \geq 0$. *If, in addition,* $\lambda([Z, Z^*]) = 0$, *then* $L(\lambda, \Delta^+)$ *is one-dimensional, and otherwise infinite-dimensional with the weights* $\lambda - \mathbb{N}_0\alpha$.

Proof. Since $\mathfrak{g}^{\alpha} = \mathbb{C}Z$, the highest weight module $L(\lambda, \Delta^+)$ is the orthogonal direct sum of the one-dimensional subspaces generated by the elements $(Z^*)^n.v_\lambda$, $n \in \mathbb{N}_0$. So it is unitary if and only if all the numbers in Lemma III.4(ii) are non-negative. Now the assertions are immediate consequences. ■

The following theorem provides the essential information that we will need in the following sections.

Theorem III.6. (Characterization of unitarity) *Let* $\mathfrak{g}$ *be a locally finite split Lie algebra with* $\Delta = \Delta_k$.

(i) *Then the highest weight module* $L(\lambda, \Delta^+)$ *of* $\mathfrak{g}$ *with respect to* Δ^+ *is unitary if and only if* $\lambda = \lambda^*$ *and* λ *is dominant integral in the sense that*

$$\lambda(\check{\alpha}) \in \mathbb{N}_0 \quad \text{for all} \quad \alpha \in \Delta^+.$$

(ii) *If* $L(\lambda, \Delta^+)$ *is unitary and* $\mathcal{R} := \mathbb{Z}[\Delta] \subseteq \mathfrak{h}^*$ *denotes the root group, then the weight system* $\mathcal{P}_\lambda$ *of* $L(\lambda, \Delta^+)$ *is given by*

$$\mathcal{P}_\lambda = \operatorname{conv}(\mathcal{W}.\lambda) \cap (\lambda + \mathcal{R}).$$

(iii) *For each* $X \in \mathfrak{g}$ *the corresponding operator on* $L(\lambda, \Delta^+)$ *is locally finite.*

Proof. (Sketch) (i) The necessity of $\lambda(\check{\alpha}) \in \mathbb{N}_0$ for all $\alpha \in \Delta^+$ follows from Proposition III.5. To see that this condition is sufficient, we first observe that we may w.l.o.g. assume that $\mathfrak{g}$ is perfect because $[\mathfrak{g}, \mathfrak{g}]$ is a subalgebra with $\mathfrak{h} + [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and the splitting Cartan subalgebra $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}] = \operatorname{span} \check{\Delta}$, and $L(\lambda, \Delta)$ also is a highest weight module for $[\mathfrak{g}, \mathfrak{g}]$ (cf. Exercise III.3).

Since $\mathfrak{g}$ is locally finite, it can be written as a directed union of finite-dimensional subalgebras $\mathfrak{g}_j$, $j \in J$, as in Proposition III.3. These subalgebras can be obtained as follows: Let $\Delta_j \subseteq \Delta$ be a finite subset which is full in the set that $\Delta_j = \Delta \cap \operatorname{span} \Delta_j$. Then we consider $\mathfrak{g}_j := \operatorname{span} \check{\Delta}_j + \sum_{\alpha \in \Delta_j} \mathfrak{g}^{\alpha}$. It is not hard to see that $\mathfrak{g}$ is a directed union of these finite-dimensional subalgebras for which $(\mathfrak{g}_j)_{\mathbb{R}}$ is a compact real form. In view of Proposition III.3, the assertion now follows from the corresponding result for finite-dimensional Lie algebras, where we already know that the fact that λ_j is dominant integral implies that $L(\lambda_j, \Delta_j^+)$ is finite-dimensional (Theorem I.1), so that the compactness of the

simply connected group U_j corresponding to $\mathfrak{u}_j := (\mathfrak{g}_j)_{\mathbb{R}}$ implies that the representation of this group on $L(\lambda_j, \Delta_j^+)$ can be made unitary by averaging a given positive definite hermitian form.
(ii) This part follows from the corresponding assertion for finite-dimensional Lie algebras (cf. [Bou90, Ch. 8]) and the trivial observation that $\mathcal{W} = \bigcup_{j\in J} \mathcal{W}_j$, where $\mathcal{W}_j \subseteq \mathcal{W}$ is the subgroup generated by the reflections r_α, $\alpha \in \Delta_j$.
(iii) Let $X \in \mathfrak{g}$ and $v \in L(\lambda, \Delta^+)$. We have to show that v is contained in a finite-dimensional X-invariant subspace. We write X as $X_h + X_s$ with $X_h \in \mathfrak{h}$ and $X_s \in [\mathfrak{g}, \mathfrak{g}]$. Now let $\mathfrak{g}_j$ be as in (iii) so large that $v \in U(\mathfrak{g}_j).v_\lambda$ and $X_s \in \mathfrak{g}_j$. Then $[X, \mathfrak{g}_j] \subseteq \mathfrak{g}_j$ implies that the finite-dimensional subspace $U(\mathfrak{g}_j).v_\lambda$ is X-invariant because $X.v_\lambda \in \mathbb{C}v_\lambda + X_s.v_\lambda \subseteq U(\mathfrak{g}_j).v_\lambda$. ■

The preceding theorem applies in particular to the Lie algebras $\mathfrak{sl}(J, \mathbb{C})$, $\mathfrak{gl}(J, \mathbb{C})$, $\mathfrak{sp}(J, \mathbb{C})$, $\mathfrak{o}(2J, \mathbb{C})$ and $\mathfrak{o}(2J + 1, \mathbb{C})$ with their natural involutions defined by $x^* = -\overline{x}^\top$.

Example III.7. The unitary highest weight modules of the Lie algebra $\mathfrak{gl}(J, \mathbb{C})$ with respect to the positive system $\Delta_{\preceq}^+ = \{\varepsilon_j - \varepsilon_k : j \prec k, j, k \in J\}$ are parametrized by functionals $\lambda = (\lambda_j)_{j\in J} \in \mathfrak{h}^* \cong \mathbb{C}^J$ which we also write as

$$\lambda = \sum_{j\in J} \lambda_j \varepsilon_j$$

or as functions $\lambda: J \to \mathbb{R}, j \mapsto \lambda_j$. If $\lambda = \lambda^*$, then the highest weight module $L(\lambda)$ of $\mathfrak{gl}(J, \mathbb{C})$ is unitary if and only if $\lambda_j - \lambda_k \in \mathbb{N}_0$ for $j \prec k$.

Given a linear order $\preceq$ on J we have particular dominant integral functionals given by

$$\varpi_M := \sum_{j\in M} \varepsilon_j,$$

where $M \subseteq J$ is a subset satisfying $M \prec J \setminus M$ (it is a lower set for the order $\preceq$). These functionals ϖ_M are called the *fundamental weights.* Note that for $M = J$ we get $\varpi_J = \mathrm{tr}$. For more details on the relation between fundamental weights and general weights we refer to the discussion in [Ne98]. ■

If $\mathfrak{g}$ is finite-dimensional, then the preceding theorem directly yields a classification of the unitary highest weight modules for the compact real form $\mathfrak{u} = \mathfrak{g}_{\mathbb{R}}$ because every simple highest weight module is finite-dimensional and every simple finite-dimensional module is isomorphic to some $L(\lambda, \Delta^+)$ for a fixed positive system Δ^+ (Theorem I.1). In the infinite-dimensional case there are many different positive systems which are not conjugate under the Weyl group $\mathcal{W}$ (cf. Remark II.9), so that we cannot expect such a simple situation. To obtain a classification of the unitary highest weight modules, we therefore have to discuss when two unitary highest weight modules $L(\lambda, \Delta^+)$ and $L(\widetilde{\lambda}, \widetilde{\Delta}^+)$ are isomorphic.

The classification of unitary highest weight modules

In this section $\mathfrak{g}$ is a locally finite split involutive Lie algebra with $\Delta = \Delta_k$.

Proposition III.8. *If V is an irreducible $\mathfrak{g}$-module with $\mathfrak{h}$-weight system $\mathcal{P}_V \subseteq \mathfrak{h}^*$, and Δ^+ a positive system such that $\lambda \in \mathcal{P}_V$ satisfies*

$$\lambda \in \mathcal{P}_V \subseteq \lambda - \mathbb{R}^+[\Delta^+],$$

then $V \cong L(\lambda, \Delta^+)$.

Proof. Let $v_\lambda \in V$ be an $\mathfrak{h}$-weight vector of weight λ. For $\alpha \in \Delta^+$ we then have $\mathfrak{g}^\alpha.v_\lambda \subseteq V^{\lambda+\alpha}$. If $\mathfrak{g}^\alpha.v_\lambda$ is non-zero, this means that

$$\lambda + \alpha \in \mathcal{P}_V \subseteq \lambda - \mathbb{R}^+[\Delta^+],$$

hence that $\alpha \in \Delta^+ \cap -\mathbb{R}^+[\Delta^+] = \emptyset$, a contradiction. Thus v_λ is a primitive element in V with respect to Δ^+, and therefore the irreducibility of V implies that V is an irreducible highest weight module of highest weight λ, i.e., isomorphic to $L(\lambda, \Delta^+)$. ■

Corollary III.9. *Two unitary highest weight modules $L(\lambda, \Delta^+)$ and $L(\lambda, \widetilde{\Delta}^+)$ are isomorphic.*

Proof. According to Theorem III.6, both modules have the same set of weights satisfying the condition of Proposition III.8, so that $L(\lambda, \widetilde{\Delta}^+) \cong L(\lambda, \Delta^+)$. ■

In view of the preceding corollary, we may define

$$L(\lambda) := L(\lambda, \Delta^+)$$

if Δ^+ is a positive system such that $L(\lambda, \Delta^+)$ is unitary because the isomorphy class of $L(\lambda, \Delta^+)$ does not depend on the choice of Δ^+. The next question is when two unitary highest weight modules $L(\lambda)$ and $L(\mu)$ are isomorphic. To answer this question, we will need the following elementary lemma:

Lemma III.10. *If E is a subset of the real vector space V, then* $\mathrm{Ext}(\mathrm{conv}\, E) \subseteq E$.

Proof. Since every element of $\mathrm{conv}(E)$ is a finite convex combination of elements of E, it clearly suffices to prove the assertion for a finite subset E.

We use induction over $|E|$. For $|E| = 1$ the assertion is trivial. If the assertion holds for set of at most n elements and $|E| = n+1$, then we write $E = E' \cup \{e\}$ with $e \in E$ and $|E'| = n$. Now

$$\mathrm{conv}(E) = \bigcup_{\lambda \in [0,1]} \big(\lambda\, \mathrm{conv}(E') + (1-\lambda)e\big),$$

and therefore $\mathrm{Ext}\,\big(\mathrm{conv}(E)\big) \subseteq \mathrm{Ext}\,\big(\mathrm{conv}(E')\big) \cup \{e\} \subseteq E' \cup \{e\} = E.$ ■

Lemma III.11. *If $L(\lambda, \Delta^+)$ is unitary, then*

$$\operatorname{Ext}\big(\operatorname{conv}(\mathcal{P}_\lambda)\big) = \mathcal{W}.\lambda.$$

Proof. In view of Proposition III.2(i), we have $\operatorname{conv}(\mathcal{P}_\lambda) \subseteq \lambda - \mathbb{R}^+[\Delta^+]$ and the convex cone $\mathbb{R}^+[\Delta^+]$ is pointed, so that $\lambda \in \operatorname{Ext}\big(\operatorname{conv}(\mathcal{P}_\lambda)\big)$. On the other hand $\mathcal{P}_\lambda$ is invariant under the Weyl group $\mathcal{W}$ (Exercise II.9), which implies that $\mathcal{W}.\lambda \subseteq \operatorname{Ext}\big(\operatorname{conv}(\mathcal{P}_\lambda)\big)$. Moreover, Theorem III.6 shows that $\operatorname{conv}(\mathcal{P}_\lambda) = \operatorname{conv}(\mathcal{W}.\lambda)$, so that Lemma III.10 leads to

$$\operatorname{Ext}\big(\operatorname{conv}(\mathcal{P}_\lambda)\big) \subseteq \mathcal{W}.\lambda.$$

This completes the proof. ∎

Lemma III.12. *Two unitary highest weight modules $L(\lambda)$ and $L(\mu)$ are isomorphic if and only if $\mu \in \mathcal{W}.\lambda$.*

Proof. If $\mu \in \mathcal{W}.\lambda$, then Theorem III.6 implies that the set of weights of $L(\lambda)$ and $L(\mu)$ coincides, hence that both are isomorphic (Proposition III.8). If, conversely, $L(\lambda) \cong L(\mu)$, then both have the same set of weights, so that Lemma III.11 yields

$$\mu \in \operatorname{Ext}\big(\operatorname{conv}(\mathcal{P}_\mu)\big) = \operatorname{Ext}\big(\operatorname{conv}(\mathcal{P}_\lambda)\big) = \mathcal{W}.\lambda. \qquad ∎$$

The remaining question is how we can see if for a functional $\lambda \in \mathfrak{h}^*$ there exists a positive system Δ^+ such that $L(\lambda, \Delta^+)$ is unitary. To answer this question we generalize a useful concept from the theory of finite root systems to our setting.

Definition III.13. A subset $\Sigma \subseteq \Delta$ is called *closed* if

$$(\Sigma + \Sigma) \cap \Delta \subseteq \Sigma.$$

It is called *parabolic* if it is closed and satisfies

$$\Sigma \cup -\Sigma = \Delta. \qquad ∎$$

Note that closed subsets correspond to subalgebras $\mathfrak{p}(\Sigma) := \mathfrak{h} + \sum_{\alpha \in \Sigma} \mathfrak{g}^\alpha$ (Exercise!).

Proposition III.14. *Every parabolic system Σ contains a positive system.*

Proof. (a) Let $\Sigma^+ := \Sigma \setminus -\Sigma$. Then

$$(\Sigma + \Sigma^+) \cap \Delta \subseteq \Sigma^+.$$

Let $\alpha \in \Sigma$ and $\beta \in \Sigma^+$ with $\alpha + \beta \in \Delta$. Since Σ is closed, we have $\alpha + \beta \in \Sigma$. If this root is not contained in Σ^+, then $-\alpha - \beta \in \Sigma$, so that the closedness of Σ leads to $-\beta = (-\alpha - \beta) + \beta \in \Sigma$, a contradiction.

(b) Using Zorn's Lemma, we find a maximal closed subset $\Gamma \subseteq \Sigma$ satisfying $\Gamma \cap -\Gamma = \emptyset$. In view of (a), we then have

$$\Big((\Gamma \cup \Sigma^+) + (\Gamma \cup \Sigma^+)\Big) \cap \Delta \subseteq \Gamma \cup \Sigma^+,$$

and further

$$(\Gamma \cup \Sigma^+) \cap -(\Gamma \cup \Sigma^+) \subseteq (\Gamma \cap -\Gamma) \cup (\Sigma \cap -\Sigma^+) \cup (-\Sigma \cap \Sigma^+) = \emptyset.$$

Therefore the maximality of Γ implies that $\Sigma^+ \subseteq \Gamma$.
(c) $\Gamma \cup -\Gamma = \Delta$. Suppose that this is not the case and pick $\alpha \in \Delta$ not in Γ or $-\Gamma$. In view of (b), we then have $\alpha \in \Sigma \cap -\Sigma$. The maximality of Γ implies that it cannot be enlarged by α, which means that there exists a finite-dimensional subspace $E \subseteq \operatorname{span} \Delta$ containing α such that the finite closed subset $\Gamma_0 := \Gamma \cap E$ cannot be put into a positive system $\widetilde{\Gamma}_0$ of $\Delta_0 := \Delta \cap E$ containing α.

The property $\Gamma_0 \cap -\Gamma_0 = \emptyset$ implies that $\mathfrak{b}_0 := \operatorname{span} \check{\Delta}_0 + \sum_{\alpha \in \Gamma_0} \mathfrak{g}^\alpha$ is a solvable subalgebra of $\mathfrak{g}_0 := \operatorname{span} \check{\Delta}_0 + \sum_{\alpha \in \Delta_0} \mathfrak{g}^\alpha$. Let $\mathfrak{b}$ be a maximal solvable subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{b}_0$. Then

$$\mathfrak{b} = \operatorname{span} \check{\Delta}_0 + \sum_{\alpha \in \Delta_0^+} \mathfrak{g}^\alpha$$

for a positive system Δ_0^+ of Δ_0 (cf. [Bou90, Ch. VIII, §3.1, Prop. 5]). Now $\alpha \in \Delta_0^+ \cup -\Delta_0^+$ leads to a contradiction which proves that $\Gamma \cup -\Gamma = \Delta$.
(d) Γ is a positive system: It suffices to show that for every finite-dimensional subspace $E \subseteq \operatorname{span} \Delta$ the set $\Gamma \cap E$ is a positive system in $\Delta_0 := \Delta \cap E$, but this follows from the existence of a linearly independent basis of Δ_0^+ ([Bou90, Ch. VIII]). ∎

Example III.15. We consider the root system $\Delta = A_J$. If $\Sigma \subseteq \Delta$ is a positive system, then

$$j \preceq_\Sigma k \quad :\Longleftrightarrow \quad \varepsilon_j - \varepsilon_k \in \Sigma \cup \{0\}$$

defines a partial order on J. Since the positive systems in Δ correspond to linear orders on J, it is easy to see that the positive systems contained in Σ correspond to the linear orderings $\preceq$ refining the partial order $\preceq_\Sigma$. In this setting Proposition III.14 means that each partial order on a set J can be refined to a linear order. ∎

Now we are ready to address the complete classification of unitary highest weight modules.

Theorem III.16. *Let $\mathcal{P} := \{\lambda \in \mathfrak{h}^* \colon \lambda^* = \lambda, (\forall \alpha \in \Delta)\, \lambda(\check{\alpha}) \in \mathbb{Z}\}$ denote the group of symmetric weights. If $L(\lambda, \Delta^+)$ is unitary, then $\lambda \in \mathcal{P}$ and, conversely, for each $\lambda \in \mathcal{P}$ there exists a positive system Δ^+ such that $L(\lambda, \Delta^+)$ is unitary. The subset $\mathcal{P} \subseteq \mathfrak{h}^*$ is invariant under the action of the Weyl group, and the map*

$\lambda \mapsto L(\lambda)$ *induces a bijection of the orbit space* $\mathcal{P}/\mathcal{W}$ *onto the set of isomorphy classes of unitary highest weight modules.*

Proof. The necessity of $\lambda \in \mathcal{P}$ follows from Theorem III.6. Now let $\lambda \in \mathcal{P}$ and consider the set

$$\Sigma_\lambda := \{\alpha \in \Delta \colon \lambda(\check{\alpha}) \in \mathbb{N}_0\}.$$

We claim that Σ_λ is a parabolic system. It is clear that $\Sigma_\lambda \cup -\Sigma_\lambda = \Delta$. Now let $\alpha, \beta \in \Sigma_\lambda$ with $\alpha + \beta \in \Delta$. Then the theory of finite root systems, applied to $\Delta_0 := \Delta \cap \operatorname{span}\{\alpha, \beta\}$, implies that

$$(\alpha + \beta)\check{} \in \mathbb{R}^+ \check{\alpha} + \mathbb{R}^+ \check{\beta}. \tag{3.2}$$

In fact, there exists a scalar product $(\cdot, \cdot)$ on $\operatorname{span} \Delta_0$ which permits us to identify this space with $\operatorname{span}\{\check{\alpha}, \check{\beta}\}$ in such a way that $\check{\gamma}$ corresponds to $\frac{2\gamma}{(\gamma,\gamma)}$ for $\gamma \in \Delta_0$ ([Bou90, Ch. VIII, §2, no. 2, Th. 2]). This implies for $\gamma = \alpha + \beta$ the relation (3.2) which in turn shows that Σ_λ is closed, hence a parabolic system. Now we use Proposition III.14 to see that there exists a positive system $\Delta^+ \subseteq \Sigma_\lambda$. Then $L(\lambda, \Delta^+)$ is unitary by Theorem III.6.

The remainder follows directly from Lemma III.12. ∎

Exercises for Section III

Exercise III.1. Let $\mathfrak{h}$ be an abelian Lie algebra and V an $\mathfrak{h}$-module which is spanned by simultaneous $\mathfrak{h}$-eigenvectors. We call V an $\mathfrak{h}$-weight module. Then the following assertions hold:

(i) $V = \bigoplus_{\mu \in \mathcal{P}_V} V^\mu$.

(ii) Every submodule $W \subseteq V$ satisfies

$$W = \bigoplus_{\mu \in \mathcal{P}_V} (W \cap V^\mu) = \bigoplus_{\mu \in \mathcal{P}_V} W^\mu.$$

(iii) Suppose that $\mathfrak{h}$ is involutive and that $\langle \cdot, \cdot \rangle$ is a contravariant hermitian form on V and let $\alpha \in \mathfrak{h}^*$. We put $\alpha^*(x) := \overline{\alpha(x^*)}$. Then $\langle V^\alpha, V^{\beta^*} \rangle = \{0\}$ for $\alpha \neq \beta$. If, in addition, $\alpha^* = \alpha$ for each weight in $\mathcal{P}_V$, then the weight decomposition of V is orthogonal with respect to $\langle \cdot, \cdot \rangle$. ∎

Exercise III.2. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra.

(i) If $\mathfrak{g}_\mathbb{R}$ is a real form defined by an involution $*$ which is compatible with a root decomposition, then the Cartan subalgebra $\mathfrak{h}_\mathbb{R} := \mathfrak{h} \cap \mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$ is *compactly embedded* in the sense that the closure of the group $e^{\operatorname{ad} \mathfrak{h}_\mathbb{R}}$ in $\operatorname{Aut}(\mathfrak{g}_\mathbb{R})$ is compact.

(ii) Find a real form of a complex semisimple Lie algebra which does not contain a compactly embedded Cartan subalgebra. Then $\mathfrak{g}_\mathbb{R}$ does not occur for any involution compatible with a root decomposition. ∎

Exercise III.3. Let $(\mathfrak{g}, \mathfrak{h})$ be a split Lie algebra.

(i) $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{h}$ and for each highest weight module V with highest weight vector v_λ we have $V = U(\mathfrak{g}).v_\lambda$.

(ii) If no root vanishes on $\mathfrak{h}_0 := \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{h}_0$ is a splitting Cartan subalgebra of $[\mathfrak{g}, \mathfrak{g}]$.

(iii) If $\mathfrak{h}_0$ is a splitting Cartan subalgebra of $[\mathfrak{g}, \mathfrak{g}]$, then a highest weight module $L(\lambda, \Delta^+)$ of $\mathfrak{g}$ is unitary if the corresponding highest weight module $L(\lambda|_{\mathfrak{h}_0}, \Delta^+)$ of $[\mathfrak{g}, \mathfrak{g}]$ is unitary. ■

Exercise III.4. We consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with $\Delta^+ = \{\alpha\}$ and the functional $\lambda \in \mathfrak{h}^*$ with $\lambda(\check{\alpha}) = n \in \mathbb{N}_0$. Let $L(\lambda, \Delta^+)$ be the corresponding $(n+1)$-dimensional simple $\mathfrak{g}$-module with the canonical basis $f^j.v_\lambda$, $j = 0, \ldots, n$. We endow $\mathfrak{g}$ with the involution with $\mathfrak{g}_\mathbb{R} = \mathfrak{su}(1,1)$. Determine the signature of the canonical hermitian on the subspaces $\mathbb{C} f^j.v_\lambda$. ■

IV. Banach–Lie groups

In this section we will briefly discuss the crucial points where infinite-dimensional Lie theory for Banach–Lie groups differs from the familiar finite-dimensional theory. Moreover, we will explain some tools that can be used to deal with specific groups quite efficiently.

General Lie theory for Banach–Lie groups

Throughout this section, we will assume some familiarity with the basic concepts and results of differential calculus in Banach spaces which does not differ very much from differential calculus in $\mathbb{R}^n$: If U is an open subset of the Banach space E, and F a Banach space, then a map $f: U \to F$ is said to be differentiable in $p \in U$ if there exists a continuous linear map $df(p): E \to F$ such that

$$f(p+v) = f(p) + df(p).v + o(\|v\|).$$

We call f a C^1-map or *continuously differentiable* if it is differentiable in every point of U and the map $df: U \to B(E,F)$ is continuous. We call f a C^2-map if df is C^1 etc. We say that f is C^∞ or *smooth* if f is C^n for every $n \in \mathbb{N}$. So essentially everything works as in $\mathbb{R}^n$, provided it is formulated in a coordinate free way. This holds in particular for the definition of manifolds, submanifolds, tangent bundles and vector fields (which are always viewed as smooth sections of the tangent bundle). For the details we refer to [La99].

Definition IV.1. A *Banach–Lie group* G is a manifold modeled over a Banach space such that the multiplication map $G \times G \to G, (x,y) \mapsto xy$ and the inversion $G \to G, x \mapsto x^{-1}$ are smooth maps. We write $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G ([La99, §VI.5]).

The Lie algebra $\mathfrak{g}$ of G can be obtained as in the finite-dimensional case: Each $X \in T_{\mathbf{1}}(G)$ (the tangent space in the identity element $\mathbf{1}$) corresponds to a unique left invariant vector field X_l with

$$X_l(g) := d\lambda_g(\mathbf{1}).X, \quad g \in G.$$

The space of left invariant vector fields is closed under the Lie bracket of vector fields ([La99, Prop. III.5.1]), hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[X,Y]_l = [X_l, Y_l]$. To emphasize the functorial dependence of $\mathfrak{g}$ of G, we frequently write $\mathbf{L}(G)$ for the Lie algebra of G. If $\|\cdot\|$ is a norm on $\mathfrak{g}$ defining the topology, then the continuity of the Lie bracket means that there exists a constant $C > 0$ with

$$\|[X,Y]\| \leq C\|X\|\,\|Y\| \quad \text{for all} \quad X, Y \in \mathfrak{g}$$

(Exercise IV.10). A Banach space $(\mathfrak{g}, \|\cdot\|)$ which at the same time is a Lie algebra with a continuous Lie bracket is called a *Banach–Lie algebra*.

The existence and uniquenes results for ordinary differential equations also hold in the setting of Banach spaces (cf. [La99, §§IV.1/2]). By integrating the flow of a left-invariant vector field X_l, we therefore obtain the *exponential function*

$$\exp\colon \mathfrak{g} \to G, \quad \exp(X) := \gamma_X(1),$$

where $\gamma_X\colon \mathbb{R} \to G$ is a solution of the initial value problem

$$\gamma_X'(t) = X_l(\gamma_X(t)), \quad \gamma_X(0) = \mathbf{1}.$$

The exponential function is a smooth map with $d\exp(0) = \mathrm{id}_{\mathfrak{g}}$. In view of the Inverse Function Theorem, this implies that one can use the exponential function to construct canonical charts of G. As for finite-dimensional groups, one can show that these charts define on G the structure of an analytic Lie group (the transition maps in charts are analytic).

The left invariance of the vector field X_l implies in particular that the integral curve $\gamma_X\colon \mathbb{R} \to G$ is a Lie group homomorphism $(\mathbb{R}, +) \to G$. It can be shown that all continuous Lie group homomorphism are of this type, so that we have a natural bijection

$$\mathfrak{g} \to \mathrm{Hom}(\mathbb{R}, G), \quad X \mapsto \gamma_X. \qquad \blacksquare$$

Since essentially all Lie groups arising in these notes will be Banach–Lie groups, we will simply call them Lie groups.

The following results carry over from finite-dimensional Lie theory:

Theorem IV.2. *Let G and H be Banach–Lie groups.*
(a) *For $X, Y \in \mathbf{L}(G)$ we have the Trotter product formula*

$$\exp(X+Y) = \lim_{n\to\infty} \Big(\exp(\tfrac{1}{n}X)\exp(\tfrac{1}{n}Y)\Big)^n$$

and the commutator formula

$$\exp([X,Y]) = \lim_{n\to\infty} \Big(\exp(\tfrac{1}{n}X)\exp(\tfrac{1}{n}Y)\exp(-\tfrac{1}{n}X)\exp(-\tfrac{1}{n}Y)\Big)^{n^2}.$$

(b) *Let $\varphi\colon G \to H$ be a continuous homomorphism between Banach–Lie groups. Then φ is smooth and $\mathbf{L}(\varphi) := d\varphi(\mathbf{1})\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a continuous homomorphism of Banach–Lie algebras.*
(c) *If, conversely, $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a continuous homomorphism of Lie algebras and G is connected and simply connected, then there exists a unique continuous homomorphism $\varphi\colon G \to H$ with $\mathbf{L}(\varphi) = \psi$.*

Proof. (Sketch) (a) This follows from analyzing the the product

$$X * Y := \exp|_V^{-1}(\exp X \exp Y) = X + Y + \frac{1}{2}[X,Y] + \ldots$$

on an open neighborhood $W \subseteq \mathfrak{g}$ for which there exists an open 0-neighborhood $V \subseteq W$ for which $\exp|_V: V \to \exp(V)$ is a diffeomorphism and

$$\exp(W)\exp(W) \subseteq \exp(V).$$

(b) For every $X \in \mathfrak{g}$ the homomorphism $\varphi \circ \gamma_X: \mathbb{R} \to H$ is a continuous one-parameter group, hence can be written as

$$\varphi \circ \gamma_X = \gamma_{\psi(X)}, \quad \psi(X) \in \mathbf{L}(H)$$

(Definition IV.1). We conclude that

$$\varphi \circ \exp_G = \exp_H \circ \psi.$$

Using (a), one shows that ψ is linear and a Lie algebra homomorphism. Since $\exp_G$ and $\exp_H$ are local diffeomorphisms, it follows that $\psi: \mathbf{L}(G) \to \mathbf{L}(H)$ is continuous.
(c) This is done as in the finite-dimensional case. ∎

The fact that Banach–Lie groups are locally contractible implies in particular that for each Banach–Lie group G there exists a simply connected covering group $\widetilde{G}$, which also carries a unique Banach–Lie group structure such that the map $q: \widetilde{G} \to G$ is a covering homomorphism of Lie groups. In the light of this fact, Theorem IV.2(c) is a very important tool to "integrate" Lie algebra representations to group representations.

Corollary IV.3. *For every closed subgroup $H \subseteq G$ the subset*

$$\mathbf{L}(H) := \{X \in \mathfrak{g}: \exp(\mathbb{R}X) \subseteq H\}$$

is a closed Lie subalgebra of $\mathfrak{g}$.

Proof. This is a direct consequence of Theorem IV.2(a). ∎

Remark IV.4. (Lie subgroups) (a) Let G be a Lie group with Lie algebra $\mathfrak{g}$. There exist various notions of Lie subgroups in the literature. The weakest one is that of Maissen ([Ma62]) who shows that for every *closed subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$ there exists a connected Lie group H_L with Lie algebra $\mathfrak{h}$ and an injective homomorphism of Lie groups

$$\eta: H_L \to G$$

with $\eta(H_L) = H := \langle \exp \mathfrak{h} \rangle$. The main idea is to refine the topology on the subgroup H in such a way that the exponential function $\exp: \mathfrak{h} \to H_L$ yields a local homeomorphism. The same approach is discussed in a slightly more restricted context in Theorem 5.52 in [HoMo98], where it is shown that for separable subalgebras $\mathfrak{h}$ we have

$$\mathbf{L}(H) := \{X \in \mathfrak{g}: \exp(\mathbb{R}X) \subseteq H\} = \mathfrak{h}.$$

For non-separable subalgebras $\mathfrak{h}$ this is no longer true in general, as the following counterexample shows ([HoMo98, p.157]): We consider the abelian Lie group $\mathfrak{g} := l^1(\mathbb{R}, \mathbb{R}) \times \mathbb{R}$, where the group structure is given by the addition. We write $(e_r)_{r\in\mathbb{R}}$ for the canonical topological basis elements of $l^1(\mathbb{R}, \mathbb{R})$ (cf. Exercise IV.11). Then the subgroup D generated by the pairs $(e_r, -r)$, $r \in \mathbb{R}$, is closed and discrete, so that $G := \mathfrak{g}/D$ is an abelian Lie group. Now we consider the closed subalgebra $\mathfrak{h} := l^1(\mathbb{R}, \mathbb{R})$ of $\mathfrak{g}$. As $\mathfrak{h} + D = \mathfrak{g}$, we have $H := \exp \mathfrak{h} = G$, and therefore

$$(0, 1) \in \mathbf{L}(H) \setminus \mathfrak{h}.$$

(b) In [La99] S. Lang calls a subgroup $H \subseteq G$ a *Lie subgroup* if H carries a Lie group structure for which there exists an immersion $\eta: H \to G$. In view of the definition of an immersion, this concept requires the Lie algebra $\mathfrak{h}$ of $\mathfrak{g}$ to be a closed subalgebra of $\mathfrak{g}$ which is complemented in the sense that there exists a closed vector space complement. Conversely, it is shown in [La99] that for every complemented closed subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ there exists a Lie subgroup in this sense ([La99, Th. VI.5.4]). For a finite-dimensional Lie group G, this concept describes the analytic subgroups of G because every subalgebra of a finite-dimensional Lie algebra is closed and complemented. As the dense wind in the two-dimensional torus $G = \mathbb{T}^2$ shows, subgroups of this type need not be closed. We also note that the closed subspace

$$c_0(\mathbb{N}, \mathbb{R}) \subseteq l^\infty(\mathbb{N}, \mathbb{R})$$

of sequences converging to 0 is not complemented ([Wil78, Ex. 14-4-9]; see also [We95, Satz IV.6.5] for an elementary proof), hence not a Lie subgroup in the sense of Lang, but a Lie subgroup in the sense of Maissen.

(c) The stronges concept is the one used in [Bou90, Ch. 3]. Here a Lie subgroup H is required to be a submanifold which implies in particular that it is locally closed and therefore closed. On the other hand this implies that the quotient space G/H has a natural manifold structure for which the quotient map $q: G \to G/H$ is a submersion ([Bou90, Ch. 3, §1.6, Prop. 11]).

(d) For finite-dimensional Lie groups closed subgroups are Lie subgroups, but for Banach–Lie groups this is no longer true. What remains true is that locally compact subgroups are Lie subgroups (cf. [HoMo98, Th. 5.41(vi)]). How bad closed subgroups can be is illustrated by the following example due to K. H. Hofmann: We consider the real Hilbert space $G := L^2([0, 1], \mathbb{R})$ as a Banach–Lie group. Then the subgroup $H := L^2([0, 1], \mathbb{Z})$ of all those functions which almost everywhere take values in $\mathbb{Z}$ is a closed subgroup. Since the one-parameter subgroups of G are of the form $\mathbb{R}f$, $f \in G$, we have $\mathbf{L}(H) = \{0\}$. On the other hand, the group H is arcwise connected and even contractible because the map $F: [0, 1] \times H \to H$ given by

$$F(t, f)(x) := \begin{cases} f(x) & 0 \leq x \leq t \\ 0 & t < x \leq 1 \end{cases}$$

is continuous with $F(1, f) = f$ and $F(0, f) = 0$. ■

The following lemma is a useful criterion to verify that subgroups of given Lie groups are Lie groups.

Proposition IV.5. *Let G be a Lie group and $H \subseteq G$ a closed subgroup for which there exists an open 0-neighborhood $V \subseteq \mathfrak{g}$ such that $\exp|_V \colon V \to U := \exp(V)$ is a diffeomorphism and*

$$\exp(V \cap \mathbf{L}(H)) = U \cap H.$$

Then H carries a natural Lie group structure such that $\mathbf{L}(H)$ is the Lie algebra of H and the exponential map of H is given by the restriction

$$\exp_H = \exp_G|_{\mathbf{L}(H)} \colon \mathbf{L}(H) \to H.$$

If, in addition, $\mathfrak{g}$ contains a closed subspace E complementing $\mathbf{L}(H)$, then H is a submanifold of G and the homogeneous space G/H carries a natural manifold structure such that the canonical map $\pi \colon G \to G/H$ is a submersion.

Proof. (Sketch) We put $\mathfrak{h} := \mathbf{L}(H)$. The idea of the proof is to use the exponential function to define an atlas of H. This is done by first observing that the restriction of exp to a suitable open 0-neighborhood $V_{\mathfrak{h}}$ in $\mathfrak{h}$ yields a homeomorphism $\varphi \colon V_{\mathfrak{h}} \to \varphi(V_{\mathfrak{h}})$ onto a $\mathbf{1}$-neighborhood $\varphi(V_{\mathfrak{h}}) \subseteq H$. Now one proceeds as in the finite-dimensional case (see also Maissen's approach, Remark IV.4).

If, in addition, a closed complement E exists for $\mathfrak{h}$, then H is a Lie subgroup in the sense of Lang and the inclusion map $\eta \colon H \to G$ is an immersion. This implies that there exists an open $\mathbf{1}$-neighborhood $U_H \subseteq H$ such that U_H is a submanifold of G. Choosing U_H such that it is contained in U, we see that $\mathbf{1}$ has an open neighborhood U' such that $H \cap U'$ is a submanifold of G. In view of the homogeneity of G, it follows that H is a submanifold of G in the sense of Bourbaki. ∎

Definition IV.6. Let G be a Lie group and $\mathfrak{h} \subseteq \mathfrak{g}$ a closed subalgebra. We call the subgroup $H := \langle \exp \mathfrak{h} \rangle$ generated by the exponential image of $\mathfrak{h}$ the corresponding *analytic subgroup* of G. According to Maissen's results, this group has a natural Lie group structure such that the map $H \hookrightarrow G$ is a morphism of Lie groups (see also [HoMo98, Cor. 5.34]).

For a closed subgroup $H \subseteq G$ we consider the closed Lie subalgebra

$$\mathfrak{h} := \mathbf{L}(H) = \{X \in \mathfrak{g} \colon \exp(\mathbb{R}X) \subseteq H\}$$

of $\mathfrak{g}$ (Corollary IV.3) and say that H is a *Lie subgroup* if there exists an open 0-neighborhood $V \subseteq \mathfrak{g}$ such that $\exp|_V$ is a diffeomorphism onto an open subset $\exp(V)$ and $\exp(V \cap \mathfrak{h}) = (\exp V) \cap H$. Then Proposition IV.5 implies that H carries a natural Lie group structure such that the map $H \hookrightarrow G$ is a homomorphism of Lie groups which is a homeomorphism onto its image.

We call a Lie subgroup H *complemented* if $\mathfrak{g}$ contains a closed subspace E complementing the closed subalgebra $\mathfrak{h}$. If this condition is satisfied, then H is a submanifold in the sense of Bourbaki, and in particular the homogeneous space G/H carries a natural manifold structure such that the canonical map $\pi \colon G \to G/H$ is a submersion (Proposition IV.5). ∎

Remark IV.7. If G is a *Hilbert–Lie group*, i.e., the topology on $\mathfrak{g}$ comes from a real Hilbert space structure on $\mathfrak{g}$, then every closed subspace $\mathfrak{h} \subseteq \mathfrak{g}$ has a closed complement, so that every Lie subgroup is complemented. ■

Linear Lie groups

Now the natural question is how to find infinite-dimensional Lie groups. In the finite-dimensional context the most natural examples are matrix groups, i.e., groups of operators on finite-dimensional vector spaces. In the infinite-dimensional context the situation is similar. The most natural examples are groups of operators on Banach spaces.

Definition IV.8. A Banach algebra is a Banach space A endowed with an associative algebra structure such that the norm on A is submultiplicative:

$$\|xy\| \leq \|x\| \cdot \|y\|, \quad x, y \in A.$$

We call A unital if A contains an identity element $\mathbf{1}$. In this case we write

$$G(A) := \{a \in A \colon (\exists b \in A)\, ab = ba = \mathbf{1}\}$$

for the *group of units of* A. ■

Proposition IV.9. *If A is a unital Banach algebra, then $G(A)$ is a Lie group with Lie algebra A (endowed with the commutator bracket) and the exponential function*

$$\exp\colon A \to G(A), \quad \exp(x) = e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Proof. First we observe that for $\|x\| < 1$ the Neumann series $\sum_{n=0}^{\infty} x^n$ converges to an element $y \in A$ satisfying $y(\mathbf{1} - x) = (\mathbf{1} - x)y = \mathbf{1}$. We conclude that

$$U := \{g \in A \colon \|g - \mathbf{1}\| < 1\} \subseteq G(A),$$

and that on U the inversion is given by the convergent power series

$$g^{-1} = \sum_{n=0}^{\infty} (\mathbf{1} - g)^n,$$

hence an analytic function and therefore in particular smooth.

For every element $g \in G$ the multiplication $\lambda_g \colon A \to A$ is a homeomorphism, so that $\lambda_g.U = gU$ is an open neighborhood of g in A which is contained in $G(A)$. This proves that $G(A)$ is open. Since the multiplication $m\colon A \times A \to A$ is bilinear, it restricts to a smooth map $G(A) \times G(A) \to G(A)$. To see that the inversion is a smooth function, we observe that for $u \in U$ we have

$$(gu)^{-1} = u^{-1} g^{-1},$$

so that the smoothness of the inversion on gU follows from the smoothness on U.

To see that $\exp$ is the exponential function of the Lie group $G(A)$, we observe that the left invariant vector fields on $G(A)$ are given by $X_l(g) = gX$, so that the corresponding integral curves starting in $\mathbf{1}$ are $\gamma_X(t) = e^{tX}$, and this implies that $\exp(X) = \gamma_X(1) = e^X$. ■

Corollary IV.10. *If E is a Banach space, then the group* $\mathrm{GL}(E)$ *of invertible bounded linear maps $E \to E$ is a Lie group with Lie algebra $B(E)$, the algebra of all bounded operators on E.*

Proof. The group $\mathrm{GL}(E)$ is the unit group of the unital Banach algebra $B(E)$. ■

Let E be a Banach space. We call a Lie subgroup $H \subseteq \mathrm{GL}(E)$ a *linear Lie group* (cf. [HoMo98, Ch. V], where linear Lie groups are discussed in a quite elementary fashion). The following lemma is a useful criterion to see that certain closed subgroups are Lie subgroups.

Lemma IV.11. *If $\varphi: G_1 \to G$ is a continuous homomorphism of Lie groups and $H \subseteq G$ a Lie subgroup, then $H_1 := \varphi^{-1}(H)$ also is a Lie subgroup. In particular* $\ker\varphi$ *is a Lie subgroup of G_1.*

Proof. We choose an open 0-neighborhood $V \subseteq \mathfrak{g}$ such that $\exp_G|_V$ is a diffeomorphism onto the open subset $U := \exp_G V$ of G, and $\exp_G(V \cap \mathfrak{h}) = U \cap H$. Then we choose an open 0-neighborhood $V_1 \subseteq \mathbf{L}(\varphi)^{-1}(V)$ such that $\exp_{G_1}|_{V_1}$ is a diffeomorphism onto $U_1 := \exp_{G_1}(V_1)$. We put $H_1 := \varphi^{-1}(H)$.

Let $X \in V_1$ with $\exp_{G_1} X \in U_1 \cap H_1$. Then

$$\varphi(\exp_{G_1} X) = \exp_G(\mathbf{L}(\varphi).X) \in U \cap H$$

with $\mathbf{L}(\varphi).X \in V$. Hence $\mathbf{L}(\varphi).X \in \mathfrak{h}$ and therefore

$$X \in \mathfrak{h}_1 = \{Y \in \mathfrak{g}_1 : \exp(\mathbb{R}Y) \subseteq H_1\},$$

which is the closed Lie subalgebra corresponding to the closed subgroup H_1 of G_1 (cf. Corollary IV.3). This implies that $U_1 \cap H_1 \subseteq \exp(V_1 \cap \mathfrak{h}_1)$ and therefore equality because the converse inclusion is trivial. ■

Lemma IV.12. *Let E be a Banach space and $F \subseteq E$ a closed subspace. Then*

$$H := \{g \in \mathrm{GL}(E) : g.F \subseteq F\}$$

is a Lie subgroup of $\mathrm{GL}(E)$.

Proof. Let $V \subseteq \mathfrak{g}$ be an open 0-neighborhood such that $\exp|_V : V \to \exp V$ is a diffeomorphism and $\|\exp x - \mathbf{1}\| < 1$ for all $x \in V$. Then the inverse function

$$\log := (\exp|_V)^{-1} : \exp V \to \mathfrak{g}$$

is given by the converging power series

$$\log(g) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(g - \mathbf{1})^n$$

(this requires a proof!). For $g = \exp X \in (\exp V) \cap H$ we then obtain $X.F \subseteq F$ directly from the power series. ■

Algebraic Lie subgroups

We will now discuss a very convenient criterion which in many concrete cases can be used to verify that a closed subgroup H of a Lie group is a Lie subgroup. To this end, we will need the concept of a polynomial function and of an algebraic subgroup.

Definition IV.13. (a) Let E and V be Banach spaces. A function $f: E \to V$ is called a *homogeneous polynomial of degree* k if there exists a symmetric k-linear function $\widetilde{f}: E^k \to V$ with

$$f(x) = \widetilde{f}(x, \ldots, x) \quad \text{for all} \quad x \in E.$$

Polynomial functions of degree 0 are constant functions and polynomial functions of degree 1 are linear maps. Polynomial functions of degree 2 are also called quadratic maps. In this case $\widetilde{f}$ can be obtained quite directly by

$$\widetilde{f}(x, y) = \frac{1}{4}\big(f(x+y) - f(x-y)\big) = \frac{1}{8}\big(f(x+y) - f(x-y) - f(-x+y) + f(-x-y)\big).$$

For polynomials of degree k we have the general formula

$$\widetilde{f}(h_1, \ldots, h_n) = \frac{1}{2^n n!} \sum_{\varepsilon \in \{1,-1\}^n} (\varepsilon_1 \cdots \varepsilon_n) f(\varepsilon_1 h_1 + \ldots + \varepsilon_n h_n).$$

We write $P_k(E, F)$ for the space of continuous F-valued homogeneous polynomials of degree k on E. A *polynomial* is a finite sum of homogeneous polynomials, so that $P(E, F) := \bigoplus_{k=0}^{\infty} P_k(E, F)$ is the space of continuous F-valued polynomials on E. If $f = \sum_k f_k$ is a polynomial, then we say that f is of *degree* d if $f_d \neq 0$ and $f_k = 0$ for $k > d$.
(b) Let A be a Banach algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A subgroup $G \subseteq G(A)$ is called *algebraic* if there exists a $d \in \mathbb{N}_0$ and a set $\mathcal{F}$ of Banach space valued polynomial functions on $A \times A$ of degree $\leq d$ such that

$$G = \{g \in G(A): (\forall f \in \mathcal{F}) f(g, g^{-1}) = 0\}. \qquad \blacksquare$$

Proposition IV.14. (Harris/Kaup) *Every algebraic subgroup $G \subseteq G(A)$ is a Lie subgroup.*

Proof. In view of the Hahn-Banach Theorem, we may assume that

$$\mathcal{F} \subseteq P := \bigoplus_{k=0}^{d} P_k(A \times A, \mathbb{K}),$$

the Banach space of scalar-valued continuous polynomials on $A \times A$ of degree $\leq d$. The space P carries a natural Banach space structure such that the action of $G(A)$ on P given by

$$\big(\pi(g).f\big)(x,y) := f(xg, g^{-1}y)$$

yields a continuous homomorphism $\pi\colon G(A) \to \mathrm{GL}(P)$ (Exercise IV.6(d)).

Replacing $\mathcal{F}$ by

$$F := \{f \in P\colon (\forall g \in G)\, f(g, g^{-1}) = 0\},$$

we may assume that $\mathcal{F} = F$. The space F is a closed subspace of P. We claim that

$$G = \{g \in G(A)\colon \pi(g).F \subseteq F\}.$$

In fact, if $g, x \in G$ and $f \in F$, then

$$\big(\pi(g).f\big)(x, x^{-1}) = f(xg, g^{-1}x^{-1}) = 0,$$

showing that $\pi(g).f \in F$. If, conversely, $g \notin G$, then there exists an $f \in F$ with

$$0 \neq f(g, g^{-1}) = \big(\pi(g).f\big)(\mathbf{1}, \mathbf{1}).$$

It follows in particular that $\pi(g).f \notin F$.

We conclude that

$$G = \pi^{-1}(\{g \in \mathrm{GL}(P)\colon \pi(g).F \subseteq F\}),$$

so that the assertion follows from Lemma IV.11 combined with Lemma IV.12. ∎

Examples IV.15. (a) If A is a unital Banach algebra and $\mathbb{M}(n, A)$ is the algebra of $(n \times n)$-matrices with entries in A, then $\mathbb{M}(n, A)$ also is a Banach algebra. In fact, on the space $A^n = A \times \ldots \times A$ we consider the norm given by

$$\|x\| := \max\{\|x_1\|, \ldots, \|x_n\|\}.$$

Then A^n is a Banach space and we have a natural embedding

$$\mathbb{M}(n, A) \hookrightarrow B(A^n)$$

which we use to define a norm on $\mathbb{M}(n, A)$. It is not hard to verify that $\mathbb{M}(n, A)$ is closed in $B(A^n)$, hence a Banach algebra. We write $\mathrm{GL}(n, A) := G(\mathbb{M}(n, A))$ for the unit group of this Banach algebra.

(b) As we will see below, it sometimes is convenient to refine the construction in (b) as follows. Let $J \trianglelefteq A$ be an ideal which is a Banach algebra in its own right such that the multiplication map

$$A \times J \to J, \quad (a, b) \mapsto ab$$

is continuous, i.e., there exists a $C > 0$ with $\|ab\|_J \leq C\|a\|_A\|b\|_J$ for $a \in A$ and $b \in J$. After replacing the norm on A by the equivalent norm

$$\|a\|' := \max(\|a\|_A, \sup\{\|ab\|_J \colon \|b\|_J \leq 1\}) \leq \max(1, C)\|a\|_A,$$

we may assume that $\|ab\|_J \leq \|a\|_A\|b\|_J$ holds for $a \in A$ and $b \in J$.

We consider the algebra

$$\mathbb{M}(2, A, J) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in A, b, c \in J \right\}$$

endowed with the norm

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| := 2 \max\{\|a\|_A, \|b\|_J, \|c\|_J, \|d\|_A\}.$$

Then $\|xy\| \leq \|x\|\|y\|$ holds for $x, y \in \mathbb{M}(2, A, J)$ (Exercise!), so that $\mathbb{M}(2, A, J)$ is a Banach algebra.

A similar construction works for $(n \times n)$-matrices, where one defines

$$\mathbb{M}(n, A, J) := \{(x_{ij})_{i,j=1,\ldots,n} \in \mathbb{M}(n, A) \colon i \neq j \Rightarrow x_{ij} \in J\}$$

and

$$\|(x_{ij})\| := n \max\{\|x_{ij}\|_J, i \neq j; \|x_{ii}\|_A, i = 1, \ldots, n\}.$$

We write $\mathrm{GL}(n, A, J)$ for the unit group of this Banach algebra.

(c) If A is a Banach algebra without a unit element, then we endow the space

$$\widetilde{A} := A \oplus \mathbb{C}$$

with the Banach algebra structure given by

$$\|(a, z)\| := \|a\| + |z| \quad \text{and} \quad (a, z)(a', z') := (aa' + za' + z'a, zz').$$

Then $A \cong A \times \{0\}$ is a closed ideal in $\widetilde{A}$, and we have an algebra homomorphism $\varepsilon \colon \widetilde{A} \to \mathbb{C}$ given by $\varepsilon(a, z) = z$. We define

$$S(A) := \varepsilon^{-1}(1) \cap G(\widetilde{A}).$$

This is a closed subgroup of $G(\widetilde{A})$ and $\{(a, 1) \colon \|a\| < 1\}$ is an open $\mathbf{1}$-neighborhood in $S(A)$. Therefore $S(A)$ is a Lie subgroup of $G(\widetilde{A})$ with the Lie algebra A and the exponential function

$$\exp \colon A \to S(A), \quad x \mapsto e^x = (e^x - \mathbf{1}, \mathbf{1}).$$

(d) If $J \trianglelefteq A$ is an ideal and A is a unital Banach algebra, then

$$S(J) := G(A) \cap (\mathbf{1} + J)$$

is the kernel of the homomorphism

$$G(A) \to G(A/J), \quad g \mapsto g + J,$$

where $G(A/J)$ denotes the unit group of the unital algebra A/J which is not required to carry a natural Banach space structure. Moreover, we have for $\widetilde{J} := \mathbb{C}\mathbf{1} + J \subseteq A$ the relation

$$G(\widetilde{J}) = G(A) \cap \widetilde{J} = \mathbb{C}^\times \cdot S(J).$$

If, in addition, J is closed, then $S(J)$ is a closed subgroup of $G(A)$ and a Lie group with Lie algebra J. Moreover, A/J carries a natural Banach algebra structure given by

$$\|a + J\| := \inf\{\|a + b\|: b \in J\},$$

and the quotient map $A \to A/J$ is continuous, so that we have an exact sequence of Lie groups

$$\{\mathbf{1}\} \to S(J) \hookrightarrow G(A) \to G(A/J).$$

Here the map on the right hand side need not be surjective. A typical example is $A = B(H)$ and $J = K(H)$ (the ideal of compact operators) for an infinite-dimensional Hilbert space (there are Fredholm operators with non-vanishing index). It is easy to see that for every norm on J for which J is a Banach algebra, the group $S(J)$ coincides, as a set, with the invertible elements in the algebra $\widetilde{J}$ from (d) above. In this sense both constructions lead to the same objects. ■

Classical Banach–Lie groups of operators

In this subsection we will introduce various types of groups of operators on a Hilbert space generalizing the finite-dimensional classical groups on real, complex and quaternionic vector spaces.

Definition IV.16. (Complex classical groups) Let H be a complex Hilbert space. Then we have the following three types of complex *classical groups.*
(1) The full linear group $\mathrm{GL}(H) := G(B(H))$.
(2) Let $I: H \to H$ be an antilinear isometry with $I^2 = \mathbf{1}$. The corresponding *orthogonal group* is defined by

$$\mathrm{O}(H, I) := \{g \in \mathrm{GL}(H): g^{-1} = Ig^*I^{-1}\}.$$

Applying Proposition IV.14 with the complex linear function

$$f: B(H) \to B(H), \quad f(x, y) := Ix^*I^{-1} - y$$

shows that $\mathrm{O}(H, I)$ is a Lie group with the Lie algebra

$$\mathfrak{o}(H, I) := \{X \in B(H): IX^*I^{-1} + X = 0\}$$

(Exercise!). This group can also be described as an isometry group of the symmetric complex bilinear form $\beta(v, w) := \langle v, I.w \rangle$ because

$$\mathrm{O}(H, I) = \mathrm{O}(H, \beta) := \{g \in \mathrm{GL}(H): (\forall v, w \in H)\, \beta(g.v, g.w) = \beta(v, w)\}$$

(cf. Exercise IV.12).
(3) Let $I: H \to H$ be an antilinear isometry with $I^2 = -\mathbf{1}$. The corresponding *symplectic group* is defined by

$$\mathrm{Sp}(H, I) := \{g \in \mathrm{GL}(H): g^{-1} = Ig^*I^{-1}\}.$$

As in (2), this is a Lie group with Lie algebra

$$\mathfrak{sp}(H, I) := \{X \in B(H): IX^*I^{-1} + X = 0\}.$$

This group can also be described as an isometry group of the skew-symmetric complex bilinear form $\beta(v, w) := \langle v, I.w \rangle$ because

$$\mathrm{Sp}(H, I) = \mathrm{Sp}(H, \beta) := \{g \in \mathrm{GL}(H): (\forall v, w \in H)\, \beta(g.v, g.w) = \beta(v, w)\}.$$

(cf. Exercise IV.12). ■

Definition IV.17. (Unitary real forms of the complex classical groups) That there are three types of complex classical groups is related to the fact that there are three finite-dimensional real skew-fields: $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Here the group $\mathrm{GL}(H)$ is related to $\mathbb{C}$. For an antilinear involution I the subspace

$$H_{\mathbb{R}} := \{v \in H: I.v = v\}$$

is a real form of the complex space H, and for an antilinear involution I with $I^2 = -\mathbf{1}$ the algebra $\mathbb{C}\mathbf{1} + \mathbb{C}I$ is isomorphic to $\mathbb{H}$, so that we obtain a quaternionic structure on H. In this case we also write $H_{\mathbb{H}}$ for the pair (H, I) meaning the complex Hilbert space H endowed with an antilinear isometric involution, i.e., a quaternionic structure.

Accordingly we obtain the following groups of $\mathbb{K}$-linear invertible isometries:
(1) For $\mathbb{K} = \mathbb{C}$ we get the *unitary group*

$$\mathrm{U}(H) := \{g \in \mathrm{GL}(H): gg^* = g^*g = \mathbf{1}\}.$$

This is a real algebraic subgroup of $\mathrm{GL}(H)$ with Lie algebra

$$\mathfrak{u}(H) := \{X \in B(H): X + X^* = 0\}.$$

(2) For $\mathbb{K} = \mathbb{R}$ we get the *orthogonal group of the real Hilbert space* $H_{\mathbb{R}}$:

$$\begin{aligned} \mathrm{O}(H_{\mathbb{R}}) &:= \mathrm{O}(H, I) \cap \mathrm{U}(H) = \{g \in \mathrm{U}(H) : gI = Ig\} \\ &\cong \{g \in \mathrm{GL}(H_{\mathbb{R}}) : gg^\top = g^\top g = \mathbf{1}\}. \end{aligned}$$

This is an algebraic subgroup of $\mathrm{U}(H)$ with Lie algebra

$$\mathfrak{o}(H_{\mathbb{R}}) := \{X \in B(H_{\mathbb{R}}) : X + X^\top = 0\}.$$

(3) For $\mathbb{K} = \mathbb{H}$ we finally get the *quaternionic unitary group*:

$$\begin{aligned} \mathrm{Sp}(H_{\mathbb{H}}) &:= \mathrm{Sp}(H, I) \cap \mathrm{U}(H) \cong \{g \in \mathrm{U}(H) : gI = Ig\} \\ &\cong \{g \in \mathrm{GL}(H_{\mathbb{H}}) : gg^* = g^* g = \mathbf{1}\}. \end{aligned}$$

This is an algebraic subgroup of $\mathrm{U}(H)$ with Lie algebra

$$\mathfrak{sp}(H_{\mathbb{R}}) := \{X \in B(H_{\mathbb{H}}) : X + X^* = 0\}.$$

The groups $\mathrm{U}(H)$, $\mathrm{O}(H_{\mathbb{R}})$, resp., $\mathrm{Sp}(H_{\mathbb{H}})$, are called the *unitary real forms* of the complex Lie groups $\mathrm{GL}(H)$, $\mathrm{O}(H, I)$, resp., $\mathrm{Sp}(H, I)$. ■

Remark IV.18. Other real forms can be constructed as follows:
(1) For $\mathrm{GL}(H)$: The groups $\mathrm{U}(H_+, H_-)$ corresponding to indefinite hermitian forms of the type

$$\beta(x_+ + x_-, y_+ + y_-) = \langle x_+, y_+ \rangle - \langle x_-, y_- \rangle,$$

where $H = H_+ \oplus H_-$ is an orthogonal decomposition and $x_+, y_+ \in H_+$, $x_-, y_- \in H_-$.
(2) For $\mathrm{O}(H, I)$: The groups $\mathrm{O}(H_{\mathbb{R}}^+, H_{\mathbb{R}}^-)$ corresponding to indefinite symmetric bilinear forms of the type

$$\beta(x_+ + x_-, y_+ + y_-) = \langle x_+, y_+ \rangle - \langle x_-, y_- \rangle$$

on a real Hilbert space $H_{\mathbb{R}}$ with the direct sum decomposition $H_{\mathbb{R}} = H_{\mathbb{R},+} \oplus H_{\mathbb{R},-}$ and $x_+, y_+ \in H_{\mathbb{R},+}$, $x_-, y_- \in H_{\mathbb{R},-}$.
(3) For $\mathrm{Sp}(H, I)$ ($H \cong l^2(2J, \mathbb{C})$) the subgroup $\mathrm{Sp}(H_{\mathbb{R}}, I)$ preserving the subspace $l^2(2J, \mathbb{R})$ which coincides with the group $\{g \in \mathrm{Sp}(H, I) : g\sigma = \sigma g\}$, where $\sigma(x) = \overline{x}$. ■

Smaller classical groups

Definition IV.19. Let H be a complex Hilbert space. We write $B_{\rm fin}(H) \trianglelefteq B(H)$ for the ideal of all operators of finite rank. This space is spanned by the *rank one operators* $P_{x,y}$, $x, y \in H$, which are given by

$$P_{x,y}(v) := \langle v, y \rangle x.$$

We also put $P_x := P_{x,x}$. We define the *trace of a finite rank operator* A by

$$\operatorname{tr}(A) := \sum_{j=1}^n \langle A.e_j, e_j \rangle = \operatorname{tr}(A\,|_{A(H)}),$$

where $e_1, \ldots, e_n$ is an orthonormal basis of the finite-dimensional subspace $A(H) \subseteq H$. For a rank-one operator we get

$$\operatorname{tr} P_{x,y} = \langle x, y \rangle.$$

We define the *trace norm*

$$\|\cdot\|_1 \colon B(H) \to [0, \infty], \quad \|A\|_1 := \sup\{|\operatorname{tr}(AB)| \colon B \in B_{\rm fin}(H), \|B\| \leq 1\}.$$

Note that the right hand side is well defined because $AB \in B_{\rm fin}(H)$. It turns out that

$$B_1(H) := \{A \in B(H) \colon \|A\|_1 < \infty\}$$

is an ideal of $B(H)$ on which $\|\cdot\|_1$ is a complete norm satisfying

$$|\operatorname{tr}(AB)| \leq \|A\|_1 \|B\|, \quad A \in B_1(H), B \in B(H)$$

(cf. [RS78]). The elements of $B_1(H)$ are called *trace class operators*. Important properties of this space are:
(a) The trace extends to a continuous linear functional $\operatorname{tr} \colon B_1(H) \to \mathbb{C}$ such that

$$\operatorname{tr}(ab) = \operatorname{tr}(ba), \quad a \in B_1(H), b \in B(H)$$

and

$$\operatorname{tr}(a) = \|a\|_1 \quad \text{for positive} \quad a.$$

(b) If $(e_j)_{j \in J}$ is an orthonormal basis of H and $A \in B_1(H)$, then

$$\operatorname{tr}(A) = \sum_{j \in J} \langle A.e_j, e_j \rangle.$$

With the aid of the trace norm we can define a continuous scale of ideals of $B(H)$ as follows. For every $p \in [1, \infty[$ the subsets

$$B_p(H) := \{X \in B(H) \colon \|(XX^*)^{\frac{p}{2}}\|_1 < \infty\},$$

are ideals of $B(H)$ which are Banach spaces with respect to the norms

$$\|X\|_p := \|(XX^*)^{\frac{p}{2}}\|_1^{\frac{1}{p}}.$$

For $p = 1$ this leads to another formula

$$\|X\|_1 = \sqrt{\mathrm{tr}(\sqrt{XX^*})}$$

for the trace norm. The spaces $B_p(H)$ are called the *Schatten ideals* and its elements *operators of Schatten class* p. (Compare this definition with the definition of the spaces $L^p(X, \mathfrak{S}, \mu)$ for a measure space $(X, \mathfrak{S}, \mu)$.)

For $p = 2$ we obtain the particularly important space $B_2(H)$ of *Hilbert-Schmidt operators.* The norm on this space satisfies

$$\|X\|_2^2 = \|XX^*\|_1 = \mathrm{tr}(XX^*),$$

showing that it is defined by the scalar product

$$\langle X, Y \rangle := \mathrm{tr}(XY^*)$$

which indeed turns $B_2(H)$ into a Hilbert space. If $(e_j)_{j \in J}$ is an orthonormal basis of H and $A \in B_2(H)$, then

$$\langle X, Y \rangle = \sum_{j \in J} \langle Y^* X.e_j, e_j \rangle = \sum_{j \in J} \langle X.e_j, Y.e_j \rangle = \sum_{j,k \in J} \langle X.e_j, e_k \rangle \langle e_k, Y.e_j \rangle$$

and in particular

$$\|X\|_2^2 = \sum_{j \in J} \|X.e_j\|^2.$$

We write $B_\infty(H) := K(H)$ for the ideal of compact operators in $B(H)$ endowed with the operator norm. We then have for $1 \leq p \leq q$

$$B_1(H) \subseteq B_p(H) \subseteq B_q(H) \subseteq B_\infty(H) = K(H),$$

and moreover

$$\|xy\|_p \leq \|x\| \|y\|_p, \quad \|xy\|_p \leq \|x\|_p \|y\|, \quad \text{and} \quad \|x\| \leq \|x\|_p$$

for $x, y \in B(H)$.

For a more detailed discussion of these operator ideals and their norms we refer to [RS78]. ■

Definition IV.20. The constructions of Examples IV.15(c),(d) lead to Lie groups

$$\mathrm{GL}_p(H) := \mathrm{GL}(H) \cap (\mathbf{1} + B_p(H))$$

with Lie algebra $\mathfrak{gl}_p(H) := B_p(H)$. The group $\mathrm{GL}_\infty(H)$ is called the *Fredholm group.* The group

$$\mathrm{U}_p(H) := \mathrm{U}(H) \cap (\mathbf{1} + B_p(H))$$

is a Lie subgroup with Lie algebra

$$\mathfrak{u}_p(H) := \mathfrak{u}(H) \cap B_p(H) = \{X \in B_p(H) \colon X^* = -X\}.$$

With

$$\mathrm{Herm}_p(H) := \mathrm{Herm}(H) \cap B_p(H)$$

we then have

$$\mathfrak{gl}_p(H) = \mathfrak{u}_p(H) \oplus \mathrm{Herm}_p(H) = \mathfrak{u}_p(H) \oplus i\mathfrak{u}_p(H). \qquad \blacksquare$$

Determinant functions

We consider the Banach–Lie algebra $\mathfrak{g} := B_1(H) = \mathfrak{gl}_1(H)$ and the corresponding Banach–Lie group $G := \mathrm{GL}_1(H)$ introduced in Definition IV.20.

Proposition IV.21. *There exists a unique holomorphic character*

$$\det\colon \mathrm{GL}_1(H) \to \mathbb{C}^\times$$

with $\mathbf{L}(\det) = \mathrm{tr}$. *Let*

$$\mathrm{SL}(H) := \ker \det$$

and define for a unit vector $v \in H$ *a holomorphic homomorphism*

$$\gamma\colon \mathbb{C}^\times \to \mathrm{GL}_1(H), \qquad \gamma(z)(w) := \begin{cases} zw & \text{for } w \in \mathbb{C}v \\ w & \text{for } w \in v^\perp. \end{cases}$$

Then $\det \circ \gamma = \mathrm{id}_{\mathbb{C}}$ *and*

$$\mathrm{GL}_1(H) = \mathrm{SL}(H)\gamma(\mathbb{C}^\times) \cong \mathrm{SL}(H) \rtimes \mathbb{C}^\times.$$

Moreover, the group $\mathrm{SL}(H)$ *is simply connected.*

Proof. (a) First we prove the existence of the determinant function $\det$. Let $q\colon \widetilde{\mathrm{GL}}_1(H) \to \mathrm{GL}_1(H)$ denote the universal covering group. Since $\mathrm{tr}\colon \mathfrak{gl}_1(H) \to \mathbb{C}$ vanishes on commutators, it is a Lie algebra homomorphism. Its continuity follows from $|\,\mathrm{tr}(X)| \leq \|X\|_1$ for $X \in \mathfrak{gl}_1(H)$. Hence there exists a holomorphic character

$$\widetilde{\det}\colon \widetilde{\mathrm{GL}}_1(H) \to \mathbb{C}^\times \quad \text{with} \quad \mathbf{L}(\widetilde{\det}) = \mathrm{tr}.$$

It remains to show that $\widetilde{\det}$ factors through the covering map q. Let $v \in H$ be a unit vector and define γ as above. Then it follows from Theorem A.10 that γ induces an isomorphism

$$\pi_1(\gamma)\colon \pi_1(\mathbb{C}^\times) \cong \mathbb{Z} \to \pi_1(\mathrm{GL}_1(H)).$$

In particular its natural lift $\widetilde{\gamma}\colon \mathbb{C} \cong \widetilde{\mathbb{C}}^\times \to \widetilde{\mathrm{GL}}_1(H)$ satisfies

$$\widetilde{\gamma}(\pi_1(\mathbb{C}^\times)) = \pi_1(\mathrm{GL}_1(H)).$$

In view of $\operatorname{tr} \circ \mathbf{L}(\gamma) = \mathrm{id}_{\mathbb{C}}$, we have

$$\widetilde{\det} \circ \widetilde{\gamma} = \exp_{\mathbb{C}^\times}\colon \mathbb{C} \to \mathbb{C}^\times,$$

showing that $\pi_1(\mathrm{GL}_1(H)) \subseteq \ker \widetilde{\det}$, and therefore there exists a unique holomorphic homomorphism

$$\det\colon \mathrm{GL}_1(H) \to \mathbb{C}^\times$$

with $\mathbf{L}(\det) = \operatorname{tr}$.

(b) In view of Lemma IV.11, $\mathrm{SL}(H)$ is a Lie subgroup of $\mathrm{GL}_1(H)$ whose Lie algebra is given by

$$\mathfrak{sl}(H) := \{X \in B_1(H)\colon \operatorname{tr} X = 0\}.$$

We claim that the mapping

$$\Phi\colon \mathrm{SL}(H) \rtimes \mathbb{C}^\times \to \mathrm{GL}_1(H), \quad (A, z) \mapsto \gamma(z)A$$

is a biholomorphic isomorphism of Banach Lie groups, where the semidirect product structure is given by the conjugation action of $\gamma(z)$ on the normal subgroup $\mathrm{SL}(H)$.

The preceding argument implies that $\det \circ \gamma = \mathrm{id}_{\mathbb{C}^\times}$ which shows that Φ is surjective and $\mathrm{SL}(H) \cap \gamma(\mathbb{C}^\times) = \{\mathbf{1}\}$, which means that Φ is a bijection. It is clear that Φ is holomorphic, and since $\Phi^{-1}(g) = \big(g\gamma(\det g)^{-1}, \det g\big)$, the mapping Φ is biholomorphic.

(c) To see that $\mathrm{SL}(H)$ is simply connected, we only have to use the product decomposition

$$\mathrm{GL}_1(H) \cong \mathrm{SL}(H)\gamma(\mathbb{C}^\times) \cong \mathrm{SL}(H) \rtimes \mathbb{C}^\times$$

and to recall that $\pi_1(\gamma)\colon \mathbb{Z} \to \pi_1(\mathrm{GL}_1(H))$ is surjective. ■

Notes on Section IV

For a more detailed discussion of the Schatten ideals and the determinant function we refer to [RS78] and in particular [RS78, Th. XIII.105].

Exercises for Section IV

Exercise IV.1. (a) Let $m: G \times G \to G$ be a smooth associative multiplication on the manifold G with identity element $\mathbf{1}$. Show that the differential in $(\mathbf{1}, \mathbf{1})$ is given by

$$dm(\mathbf{1}, \mathbf{1}): T_{\mathbf{1}}(G) \times T_{\mathbf{1}}(G) \to T_{\mathbf{1}}(G), \quad (v, w) \mapsto v + w.$$

(b) Show that the smoothness of the inversion in the definition of a Banach–Lie group is redundant because the Inverse Function Theorem can be applied to the map

$$G \times G \to G \times G, \quad (x, y) \mapsto (x, xy)$$

whose differential in $(\mathbf{1}, \mathbf{1})$ is given by the map $(v, w) \mapsto (v, v + w)$. ■

Exercise IV.2. Let E be a Banach space. Show that every continuous group homomorphism $\gamma: (\mathbb{R}, +) \to (E, +)$ can be written as $\gamma(t) = tv$ for some $v \in E$.■

Exercise IV.3. Let E be a Banach space.
(1) If F is a closed subspace of E and $H := \{g \in \mathrm{GL}(E): g.F \subseteq F\}$, then

$$\mathbf{L}(H) = \{Y \in B(E): Y.F \subseteq F\}.$$

(2) For each $v \in E$ and $H := \{g \in \mathrm{GL}(E): g.v = v\}$ we have

$$\mathbf{L}(H) = \{Y \in B(E): Y.v = 0\}.$$ ■

Exercise IV.4. Let A be a Banach space and $m: A \times A \to A$ a continuous linear map. Then the group

$$\mathrm{Aut}(A, m) := \{g \in \mathrm{GL}(A): (\forall a, b \in A)\ m(g.a, g.b) = g.m(a, b)\}$$

of automorphisms of the "algebra" (A, m) is a Lie group whose Lie algebra is the space

$$\mathrm{der}(A, m) := \{X \in B(A): (\forall a, b \in A)\, X.m(a, b) = m(X.a, b) + m(a, X.b)\}$$

of derivations of (A, m). ■

Exercise IV.5. (a) Let G and N be Lie groups and $\varphi: G \to \mathrm{Aut}(N)$ be a homomorphism such that the map $G \times N \to N, (g, n) \mapsto \varphi(g)(n)$ is smooth. Then the semidirect product group $G \rtimes N$ with the multiplication

$$(n, g)(n', g') := (n\varphi(g)(n'), gg')$$

is a Lie group with Lie algebra $\mathfrak{n} \rtimes \mathfrak{g}$.
(b) Let H be a Hilbert space. Show that the motion group

$$\mathrm{Mot}(H) := H \rtimes \mathrm{U}(H)$$

is a Lie group with Lie algeba $H \rtimes \mathfrak{u}(H)$. ■

Exercise IV.6. Let E and F be Banach spaces and $B^k(E,F)$ be the space of continuous k-linear maps $E^k \to F$.
(a) Then $B^k(E,F)$ is a Banach space with respect to the norm

$$\|f\| := \sup\{\|f(x_1,\ldots,x_k)\|: x_i \in E, \|x_1\|,\ldots,\|x_k\| \leq 1\}.$$

(b) The assignment

$$\big(\pi(g).f\big)(x_1,\ldots,x_k) := f(g^{-1}.x_1,\ldots,g^{-1}.x_k)$$

defines a continuous homomorphism $\pi: \mathrm{GL}(E) \to \mathrm{GL}(B^k(E,F))$. Hint: The map $\eta: B(E) \to B(B^k(E,F))$ with

$$\big(\eta(A).f\big)(x_1,\ldots,x_k) := f(A.x_1,\ldots,A.x_k)$$

is a continuous k-linear map.
(c) Calculate the derived Lie algebra representation $d\pi: B(E) \to B(B^k(E,F))$.
(d) We identify the space $P_k(E,F)$ of F-valued continuous polynomial functions of degree k on E with the closed subspace $\mathrm{Sym}^k(E,F) \subseteq B^k(E,F)$. Then the norm on this space is given by

$$\|f\| = \sup\{\|f(x)\|: \|x\| \leq 1\}$$

and the assignment

$$\big(\pi(g).f\big)(x) := f(g^{-1}.x)$$

defines a continuous homomorphism $\pi: \mathrm{GL}(E) \to \mathrm{GL}(P_k(E,F))$. ∎

Exercise IV.7. (a) Let H be a complex Hilbert space. Show that there exists an antilinear isometric map $I: H \to H$ with $I^2 = \mathbf{1}$.
(b) If I_1 and I_2 are two such maps, then there exists a unitary operator $g \in \mathrm{U}(H)$ with $I_2 = gI_1g^{-1}$.
(c) Show that for a fixed complex Hilbert space H all groups $\mathrm{O}(H,I)$ are isomorphic. ∎

Exercise IV.8. (a) Let H be an infinite-dimensional or even-dimensional complex Hilbert space. Show that there exists an antilinear isometric map $I: H \to H$ with $I^2 = -\mathbf{1}$.
(b) If I_1 and I_2 are two such maps, then there exists a unitary operator $g \in \mathrm{U}(H)$ with $I_2 = gI_1g^{-1}$.
(c) Show that for a fixed complex Hilbert space H all groups $\mathrm{Sp}(H,I)$ are isomorphic. ∎

Exercise IV.9. Let H be a complex Hilbert space and I an antilinear isometry with $I^2 = \pm\mathbf{1}$. We consider the complex bilinear form

$$\beta(v,w) := \langle v, I.w\rangle.$$

(1) β is symmetric (skew-symmetric) if $I^2 = \mathbf{1}$ ($I^2 = -\mathbf{1}$).

(2) For $I^2 = \mathbf{1}$ we have

$$\mathrm{O}(H,I) = \{g \in \mathrm{GL}(H) \colon (\forall v,w \in H)\, \beta(g.v, g.w) = \beta(v,w)\}$$

and the Lie algebra of this subgroup is

$$\begin{aligned}\mathfrak{o}(H,I) &= \{X \in B(H) \colon IX^*I^{-1} + X = 0\} \\ &= \{X \in B(H) \colon (\forall v,w \in H)\, \beta(X.v, w) + \beta(v, X.w) = 0\}.\end{aligned}$$

(3) For $I^2 = -\mathbf{1}$ we have

$$\mathrm{Sp}(H,I) = \{g \in \mathrm{GL}(H) \colon (\forall v,w \in H)\, \beta(g.v, g.w) = \beta(v,w)\}$$

and the Lie algebra of this subgroup is

$$\begin{aligned}\mathfrak{sp}(H,I) &= \{X \in B(H) \colon IX^*I^{-1} + X = 0\} \\ &= \{X \in B(H) \colon (\forall v,w \in H)\, \beta(X.v, w) + \beta(v, X.w) = 0\}.\end{aligned}$$

(4) If $I^2 = \mathbf{1}$ and $\dim H = \infty$, there exists an orthonormal basis $(e_j)_{j\in 2J}$ of H with $I.e_j = e_{-j}$, $j \in 2J$. Then

$$H \cong l^2(2J,\mathbb{C}) \cong l^2(J,\mathbb{C}) \oplus l^2(-J,\mathbb{C}) \cong l^2(J,\mathbb{C}) \oplus l^2(J,\mathbb{C}),$$

and with respect to this decomposition, we write elements of $B(H)$ as 2×2-block matrices. For $Q.(v,w) = (w,v)$ we then have

$$\mathrm{O}(H,I) = \{g \in \mathrm{GL}(H) \colon g^{-1} = Qg^\top Q\}$$

and for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ this means that

$$cb^\top + da^\top = \mathbf{1}, \quad cd^\top + dc^\top = 0 \quad \text{and} \quad ab^\top + ba^\top = 0.$$

(5) If $I^2 = -\mathbf{1}$, then there exists an orthonormal basis $(e_j)_{j\in 2J}$ of H with

$$I.e_j = \begin{cases} e_{-j}, & j \in J, \\ -e_{-j}, & j \in -J. \end{cases}$$

Then

$$\mathrm{Sp}(H,I) = \{g \in \mathrm{GL}(H) \colon g^{-1} = -Qg^\top Q\} \quad \text{with} \quad Q = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

and for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ this means that

$$c^\top a = a^\top c, \quad d^\top b = b^\top d \quad \text{and} \quad a^\top d - c^\top b = \mathbf{1}. \qquad \blacksquare$$

Exercise IV.10. Let E, F and G be Banach spaces. Show that for a bilinear map $\beta\colon E \times F \to G$ the following are equivalent:

(1) β is continuous.

(2) β is continuous in $(0,0)$.

(3) $(\exists C > 0)(\forall x \in E)(\forall y \in F)\, \|\beta(x,y)\| \leq C\|x\|\,\|y\|$. $\blacksquare$

Exercise IV.11. Let J be a set. For a tuple $x = (x_j)_{j \in J} \in (\mathbb{R}^+)^J$ we define

$$\sum_{j \in J} x_j := \sup \Big\{ \sum_{j \in F} x_j : F \subseteq J \text{ finite} \Big\}.$$

Show that

$$l^1(J, \mathbb{R}) := \Big\{ x = (x_j)_{j \in J} : \sum_{j \in J} |x_j| < \infty \Big\}$$

is a Banach space with respect to

$$\|x\|_1 := \sum_{j \in J} |x_j|.$$

Define $e_j \in l^1(J, \mathbb{R})$ by $(e_j)_i = \delta_{ij}$. Show that the subgroup Γ generated by $\{e_j : j \in J\}$ is discrete. ■

V. Holomorphic representations of classical Banach–Lie groups

We have seen in Section III how the unitary highest weight modules of an involutive split locally finite Lie algebra $\mathfrak{g}$ with respect to a "compact" real form $\mathfrak{u} = \mathfrak{g}_{\mathbb{R}}$ can be classified. Our goal is to realize such representations by holomorphic sections of a holomorphic line bundle over some coadjoint orbit which at the same time is a complex Kähler manifold.

In the preceding section we have discussed several aspects of the general theory of Banach–Lie groups and in particular the groups $\mathrm{GL}_p(H)$ with the Lie algebras $B_p(H)$ which for $H = l^2(J, \mathbb{C})$ can be viewed as Banach versions of the locally finite Lie algebra $\mathfrak{gl}(J, \mathbb{C})$, which is a completely algebraic object. In this section we will make our first step towards a geometric realization of the representations $(\rho_\lambda, L(\lambda))$ of $\mathfrak{g}$ (mainly for $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$) by discussing conditions under which they can be integrated to holomorphic representations of certain complex Lie groups ($\mathrm{GL}_1(H)$ for $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$). In the next section we will then discuss coadjoint orbits of Banach–Lie groups and how one can construct holomorphic line bundles thereon.

For the sake of simplicity of the exposition, we state several results in this section only for $\mathfrak{gl}(J, \mathbb{C})$ and the corresponding groups. One can develop the whole theory in the context of groups associated to L^*-algebras which then makes it possible to deal with all special cases simultaneously, but this theory requires a more elaborate background which is superfluous in the special case of $\mathfrak{gl}(J, \mathbb{C})$.

The norm function of a unitary highest weight module

Let $(\mathfrak{g}, \mathfrak{h})$ be a split locally finite involutive Lie algebra with $\Delta = \Delta_k$. Since $\mathfrak{g}$ is locally finite, each element $X \in \mathfrak{g}$ defines an inner automorphism $e^{\mathrm{ad}\, X}$ of $\mathfrak{g}$ because for each $Y \in \mathfrak{g}$ the series

$$e^{\mathrm{ad}\, X}.Y := \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}\, X)^n.Y$$

converges since X and Y generate a finite-dimensional subalgebra of $\mathfrak{g}$. We call the group

$$\mathrm{Inn}(\mathfrak{g}_{\mathbb{R}}) := \langle \{e^{\mathrm{ad}\, Y} \colon Y^* = -Y\} \rangle$$

the group of *unitary inner automorphisms of* $\mathfrak{g}$.

Lemma V.1. *For a unitary highest weight representation* $(\rho_\lambda, L(\lambda))$ *of* $\mathfrak{g}$ *the following assertions hold:*

(i) *For each* $X \in \mathfrak{g}$ *the operator* $e^{\rho_\lambda(X)}$ *on* $L(\lambda)$ *is well defined and satisfies*

$$(e^{\rho_\lambda(X)})^* = (e^{\rho_\lambda(X^*)}). \tag{5.1}$$

In particular $e^{\rho_\lambda(X)}$ *is unitary for* $X^* = -X$, *i.e.,* $X \in \mathfrak{u} = \mathfrak{g}_{\mathbb{R}}$.

(ii) *For* $X, Y \in \mathfrak{g}$ *we have*

$$e^{\rho_\lambda(X)}\rho_\lambda(Y)e^{-\rho_\lambda(X)} = \rho_\lambda(e^{\mathrm{ad}\, X}.Y). \tag{5.2}$$

(iii) *The function* $s\colon \mathfrak{g} \to [0, \infty], s(X) := \|\rho_\lambda(X)\|$ *is a convex function which is positively homogeneous in the sense that*

$$s(zX) = |z|s(X) \quad \text{for} \quad X \in \mathfrak{g}, z \in \mathbb{C},$$

and it is invariant under the involution $*$ *and the group* $\mathrm{Inn}(\mathfrak{g}_{\mathbb{R}})$. *For* $X \in \mathfrak{h}$ *we have*

$$s(X) = \sup |\langle \mathcal{P}_\lambda, X\rangle| = \sup |\langle \mathcal{W}.\lambda, X\rangle|.$$

Proof. (i) For each element $X \in \mathfrak{g}$ the operator $\rho_\lambda(X)$ is locally finite on $L(\lambda)$ (Theorem III.6(iii)), so that we find for each $v \in L(\lambda)$ a finite-dimensional $\rho_\lambda(X)$-invariant subspace E containing v. Now

$$e^{\rho_\lambda(X)}.v := \sum_{n=0}^{\infty} \frac{1}{n!}\rho_\lambda(X)^n.v$$

converges because the series for $e^{\rho_\lambda(X)|_E}$ converges in $\mathrm{End}(E)$.

An easy verification shows that, as an operator on the pre-Hilbert space $L(\lambda)$, we have $\rho_\lambda(X)^* = \rho_\lambda(X^*)$ because $L(\lambda)$ is a unitary $\mathfrak{g}$-module. This implies that for $v, w \in L(\lambda)$ we have

$$\langle e^{\rho_\lambda(X)}.v, w\rangle = \langle v, e^{\rho_\lambda(X^*)}.w\rangle$$

which means that the operator $e^{\rho_\lambda(X)}$ has an adjoint given by (5.1).

(ii) Now let $X, Y \in \mathfrak{g}$ and $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be a finite-dimensional subalgebra containing both. Then $e^{\mathrm{ad}\, Y}.X$ is well defined. Since each $v \in L(\lambda)$ is contained in a finite-dimensional $\mathfrak{g}_0$-invariant subspace, we now easily obtain

$$e^{\rho_\lambda(X)}\rho_\lambda(Y)e^{-\rho_\lambda(X)} = \rho_\lambda(e^{\mathrm{ad}\, X}.Y).$$

(iii) That s is convex, positively homogeneous and $*$-invariant follows from the corresponding properties of the norm function on the algebra

$$\{A \in \mathrm{End}(L(\lambda))\colon (\exists A^* \in \mathrm{End}(L(\lambda)))(\forall v, w \in L(\lambda))\ \langle A.v, w\rangle = \langle v, A^*.w\rangle\}$$

(cf. [Ne99a, Prop. II.3.5]). For each element $Y = -Y^* \in \mathfrak{g}$ we further get by combining (i) and (ii):

$$s(e^{\mathrm{ad}\, Y}.X) = \|\rho_\lambda(e^{\mathrm{ad}\, Y}.X)\| = \|e^{\rho_\lambda(Y)}\rho_\lambda(X)e^{-\rho_\lambda(Y)}\| = \|\rho_\lambda(X)\| = s(X),$$

showing that s is $\mathrm{Inn}(\mathfrak{g}_{\mathbb{R}})$-invariant.

The formula for $s(X)$, $X \in \mathfrak{h}$, follows directly from the weight decomposition of $L(\lambda)$ and the description of the set of weights in Theorem III.6. ∎

Proposition V.2. *Assume that $\mathfrak{g}$ is simple. Then for a unitary highest weight representation the following are equivalent:*

(1) *There exists an element $X \in \mathfrak{g}$ with $\|\rho_\lambda(X)\| < \infty$.*

(2) *$\|\rho_\lambda(X)\| < \infty$ for all $X \in \mathfrak{g}$.*

(3) *$\lambda(\check{\Delta})$ is a bounded subset of $\mathbb{Z}$.*

Proof. (1) $\Rightarrow$ (2): We consider the subset

$$\mathfrak{g}_s := \{X \in \mathfrak{g} : s(X) < \infty\}.$$

The properties of the function $s: \mathfrak{g} \to [0, \infty]$ (Lemma V.1(iii)) imply that $\mathfrak{g}_s$ is a complex subspace of $\mathfrak{g}$ which is invariant under the group $\mathrm{Inn}(\mathfrak{g}_\mathbb{R})$. For $X \in \mathfrak{g}_s$ and $Y^* = -Y$ the curve $e^{\mathbb{R}\,\mathrm{ad}\,Y}.X$ lies in a finite-dimensional subspace of $\mathfrak{g}_s$, so taking derivatives leads to $[Y, X] \in \mathfrak{g}_s$ and therefore to $[\mathfrak{g}_\mathbb{R}, \mathfrak{g}_s] \subseteq \mathfrak{g}_s$ which in turn implies that $[\mathfrak{g}, \mathfrak{g}_s] \subseteq \mathfrak{g}_s$. This means that $\mathfrak{g}_s$ is a non-zero ideal of $\mathfrak{g}$ and therefore equal to $\mathfrak{g}$ because $\mathfrak{g}$ was assumed to be simple.

(2) $\Rightarrow$ (3): As in the finite-dimensional case, one shows that the Weyl group $\mathcal{W}$ has at most two orbits in Δ and likewise in $\check{\Delta}$ for its natural action on $\mathfrak{h}$ given by

$$r_\alpha.x := x - \alpha(x)\check{\alpha}, \quad x \in \mathfrak{h}$$

(Exercise V.2). We write $\Delta = \mathcal{W}.\{\alpha_1, \alpha_2\}$. Then we also have $\check{\Delta} = \mathcal{W}.\{\check{\alpha}_1, \check{\alpha}_2\}$. For $\alpha \in \Delta$ we now obtain

$$\begin{aligned}|\lambda(\check{\alpha})| &\leq \sup\{|\lambda(w.\check{\alpha}_j)| : w \in \mathcal{W}, j = 1, 2\}| \leq \sup\{|\langle w.\lambda, \check{\alpha}_j\rangle| : w \in \mathcal{W}, j = 1, 2\}| \\ &\leq \max(\|\rho_\lambda(\check{\alpha}_1)\|, \|\rho_\lambda(\check{\alpha}_2)\|).\end{aligned}$$

(3) $\Rightarrow$ (1): Let $\alpha \in \Delta$. Then

$$\|\rho_\lambda(\check{\alpha})\| = \sup|\langle \mathcal{W}.\lambda, \check{\alpha}\rangle| = \sup|\langle \lambda, \mathcal{W}.\check{\alpha}\rangle| \leq \sup|\langle \lambda, \check{\Delta}\rangle| < \infty. \qquad \blacksquare$$

Corollary V.3. *Assume that $\mathfrak{g}$ is reductive. If $\lambda(\check{\Delta})$ is a bounded subset of $\mathbb{Z}$, then $\|\rho_\lambda(X)\| < \infty$ for all $X \in \mathfrak{g}$.*

Proof. The assumption that $\mathfrak{g}$ is reductive means that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{z}(\mathfrak{g})$ acts by scalar multiples of the identity on $L(\lambda)$ (Proposition III.2(i)), all the operators in $\rho_\lambda(\mathfrak{z}(\mathfrak{g}))$ are bounded anyway. Therefore it suffices to assume that $\mathfrak{g}$ is semisimple. The proof of (3) $\Rightarrow$ (1) in Proposition V.2 implies that $\|\rho_\lambda(\check{\alpha})\| < \infty$ for all $\alpha \in \Delta$. Since $\mathfrak{g}$ coincides with the smallest $\mathrm{Inn}(\mathfrak{g}_\mathbb{R})$-invariant complex subspace containing $\check{\Delta}$, the fact that $s^{-1}(\mathbb{R})$ is such a subspace implies the assertion. $\blacksquare$

The boundedness of $\lambda(\check{\Delta})$ is sufficient, but not necessary for all the operators $\rho_\lambda(X)$, $X \in \mathfrak{g}$, to be bounded. In fact, $\mathfrak{g}$ might be an infinite direct sum of simple ideals $\mathfrak{g}_i$, $i \in I$, such that

$$\sup|\langle \lambda, \check{\Delta}_i\rangle| < \infty$$

for each individual $i \in I$, but

$$\sup|\langle \lambda, \check{\Delta}\rangle| = \infty.$$

Then we still have $\|\rho_\lambda(X)\| < \infty$ for each $X \in \mathfrak{g}$.

Holomorphic highest weight representations of $\mathrm{GL}_1(H)$

In this subsection we apply the general results of the preceding section to holomorphic highest weight representations of the complex Banach–Lie group $\mathrm{GL}_1(H)$.

Let $L(\lambda)$ be a unitary highest weight module of $\mathfrak{gl}(J,\mathbb{C})$. As in Example III.7, we represent λ by an element of $\mathbb{C}^J \cong \mathfrak{h}^*$.

Lemma V.4. *For a unitary highest weight module $L(\lambda)$ of $\mathfrak{gl}(J,\mathbb{C})$ the following are equivalent:*

(1) $\lambda(\check{\Delta})$ *is bounded.*

(2) $\lambda = (\lambda)_{j\in J}$ *is bounded as an element of* $\mathbb{C}^J$, *hence an element of* $l^\infty(J,\mathbb{C})$.

(3) $\|\rho_\lambda(X)\| \leq 2\|\lambda\|_\infty \|X\|_1$ *for all* $X \in \mathfrak{g}$, *where* $\|\cdot\|_1$ *denotes the trace norm.*

Proof. (1) $\Rightarrow$ (2): Let $\alpha \in \Delta$. The values of λ on the coroots $E_{jj} - E_{kk}$ are given by $\lambda_j - \lambda_k$, $j,k \in J$. It is clear that λ is bounded if the set of all these numbers is bounded.

(2) $\Rightarrow$ (3): Let $\|\lambda\|_\infty := \sup\{|\lambda_j| : j \in J\}$. Then for each $X \in \mathfrak{h}$ the relation $\mathrm{conv}(\mathcal{P}_\lambda) = \mathrm{conv}(\mathcal{W}.\lambda)$ (Theorem III.6) and the fact that $\mathcal{W}$ acts isometrically on $l^\infty(J)$ imply for $X \in \mathfrak{h}$ that

$$\|\rho_\lambda(X)\| \leq \|\lambda\|_\infty \|X\|_1.$$

For $X = X^* \in \mathfrak{gl}(J,\mathbb{C})$ there exists a $g \in \mathrm{U}(J,\mathbb{C})$ with $\mathrm{Ad}(g).X = gXg^{-1} \in \mathfrak{h}$ (Exercise V.3). We now obtain

$$\|\rho_\lambda(X)\| = \|\rho_\lambda(gXg^{-1})\| \leq \|\lambda\|_\infty \|gXg^{-1}\|_1 = \|\lambda\|_\infty \|X\|_1.$$

For a general element $X \in \mathfrak{g}$ this leads to

$$\begin{aligned} s(X) &= s\Big(\frac{X+X^*}{2} + \frac{X-X^*}{2}\Big) \leq \frac{1}{2}\big(s(X+X^*) + s(i(X-X^*))\big) \\ &\leq \frac{1}{2}\|\lambda\|_\infty \big(\|X+X^*\|_1 + \|X-X^*\|_1\big) \\ &\leq \|\lambda\|_\infty (\|X\|_1 + \|X^*\|_1) = 2\|\lambda\|_\infty \|X\|_1. \end{aligned}$$

(3) $\Rightarrow$ (1): We have $\|\check{\alpha}\|_1 = 2$ for every coroot $\check{\alpha} = E_{ii} - E_{jj}$ for $\alpha = \varepsilon_i - \varepsilon_j$. Therefore (3) implies that

$$|\lambda(\check{\alpha})| \leq \|\rho_\lambda(\check{\alpha})\| \leq 2\|\lambda\|_\infty \|\check{\alpha}\|_1 = 4\|\lambda\|_\infty.$$ ∎

Definition V.5. We call a holomorphic representation $\pi\colon \mathrm{GL}_p(H) \to \mathrm{GL}(\mathcal{H})$ a *highest weight representation* if

(1) $\mathcal{H}$ contains a dense subspace which is a highest weight module for the Lie algebra $\mathfrak{gl}(J,\mathbb{C}) \subseteq \mathfrak{gl}_p(H)$ (for $H \cong l^2(J,\mathbb{C})$), and

(2) $\pi(g^*) = \pi(g)^*$ for all $g \in \mathrm{GL}_p(H)$.

The preceding condition means in particular that the subgroup $U_p(H)$ acts unitarily on $\mathcal{H}$. ∎

We now use Lemma V.4 to obtain a classification of all holomorphic highest weight representations of the group $\mathrm{GL}_1(H)$ and its universal covering group $\widetilde{\mathrm{GL}}_1(H)$.

Theorem V.6. *Let $\Delta^+_{\preceq} \subseteq \Delta$ be a positive system and $\lambda = \lambda^* = \sum_j \lambda_j \varepsilon_j \in \mathfrak{h}^*$. Then λ is the highest weight of a holomorphic highest weight representation of $\widetilde{\mathrm{GL}}_1(H)$ if and only if the following conditions are satisfied:*

(i) *λ is dominant integral, i.e., $\lambda_j - \lambda_k \in \mathbb{N}_0$ for $j \prec k$, and*

(ii) *λ is bounded.*

The corresponding representation factors through $\mathrm{GL}_1(H)$ if and only if, in addition,

(iii) *$\lambda_j \in \mathbb{Z}$ for all $j \in J$.*

Proof. Let $\mathcal{H}_\lambda$ denote the completion of the pre-Hilbert space $L(\lambda)$. First we observe that a unitary highest weight representation of $\mathfrak{gl}(J, \mathbb{C})$ extends to a continuous Lie algebra representation

$$\rho_\lambda : \mathfrak{gl}_1(H) \to B(\mathcal{H}_\lambda)$$

if and only if λ is bounded (Lemma V.4). Therefore the first part of the theorem follows from Theorem III.6 (see also Example III.7) because the continuous representations $\rho_\lambda : \mathfrak{gl}_1(H) \to B(\mathcal{H}_\lambda)$ are in one-to-one correspondence with the holomorphic representations $\pi_\lambda : \widetilde{\mathrm{GL}}_1(H) \to \mathrm{GL}(\mathcal{H}_\lambda)$ with $\mathbf{L}(\pi_\lambda) = \rho_\lambda$ (Theorem IV.2(c)).

So let us assume that λ is bounded and that $L(\lambda)$ is unitary. Pick $j \in J$ and consider the holomorphic homomorphism

$$\gamma : \mathbb{C} \to \mathrm{GL}_1(H), \quad z \mapsto e^{zE_{jj}}.$$

In view of Proposition IV.21, the canonical lift

$$\widetilde{\gamma} : \mathbb{C} \to \widetilde{\mathrm{GL}}_1(H), \quad z \mapsto \exp(zE_{jj})$$

satisfies

$$\widetilde{\gamma}(2\pi i \mathbb{Z}) = \pi_1(\mathrm{GL}_1(H)).$$

We conclude that π_λ factors through $\mathrm{GL}_1(H)$ if and only if

$$\mathbf{1} = e^{2\pi i \rho_\lambda(E_{jj})} = \pi_\lambda(\exp 2\pi i E_{jj}),$$

which, in view of Theorem III.6, is equivalent to $\lambda_j \in \mathbb{Z}$. This is equivalent to $\lambda_k \in \mathbb{Z}$ for all $k \in J$. ∎

Remark V.7. Let J be a set and $H = l^2(J, \mathbb{C})$.

(a) If the order $\preceq$ on J is such that J has m smallest elements $j_1 \prec \ldots \prec j_m$, then the fundamental weights $\varpi_k = \varepsilon_{j_1} + \ldots + \varepsilon_{j_k}$, $k \leq m$ correspond to the irreducible representations $\big(\Lambda^k, \Lambda^k(H)\big)$ of $\mathrm{GL}(H)$ on the space $\Lambda^k(H)$ given by

$$\Lambda^k(g)(v_1 \wedge \ldots \wedge v_k) = (g.v_1) \wedge \ldots \wedge (g.v_k).$$

A primitive element for $\mathfrak{gl}(J,\mathbb{C})$ is given by $e_{j_1} \wedge \ldots \wedge e_{j_k}$. For $k = 1$ we obtain the identical representation of $B(H)$ on H. For $J = \mathbb{N}$ with the natural order we have in particular $\varpi_k = \varepsilon_1 + \ldots + \varepsilon_k$ for each $k \in \mathbb{N}$.

If $M \subseteq J$ satisfies $M \prec M^c := J \setminus M$ and $J \setminus M$ is finite, then we can describe the representation corresponding to $\lambda = \varpi_M - \varpi_J$ as follows. Let $\tau\colon H \to H$ be the antilinear isometry given by $\tau(\sum_{j\in J} x_j e_j) = \sum_{j \in J} \overline{x_j} e_j$. For $A \in B(H)$ we put $A^\top := \tau A^* \tau$ and note that $B(H) \to B(H), A \mapsto A^\top$, is a linear antiautomorphism of the algebra $B(H)$.

We claim that for $k := |M^c| < \infty$ the representation of $\mathrm{GL}(H)$ on $\Lambda^k(H)$ given by $\widetilde\Lambda^k(g) := \Lambda^k(g^\top)^{-1}$ is a holomorphic highest weight representation with highest weight $\lambda := \varpi_M - \varpi_J = -\varpi_{M^c}$ (cf. Example III.7). In fact, let $M^c = \{j_1, \ldots, j_k\}$ and put $v_\lambda := e_{j_1} \wedge \ldots \wedge e_{j_k}$. It is clear that v_λ generates a dense $\mathfrak{gl}(J,\mathbb{C})$-submodule of $\Lambda^k(H)$. Furthermore for $X \in \mathfrak{h}$ we have

$$X.v_\lambda = -d\Lambda^k(X^\top).v_\lambda = -d\Lambda^k(X).v_\lambda = \varpi_{M^c}(X)v_\lambda.$$

Hence $E_{jk}^\top = E_{kj}$ leads to $j \prec k$ to $E_{jk}.v_\lambda = 0$. This means that v_λ is a primitive element with respect to $\Delta_\prec^+$, and thus $\big(\widetilde\Lambda^k, \Lambda^k(H)\big)$ is a holomorphic highest weight representation of $\mathrm{GL}(H)$ with highest weight $-\varpi_{M^c}$.

(b) (The infinite wedge representations) A particular interesting case covered by the preceding theorem is given by $J = \mathbb{Z}$ endowed with the natural order and $M = \{m \in \mathbb{Z}\colon m \leq k\}$. In this case

$$\lambda = \varpi_M = \sum_{j=-\infty}^{k} \varepsilon_j.$$

Here $\mathcal{H}_{\varpi_M}$ can be identified with a the Hilbert space with the orthonormal basis

$$e_{i_k} \wedge e_{i_{k-1}} \wedge e_{i_{k-2}} \wedge \ldots, \quad \text{where} \quad i_k > i_{k-1} > i_{k-2} > \ldots,$$

and there exists $j_0 \in \mathbb{Z}$ with $i_j = j$ for $j \leq j_0$. Then the dense subspace spanned by these basis vectors carries a unitary highest weight representation of the Lie algebra $\mathfrak{gl}(\mathbb{Z},\mathbb{C})$ of finite $\mathbb{Z} \times \mathbb{Z}$-matrices which is one of the "infinite wedge representations" described in [KR87].

(c) For each $s \in \mathbb{R}$ the functional $s\varpi_J = s\,\mathrm{tr}$ is dominant integral and bounded. The corresponding representation of $\widetilde{\mathrm{GL}}_1(H)$ is given by the character

$$\det{}^s\colon \widetilde{\mathrm{GL}}_1(H) \to \mathbb{C}^\times, \quad g \mapsto \det(g)^s$$

(see the proof of Proposition IV.21).

Suppose that λ satisfies conditions (i) and (ii) in Theorem V.6 and put $s := \lambda_j$ for some $j \in J$. Then $\lambda - s\,\mathrm{tr}$ satisfies (i)–(iii), hence defines a holomorphic representation of $\mathrm{GL}_1(H)$. This shows that the representation $\pi_\lambda \otimes \det^{-s}$ of $\widetilde{\mathrm{GL}}_1(H)$ factors to $\mathrm{GL}_1(H)$. So apart from the real powers of the determinant the holomorphic highest weight representations of $\widetilde{\mathrm{GL}}_1(H)$ are

more or less the same than the holomorphic highest weight representations of $\mathrm{GL}_1(H)$.
(d) Theorem V.6 can also be used to obtain a classification of all holomorphic highest weight representations of $\widetilde{\mathrm{GL}}_1(H)$ in the same spirit as in the finite dimensional case. Suppose that λ satisfies (i) and (ii).

Pick $j_0 \in J$ and put $m := \min\{\lambda_j : j \in J\}$. If $M_k := \{j \in J : \lambda_j \geq m + k\}$, then $M_k \prec J \setminus M_k$ for all $k \in \mathbb{Z}$, and an elementary consideration leads to the representation

$$\lambda = m\varpi_J + \sum_{k=1}^{n} \varpi_{M_k}$$

of λ as a finite sum of fundamental weights, where $n > \max\{\lambda_j : j \in J\} - m$. We conclude in particular that $L(\lambda)$ and also $\mathcal{H}_\lambda$ can be realized in a finite tensor product of the Hilbert spaces corresponding to the fundamental weights. ■

Remark V.8. (a) The constructions in this section can also be carried out for the other three types of simple split locally finite Lie algebras, where the boundedness of λ leads to a holomorphic representation of the corresponding groups

$$\mathrm{Sp}_1(H, I) := \mathrm{Sp}(H, I) \cap \mathrm{GL}_1(H) \quad \text{and} \quad \mathrm{O}_1(H, I) := \mathrm{O}(H, I) \cap \mathrm{GL}_1(H)$$

(cf. [Ne98]).
(b) In [Ne98] one also finds a classification of *all* holomorphic representations of the groups $\mathrm{GL}_p(H)$, $p > 1$. These representations are direct sums of highest weight representations with finite highest weights. For $p = 1$ the situation is more complicated in three respects:

(1) First one has much more highest weight representations because the boundedness condition is much weaker than the condition that at most finitely many λ_j are non-zero.

(2) Second the global holomorphic representation theory of the group $\mathrm{GL}_1(H)$ is more complicated in the sense that it also has holomorphic factor representations of type II and III. These are not direct sums of irreducible representations.

(3) There are irreducible holomorphic representations which are not highest weight representations. This is discussed in the following example. ■

Example V.9. We consider the Lie algebra $\mathfrak{g} := \mathfrak{gl}(\mathbb{N}, \mathbb{C})$ as the union of the subalgebras $\mathfrak{g}_n := \mathfrak{gl}(2n, \mathbb{C})$, $n \in \mathbb{N}$, and fix the standard positive system $\Delta^+ := \{\varepsilon_j - \varepsilon_k : j < k\}$. For each $n \in \mathbb{N}$ we consider the dominant integral weight

$$\lambda_n := (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, \underbrace{-1, -1, \ldots, -1}_{n \text{ times}})$$

with respect to $\Delta_n^+ := \Delta_n \cap \Delta^+$ and $\Delta_n := \{\alpha \in \Delta : \mathfrak{g}^\alpha \subseteq \mathfrak{g}_n\}$. Then the set $\mathcal{P}_{\lambda_n}$ of weights of the highest weight module $L(\lambda_n, \mathfrak{g}_n)$ of $\mathfrak{g}_n$ is given by

$$\mathcal{P}_{\lambda_n} = \Big\{ \sum_{j=1}^{2n} a_j \varepsilon_j : a_j \in \{-1, 0, 1\}, \sum_{j=1}^{2n} a_j = 0 \Big\},$$

as follows easily from $\mathcal{P}_{\lambda_n} = \mathrm{conv}(\mathcal{W}_n.\lambda_n) \cap (\lambda_n + \mathbb{Z}[\Delta_n])$, where $\Delta_n \subseteq \mathfrak{h}_n^*$ denotes the roots of $\mathfrak{g}_n$. In particular each weight $\alpha \in \mathcal{P}_{\lambda_n}$ can be written as

$$\alpha = \sum_{j \in N_1} \varepsilon_j - \sum_{j \in N_2} \varepsilon_j, \quad \text{where} \quad |N_1| = |N_2| \leq n \quad \text{and} \quad N_1 \cap N_2 = \emptyset.$$

We see in particular that λ_{n-1} is contained in $\mathcal{P}_{\lambda_n}$, and that the corresponding weight space generates a $\mathfrak{g}_{n-1}$-submodule of highest weight λ_{n-1}. Using a fixed choice of embeddings

$$L(\lambda_n, \mathfrak{g}_n) \hookrightarrow L(\lambda_{n+1}, \mathfrak{g}_{n+1}), \qquad n \in \mathbb{N},$$

we obtain a simple weight module $V := \varinjlim L(\lambda_n, \mathfrak{g}_n)$ of $\mathfrak{g}$. The weight system of this module is given by

$$\mathcal{P}_V = \bigcup_{n \in \mathbb{N}} \mathcal{P}_{\lambda_n} = \Big\{ \sum_{j=1}^m a_j \varepsilon_j \colon m \in \mathbb{N}, a_j \in \{-1, 0, 1\}, \sum_{j=1}^m a_j = 0 \Big\}.$$

If $\alpha \in \mathcal{P}_V$ is an extreme point of $\mathrm{conv}(\mathcal{P}_V)$, then there exists an $n \in \mathbb{N}$ with $\alpha = \sum_{j=1}^{2n} a_j \varepsilon_j \in \mathcal{P}_{\lambda_n}$. Then $\alpha \in \mathrm{Ext}(\mathrm{conv}\, \mathcal{P}_{\lambda_n}) = \mathcal{W}_n.\lambda_n$. This means that $|\{j \colon a_j = 1\}| = n$. Then α is not extremal in $\mathrm{conv}(\mathcal{P}_{\lambda_{n+1}})$, hence not in $\mathrm{conv}(\mathcal{P}_V)$. This contradiction shows that the set $\mathrm{Ext}\big(\mathrm{conv}(\mathcal{P}_V)\big)$ of extreme points of $\mathrm{conv}(\mathcal{P}_V)$ is empty, and hence that V is not a highest weight module (cf. Lemma III.10).

The fact that all the highest weight modules $L(\lambda_n, \mathfrak{g}_n)$ are unitary implies that the embeddings $L(\lambda_n, \mathfrak{g}_n) \hookrightarrow L(\lambda_{n+1}, \mathfrak{g}_{n+1})$ can be turned into isometric embeddings, so that we obtain on V the structure of a unitary $\mathfrak{g}$-module.

As the set $\mathcal{P}_V$ is bounded in $l^\infty(\mathbb{N}, \mathbb{C})$, a similar argument as in the proof of Lemma V.4 shows that there exists a constant $C > 0$ with

$$\|\rho_V(X)\| \leq C \|X\|_1,$$

where $\|\cdot\|_1$ is the trace norm. Then ρ_V integrates to a holomorphic representation

$$\pi_V \colon \mathrm{GL}_1(H) \to \mathrm{GL}(\mathcal{H}_V),$$

where $\mathcal{H}_V$ is the completion of V with respect to the inner product. As the construction shows, the representation π_V is not a highest weight representation. ■

One should observe that our construction of representations always assumed a fixed choice of a splitting Cartan subalgebra. Although Cartan subalgebras of $\mathfrak{gl}(J, \mathbb{C})$ are conjugate under the group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$, not every such automorphism fixes the highest weight representations, i.e., induces an operator on the corresponding representation space. Therefore it is an interesting question how unitary highest weight representations with respect to one Cartan subalgebra $\mathfrak{h}$ behave with respect to another Cartan subalgebra $\widetilde{\mathfrak{h}}$. One possible strategy is to attach unitary highest weight representation to certain coadjoint orbits and then to study the geometry of these orbits and how bigger groups of automorphisms permute the orbits. Some results related to this approach will be discussed in the next sections.

Notes on Section V

Most of the material of this section has been adapted from [Ne98]. A discussion of the boundedness condition for highest weight representations of inductive limit groups can also be found in [NRW99, Prop. 3.14]. The arguments used there are quite different from ours. Also related is the approach of Neretin to realize the spin representation of the infinite-dimensional orthogonal group in a Fréchet space ([Ner87]).

The problem of integrating representations of infinite-dimensional Lie algebras to group representations becomes quite difficult if the Lie algebra acts by unbounded operators. Laredo has recently made significant progress on this problem ([Lar99]). The case of unitary highest weight representations of loop groups and the Virasoro group is due to Goodman and Wallach ([GW84] and [GW85]; see also [Se81]).

In the literature one finds many results on representations of the unitary groups $U_p(H)$. In [Se57] I. E. Segal studies unitary representations of the full group $U(H)$ called *physical representations* which are characterized by the condition that their differential maps finite rank hermitian projections to positive operators. Segal shows that physical representations decompose discretely into irreducible physical representations which are precisely those occuring in the decomposition of finite tensor products of the identity representation. Later A. A. Kirillov ([Ki73]) and also G. I. Ol'shanskiĭ ([Ol78, Th. 1.11]) proved that all strongly continuous representations of the Banach–Lie group $U_\infty(H)$, H separable, are type I, they even decompose as direct sums of irreducible representations.

We have mentioned in Remark V.8 that the group $\mathrm{GL}_1(H)$ has holomorphic representations of type II and III. The same is true for unitary strongly continuous representations of the group $U_2(H)$ ([Bo90]). In the same paper Boyer develops a Borel–Weil theory for the (linear) coadjoint orbits of the group $U_2(H)$, which only leads to those highest weight representations where the highest weight has finitely many non-zero entries.

Highest weight representations of particularly interest in physics are the spin representations of the group $\mathrm{O}_1(H)$. For a detailed discussion of these representations we refer to [CP89], [PS86], [Ot95] and [Ner96].

Exercises for Section V

Exercise V.1. Show that for a split Lie algebra $(\mathfrak{g}, \mathfrak{h})$ there exists an action of the Weyl group $\mathcal{W}$ on $\mathfrak{h}$ satisfying $r_\alpha(x) = x - \alpha(x)\check{\alpha}$ for all $\alpha \in \Delta_i$. Hint: Consider $\mathfrak{h}$ as a subspace of $(\mathfrak{h}^*)^*$. ■

Exercise V.2. Show that for a locally finite split simple Lie algebra $\mathfrak{g}$ the set Δ contains at most two $\mathcal{W}$-orbits. Hint: Assume that $\mathcal{W}$ does not act transitively and pick two non-conjugate roots $\{\alpha_1, \alpha_2\}$. Then for each $\alpha \in \Delta$ there exists a finite-dimensional subalgebra $\mathfrak{g}_0$ with $\{\alpha_1, \alpha_2, \alpha\} \subseteq \Delta_0$. Now use the finite-dimensional result. ■

Exercise V.3. Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite split Lie algebra with $\Delta = \Delta_k$. Show that for each element $Y = Y^* \in \mathfrak{g}$ there exists an element $X = -X^* \in \mathfrak{g}$ with $e^{\operatorname{ad} X}.Y \in \mathfrak{h}$. Hint: Choose a finite-dimensional $*$-invariant split subalgebra containing X and Y and then argue with the corresponding result for compact Lie algebras. ■

VI. Geometry of coadjoint orbits of Banach–Lie groups

In the preceding section we have seen that for each bounded highest weight λ, for which $L(\lambda)$ is unitary, we obtain a holomorphic highest weight representation of $G_1 := \mathrm{GL}_1(H)$ on the corresponding Hilbert space $\mathcal{H}_\lambda$. A closer inspection of this situation would show that if one considers the constructions of Section I for complex homogeneous space of the type G_1/P_1, then one would obtain a realization of $\mathcal{H}_\lambda$ by a holomorphic section in a line bundle over a space which is not a Hilbert manifold and therefore cannot be a strong Kähler manifold. Motivated by this observation, we now take a closer look at the geometry of coadjoint orbits of Banach–Lie groups to find the appropriate Kähler manifolds. For our guiding example $\mathfrak{gl}(J, \mathbb{C})$, this will lead to rather complete information in the case $G = U_2(H) = U(H) \cap (\mathbf{1} + B_2(H))$.

Let G be a real connected Banach–Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the topological dual space consisting of the continuous linear functionals on $\mathfrak{g}$. As for finite-dimensional groups, the *coadjoint representation*

$$\mathrm{Ad}^*\colon G \to \mathrm{GL}(\mathfrak{g}^*), \quad \mathrm{Ad}^*(g).\beta := \beta \circ \mathrm{Ad}(g)^{-1}$$

of G plays a crucial role in the process of obtaining natural realizations of representations of G. For finite-dimensional groups, coadjoint orbits

$$\mathcal{O}_f := \mathrm{Ad}^*(G).f$$

always carry a natural manifold structure by identifying them with the homogeneous space G/G_f, where $G_f := \{g \in G\colon \mathrm{Ad}^*(g).f = f\}$ is the stabilizer of f in G. Unfortunately, for an infinite-dimensional group G, the topological space G/G_f need not have a natural manifold structure. This problem suggests that in many cases in which G/G_f does not exist as a manifold its geometry should be reflected by geometric objects directly related to G. In fact, this point of view turns out to be quite successful in many respects.

In this spirit a G-invariant symplectic structure on G/G_f should correspond to a left invariant closed 2-form on G which is degenerate along the cosets of G_f. So we first take a closed look at left-invariant 2-forms on G.

The tangent bundle TG of G is trivial, and a convenient trivialization is given by the map

$$\Psi\colon G \times \mathfrak{g} \to TG, \quad \Psi(g)(X) := d\lambda_g(\mathbf{1}).X,$$

where $\lambda_g(x) = gx$ is the left multiplication on G. We therefore obtain a bijection $\omega \mapsto \omega_l$ assigning to each continuous alternating p-form $\omega \in \mathrm{Alt}^p(\mathfrak{g}; \mathbb{R})$ a left-invariant differential p-form ω_l on G given by

$$\omega_l(g)(d\lambda_g(\mathbf{1}).X_1, \ldots, d\lambda_g(\mathbf{1}).X_p) = \omega(X_1, \ldots, X_p), \quad g \in G, X_1, \ldots, X_p \in \mathfrak{g}.$$

Lemma VI.1. *Let* $\omega \in \mathrm{Alt}^p(\mathfrak{g}, \mathbb{R})$. *The left-invariant* p*-form* ω_l *on* G *is closed if and only if*

$$\sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{p+1}) = 0$$

holds for all $X_1, \ldots, X_{p+1} \in \mathfrak{g}$.

Proof. For $X \in \mathfrak{g}$ we write $X_l \in \mathcal{V}(G)$ for the corresponding left invariant vector field on G, i.e., $X_l(g) = d\lambda_g(\mathbf{1}).X$. A left invariant p-form ω_l on G is closed if and only if $d\omega_l$ vanishes on $p+1$-tuples of left invariant vector fields. Moreover, $d\omega_l((X_1)_l, \ldots, (X_{p+1})_l)$ is a constant function. To calculate its value in the identity, we observe that each function

$$\omega_l\big((X_1)_l, \ldots, \widehat{(X_i)_l}, \ldots, (X_{p+1})_l\big)$$

is constant, so that

$$\begin{aligned}
&\Big(d\omega_l((X_1)_l, \ldots, (X_{p+1})_l)\Big)(\mathbf{1}) \\
&= \sum_{i=1}^{p+1}(-1)^i (X_i)_l.\omega_l((X_1)_l, \ldots, \widehat{(X_i)_l}, \ldots, (X_{p+1})_l)(\mathbf{1}) \\
&\quad + \sum_{i<j}(-1)^{i+j}\omega_l([(X_i)_l, (X_j)_l], (X_1)_l, \ldots, \widehat{(X_i)_l}, \ldots, \widehat{(X_j)_l}, \ldots, (X_{p+1})_l)(\mathbf{1}) \\
&= \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{p+1}).
\end{aligned}$$

■

Definition VI.2. We conclude in particular that a bilinear form $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defines a closed left invariant 2-form $\Omega := \omega_l$ on G if and only if ω is a *Lie algebra* 2*-cocycle*, i.e.,

$$\omega(x, [y, z]) + \omega(y, [z, x]) + \omega(z, [x, y]) = 0, \quad x, y, z \in \mathfrak{g}.$$

We write $Z^2_c(\mathfrak{g}, \mathbb{R})$ for the space of continuous real-valued 2-cocycles on $\mathfrak{g}$.

Let $\Omega \in \Omega^2(G)$ be a left invariant closed 2-form defined by $\omega \in Z^2_c(\mathfrak{g}, \mathbb{R})$. Then there exists a left invariant 1-form β with $d\beta = \Omega$ if and only if there exists a continuous linear functional $f \in \mathfrak{g}^*$ with $\omega(x, y) = f([x, y])$ for $x, y \in \mathfrak{g}$, i.e., the 2-cocycle ω is a 2*-coboundary.* The space of continuous 2-coboundaries is denoted $B^2_c(\mathfrak{g}, \mathbb{R})$, and the quotient space

$$H^2_c(\mathfrak{g}, \mathbb{R}) := Z^2_c(\mathfrak{g}, \mathbb{R})/B^2_c(\mathfrak{g}, \mathbb{R})$$

is the *second continuous real-valued Lie algebra cohomology of* $\mathfrak{g}$. ■

Below we will discuss modifications of the coadjoint action to certain affine actions, so we first have a closer look at the affine group of a Banach space.

Definition VI.3. (a) Let V be a Banach space. We consider the *affine group* $\mathrm{Aff}(V) \cong V \rtimes \mathrm{GL}(V)$ which acts on V by $(x,g).v = g.v + x$. On the space $\widetilde{V} := V \times \mathbb{R}$ the group $\mathrm{Aff}(V)$ acts by linear maps $(x,g).(v,z) := (g.v + zx, z)$, and we thus obtain a realization of $\mathrm{Aff}(V)$ as a linear Lie subgroup of $\mathrm{GL}(\widetilde{V})$. The corresponding Lie algebra is $\mathfrak{aff}(V) \cong V \rtimes \mathfrak{gl}(V)$ with the bracket

$$[(v,A),(v',A')] = (A.v' - A'.v, [A,A']).$$

(b) A homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{aff}(V)$ is therefore given by a pair (ρ_l, θ) consisting of a linear representation $\rho_l\colon \mathfrak{g} \to \mathfrak{gl}(V)$ and map $\theta\colon \mathfrak{g} \to V$ satisfying

$$\theta([x,y]) = \rho_l(x).\theta(y) - \rho_l(y).\theta(x), \quad \text{for} \quad x,y \in \mathfrak{g}. \tag{6.1}$$

A map $\theta\colon \mathfrak{g} \to V$ satisfying (6.1) is called a 1-*cocycle* with values in the $\mathfrak{g}$-representation (ρ_l, V).

(c) On the group level a homomorphism $\rho\colon G \to \mathrm{Aff}(V)$ is given by a pair (ρ_l, Θ) of a linear representation $\rho_l\colon G \to \mathrm{GL}(V)$ and map $\Theta\colon G \to V$ satisfying

$$\Theta(g_1 g_2) = \rho_l(g_1).\Theta(g_2) + \Theta(g_1) \quad \text{for} \quad g_1, g_2 \in G. \tag{6.2}$$

A map $\Theta\colon G \to V$ satisfying (6.2) is called a 1-*cocycle* with values in the G-representation (ρ_l, V). Typical examples of 1-cocycles are maps of the form

$$\Theta(g) := \rho_l(g).v - v, \quad v \in V.$$

These cocycles are called trivial (coboundaries). ∎

Lemma VI.4. *Let $\omega \in Z^2(\mathfrak{g}, \mathbb{R})$ be a continuous 2-cocycle. Then $\theta(x)(y) := \omega(x,y)$ is a 1-cocycle with values in the coadjoint representation $(\mathrm{ad}^*, \mathfrak{g}^*)$, where $\mathrm{ad}^*(x).\beta = -\beta \circ \mathrm{ad}\, x$. If G is simply connected, then there exists a unique affine representation*

$$\mathrm{Ad}^*_\omega\colon G \to \mathrm{Aff}(\mathfrak{g}^*), \quad \mathrm{Ad}^*_\omega(g) = (\Theta(g), \mathrm{Ad}^*(g))$$

with $d\Theta(\mathbf{1}) = \theta$.

Proof. That θ is a 1-cocycle with values in the coadjoint representation $(\mathrm{ad}^*, \mathfrak{g}^*)$ follows from

$$\begin{aligned}\theta([x,y])(z) &= \omega([x,y],z) = -\omega(y,[x,z]) + \omega(x,[y,z]) \\ &= (\mathrm{ad}^*(x).\theta(y))(z) - (\mathrm{ad}^*(y).\theta(x))(z).\end{aligned}$$

We therefore obtain an affine representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ given by

$$\mathrm{ad}^*_\omega\colon \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g}^*), \quad \mathrm{ad}^*_\omega(x) = (\theta(x), \mathrm{ad}^*(x)).$$

If G is simply connected, then Theorem IV.2(c) implies that the affine representation ad^*_ω integrates to an affine representation

$$\mathrm{Ad}^*_\omega\colon G \to \mathrm{Aff}(\mathfrak{g}^*), \quad \mathrm{Ad}^*_\omega(g) = (\Theta(g), \mathrm{Ad}^*(g)),$$

where $\Theta\colon G \to \mathfrak{g}^*$ is a smooth 1-cocycle for G with values in the coadjoint representation. The group cocycle is related to the Lie algebra cocycle θ by $d\Theta(\mathbf{1}) = \theta$ and the uniqueness of Θ follows from the uniqueness assertion in Theorem IV.2(c). ∎

The affine actions of G on $\mathfrak{g}^*$ obtained by this process are generalizations of the coadjoint action. The action Ad^*_ω of G on $\mathfrak{g}^*$ is equivalent to a linear representation if and only if it has a fixed point $f \in \mathfrak{g}^*$. This means that $\mathrm{ad}^*_\omega(\mathfrak{g}).f = \{0\}$, i.e., $\omega(x, \cdot) = f \circ \mathrm{ad}\, x$ for all $x \in \mathfrak{g}$, which in turn means that $\omega(x, y) = f([x, y])$, i.e., ω is a coboundary.

Remark VI.5. (a) The cocycle $\omega \in Z^2(\mathfrak{g}, \mathbb{R})$ defines a Lie algebra structure on $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{R}$ by

$$[(x, z), (x', z')] = ([x, x'], \omega(x, x')),$$

which is a central extension of $\mathfrak{g}$ by $\mathbb{R}$. The affine action Ad^*_ω of G on $\mathfrak{g}^*$ corresponds to a linear action on $\widehat{\mathfrak{g}}^*$ given by

$$\mathrm{Ad}^*_{\widehat{\mathfrak{g}}}(g).(\alpha, \lambda) = (\mathrm{Ad}^*(g).\alpha + \lambda\Theta(g), \lambda).$$

This means that the affine action of G on $\mathfrak{g}^*$ is equivalent to a linear action on the affine hyperplane $\mathfrak{g}^* \times \{1\}$ in $\widehat{\mathfrak{g}}^*$.

(b) The main reason for preferring the affine action is that to understand the action on $\widehat{\mathfrak{g}}^*$ in a proper sense as a coadjoint action, we would need a group $\widehat{G}$ with $\mathbf{L}(\widehat{G}) = \widehat{\mathfrak{g}}$, but such groups need not exist (see the example below). On the other hand, one knows that in all cases the action of the simply connected group with Lie algebra $\mathfrak{g}$ has a natural linear action on $\widehat{\mathfrak{g}}$, even if G is not a Banach–Lie group (cf. [Ne00c]).

One of the most simple examples of a Banach–Lie algebra for which no corresponding group exists is the quotient

$$\mathfrak{g} := \big(\mathfrak{u}(H) \oplus \mathfrak{u}(H)\big)/\mathfrak{z},$$

where $\mathfrak{z} := \mathbb{R}(i\mathbf{1}, \sqrt{2}i\mathbf{1})$ and H is an infinite-dimensional complex Hilbert space (cf. [EK64]). ■

Now we turn to the geometric structure of orbits of the action Ad^*_ω. The following theorem generalized the observation of Kirillov, Kostant and Souriau, that every coadjoint orbit of a finite-dimensional Lie group carries a natural invariant symplectic structure.

We recall that a *weakly symplectic manifold* is a pair (M, Ω) of a manifold M and a non-degenerate closed 2-form Ω. It is called *strongly symplectic* if for each $p \in M$ the injective map

$$T_p(M) \to T_p(M)^*, \quad v \mapsto \Omega_p(v, \cdot)$$

is surjective. Note that each finite-dimensional weakly symplectic manifold is also strongly symplectic (cf. Exercise VI.1).

Theorem VI.6. *Suppose that the affine representation Ad^*_ω of G on $\mathfrak{g}^*$ exists. Let $f \in \mathfrak{g}^*$ and $\omega \in Z^2_c(\mathfrak{g}, \mathbb{R})$. If the stabilizer G_f is complemented, then the orbit $\mathcal{O}_f = \mathrm{Ad}^*_\omega(G).f$ carries the structure of a weakly symplectic manifold.*

Proof. Let $f \in \mathfrak{g}^*$. Then the stabilizer $\mathfrak{g}_f$ of f is given by

$$\begin{aligned}\mathfrak{g}_f &= \{x \in \mathfrak{g} \colon \mathrm{ad}_\omega^*(x).f = 0\} = \{x \in \mathfrak{g} \colon \theta(x) = f \circ \mathrm{ad}\, x\} \\ &= \{x \in \mathfrak{g} \colon \omega(x, \cdot) - f([x, \cdot]) = 0\}.\end{aligned} \tag{6.3}$$

The corresponding subgroup $G_f \subseteq G$ is a Lie subgroup (Exercise IV.3). Let us assume, in addition, that G_f is complemented (which is the case if G is a Hilbert–Lie group). Then we identify $\mathcal{O}_f$ with G/G_f and obtain on $\mathcal{O}_f$ a natural manifold structure such that G acts smoothly and transitively on $\mathcal{O}_f$. The tangent space $T_f(\mathcal{O}_f)$ can be identified with $\mathfrak{g}/\mathfrak{g}_f$ on which we have the skew-symmetric bilinear form given by

$$\Omega_f(\mathrm{ad}_\omega^*(x).f, \mathrm{ad}_\omega^*(y).f) := \omega(x, y) - f([x, y])$$

which is well defined, non-degenerate (see (6.3)) and G_f-invariant (Exercise!). Hence there exists a G-invariant 2-form Ω on $\mathcal{O}_f$ which coincides with this form on $T_f(\mathcal{O}_f)$. Let $\pi \colon G \to \mathcal{O}_f \cong G/G_f$ denote the orbit map. Then $\pi^*\Omega$ is the left invariant 2-form on G which in $\mathbf{1}$ coincides with $\omega + df \in Z_c^2(\mathfrak{g}, \mathbb{R})$, where $df(x, y) := f([y, x])$. This implies that

$$d\pi^*\Omega = \pi^* d\Omega = 0,$$

showing that $d\Omega = 0$, i.e., Ω is closed. ■

Remark VI.7. If $\mathfrak{g}$ is topologically isomorphic to a Hilbert space, then the assumption of Theorem VI.6 is automatically satisfied. ■

Remark VI.8. Let $\omega \in Z_c^2(\mathfrak{g}, \mathbb{R})$ and Ad_ω^* be as above. For $\alpha \in \mathfrak{g}^*$ we consider the equivalent cocycle

$$\widetilde{\omega} := \omega + d\alpha, \quad \widetilde{\omega}(x, y) = \omega(x, y) - \alpha([x, y]).$$

Then $\widetilde{\theta}(x) = \theta(x) + \mathrm{ad}^*(x).\alpha$, and therefore

$$\widetilde{\Theta}(g) = \Theta(g) + \mathrm{Ad}^*(g).\alpha - \alpha.$$

This implies that the translation map $\tau_\alpha \colon \mathfrak{g}^* \to \mathfrak{g}^*, \beta \mapsto \beta + \alpha$ satisfies

$$\mathrm{Ad}_\omega^*(g) \circ \tau_\alpha = \tau_\alpha \circ \mathrm{Ad}_{\widetilde{\omega}}^*(g). \tag{6.4}$$

Therefore the two affine actions Ad_ω^* and $\mathrm{Ad}_{\widetilde{\omega}}^*$ are equivalent. For the corresponding orbits this means that

$$\mathcal{O}_\beta = \widetilde{\mathcal{O}}_{\beta - \alpha} + \alpha,$$

and one easily checks that this isomorphism preserves the symplectic structure, so that it suffices to study the orbits of the type

$$\mathcal{O}_\omega := \mathcal{O}_0 := \mathrm{Ad}_\omega^*(G).0 = \Theta(G) \subseteq \mathfrak{g}^*.$$

If, in addition, α vanishes on the commutator algebra, then $d\alpha = 0$, so that $\widetilde{\omega} = \omega$. In this case (6.4) means that τ_α commutes with the affine action Ad_ω^*. ■

Complex structures on homogeneous spaces

The most direct way to obtain complex structures on homogeneous spaces is to realize them as open submanifolds of complex manifolds. To see how to find such embeddings systematically, let us assume that $M = G/H$ is a homogeneous space of a Banach–Lie group G for which H is a complemented Lie subgroup, so that M carries a natural real manifold structure.

Remark VI.8. Let us assume that M also carries an invariant complex structure, i.e., M is a complex manifold such that G acts by biholomorphic mappings. Then the natural Lie algebra homomorphism $\dot\sigma\colon \mathfrak{g} \to \mathcal{V}(M)$ given by

$$\dot\sigma(X)(p) := \frac{d}{dt}\Big|_{t=0} \exp(-tX).p$$

extends to a complex linear homomorphism $\dot\sigma_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathcal{V}_{\mathrm{hol}}(M)$, where $\mathcal{V}_{\mathrm{hol}}(M)$ denote the Lie algebra of holomorphic vector fields on M. Let

$$\mathfrak{p} := \{X \in \mathfrak{g}_{\mathbb{C}} \colon \dot\sigma_{\mathbb{C}}(X)(x_0) = 0\},$$

where $x_0 = \mathbf{1}H \in M$ is the base point. We write $X \mapsto \overline{X}$ for the complex conjugation on $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{p}$ has the following properties:

(C1) $\mathfrak{p}$ is a closed $\mathrm{Ad}(H)$-invariant subalgebra of the Banach–Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

(C2) $\mathfrak{p} \cap \overline{\mathfrak{p}} = \mathfrak{h}_{\mathbb{C}}$, and

(C3) $\mathfrak{p} + \overline{\mathfrak{p}} = \mathfrak{g}_{\mathbb{C}}$.

(C1) follows from the relation

$$\dot\sigma_{\mathbb{C}}(\mathrm{Ad}(h).X)(x_0) = \pi_{\mathrm{iso}}(h).\dot\sigma_{\mathbb{C}}(X)(x_0),$$

where $\pi_{\mathrm{iso}}\colon H \to \mathrm{GL}(T_{x_0}(M))$ is the isotropy representation of H in x_0. To verify (C2), we observe that the complex Lie algebra $\mathfrak{p} \cap \overline{\mathfrak{p}}$ is conjugation-invariant, hence satisfies

$$\mathfrak{p} \cap \overline{\mathfrak{p}} = (\mathfrak{p} \cap \overline{\mathfrak{p}} \cap \mathfrak{g})_{\mathbb{C}} = (\mathfrak{p} \cap \mathfrak{g})_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}.$$

For condition (C3) we note that $M = G/H$ implies that

$$T_{x_0}(M) = \dot\sigma(\mathfrak{g})(x_0),$$

so that $\mathfrak{g}_{\mathbb{C}} \subseteq \mathfrak{g} + \mathfrak{p}$. This in turn means that

$$\{X - \overline{X} \colon X \in \mathfrak{g}_{\mathbb{C}}\} = \{X - \overline{X} \colon X \in \mathfrak{p}\} = \{X - \overline{X} \colon X \in \mathfrak{p} + \overline{\mathfrak{p}}\},$$

so that (C3) follows because $\mathfrak{p} \cap \overline{\mathfrak{p}}$ is the complexification of the subspace of the purely imaginary elements it contains.

In the finite-dimensional case one can show that a subalgebra $\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying (1)–(3) is already enough to obtain on $M = G/H$ an invariant complex structure (cf. [Ki76, p.203]). We do not expect that (C1)–(C3) would be enough in the infinite-dimensional case. For the arguments in [Ki76] to work one needs at least the additional assumption that $\mathfrak{p}$ is complemented in $\mathfrak{g}$, and this should be enough. Since we will need complex structures only in quite specific situations, let us formulate a sufficient set of conditions for the existence of a complex structure.

Proposition VI.9. *We assume that G is contained as a Lie subgroup in a complex Banach–Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on which we have an antiholomorphic automorphism σ such that*

$$\mathbf{L}(\sigma)(X) = \overline{X}, \quad X \in \mathfrak{g}_{\mathbb{C}}.$$

Let H be a complemented Lie subgroup of the connected Lie group G, $\mathfrak{g} := \mathbf{L}(G)$, $\mathfrak{h} := \mathbf{L}(H)$, and $\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$ a closed complex subalgebra for which the following assertions hold:

(1) $P := \langle \exp \mathfrak{p} \rangle$ *is a complemented Lie subgroup of $G_{\mathbb{C}}$.*

(2) $P \cap G = H$.

(3) $\mathfrak{p} + \mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$.

Then the orbit mapping $G \to G_{\mathbb{C}}/P, g \mapsto gP$, induces an open embedding of G/H as an open G-orbit in the complex manifold $G_{\mathbb{C}}/P$.

Proof. We consider the orbit map $\eta\colon G \to G_{\mathbb{C}}/P, g \mapsto g.x_0$, where $x_0 = \mathbf{1}P$ is the base point in $G_{\mathbb{C}}/P$. This is a smooth map which is constant on the H-left cosets gH in G, hence factors to a smooth map $\overline{\eta}\colon M \to G_{\mathbb{C}}/P$ which is injective because of (2). Its differential in x_0 corresponds to the canonical map

$$\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$$

which, in view of (2), is injective, and, according to (3), is surjective. Therefore the Inverse Function Theorem shows that $\overline{\eta}$ is a local diffeomorphism in x_0. Since $\overline{\eta}$ is also G-equivariant, it follows that η is an open embedding of manifolds. ■

The assumption (2) in Proposition VI.9 implies $\mathfrak{p} \cap \mathfrak{g} = \mathfrak{h}$ which is equivalent to (C2), and (3) is easily seen to be equivalent to (C3).

Complex structures on coadjoint orbits

Now we turn more specifically to coadjoint orbits in the sense of Theorem VI.6. So we consider a homogeneous space $M = G/H$ which is a coadjoint orbit of the type $\mathcal{O}_\omega$ considered in Theorem VI.6. Then we want, in addition, that the complex structure I (viewed as the multiplication by i in each tangent space), preserves the symplectic form. Taking the homogeneity of M into account, it suffices to verify this condition in the base point $x_0 = 0$. The tangent space $T_{x_0}(M) = T_0(\mathcal{O}_\omega)$ can naturally be identified with $\mathfrak{g}/\mathfrak{h}$ by the map

$$\mathfrak{g}/\mathfrak{h} \to T_0(\mathcal{O}_\omega), \quad X + \mathfrak{h} \mapsto \mathrm{ad}^*_\omega(X).0 = \theta(X).$$

Let us write $Z^* := -\overline{Z}$ for $Z \in \mathfrak{g}_{\mathbb{C}}$. In view of $\mathfrak{p} + \overline{\mathfrak{p}} = \mathfrak{g}_{\mathbb{C}}$, we may write each element $X \in \mathfrak{g}$ as $Z - Z^*$, $Z \in \overline{\mathfrak{p}}$ (the map $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}$, $Z \mapsto Z - Z^*$ is surjective and $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ have the same image). Suppose that M carries a complex structure defined by $\mathfrak{p}$. Then $T_{x_0}(M) \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$ describes the complex structure on the tangent space. We write $\theta\colon \mathfrak{g}_{\mathbb{C}} \to T_0(\mathcal{O}_\omega)$ for the complex linear extension of

$\mathfrak{g}$ with respect to the complex structure on $T_0(\mathcal{O}_\omega)$. Writing X as $Z - Z^*$ for $Z \in \overline{\mathfrak{p}}$, we obtain for the multiplication with i on $T_0(\mathcal{O}_\omega)$ the formula

$$I.\theta(X) = I.\theta(Z - Z^*) = I.\theta(Z) = \theta(iZ) = \theta(i(Z + Z^*))$$

with $i(Z + Z^*) \in \mathfrak{u}$. Now

$$\begin{aligned} &\Omega_0(\theta(Z - Z^*), I.\theta(W - W^*)) \\ &= \omega(Z - Z^*, i(W + W^*)) \\ &= \omega(iZ, W^*) + \omega(iW, Z^*) + \omega(iZ, W) + \omega(-iZ^*, W^*), \\ &= \underbrace{\omega(iZ, W^*) + \omega(iW, Z^*)}_{\text{symmetric}} + \underbrace{2\,\mathrm{Re}\,\big(\omega(iZ, W)\big)}_{\text{skew}-\text{symmetric}}, \end{aligned}$$

so that the requirement that this form is symmetric means that $\mathrm{Re}\,\omega$ vanishes on $\overline{\mathfrak{p}} \times \overline{\mathfrak{p}}$, which is the same as
(C4) $\omega(\mathfrak{p} \times \mathfrak{p}) = \{0\}$.

If $\mathfrak{p}$ satisfies (C1)–(C4), we call it a *complex polarization in* ω. Our calculation above has shown that this condition means that if a complex structure is obtained from Proposition VI.9, then (C4) guarantees that the complex structure is compatible with the symplectic structure in the sense that multiplication by I is a symplectic isomorphism in each tangent space. If (M, Ω) is a weakly symplectic manifold endowed with a complex manifold structure for which I satisfies this condition, we call the triple (M, Ω, I) a *pseudo-Kähler manifold.*

We call it a *Kähler manifold* if, in addition, we have $0 < \Omega_0(v, I.v)$ for $0 \neq v$. For a complex polarization this means that for $Z \in \mathfrak{p} \setminus \mathfrak{h}_{\mathbb{C}}$ we have

$$0 < \Omega_0(\mathrm{ad}_\omega^*(Z - Z^*).\beta, I\,\mathrm{ad}_\omega^*(Z - Z^*).\beta) = \omega(Z - Z^*, -iZ - iZ^*) = -2i\omega(Z, Z^*).$$

So we formulate an additional condition on $\mathfrak{p}$:
(C5) For all $Z \in \mathfrak{p} \setminus \mathfrak{h}_{\mathbb{C}}$ we have $-i\omega(Z, Z^*) > 0$.

Notes on Section VI

For finite-dimensional split involutive Lie algebras Kähler structures on coadjoint orbits have been studied in some detail [Ne95a]. For finite-dimensional Lie algebras coadjoint orbits of the highest weights of unitary highest weight representations are always coadjoint Kähler orbits ([Ne95b]). For further material on Kähler orbits for compact groups, we refer to [GS84]. A detailed analysis of homogeneous Kähler manifolds is undertaken in [DoNa88] by Dorfmeister and Nakajima, where they prove the Fundamental Conjecture for Homogeneous Kähler Manifolds which essentially leads to a classification of all homogeneous Kähler manifolds. All infinite-dimensional homogeneous strongly Kähler manifolds we are aware of have the same fibration structure given in the classification for the finite-dimensional case. It seems that the geometry becomes much less controllable for weakly Kähler manifolds.

Exercises for Section VI

Exercise VI.1. We consider the Banach space $E := l^1(\mathbb{Z} \setminus \{0\}, \mathbb{R})$ with the continuous alternating bilinear form

$$\omega(x, y) := \sum_{j>0} x_j y_{-j} - x_{-j} y_j.$$

We then define a constant 2-form Ω on E by $\Omega_p := \omega$ for all $p \in E$. Show that (E, Ω) is a weakly symplectic manifold which is not strongly symplectic. Hint: The image of the natural map $E \to E^* \cong l^\infty(\mathbb{Z} \setminus \{0\}, \mathbb{R})$ is the subspace $l^1(\mathbb{Z} \setminus \{0\}, \mathbb{R})$. ■

Exercise VI.2. It is instructive to visualize the constructions in this section for the case of abelian Lie algebras. Let $\mathfrak{g}$ be an abelian Banach–Lie algebra which we also consider as a group $G = \mathfrak{g}$ with $\exp = \mathrm{id}$.

(a) $Z^2_c(\mathfrak{g}, \mathbb{R}) = \mathrm{Alt}^2(\mathfrak{g}; \mathbb{R})$ is the space of continuous alternating bilinear forms ω on $\mathfrak{g}$.

(b) The affine action of G on $\mathfrak{g}^*$ corresponding to ω is given by $\mathrm{Ad}^*_\omega(x).\beta = \beta + \omega(x, \cdot)$. Its orbits are affine subspaces of $\mathfrak{g}^*$.

(c) Suppose that ω is non-degenerate and that $\mathfrak{g}$ has a complex structure I for which there exists a real subspace $\mathfrak{n} \subseteq \mathfrak{g}$ satisfying:

  (1) $\mathfrak{g} = \mathfrak{n} \oplus I\mathfrak{n}$.

  (2) ω vanishes on $\mathfrak{n} \times \mathfrak{n}$.

  (3) ω is I-invariant.

  Show that the complex subspace

$$\mathfrak{p} := \{v - iIv : v \in \mathfrak{n}\} \subseteq \mathfrak{g}_{\mathbb{C}}$$

is a complex polarization in ω which is complemented. When is it a Kähler polarization? ■

Exercise VI.3. Let (V, Ω) be a symplectic vector space, i.e., $\Omega: V \times V \to \mathbb{R}$ is a non-degenerate alternating bilinear form. For a complex structure I on V the following are equivalent:

(a) Ω is I-invariant, i.e., $I \in \mathrm{Sp}(V, \Omega)$.

(b) $\Omega(v, I.w) = \Omega(w, I.v)$ for $v, w \in V$.

(c) The complex bilinear extension $\Omega: V_{\mathbb{C}} \times V_{\mathbb{C}} \to V$ of Ω satisfies $\Omega(v, w) = 0$ for $v, w \in V^+ := \{x - iIx : x \in V\}$. ■

VII. Coadjoint orbits and complex line bundles for $U_2(H)$

In this section we complete the picture for the special case of the group $U := U_2(H)$. This means that we will describe Kähler coadjoint orbits of this group and realize all unitary highest weight representations $L(\lambda)$ of $\mathfrak{gl}(J, \mathbb{C})$ (for $H = l^2(J, \mathbb{C})$) with bounded λ in a space of holomorphic sections of a complex line bundle over such orbits. This picture will show in particular that the group $\mathrm{GL}_1(H)$ acting on the Hilbert space $\mathcal{H}_\lambda$ is far from being maximal.

Although the material in this section is formulated, for simplicity, only for the group $U_2(H)$, it works in the more general setting of L^*-algebras.

Coadjoint Kähler orbits for $\mathfrak{u}_2(H)$

To fix the notation, we write

$$U := U_2(H), \quad G := \mathrm{GL}_2(H), \quad \mathfrak{u} = \mathbf{L}(U) = \mathfrak{u}_2(H), \quad \mathfrak{g} = \mathbf{L}(G) = B_2(H).$$

We also identify $\mathfrak{u}$ with $\mathfrak{u}^*$ using the trace form $(x, y) := \mathrm{tr}(xy)$. Then the coadjoint action is given by $\mathrm{Ad}^*(g).x = \mathrm{Ad}(g).x = gxg^{-1}$. As we have seen in Section VI, to understand the affine coadjoint actions from a higher viewpoint, we first have to describe the space $Z^2_c(\mathfrak{u}, \mathbb{R})$ or real-valued 2-cocycles.

Lemma VII.1. *Every continuous cocycle $\omega \in Z^2_c(\mathfrak{u}, \mathbb{R})$ can be written as $\omega(x, y) = \mathrm{tr}(A[x, y])$ for some $A \in B(H)$ with $A^* = -A$.*

Proof. In [dlH72, Prop. II.9] it is shown that the complex bilinear extension $\omega_{\mathbb{C}} \in Z^2_c(\mathfrak{g}, \mathbb{C})$ can be written as $\omega_{\mathbb{C}}(x, y) = \mathrm{tr}(B[x, y])$ for some $B \in B(H)$. For $C^* = C$ and $x \in \mathfrak{u}$ we have $\mathrm{tr}(Cx) \in i\mathbb{R}$, so that $\omega(x, y) \in \mathbb{R}$ for $x, y \in \mathfrak{u}$ implies that $\mathrm{tr}((B + B^*)[x, y]) = 0$ for all $x, y \in \mathfrak{u}$. Hence we obtain with $A := \frac{1}{2}(B - B^*)$ the relation $\omega(x, y) = \mathrm{tr}(A[x, y])$. ■

In the following we assume that $\omega \in Z^2_c(\mathfrak{u}, \mathbb{R})$ is given by $A = -A^* \in B(H)$ as in Lemma VII.1 by

$$\omega(x, y) = \mathrm{tr}(A[x, y]) = \mathrm{tr}([A, x]y).$$

This means that the corresponding Lie algebra cocycle $\theta \colon \mathfrak{u} \to \mathfrak{u}$ is given by

$$\theta(x) = [A, x].$$

Note that $\mathrm{ad}\, A \colon B_2(H) \to B_2(H)$ is a continuous map and that $B_2(H)$ is an ideal in $B(H)$. Having such a concrete formula for the cocycle, it is easy to describe the corresponding group cocycle which exists although the group U is not simply connected (in view of Proposition A.4 and Theorems A.10/11 we have $\pi_1(U) \cong \mathbb{Z}$):

$$\Theta: U \to \mathfrak{u}, \quad \Theta(g) = gAg^{-1} - A.$$

Note that for $g \in U = U_2(H)$ and $A \in B(H)$ we have

$$gAg^{-1} = (g - \mathbf{1})Ag^{-1} + A(g^{-1} - \mathbf{1}) \in B_2(H)$$

and that for $A^* = -A$ we also get $\Theta(g)^* = -\Theta(g)$.

Remark VII.2. In this case it would also be possible to work with a central extension of the group U. In view of Proposition A.4 and Theorems A.10/11, the group $\pi_2(U)$ is trivial, so that the results in [Ne00c] imply the existence of a central extension

$$\mathbb{R} \to \widehat{U} \to U$$

corresponding to the Lie algebra extension $\widehat{\mathfrak{u}} = \mathfrak{u} \oplus_\omega \mathbb{R}$ defined by $\omega \in Z^2_c(\mathfrak{u}, \mathbb{R})$ with the bracket

$$[(x, z), (x', z')] = ([x, x'], \omega(x, x')). \qquad \blacksquare$$

The next problem is to find the geometrically well behaved coadjoint orbits in $\mathfrak{u}$. As we have seen in Section VI, it suffices to consider the orbit

$$\mathcal{O}_\omega := \Theta(G) \subseteq \mathfrak{u}$$

of 0 (Remark VI.8). Since we are looking for strong Kähler orbits, a natural question is when these orbits are submanifolds of $\mathfrak{u}$. Infinitesimally this leads to the question when the tangent space $T_0(\mathcal{O}_\omega) = \theta(\mathfrak{u}) = [A, \mathfrak{u}]$ is a closed subspace of $\mathfrak{u}$.

For a normal operator $A \in B(H)$ we write $A = A_c \oplus A_d$ (continuous and discrete part of A) according to the orthogonal decomposition $H = H_c \oplus H_d$, where H_d is the Hilbert space direct sum of the eigenspaces of A, $H_c = H_d^\top$, $A_d = A|_{H_d}$, and $A_c = A|_{H_c}$.

Lemma VII.3. *For $A = -A^* \in B(H)$ the following assertions hold:*

(i) *If $X \in \mathfrak{u}$ commutes with A, then $XA_c = A_cX = 0$, and X vanishes on H_c. Moreover,* $\ker(\operatorname{ad} A) \subseteq B_2(H_d)$.

(ii) *The map $\operatorname{ad} A: \mathfrak{u} \to \mathfrak{u}$ has closed range if and only if A is diagonalizable with finite spectrum.*

Proof. (i) The stabilizer of $0 \in \mathcal{O}_\omega$ in $\mathfrak{u}$ is the centralizer of A in $\mathfrak{u}$. We recall that every operator $X \in \mathfrak{u}$ is compact and normal, hence diagonalizable and all eigenspaces corresponding to non-zero eigenvalues are finite-dimensional. If $X \in \mathfrak{u}$ commutes with A, then it preserves the eigenspaces of A, hence commutes with A_c and A_d. Therefore A_c preserves the finite-dimensional non-trivial eigenspaces of X on which it acts trivially because it does not have any eigenvector for a non-zero eigenvalue. Therefore $XA_c = A_cX = 0$. Since $A(H_c)$ is dense in H_c, it follows that $H_c \subseteq \ker X$ and $\operatorname{im} X \subseteq (\ker X)^\perp \subseteq H_d$, so

that we can identify X with an element of $B_2(H_d)$. In this sense we have $\ker(\operatorname{ad} A) \subseteq B_2(H_d)$.
(ii) We consider the hermitian operator $\operatorname{ad} A$ on the complex Hilbert space $\mathfrak{g} = B_2(H)$. Then $\operatorname{im}(\operatorname{ad} A) = [A, \mathfrak{g}]$ is the complexification of the space $[A, \mathfrak{u}]$, showing that this space is a closed subspace of $\mathfrak{u}$ if and only if $\operatorname{ad} A$ has closed range on $\mathfrak{g}$. Since $\operatorname{ad} A$ is normal, this condition is equivalent to 0 being an isolated point in the spectrum $\operatorname{Spec}(\operatorname{ad} A)$ of $\operatorname{ad} A$ (Exercise VII.1).

We note that $B_2(H_c) \subseteq (\ker \operatorname{ad} A)^\perp = \overline{\operatorname{im} \operatorname{ad} A}$ follows directly from (i). The trivial fact that A_d commutes with $B_2(H_c)$ entails that $\operatorname{ad} A$, resp., $\operatorname{ad} A_c$ restricts to an invertible operator $B_2(H_c) \to B_2(H_c)$. Let E be the spectral measure of A_c and $\lambda \neq \mu \in \operatorname{Spec}(A_c)$. Then there exist disjoint compact ε-neighborhoods U_λ of λ and U_μ of μ in $\mathbb{C}$. We recall the rank-one operators $P_{v,w}(x) = \langle x, v\rangle w$ on H. For unit vectors $v \in E(U_\lambda)$ and $w \in E(U_\mu)$ we now get

$$\begin{aligned}\|[A_c, P_{v,w}]\|^2 &= \langle P_{A_c.v,w} - P_{v,A_c.w}, P_{A_c.v,w} - P_{v,A_c.w}\rangle \\ &= \|A_c.v\|^2 + \|A_c.w\|^2 - 2\langle A_c.v, v\rangle\langle A_c.w, w\rangle.\end{aligned}$$

If ε tends to 0, then this number tends to $\lambda^2 + \mu^2 - 2\lambda\mu = (\lambda - \mu)^2$. Since λ is not isolated in the spectrum of A_c, we conclude that the expression $\|[A_c, P_{v,w}]\|^2$ can be arbitrarily small, contradicting the invertibility of $\operatorname{ad} A_c$ on $B_2(H_c)$. We conclude that $A_c = 0$, i.e., $A = A_d$ is diagonalizable on H.

Now we apply the same argument with eigenvectors v, resp., w of A corresponding to the eigenvalues λ, resp., μ, and obtain

$$[A, P_{v,w}] = (\lambda - \mu)P_{v,w}.$$

Since 0 is isolated in $\operatorname{Spec}(\operatorname{ad} A)$, we conclude that every point in $\operatorname{Spec}(A)$ is isolated, and hence that this compact set is finite. ■

Motivated by Lemma VII.3, we now restrict our attention to those cocycles ω for which A is diagonalizable with finite spectrum. Let $\lambda_1 > \ldots > \lambda_k$ denote the eigenvalues of the hermitian operator iA and $H_j := \ker(iA - \lambda_j \mathbf{1})$ be the corresponding eigenspace. We then have an orthogonal decomposition

$$H = H_1 \oplus \ldots \oplus H_k.$$

Accordingly we write operators $B \in B(H)$ as block $k \times k$-matrices with entries $b_{ij} \in B(H_j, H_i)$.

The stabilizer of $0 \in \mathcal{O}_\omega$ coincides with the centralizer of A, hence is isomorphic to

$$U^0 := U_2(H_1) \times \ldots \times U_2(H_k) = \{u \in U : (\forall i \neq j) u_{ij} = 0\}.$$

We want to show that $\mathcal{O}_\omega$ carries a natural structure of a strong Kähler manifold compatible with the symplectic structure. So we have to find a Kähler polarization $\mathfrak{p}$ in ω.

Lemma VII.4. *The closed subalgebra*

$$\mathfrak{p} := \{X = (X_{ij})_{i,j=1,\ldots,k} \in \mathfrak{g} \colon i < j \Rightarrow X_{ij} = 0\}$$

is a complemented subalgebra which is a complex Kähler polarization in the cocycle ω *in the sense that (CP1)-(CP5) are satisfied.*

For $k = 3$ the elements of $\mathfrak{p}$ have the form

$$\begin{pmatrix} X_{11} & 0 & 0 \\ X_{21} & X_{22} & 0 \\ X_{31} & X_{32} & X_{33}. \end{pmatrix}$$

Proof. That $\mathfrak{p}$ is complemented is clear because

$$\mathfrak{n} := \{X = (X_{ij})_{i,j=1,\ldots,k} \in \mathfrak{g} \colon i \geq j \Rightarrow X_{ij} = 0\}$$

is a closed subspace of $\mathfrak{g}$ complementing $\mathfrak{p}$.

We have to verify conditions (C1)–(C5) from Section VI. From the explicit description of the stabilizer group U^0 we immediately derive that $\mathfrak{p}$ is $\mathrm{Ad}(U^0)$-invariant, which is (C1). The relations

$$\mathfrak{p} \cap \overline{\mathfrak{p}} = \mathfrak{u}^0_{\mathbb{C}} \quad \text{and} \quad \mathfrak{p} + \overline{\mathfrak{p}} = \mathfrak{g}$$

are also trivially satisfied. To verify (C4), let $X, Y \in \mathfrak{p}$. Then

$$\omega(X, Y) = \mathrm{tr}(A[X, Y]) = \sum_{j=1}^{k} -i\lambda_j \,\mathrm{tr}([X_{jj}, Y_{jj}]) = 0$$

follows from the fact that $[B_2(H_j), B_2(H_j)] \subseteq \mathfrak{sl}(H_j)$ for each j (Exercise VII.2). For (C5) we calculate for $Z \in \mathfrak{p}$:

$$\begin{aligned} -i\omega(Z, Z^*) &= -i\,\mathrm{tr}(A[Z, Z^*]) = -i\,\mathrm{tr}([A, Z], Z^*) = -i \sum_{j \geq k} \mathrm{tr}([A, Z_{jk}]Z^*_{kj}) \\ &= -i \sum_{j \geq k} \mathrm{tr}\big(-i(\lambda_j - \lambda_k) Z_{jk} Z^*_{kj}\big) = \sum_{j \geq k} (\lambda_k - \lambda_j)\,\mathrm{tr}(Z_{jk} Z^*_{kj}) \\ &= \sum_{j \geq k} (\lambda_k - \lambda_j) \|Z_{jk}\|_2^2 > 0 \end{aligned}$$

for $Z \notin \mathfrak{u}^0_{\mathbb{C}}$. ■

Lemma VII.5. *Let* $P := \langle \exp \mathfrak{p} \rangle \subseteq G$ *denote the analytic subgroup corresponding to* $\mathfrak{p}$*. Then*

$$P = \{(g_{ij}) \in \mathrm{GL}_2(H) \colon i < j \Rightarrow g_{ij} = 0\}.$$

In particular P *is a complemented Lie subgroup of* G*.*

Proof. We have to show that the group P' on the right hand side is connected. It is a semidirect product $\overline{N} \rtimes G^0$, where

$$G^0 := \{(g_{ij}) \in \mathrm{GL}_2(H): i \neq j \Rightarrow g_{ij} = 0\} \cong \prod_{j=1}^{k} \mathrm{GL}_2(H_j)$$

and

$$\overline{N} := \{(g_{ij}) \in \mathrm{GL}_2(H): g_{jj} = \mathbf{1}, i < j \Rightarrow g_{ij} = 0\}.$$

For the group $\overline{N}$ the exponential function $\exp: \overline{\mathfrak{n}} \to \overline{N}$ is a diffeomorphism whose inverse is given by

$$\log: \overline{N} \to \overline{\mathfrak{n}}: \quad \log(g) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(g-\mathbf{1})^n = \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n}(g-\mathbf{1})^n.$$

Therefore the connectedness of the right hand side above follows from the connectedness of the groups $\mathrm{GL}_2(H_j)$ (Theorem A.10) or directly from the observation that $\overline{N} = \mathbf{1} + \overline{\mathfrak{n}}$. ∎

Theorem VII.6. *If A is diagonalizable with discrete spectrum and $\omega(x,y) = \mathrm{tr}(A[x,y])$, then the coadjoint orbit $\mathcal{O}_\omega$ is a strong Kähler orbit, i.e., a Kähler orbit which is a strongly symplectic manifold.*

Proof. In view of Lemmas VII.4 and VII.5, we have exactly the situation asked for in Proposition VI.9, so that we obtain an open embedding

$$\mathcal{O}_\omega \cong U/U^0 \hookrightarrow G/P,$$

which yields on $\mathcal{O}_0$ the structure of a complex manifold. Since $\mathfrak{p}$ is a Kähler polarization, we see that $\mathcal{O}_\omega$ is a Kähler manifold.

It remains to show that the symplectic structure on the tangent space $T_0(\mathcal{O}_\omega)$ yields an isomorphism to the dual space. We identify $T_0(\mathcal{O}_\omega)$ with

$$\mathfrak{g}/\mathfrak{p} \cong \mathfrak{n} = \{X = (X_{ij})_{i,j=1,\ldots,k} \in \mathfrak{g}: i \geq j \Rightarrow X_{ij} = 0\}$$

as in the proof of Lemma VII.4. Then the real scalar product corresponding to the Kähler structure is given for $Z \in \mathfrak{n}$ by

$$-i\omega(Z^*, Z) = \sum_{k<j} (\lambda_k - \lambda_j)\|Z_{jk}^*\|_2^2 = \sum_{j<k} (\lambda_j - \lambda_k)\|Z_{jk}\|_2^2.$$

The fact that the differences $\lambda_j - \lambda_k$, $j < k$, are all positive shows that $\mathfrak{n}$ is a complex Hilbert space with respect to the above scalar product, hence that $\mathcal{O}_\omega$ is a strong Kähler orbit. ∎

We will see in Remark VII.19 below that the natural inclusion map $U/U^0 \hookrightarrow G/P$ is in fact a bijection, i.e., U acts transitively on G/P.

So far our geometric approach has provided us with a certain set of Kähler orbits of the Lie algebra $\mathfrak{u}_2(H)$ in the sense of affine coadjoint actions. These orbits are coadjoint orbits in the usual sense if and only if $\mathrm{im}(A) = \sum_{\lambda_j \neq 0} H_j$ is finite-dimensional, which is quite restrictive. In the next subsection we turn to the construction of the corresponding holomorphic line bundles and show that we can realize all holomorphic unitary highest weight representations in Hilbert spaces of holomorphic sections of such bundles.

Construction of the complex line bundles

In this subsection we start with an orthogonal decomposition

$$H = H_1 \oplus \ldots \oplus H_k$$

of the complex Hilbert space H. We consider the Banach algebra

$$\begin{aligned} &B_{r,1}(H_1, \ldots, H_k) \\ &:= \{A = (a_{ij}) \in B(H) \colon (\forall i \neq j)\, a_{ij} \in B_2(H_j, H_i), (\forall j)\, a_{jj} \in B_1(H_j)\} \end{aligned}$$

with the norm

$$\|X\| := \max\{\|a_{jj}\|_1, j = 1, \ldots, k; \|a_{jl}\|, j \neq l\}$$

(Example VII.3) and

$$B_{\rm res}(H_1, \ldots, H_k) := \{A = (a_{ij}) \in B(H) \colon (\forall i \neq j)\, a_{ij} \in B_2(H_j, H_i)\}$$

with the norm

$$\|X\| := \max\{\|a_{jj}\|, j = 1, \ldots, k; \|a_{jl}\|, j \neq l\}$$

(Example VII.4).

Lemma VII.7. (a) $\mathrm{GL}_{\rm res} := \mathrm{GL}(H) \cap B_{\rm res}(H_1, \ldots, H_k)$ *is a group, and*

$U_{\rm res} := \mathrm{GL}_{\rm res} \cap U(H) = \{(g_{ij}) \in U(H) \colon (\forall i > j)\, g_{ij} \in B_2(H_j, H_i), g_{jj}$ Fredholm$\}$.

(b) $G_{r,1} := \mathrm{GL}(H) \cap \big(\mathbf{1} + B_{r,1}(H_1, \ldots, H_k)\big)$ *is a group.*

Proof. (a) Let $g \in G_{r,1}$. For the first assertion we only have to show that $(g^{-1})_{il} \in B_2(H_l, H_i)$ holds for $i \neq l$. First we observe that

$$\mathbf{1} = g_{ii}(g^{-1})_{ii} + \sum_{j \neq i} g_{ij}(g^{-1})_{ji} \in g_{ii}(g^{-1})_{ii} + B_2(H_i).$$

We also have

$$g_{ii}(g^{-1})_{il} = -\sum_{j \neq i} g_{ij}(g^{-1})_{jl}.$$

Multiplying this equation with $(g^{-1})_{ii}$, we obtain

$$(g^{-1})_{ii} g_{ii}(g^{-1})_{il} = -\sum_{j \neq i} (g^{-1})_{ii} g_{ij}(g^{-1})_{jl} \in B_2(H_l, H_i),$$

so that

$$(g^{-1})_{il} \in (g^{-1})_{ii} g_{ii} (g^{-1})_{il} + B_2(H_l, H_i) \subseteq B_2(H_l, H_i).$$

For the second part we first observe that each element $g \in U_{\rm res}$ trivially satisfies $g_{ij} \in B_2(H_j, H_i)$ for $i > j$. Let us assume, conversely, that these conditions are satisfied. From $g^* g = \mathbf{1}$ we then get for $n, m \in \{1, \ldots, k\}$ the relations

$$\delta_{nm} \mathbf{1} = \sum_l g_{ln}^* g_{lm}.$$

For $m = 1 < n$ this leads to

$$0 = g_{1n}^* g_{11} + \underbrace{g_{2n}^* g_{21} + \ldots + g_{kn}^* g_{k1}}_{\in B_2(H_1, H_n)}.$$

Since g_{11} is a Fredholm operator (it has finite-dimensional kernel and cokernel), we derive that $g_{1n} \in B_2(H_n, H_1)$ for $n > 1$ (Example VII.7). For $m < n$ we now assume that $g_{ln} \in B_2(H_n, H_l)$ for $l < m$. Then we obtain

$$0 = \underbrace{\sum_{l<m} g_{ln}^* g_{lm}}_{\in B_2(H_m, H_n)} + g_{mn}^* g_{mm} + \underbrace{\sum_{l>m} g_{ln}^* g_{lm}}_{\in B_2(H_m, H_n)},$$

so that $g_{mn}^* g_{mm} \in B_2(H_m, H_n)$, and we see as above that g_{mn} is Hilbert–Schmidt.

(b) Let $g \in G_r$. Then (a) implies that for $j \neq l$ we have $(g^{-1})_{jl} \in B_2(H_l, H_j)$. We further have

$$\mathbf{1} = g_{ii}(g^{-1})_{ii} + \sum_{j \neq i} g_{ij}(g^{-1})_{ji} \in (\mathbf{1} + B_1(H_i))(g^{-1})_{ii} + B_1(H_i) \subseteq (g^{-1})_{ii} + B_1(H_i),$$

so that $g^{-1} \in \mathbf{1} + B_{r,1}(H_1, \ldots, H_k)$. ∎

We recall from Lemma VII.5 the subgroup $P \subseteq G$ which we write in the canonical way as a semidirect product

$$P \cong \overline{N} \rtimes G^0,$$

where

$$\overline{N} := \{(g_{ij}) \in \mathrm{GL}_2(H) \colon g_{jj} = \mathbf{1}, i < j \Rightarrow g_{ij} = 0\}.$$

Accordingly we put

$$N := \{(g_{ij}) \in \mathrm{GL}_2(H) \colon g_{jj} = \mathbf{1}, i > j \Rightarrow g_{ij} = 0\}.$$

We also consider the corresponding subgroup of $G_{r,1}$:

$$P_{r,1} := P \cap G_{r,1} \cong \overline{N} \rtimes G_1^0.$$

Since the group $P_{r,1}$ is a submanifold of the Banach–Lie group $G_{r,1}$, the quotient space $M := G_{r,1}/P_{r,1}$ carries a natural complex manifold structure modeled over the Hilbert space $\mathfrak{n}$. It is clear that for $G_1 := \mathrm{GL}_1(H)$ and $P_1 := P \cap \mathrm{GL}_1(H)$ we obtain an injection $G_1/P_1 \hookrightarrow G_{r,1}/P_{r,1}, gP_1 \mapsto gP_{r,1}$.

We construct holomorphic line bundles on M as follows. Since the group G_1^0 is isomorphic to the product $\prod_{j=1}^k \mathrm{GL}_1(H_j)$, we have for each

$$\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$$

a holomorphic character

$$\chi: G_1^0 \to \mathbb{C}^\times, \quad \chi(g) = \prod_{j=1}^k \det(g_{jj})^{\lambda_j}$$

which we immediately extend to a holomorphic character $\chi: P_{r,1} \to \mathbb{C}^\times$ with $\overline{N} \subseteq \ker \chi$. Actually every holomorphic character of $P_{r,1}$ has this form (Exercise VII.5).

Now we define an action of $P_{r,1}$ on $G_{r,1} \times \mathbb{C}$ by

$$p.(g,z) := (gp^{-1}, \chi(p)z), \quad p \in P_{r,1}, z \in \mathbb{C}, g \in G_{r,1}$$

and obtain a the homogeneous complex line bundle

$$\mathcal{L}_\chi := G_{r,1} \times_{P_{r,1}} \mathbb{C} \to M$$

as the quotient manifold with respect to this action (the same arguments as in Section I apply). We write $[g,z]$ for the element of $\mathcal{L}_\chi$ corresponding to the orbit of (g,z) under the action of $P_{r,1}$ and $\Gamma(\mathcal{L}_\chi)$ for the space of holomorphic sections.

We will now address the question when the bundle $\mathcal{L}_\chi$ has non-zero holomorphic sections. First we will see that a simple SL_2-reduction argument yields a necessary condition of which we will see later that it also is sufficient.

Lemma VII.8. *If $\Gamma(\mathcal{L}_\chi) \neq \{0\}$, then*

$$\lambda_1 \geq \ldots \geq \lambda_k. \tag{7.1}$$

Proof. We assume that $i < j$ and pick unit vectors $v \in H_i$ and $w \in H_j$. Then

$$h := P_{v,v} - P_{w,w}, \quad e := P_{v,w} \quad \text{and} \quad f := P_{w,v}$$

satisfy the commutator relations of $\mathfrak{sl}(2,\mathbb{C})$ (Example II.2), so that

$$\mathfrak{g}(v,w) := \mathrm{span}\{P_{v,w}, P_{w,v}, P_{v,v} - P_{w,w}\} \cong \mathfrak{sl}(2,\mathbb{C}).$$

We put $G(v,w) := \langle \exp \mathfrak{g}(v,w) \rangle \cong \mathrm{SL}(2,\mathbb{C}) \subseteq G_{r,1}$. Then

$$P(v,w) := P_{r,1} \cap G(v,w)$$

is a parabolic subgroup of $G(v,w)$ with Lie algebra

$$\mathfrak{p}(v,w) = \mathbb{C}h + \mathbb{C}f,$$

and the restriction of the character χ to $\mathfrak{p}(u,w)$ satisfies

$$d\chi(h) = \lambda_i - \lambda_j.$$

If $\mathcal{L}_\chi$ has non-zero holomorphic sections, then it has a non-zero holomorphic section not vanishing in the base point, and therefore the bundle $\mathcal{L}_{\lambda_i - \lambda_j}$ over $G(v,w)/P(v,w)$ has non-zero holomorphic sections (cf. the proof of Theorem I.5). In view of Theorem I.4, this implies $\lambda_i - \lambda_j \in \mathbb{N}_0$. ∎

The next step is to show that if λ satisfies (7.1) (we call such a λ dominant), then $\mathcal{L}_\chi$ is non-zero. In [HH94a] this is done by a direct construction of holomorphic sections. Here we will give a general argument which is universal for all types of groups U coming from L^*-algebras $\mathfrak{g}$ (see the comments at the end of this section).

We start with the information we have from Section V. We choose an orthonormal basis $(e_j)_{j\in J}$ in H subordinated to the decomposition of H into the subspaces $H_1, \ldots, H_k$. If λ satisfies (7.1), we can view it as an element of $l^\infty(J,\mathbb{C})$, so that we obtain with Theorem V.6 a holomorphic highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $G_1 := \mathrm{GL}_1(H)$ with highest weight λ.

Pick a highest weight vector $v_\lambda \in \mathcal{H}_\lambda$. We define $\delta \in \mathcal{H}_\lambda^*$ by $\delta(v) := \langle v, v_\lambda\rangle$ and consider on $\mathcal{H}_\lambda^*$ the holomorphic representation defined by $(g.\beta)(v) := \beta(g^{-1}.v)$. For the complex Lie subgroup $P_1 := G_1 \cap P$ we then have

$$p.\delta = \chi(p)\delta \quad \text{for all} \quad p \in P.$$

We now have a map

$$\Psi\colon \mathcal{H}_\lambda \to \mathrm{Hol}(G_1), \quad \Psi(v)(g) := (g.\delta)(v) = \langle \delta, g^{-1}.v\rangle = \langle g^{-1}.v, v_\lambda\rangle.$$

Then Ψ is a G_1-equivariant linear map with respect to the natural representation of G_1 on $\mathrm{Hol}(G_1)$ given by $(g.f)(x) := f(g^{-1}x)$, and each function f in the range of Ψ satisfies

$$f(gp) = \chi(p)^{-1} f(g), \quad g \in G_1, p \in P_1.$$

Since $G_1/P_1 \subseteq G_{r,1}/P_{r,1}$ is a proper subset, these functions on G_1 are not sufficient to define holomorphic sections of $\mathcal{L}_\chi$, we first have to extend them to the bigger group $G_{r,1}$. The following lemmas prepare the holomorphic extension of the function $f_{\lambda,1} := \Psi(v_\lambda)$ to $G_{r,1}$.

Lemma VII.9. *For $g_1, g_2 \in G$ the commutator $(g_1,g_2) := g_1 g_2 g_1^{-1} g_2^{-1}$ is contained in G_1, i.e., $(G,G) \subseteq G_1$.*

Proof. We write $g_1 = \mathbf{1} + x$ and $g_2 = \mathbf{1} + y$ with $x, y \in B_2(H)$. Then we also have

$$(\mathbf{1}+x)^{-1} = \mathbf{1} + x' \quad \text{and} \quad (\mathbf{1}+y)^{-1} = \mathbf{1} + y'$$

with $x', y' \in B_2(H)$ and

$$\mathbf{1} = (\mathbf{1}+x)(\mathbf{1}+x') = \mathbf{1} + x + x' + xx',$$

so that $x + x' = -xx' \in B_1(H)$. Likewise we get $y + y' \in B_1(H)$, and therefore

$$(g_1, g_2) = (\mathbf{1}+x)(\mathbf{1}+y)(\mathbf{1}+x')(\mathbf{1}+y') \in \mathbf{1}+x+x'+y+y'+B_1(H) \subseteq \mathbf{1}+B_1(H).$$

This shows that $(g_1, g_2) \in \mathrm{GL}_1(H)$. ■

Lemma VII.10. *The map*

$$\Phi\colon N \times G_1 \times \overline{N} \to G_{r,1}, \quad (x, g, y) \mapsto xgy$$

is a surjective holomorphic submersion with the property that

$$\Phi(x, g, y) = \Phi(x', g', y') \quad \Rightarrow \quad g' \in N_1 g \overline{N}_1.$$

Proof. (1) First we show that Φ is surjective. In view of Lemma VII.9, the group $G_{r,1}/G_1$ (which we only consider as an abstract group) is abelian. Therefore the image of $N\overline{N}$ in $G_{r,1}/G_1$ is a subgroup, so that $NG_1\overline{N} = N\overline{N}G_1$ is a subgroup of $G_{r,1}$. Since it also contains the open subset $NG_1^0\overline{N}$ (this requires a generalization of Exercise I.5 to Banach–Lie groups), it is an open subgroup, so that the connectedness of $G_{r,1}$ (a similar argument as in Theorem A.10 applies) implies that $G_{r,1} = \Sigma$.
(2) Since $\overline{N}$ acts smoothly by conjugation on the group G_1, we can form the corresponding semidirect product group $G_1 \rtimes \overline{N}$. Now we consider the right action of the group $N \times (G_1 \rtimes \overline{N})$ on $G_{r,1}$ given by

$$x.(n, (g, m)) := n^{-1}xgm.$$

Then, up to the diffeomorphism $(n, g_1, n') \mapsto (n^{-1}, g_1, n')$, Φ is an orbit mapping for this action, so that it suffices to prove that $d\Phi(\mathbf{1}, \mathbf{1}, \mathbf{1})$ is surjective with splitting kernel ([La99, Prop. 2.2]). The map $d\Phi(\mathbf{1}, \mathbf{1}, \mathbf{1})$ is simply the addition map

$$\mathfrak{n} \times \mathfrak{g}_1 \times \overline{\mathfrak{n}} \to \mathfrak{g}_{r,1} = \mathfrak{n} \oplus \mathfrak{g}_1^0 \oplus \overline{\mathfrak{n}}$$

which obviously is surjective. The closed subspace $\mathfrak{n} \times \mathfrak{g}_1^0 \times \overline{\mathfrak{n}}$ of $\mathfrak{n} \times \mathfrak{g}_1 \times \overline{\mathfrak{n}}$ is a closed complement of the kernel of $d\Phi(\mathbf{1}, \mathbf{1}, \mathbf{1})$. Therefore Φ is a submersion in $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ and hence everywhere.
(3) Description of the fibers: Replacing x by $(x')^{-1}x$ and y by $y(y')^{-1}$, we may assume that $x' = y' = \mathbf{1}$. Then $g' = xgy$, so that the normality of G_1 in $G_{r,1}$ implies that $xy \in G_1$. Let $a := x - \mathbf{1} \in \mathfrak{n}$ and $b := y - \mathbf{1} \in \overline{\mathfrak{n}}$. Then

$$xy = (\mathbf{1}+a)(\mathbf{1}+b) = \mathbf{1} + a + b + ab \in \mathbf{1} + B_1(H)$$

implies that $a + b \in \mathfrak{g}_1 = B_1(H)$. Now $\mathfrak{g}_1 = \mathfrak{n}_1 \oplus \mathfrak{g}_1^0 \oplus \overline{\mathfrak{n}}_1$ implies that $a \in \mathfrak{n}_1$ and $b \in \overline{\mathfrak{n}}_1$, showing that $x \in N_1$ and $y \in \overline{N}_1$. ■

Proposition VII.11. *The function $f_{\lambda,1} = \Psi(v_\lambda)$ on G_1 extends to a holomorphic function f_λ on $G_{r,1}$ with $f_\lambda(n_1 g n_2) = f_\lambda(g)$ for $n_1 \in N$, $n_2 \in \overline{N}$ and $g \in G_{r,1}$. Moreover, we have*

$$f_\lambda(gp^{-1}) = \chi(p) f_\lambda(g), \quad g \in G_{r,1}, p \in P_{r,1}.$$

Proof. We consider the holomorphic function

$$F: N \times G_1 \times \overline{N} \to \mathbb{C}, \quad (x, g, y) \mapsto f_{\lambda,1}(g).$$

In view of $f_{\lambda,1}(n_1 g n_2) = f_{\lambda,1}(g)$ for $n_1 \in N_1$, $n_2 \in \overline{N}_1$ and $g \in G_1$, the function F is constant on the fibers of the map Φ (Lemma VII.10). Since Φ is a submersion onto $G_{r,1}$, the function F factors through Φ to a holomorphic function on $G_{r,1}$ with the required properties.

For $p = ng_1 \in P_{r,1} = \overline{N} \rtimes G_1^0$ this further leads to

$$f_\lambda(gp^{-1}) = f_\lambda(gg_1^{-1}) = \chi(g_1) f_\lambda(g) = \chi(p) f_\lambda(g),$$

first for $g \in G_1$ and then by continuity for all $g \in G_{1,r}$. ∎

The following theorem is a generalization of the geometric part of the Borel–Weil Theorem to the line bundles $\mathcal{L}_\chi$ over $G/P = G_{r,1}/P_{r,1}$.

Theorem VII.12. (Helminck and Helminck) *The bundle $\mathcal{L}_\chi$ has non-zero holomorphic sections if and only if*

$$\lambda_1 \geq \ldots \geq \lambda_k.$$

Proof. The first half follows from Lemma VII.8, and for the converse we use Proposition VII.11 to see that the space $\Gamma_{G_{r,1}}(\mathcal{L}_\chi)$ is non-zero, and therefore that $\mathcal{L}_\chi$ has non-zero holomorphic sections. ∎

Reproducing kernel Hilbert spaces

Let M be a complex manifold and $\mathcal{H} \subseteq \mathrm{Hol}(M)$ a Hilbert space of holomorphic functions such that for each $z \in M$ the evaluation map

$$\mathcal{H} \to \mathbb{C}, \quad f \mapsto f(z)$$

is continuous. In view of Riesz' Theorem, there exists an element $K_z \in \mathcal{H}$ with $f(z) = \langle f, K_z \rangle$ for all $z \in M$. We call the function

$$K: M \times M \to \mathbb{C}, \quad K(z, w) := K_w(z) = \langle K_w, K_z \rangle$$

the *reproducing kernel of the Hilbert space* $\mathcal{H}$ and $\mathcal{H}$ a reproducing kernel Hilbert space.

The function K has the following properties:

(P1) K is a *positive definite kernel*, i.e., for $z_1, \ldots, z_n \in M$ the matrix

$$(K(z_i, z_j))_{i,j=1,\ldots,n}$$

is positive semidefinite.

(P2) The functions $K_w\colon z \mapsto K(z,w)$ are holomorphic.

If, conversely, $K\colon M \times M \to \mathbb{C}$ is a function satisfying (P1), (P2) and

(P3) The function $M \to \mathbb{R}, z \mapsto K(z,z)$, is locally bounded,

then one can show that there exists a unique reproducing kernel Hilbert space $\mathcal{H}_K \subseteq \mathrm{Hol}(M)$ with reproducing kernel K (cf. [Ne99a, Prop. I.1.9(iii)]).

The main idea of the construction is to consider the space

$$\mathcal{H}_K^0 := \mathrm{span}\{K_w\colon w \in M\}$$

and show that is has a positive hermitian form $\langle \cdot, \cdot \rangle$ satisfying

$$\langle f, K_z \rangle = f(z) \quad \text{for all} \quad z \in M.$$

Next one uses (P3) to show that the completion $\mathcal{H}_K$ of $\mathcal{H}_K^0$ can also be viewed as a space of holomorphic functions on M.

Realizing $\mathcal{H}_\lambda$ in $\Gamma(\mathcal{L}_\lambda)$

So far we have shown that the bundle $\mathcal{L}_\chi$ has non-zero holomorphic sections. The next step is to see that the whole Hilbert space $\mathcal{H}_\lambda$ can be realized by holomorphic sections of $\mathcal{L}_\lambda$.

We consider $G_{r,1}$ as a complex semigroup with involution given by $g \mapsto g^*$. A function $f\colon G_{r,1} \to \mathbb{C}$ is called *positive definite* if for all $g_1, \ldots, g_n \in G_{r,1}$ the matrix $f(g_i g_j^*)_{i,j=1,\ldots,n}$ is positive semidefinite.

Lemma VII.13. *The function f_λ on $G_{r,1}$ from* Proposition VII.11 *is positive definite.*

Proof. Since G_1 is dense in $G_{r,1}$, it suffices to assume that $g_1, \ldots, g_n \in G_1$. For $x, y \in G_1$ we have

$$f_\lambda(xy^*) = \langle (xy^*)^{-1}.v_\lambda, v_\lambda \rangle = \langle x^{-1}.v_\lambda, y^{-1}.v_\lambda \rangle,$$

so that we obtain for $c_1, \ldots, c_n \in \mathbb{C}$:

$$\sum_{i,j} c_i \overline{c_j} f_\lambda(g_i g_j^*) = \sum_{i,j} c_i \overline{c_j} \langle g_i^{-1}.v_\lambda, g_j^{-1}.v_\lambda \rangle = \| \sum_i c_i g_i^{-1}.v_\lambda \|^2 \geq 0. \qquad \blacksquare$$

Lemma VII.14. *There exists a Hilbert subspace*

$$\mathcal{H}_{f_\lambda} \subseteq \Gamma_{G_{r,1}}(\mathcal{L}_\chi)$$

containing all left-translates $g.f_\lambda$, $g \in G_{r,1}$, *of* f_λ *such that for all* $v \in \mathcal{H}_{f_\lambda}$ *and* $g \in G_{r,1}$ *we have*

$$v(g) = \langle v, (g^{-1})^*.f_\lambda\rangle. \tag{7.2}$$

Proof. First we consider the kernel K on $G_{r,1}$ given by

$$K(x,y) := f_\lambda(y^*x).$$

Lemma VII.13 means that K is positive definite, and, moreover, $x \mapsto K(x,x) = f_\lambda(x^*x)$ is a locally bounded function because f_λ is holomorphic and therefore continuous. Now [Ne99a, Prop. I.19(iii)] implies the existence of a Hilbert subspace $\mathcal{H}_{f_\lambda} \subseteq \mathrm{Hol}(G_{r,1})$ containing all left-translates $g.f_\lambda$, $g \in G_{r,1}$, and satisfying (7.2).

Next we observe that Lemma VII.13 implies in particular $f_\lambda(g^*) = \overline{f_\lambda(g)}$ for $g \in G_{r,1}$ and therefore

$$(p^*.f_\lambda)(g) = f_\lambda((p^*)^{-1}g) = \overline{f_\lambda(g^*p^{-1})} = \overline{\chi(p)f_\lambda(g^*)} = \overline{\chi(p)}f_\lambda(g).$$

For each $v \in \mathcal{H}_{f_\lambda}$, $g \in G_{r,1}$ and $p \in P_{r,1}$ we now get

$$\begin{aligned} v(gp^{-1}) &= \langle v, (g^{-1})^*p^*.f_\lambda\rangle = \langle v, (g^{-1})^*\overline{\chi(p)}.f_\lambda\rangle \\ &= \chi(p)\langle v, (g^{-1})^*.f_\lambda\rangle = \chi(p)v(g). \end{aligned}$$

Therefore $\mathcal{H}_{f_\lambda} \subseteq \Gamma_{G_{r,1}}(\mathcal{L}_\chi)$. ■

Lemma VII.15. *The restriction map* $\mathrm{Hol}(G_{r,1}) \to \mathrm{Hol}(G_1)$ *induces a surjective isometry*

$$r\colon \mathcal{H}_{f_\lambda} \to \Psi(\mathcal{H}_\lambda) \cong \mathcal{H}_\lambda.$$

Proof. First we observe that r is injective because G_1 is dense in $G_{r,1}$.

For $v \in \mathcal{H}_\lambda$ and $g \in G_1$ we have

$$\begin{aligned} \Psi(v)(g) &= \langle g^{-1}.v, v_\lambda\rangle = \langle \Psi(g^{-1}.v), \Psi(v_\lambda)\rangle = \langle g^{-1}.\Psi(v), f_{\lambda,1}\rangle \\ &= \langle \Psi(v), (g^{-1})^*.f_{\lambda,1}\rangle \end{aligned}$$

and $\Psi(G_1.v_\lambda) = G_1.(f_{\lambda,1})$ is a total subset of $\Psi(\mathcal{H}_\lambda)$. This means that $\Psi(\mathcal{H}_\lambda)$ is a reproducing kernel Hilbert space with kernel

$$K_1(x,y) = \langle (y^{-1})^*.\Psi(v_\lambda), (x^{-1})^*.\Psi(v_\lambda)\rangle = f_\lambda(y^*x).$$

Now we can apply [Ne99a, Prop. I.2.1(iii)] because r is injective. ■

The outcome of this construction is that we have realized the Hilbert space $\mathcal{H}_\lambda$ in the space of holomorphic sections of $\mathcal{L}_\lambda$ in such a way that the bigger group $G_{r,1}$ acts on a dense subspace containing the highest weight vector f_λ. This picture is still not optimal because there are larger groups acting on the bundle $\mathcal{L}_\chi$ and therefore on the space of holomorphic sections.

Enlarging the groups

The group $G_b^0 := \prod_{j=1}^k \mathrm{GL}(H_j) \subseteq G_r$ acts smoothly by automorphisms on the group $G_{r,1}$, so that we can form the semidirect product Banach–Lie group $G_{r,1} \rtimes G_b^0$. We consider the identity component

$$G_r := \mathrm{GL}_{\mathrm{res}}(H_1, \ldots, H_k)_0.$$

The connected components of the group $\mathrm{GL}_{\mathrm{res}} := \mathrm{GL}_{\mathrm{res}}(H_1, \ldots, H_k)_0$ are given by the group homomorphism

$$\mathrm{ind}\colon \mathrm{GL}_{\mathrm{res}} \to \mathbb{Z}^{k_\infty}, \quad g \mapsto (\mathrm{ind}(g_{jj}))_{\dim H_j = \infty},$$

where $k_\infty := |\{j\colon \dim H_j = \infty\}|$. The image of this group homomorphism is the set of those tuples (n_j) with $\sum_j n_j = 0$, showing that

$$\pi_0(\mathrm{GL}_{\mathrm{res}}) \cong \mathbb{Z}^{k_\infty - 1}$$

(cf. [HH94b, Prop. 2.3.1]).

Lemma VII.16. *We have surjective homomorphisms*

$$\eta\colon G_{r,1} \rtimes G_b^0 \to G_r, \quad (a,d) \mapsto ad$$

and

$$\eta_U\colon U_{r,1} \rtimes U_b^0 \to U_r, \quad (a,d) \mapsto ad.$$

Proof. The inclusion $G_{r,1}G^0 \subseteq G_r$ holds trivially. For the converse, let $g \in G_r$. Then each g_{jj} is a Fredholm operator, and since G_r is connected by definition, it is a Fredholm operator of index 0 (Exercise). Hence there exists a finite rank operator b_j mapping $\ker(g_{jj})$ bijectively onto $\mathrm{im}(g_{jj})^\perp$. Then $d_j := g_{jj} + b_j \in \mathrm{GL}(H_j)$ satisfies

$$g_{jj} = g_{jj} + b_j - b_j \in (g_{jj} + b_j)(\mathbf{1} + B_1(H_j)).$$

Therefore $d := \mathrm{diag}(d_j) \in G^0$ satisfies $d^{-1}g \in G_{r,1}$.

The first part implies in particular that the group G_r is connected because G_b^0 and $G_{r,1}$ are connected, so that its polar decomposition (cf. Proposition A.5 for a related case) shows that U_r is connected. Therefore η_U is a homomorphism of connected Banach–Lie groups, and since $\mathfrak{u}_r = \mathfrak{u}_{r,1} + \mathfrak{u}_b^0$, it is open and therefore surjective. ■

The kernel of η is the subgroup

$$K := \{(a, a^{-1})\colon a \in G_1^0\}, \quad \text{where} \quad G_1^0 = G_b^0 \cap G_{r,1} \cong \prod_{j=1}^k \mathrm{GL}_1(H_j).$$

The normal subgroup $K \trianglelefteq G_{r,1} \rtimes G_b^0$ is a closed normal subgroup which is a submanifold in the sense of Banach manifolds (cf. [La99]; Corollary V.5). Therefore the quotient group $(G_{r,1} \rtimes G_b^0)/K$ carries a unique Lie group structure for which the map

$$\eta\colon (G_{r,1} \rtimes G_b^0)/K \to G_r, \quad [a,d] \mapsto ad$$

is an isomorphism.

Definition VII.17. The group

$$K \cong G_1^0 \cong \mathrm{GL}_1(H_1) \times \ldots \times \mathrm{GL}_1(H_k)$$

has a natural holomorphic homomorphism

$$\Delta\colon G_1^0 \to Z := (\mathbb{C}^\times)^k, \quad g \mapsto (\det(g_j))_{j=1,\ldots,k}.$$

Since this homomorphism is invariant under conjugation with elements of G_b^0, the graph

$$\Gamma(\Delta^{-1}) := \{(k, \Delta(k)^{-1})\colon k \in K\} \subseteq K \times Z \subseteq (G_{r,1} \rtimes G_b^0) \times Z$$

is a central subgroup which is a submanifold (Exercise VII.6), so that we may form the quotient group

$$\widehat{G}_r := \big((G_{r,1} \rtimes G_b^0) \times Z\big)/\Gamma(\Delta^{-1})$$

whose elements are written as $[a, d, z] := (a, d, z)\Gamma(\Delta^{-1})$. This group has a natural homomorphism

$$q\colon \widehat{G}_r \to G_r, \quad q([a, d, z]) := ad$$

whose kernel coincides with

$$(K \times Z)/\Gamma(\Delta^{-1}) \cong Z = (\mathbb{C}^\times)^k.$$

We thus obtain a central extension

$$Z \hookrightarrow \widehat{G}_r \xrightarrow{q} G_r$$

of G_r. On the subgroup $G_{r,1}$ the central extension has a natural splitting given by

$$\sigma\colon G_{r,1} \to \widehat{G}_r, \quad \sigma(g) := [(g, \mathbf{1}, \mathbf{1})]. \qquad \blacksquare$$

Let $\widehat{P}_r := q^{-1}(P_r)$. Then $P_r \cong N \rtimes G_b^0$ implies that

$$\widehat{P}_r \cong N \rtimes \widehat{G}_b^0 \cong N \rtimes (G_b^0 \times Z) \tag{7.3}$$

because we can use the homomorphism $\sigma\colon N \to \widehat{G}_r$ to split off this group. We define a holomorphic character

$$\widehat{\chi}\colon \widehat{P}_r \to \mathbb{C}^\times, \quad \widehat{\chi}(n, d, z) := \prod_{j=1}^k z_j^{\lambda_j}.$$

One easily verifies that $\widehat{\chi}$ is compatible with χ in the sense that $\widehat{\chi} \circ \sigma(p) = \chi(p)$ for $p \in P_{r,1}$. We form the corresponding complex line bundle

$$\widehat{\mathcal{L}}_\chi := \widehat{G}_r \times_{\widehat{P}_r} \mathbb{C}.$$

We then have a natural holomorphic map

$$\psi: \mathcal{L}_\chi \to \widehat{\mathcal{L}}_\chi, \quad [g,z] \mapsto [\sigma(g), z]$$

because for $p \in P_{r,1}$ we have

$$[\sigma(gp^{-1}), \chi(p)z] = [\sigma(g)\sigma(p)^{-1}, \widehat{\chi}(\sigma(p))z] = [\sigma(g), z].$$

Since the canonical map

$$G_{r,1}/P_{r,1} \to \widehat{G}_r/\widehat{P}_r \cong G_r/P_r$$

is biholomorphic (cf. Lemma VII.16), it easily follows that the map ψ is a biholomorphic isomorphism of complex line bundles. In particular the space

$$\Gamma(\mathcal{L}_\chi) \cong \Gamma(\widehat{\mathcal{L}}_\chi)$$

has a natural realization in $\mathrm{Hol}(\widehat{G}_r)$, and we have a natural action of the complex group $\widehat{G}_r$ on this space.

The action of the diagonal group $G_b^0 \subseteq \ker \widehat{\chi} \subseteq \widehat{P}_r$ on $\widehat{\mathcal{L}}_\chi$ satisfies

$$d.[\sigma(g), z] = [d\sigma(g), z] = [\sigma(dgd^{-1})d, z] = [\sigma(dgd^{-1}), z],$$

so that

$$d.[d^{-1}gd, f(d^{-1}gd)] = [g, f(d^{-1}gd)]$$

implies that the action of G_b^0 on $\Gamma_{G_{r,1}}(\mathcal{L}_\chi)$ is given by

$$(d.f)(g) = f(d^{-1}gd). \tag{7.4}$$

For $f \in \Gamma_{G_{r,1}}(\mathcal{L}_\chi)$ and the corresponding section $s: G_{r,1}/P_{r,1} \cong \widehat{G}_r/\widehat{P}_r \to \mathcal{L}_\chi$ the map $\eta \circ s: G_{r,1}/P_{r,1} \cong \widehat{G}_r/\widehat{P}_r \to \widehat{\mathcal{L}}_\chi$ is a holomorphic section, and the corresponding function $\widehat{f}$ on $\widehat{G}_r$ satisfies $\widehat{f}(\sigma(g)) = f(g)$ for $g \in G_{r,1}$. It is uniquely determined by its values on $G_{r,1}$ because

$$\widehat{G}_r = \sigma(G_{r,1})\widehat{G}_b^0 = \sigma(G_{r,1})\widehat{P}_r^0$$

and

$$\widehat{f}(gp^{-1}) = \chi(p)\widehat{f}(g), \quad g \in \widehat{G}_r, p \in \widehat{P}_r.$$

Now we have all means to extend the representation of G_1 on $\mathcal{H}_\lambda$ to a unitary representation of $\widehat{U}_r$ and an unbounded representation of $\widehat{G}_r$.

Theorem VII.18. *Let $L(\lambda, \Delta^+)$ be a unitary highest weight module of $\mathfrak{gl}(J,\mathbb{C})$ with $\lambda \in l^\infty(J,\mathbb{Z})$. Then the holomorphic action of the group $G_1 = \mathrm{GL}_1(H)$ on $\mathcal{H}_\lambda$ extends to a representation of the group $\widehat{G}_r$ on a dense subspace of $\mathcal{H}_\lambda$, and the action of the unitary group $\widehat{U}_r$ extends to a continuous unitary action on the whole space $\mathcal{H}_\lambda$.*

Proof. Let $\widehat{f}_\lambda \in \Gamma_{\widehat{G}_r}(\widehat{\mathcal{L}}_\chi)$ denote the function corresponding to $f_\lambda \in \Gamma_{G_1}(\mathcal{L}_\chi)$. In view of (7.4) and the fact that f_λ on $G_{r,1}$ is invariant under the action of G_b^0, the function $\widehat{f}_\lambda$ is G_b^0-invariant. It also is $\overline{N}$-invariant, so that (7.3) implies that for $p = (n,d,z) \in \widehat{P}_r$ we have

$$p^*.\widehat{f}_\lambda = \chi(z^*)f_\lambda = \overline{\chi(z)}f_\lambda = \overline{\chi(p)}f_\lambda.$$

For a sequence of elements $g_1,\ldots,g_n \in \widehat{G}_r$ we write $g_i = \sigma(x_i)p_i$ with $x_i \in G_{r,1}$ and $p_i \in \widehat{P}_r$. Then

$$\begin{aligned} f_\lambda(g_i^* g_j) &= \chi(p_j)^{-1}\big((p_i^*)^{-1}.f_\lambda\big)(\sigma(x_i)^*\sigma(x_j)) \\ &= \chi(p_j)^{-1}\overline{\chi(p_j)}^{-1} f_\lambda(\sigma(x_i)^*\sigma(x_j)), \end{aligned}$$

so that Lemma VII.13 implies that $\widehat{f}_\lambda$ is a positive definite function on $\widehat{G}_r$. Now the same arguments as for $G_{r,1}$ show that $\widehat{G}_r.\widehat{f}_\lambda \subseteq \mathcal{H}_\lambda$ (viewed as a subspace of $\Gamma(\widehat{\mathcal{L}}_\chi)$, so that $\widehat{G}_r$ has a representation on a dense subspace of $\mathcal{H}_\lambda$.

For $u \in \widehat{U}_r$ and $x, y \in \widehat{G}_r$ we have

$$\widehat{f}_\lambda((uy)^*(ux)) = \widehat{f}_\lambda(y^*u^*ux) = \widehat{f}_\lambda(y^*x),$$

which implies that the action of the group $\widehat{U}_r$ on the dense subspace $\mathrm{span}(\widehat{G}_r.\widehat{f}_\lambda)$ extends to a continuous unitary action on $\mathcal{H}_\lambda$ (cf. [Ne99a, Prop. IV.1.9]). ■

Remark VII.19. (The relation to restricted flag manifolds) Let

$$\mathcal{F} := (F_1,\ldots,F_k)$$

be a *flag in the complex Hilbert space* H, i.e.,

$$F_1 \subseteq \ldots \subseteq F_k = H$$

are closed subspaces of H. The flag $\mathcal{F}$ can also be represented by the sequence

$$\mathcal{E} := \mathcal{E}(\mathcal{F}) := (E_1,\ldots,E_k)$$

of closed subspaces defined by $E_j := F_j \cap F_{j-1}^\perp$ (where $F_0 := \{0\}$). Then $H = E_1 \oplus \ldots \oplus E_k$ is an orthogonal decomposition.

We call $\mathcal{F}$ and $\mathcal{F}'$ *close* if there exists an element $g \in G_r$ with $g.F_j = F_j'$ for all j. If this is the case, then one easily verifies that the orthogonal projections $p_{F_j}\colon F_j' \to F_j$ are Fredholm operators of index 0 and the orthogonal projections

$\widetilde{p}_{F_j}: F_j' \to F_j^\perp$ are Hilbert–Schmidt (Exercise!). Suppose, conversely, that this is the case for two flags $\mathcal{F}$ and $\mathcal{F}'$ and E_j and E_j' be as above. Then it is not hard to see that the orthogonal projections $p_{ij}: E_j' \to E_i$ are Hilbert–Schmidt for $i > j$ and Fredholm of index 0 for $i = j$ (Exercise). It follows in particular that E_j and E_j' have the same Hilbert dimension, so that there exists a unitary operator $u \in U(H)$ with $u.E_j = E_j'$ for $j = 1, \ldots, k$. Writing u as a $(k \times k)$-block matrix with respect to the decomposition $H = E_1 \oplus \ldots \oplus E_k$, we see that the diagonal blocks u_{jj} as Fredholm of index 0 and the lower diagonal blocks u_{ij}, $i > j$, are Hilbert–Schmidt. In view of Lemma VII.7(i), this implies that $u \in U_{\mathrm{res}}(E_1, \ldots, E_k)_0 = U_r$. This means that for the flag $\mathcal{F}$ corresponding to $\mathcal{E}$, we have

$$G_r.\mathcal{F} = U_r.\mathcal{F}.$$

Lemma VII.16 further implies that

$$G_r.\mathcal{F} = G_{r,1}.\mathcal{F} \quad \text{and} \quad U_r.\mathcal{F} = U_{r,1}.\mathcal{F}.$$

We conclude in particular that $U \supseteq U_{r,1}$ acts transitively on $G/P \cong G_{r,1}/P_{r,1} \cong G_{r,1}.\mathcal{F}$. This means that G/P can be identified with the coadjoint orbit $\mathcal{O}_\omega \cong U/U^0 \subseteq \mathfrak{u}^*$. ■

Concluding remarks

In this section we have seen that the Borel–Weil picture for finite-dimensional complex reductive groups carries over to the group $G = \mathrm{GL}_2(H)$, H a complex Hilbert space. Our first step was to identify H with some $l^2(J, \mathbb{C})$, so that we obtain a dense locally finite subalgebra $\mathfrak{gl}(J, \mathbb{C})$ whose unitary highest weight modules can be classified by algebraic means. Then we globalized the picture by integrating those representations with bounded highest weight λ to holomorphic representations of the group $\mathrm{GL}_1(H)$. The next step was to consider Kähler structures on (affine) coadjoint orbits of $\mathfrak{u}^*$ for $\mathfrak{u} = \mathfrak{u}_2(H)$. In this context we have seen that the condition that such a coadjoint orbit has a closed tangent space already leads to orbits defined by tuples $(\lambda_1, \ldots, \lambda_k)$ and an orthogonal decomposition of the space H. Eventually we realized the Hilbert space $\mathcal{H}_\lambda$ as a space of holomorphic sections of a complex line bundle over such a coadjoint orbit. This led us to much bigger groups such as $\widehat{G}_r$, resp., $\widehat{U}_r$, where the first group acts "holomorphically" on a dense subspace of $\mathcal{H}_\lambda$ and the latter acts unitarily on the whole space.

Although there was no time in these lectures to discuss the more general approach via L^*-algebra, let us briefly describe the main ideas. An L^*-algebra is a complex Hilbert space $\mathfrak{g}$ which at the same time is a complex Lie algebra such that the scalar product satisfies

$$\langle [x, y], z \rangle = \langle y, [x^*, z] \rangle, \quad x, y, z \in \mathfrak{g}.$$

This means that the adjoint of the operator $\mathrm{ad}\, x$ is $\mathrm{ad}\, x^*$. Typical examples of simple L^*-algebras are

$$B_2(H), \quad \mathfrak{sp}_2(H, I) \quad \text{and} \quad \mathfrak{o}_2(H, I),$$

and these are all infinite-dimensional simple L^*-algebras (cf. [CGM90], [Neh93] and [St99b]).

For each L^*-algebra there is a natural complex Lie group G and a "compact" real form U. First one determines the space $Z^2_c(\mathfrak{u}, \mathbb{R})$ and then one shows that for a coadjoint orbit $\mathcal{O}_\omega$, $\omega \in Z^2_c(\mathfrak{u}, \mathbb{R})$, the closedness of the tangent space implies that it meats the dual of a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ which is a maximal abelian $*$-invariant subalgebra. Results of Schue ([Sch60]) imply that $\mathfrak{g}$ has an orthogonal root decomposition with respect to $\mathfrak{h}$, and that the subalgebra generated by the root spaces is a locally finite semisimple Lie algebra $\mathfrak{g}_0$. Section III contains in particular a classification of all unitary highest weight modules of this Lie algebra, and there is also an analog of Section V, where $\mathfrak{g}_1$ is a natural Banach–Lie algebra which for $\mathfrak{g} = B_2(H)$ is $\mathfrak{sl}(H)$. Sections VI and VII also generalize to this context, where one simply has to verify that the arguments we have used above can be carried over.

The advantage of the L^*-approach is comparable to the advantage of considering finite-dimensional reductive Lie algebras instead of studying classical simple Lie algebras cases by case.

Notes on Section VII

In [Bo80] Boyer describes the representations of the group $U = \mathrm{U}_2(H)$ in holomorphic sections in line bundles over coadjoint orbits of this group in $\mathfrak{u}^*$. This approach is quite restrictive, because the condition that the diagonal matrix defined by λ is Hilbert-Schmidt implies that λ has finite support. Representations in spaces of holomorphic sections of associated line bundles are only constructed for the case where λ is integral and either positive or negative, but not in the mixed case. It is also shown that the norm-continuous unitary representations of U_2 are elementary in the sense that they are direct sums of highest weight representations (cf. [Ne98] and Section V).

A more general approach is described in [HH94a] and [HH94b], where for a separable Hilbert space H the homogeneous manifolds G/P considered in this section are constructed directly as restricted flag manifolds (cf. Remark VII.19). The case where $k = 2$ leads to the restricted Graßmannian G/P which has been discussed earlier in [PS86].

Exercises for Section VII

Exercise VII.1. Let H be a complex Hilbert space and A a normal bounded operator on H. Then $\mathrm{im}(A)$ is closed if and only if 0 is isolated in the spectrum $\sigma(A)$ of A. Hint: Reduce to the case where A is injective. ■

Exercise VII.2. Let H be a Hilbert space. Show that

$$[B_2(H), B_2(H)] \subseteq \mathfrak{sl}(H).$$

Hint: $\|XY\|_1 \leq \|X\|_2 \|Y\|_2$ for $X, Y \in B_2(H)$. ■

Exercise VII.3. Show that the space $B_{\mathrm{res},1}(H_1, \ldots, H_k)$ is a complex Banach-$*$-algebra with respect to the composition, the natural involution, and the norm

$$\|X\| := \max\{\|a_{jj}\|_1, j = 1, \ldots, k; \|a_{jl}\|, j \neq l\}.$$

Hint: $\|XY\|_1 \leq \|X\|_2 \|Y\|_2$ for $X, Y \in B_2(H)$. ■

Exercise VII.4. Show that the space $B_{\mathrm{res}}(H_1, \ldots, H_k)$ is a complex Banach-$*$-algebra with respect to the composition, the natural involution, and the norm

$$\|X\| := \max\{\|a_{jj}\|, j = 1, \ldots, k; \|a_{jl}\|, j \neq l\}.$$

Hint: $\|XY\|_2 \leq \|X\| \|Y\|_2$ for $X \in B(H($, $Y \in B_2(H)$. ■

Exercise VII.5. Show that each holomorphic character $\chi: P_{r,1} \cong \overline{N} \rtimes G_1^0 \to \mathbb{C}^\times$ is of the form

$$\chi(n, g) = \prod_{j=1}^{k} \det(g_{jj})^{\lambda_j}$$

for $\lambda \in \mathbb{Z}^k$. ■

Exercise VII.6. If M and N are Banach manifolds, $M_1 \subseteq M$ is a submanifold, and $f: M_1 \to N$ is a smooth map, then the graph

$$\Gamma(f) := \{(x, f(x)): x \in M_1\}$$

is a submanifold of $M \times N$. ■

Exercise VII.7. (a) If $g \in \mathrm{GL}_{\mathrm{res}}$, then each diagonal entry g_{jj}, $j = 1, \ldots, k$, is a Fredholm operator.
(b) If A is a Fredholm operator on H and $B \in B(H)$ with $AB \in B_2(H)$, then $B \in B_2(H)$. Hint: Consider $A^*AB \in B_2(H)$ and write this operator in 2×2-block form according to $\mathrm{im}(A^*A)$ and $\ker(A^*A)$. ■

Exercise VII.8. We consider the Lie algebra $\mathfrak{g}_{r,1} = B_{r,1}(H_1, \ldots, H_k)$ and define a continuous linear functional

$$\mathrm{tr}\colon \mathfrak{g}_{r,1} \to \mathbb{C}, \quad X \mapsto \sum_{j=1}^{k} \mathrm{tr}\, X_{jj}.$$

Show that tr is a Lie algebra homomorphism which integrates to a holomorphic character

$$\det\colon G_{r,1} \to \mathbb{C}.$$

Hint: $[B_2(H), B_2(H)] \subseteq \mathfrak{sl}(H)$. ■

Appendix. The topology of classical Banach–Lie groups

In this appendix we collect some useful results on the homotopy groups of groups of operators on a Hilbert space. A crucial tool for the analysis of the topology of operator groups is the polar decomposition which is discussed for several types of groups in the first subsection. A more general context for polar decompositions based on the geometric context of symmetric spaces of seminegative curvature is described in [Ne99b]. We then explain how certain general results of Palais can be used to analyze the topology of groups like $\mathrm{GL}_p(H)$.

Polar decompositions

For the following lemma we recall the definition of the *spectrum* of an element of a Banach algebra A:

$$\mathrm{Spec}(a) := \{\lambda \in \mathbb{C}\colon a - \lambda \mathbf{1} \not\in G(A)\}.$$

Lemma A.1. *Let A be a complex unital Banach algebra and*

$$\mathcal{D} := \{a \in A\colon \inf \mathrm{Re}\,\mathrm{Spec}(a) > 0\}.$$

For an $a \in \mathcal{D}$ we choose a contour Γ in $\mathbb{C}_+ := \{z \in \mathbb{C}\colon \mathrm{Re}\, z > 0\}$ surrounding the spectrum $\mathrm{Spec}(a)$ and define

$$\log(a) := \frac{1}{2\pi i}\oint_\Gamma (\log\lambda)(\lambda\mathbf{1} - a)^{-1}\,d\lambda.$$

Then we obtain a holomorphic function $\log\colon \mathcal{D} \to A$. If $*$ is an antilinear anti-automorphism of A, then we have $\log(a^*) = \log(a)^*$.

Proof. That $\mathcal{D}$ is open follows from [Ru73, Th. 10.20], and the holomorphy of log from [Ru73, Th. 10.38]. For any antilinear antiautomorphism of A we have $f(a^*) = f(a)^*$ for any real-valued polynomial $f \in \mathbb{R}[X]$, and this implies that $\log(a^*) = \log(a)^*$ because, according to Runge's Theorem, on $\mathrm{Spec}(a)$ the log-function is a uniform limit of polynomials (cf. [Ru86]). ∎

Proposition A.2. *If H is a complex Hilbert space, then the polar map*

$$p\colon \mathrm{U}(H) \times \mathrm{Herm}(H) \to \mathrm{GL}(H), \quad (u, X) \mapsto ue^X$$

is a diffeomorphism.

Proof. Let $g \in \mathrm{GL}(H)$. Then g^*g is a positive hermitian operator, so that the continuous functional calculus provides a unique hermitian operator

$$X := \frac{1}{2}\log(g^*g).$$

Let $u := ge^{-X}$. Then

$$uu^* = ge^{-2X}g^* = g(g^*g)^{-1}g^* = \mathbf{1}$$

and

$$u^*u = e^{-X}g^*ge^{-X} = e^{-X}e^{2X}e^{-X} = \mathbf{1}.$$

We conclude that every operator $g \in \mathrm{GL}(H)$ has a unique decomposition $g = ue^X$ with $X \in \mathrm{Herm}(H)$. This means that p is bijective. It is also clear that p is a smooth map.

To see that p^{-1} is also smooth, we have to verify that the function

$$\log\colon \{g \in \mathrm{GL}(H) \cap \mathrm{Herm}(H)\colon \inf \mathrm{Spec}(g) > 0\} \to \mathrm{Herm}(H)$$

is smooth. This follows directly from Lemma A.1. ∎

Remark A.3. Our proof for the polar decomposition works also for abstract C^*-algebras, where it provides a diffeomorphism

$$p\colon \mathrm{U}(A) \times \mathrm{Herm}(A) \to G(A), \quad (u, X) \mapsto ue^X,$$

where

$$\mathrm{U}(A) = \{a \in A\colon aa^* = a^*a = \mathbf{1}\}.$$

For commutative algebras $A = C(X, \mathbb{C})$, X a compact space, this is the trivial decomposition

$$C(X, \mathbb{T}) \times C(X, \mathbb{R}) \to C(X, \mathbb{C}^\times), \quad (u, f) \mapsto ue^f.$$

∎

Proposition A.4. *For every $p \in [1, \infty]$ the polar map*

$$p\colon \mathrm{U}_p(H) \times \mathrm{Herm}_p(H) \to \mathrm{GL}_p(H), \quad (u, X) \mapsto ue^X$$

is a diffeomorphism.

Proof. We consider the Banach-$*$-subalgebra

$$\widetilde{B}_p(H) = \mathbb{C}\mathbf{1} + B_p(H) \subseteq B(H).$$

In Example IV.15(d) we have seen that

$$\widetilde{B}_p(H) \cap \mathrm{GL}(H) = G(\widetilde{B}_p(H)),$$

so that the spectrum $\mathrm{Spec}_p(X)$ of an element $X \in \widetilde{B}_p(H)$ coincides with the spectrum $\mathrm{Spec}(X)$ of X as an element of $B(H)$. Therefore Lemma A.1 implies that for $g \in \mathrm{GL}_p(H)$ we have $\log(g^*g) \in B_p(H)$, and that the map

$$\log\colon \{g \in \mathrm{GL}_p(H) \cap \mathrm{Herm}_p(H)\colon \inf \mathrm{Spec}(g) > 0\} \to \mathrm{Herm}_p(H)$$

is smooth. This implies the assertion. ∎

For an orthogonal decomposition $H = H_- \oplus H_+$ we write operators on H as (2×2)-block matrices and define the unital Banach-$*$-algebra

$$B_{\mathrm{res}}(H_-, H_+) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(H) \colon b \in B_2(H_+, H_-), c \in B_2(H_-, H_+) \right\}$$

with the norm $\|x\| := 2\max\{\|a\|, \|b\|_2, \|c\|_2, \|d\|\}$ (cf. Example IV.15(b)). We further consider

$$\mathrm{GL}_{\mathrm{res}}(H_-, H_+) := \mathrm{GL}(H) \cap B_{\mathrm{res}}(H_-, H_+)$$

(cf. Lemma VII.7),

$$U_{\mathrm{res}}(H_-, H_+) := U(H) \cap B_{\mathrm{res}}(H_-, H_+)$$

and

$$\mathrm{Herm}_{\mathrm{res}}(H_-, H_+) := \mathrm{Herm}(H) \cap B_{\mathrm{res}}(H_-, H_+).$$

Proposition A.5. *For every orthogonal decomposition $H = H_- \oplus H_+$ the polar map*

$$p \colon \mathrm{U}_{\mathrm{res}}(H_-, H_+) \times \mathrm{Herm}_{\mathrm{res}}(H_-, H_+) \to \mathrm{GL}_{\mathrm{res}}(H_-, H_+), \quad (u, X) \mapsto ue^X$$

is a diffeomorphism.

Proof. In view of Lemma VII.7, the group $\mathrm{GL}_{\mathrm{res}}(H_-, H_+)$ is the unit group of the Banach algebra $A := B_{\mathrm{res}}(H_-, H_+)$. It follows in particular that the spectrum $\mathrm{Spec}_A(X)$ of an element $X \in A$ coincides with the spectrum $\mathrm{Spec}(X)$ of X as an element of $B(H)$. Therefore Lemma A.1 implies that for $g \in \mathrm{GL}_{\mathrm{res}}(H_-, H_+) = G(A)$ we have $\log(g^*g) \in A$, and that the map

$$\log \colon \{g \in G(A) \cap \mathrm{Herm}(H) \colon \inf \mathrm{Spec}(g) > 0\} \to \mathrm{Herm}_{\mathrm{res}}(H)$$

is smooth. This implies the assertion. ∎

Remark A.6. We consider the convex domain

$$\Omega := \{X \in \mathrm{Herm}(H) \colon X \gg 0, \mathbf{1} - X \in B_2(H)\}$$

which can be identified with the open convex domain

$$\{Y \in \mathrm{Herm}_2(H) \colon \mathbf{1} - Y \gg 0\} = \{Y \in \mathrm{Herm}_2(H) \colon \sup \mathrm{Spec}(Y) < 1\},$$

where $X \gg 0$ means that $\mathrm{Spec}(X) \subseteq]0, \infty[$.

The group $\mathrm{GL}(H)$ acts on $\mathrm{Herm}(H)$ by $g.A := gAg^*$. We claim that

$$\begin{aligned} G_2 &:= \{g \in \mathrm{GL}(H) \colon g.\Omega \subseteq \Omega\} = \{g \in \mathrm{GL}(H) \colon g^*g \in \mathrm{GL}_2(H)\} \\ &= \mathrm{U}(H) \exp\big(\mathrm{Herm}_2(H)\big). \end{aligned}$$

In fact, if $g.\Omega \subseteq \Omega$, then we have in particular that $g.\mathbf{1} = gg^* \in \Omega$, so that $gg^* \in \mathrm{GL}_2(H)$ and therefore also $g^*g = g^{-1}(gg^*)g \in \mathrm{GL}_2(H)$. If, conversely, $g^*g \in \mathrm{GL}_2(H)$ and $X \in \Omega$, then

$$g.X - \mathbf{1} = gXg^* - gg^* + gg^* - \mathbf{1} = g(X - \mathbf{1})g^* + gg^* - \mathbf{1} \in B_2(H).$$

The preceding calculations show that the group G_2 is the natural symmetry group of the domain Ω. Similar observations can be made for real and quaternionic Hilbert spaces.

To construct a natural Lie group structure on the group G_2, we first observe that the polar decomposition implies that

$$G_2 = \mathrm{U}(H)\exp(\mathrm{Herm}_2(H)) = \mathrm{U}(H)\,\mathrm{U}_2(H)\exp(\mathrm{Herm}_2(H)) = \mathrm{U}(H)\,\mathrm{GL}_2(H).$$

We first consider the semidirect product group

$$S := \mathrm{GL}_2(H) \rtimes \mathrm{U}(H).$$

Then

$$N := \{(g, g^{-1}) \colon g \in \mathrm{U}_2(H)\} \cong \mathrm{U}_2(H)$$

is a closed normal subgroup of S. It is the kernel of the multiplication map

$$m \colon S \twoheadrightarrow G_2, \quad (a, b) \mapsto ab$$

which is in particular continuous with respect to the uniform topology on G_2. The group S has a natural Banach–Lie group structure. The group S is diffeomorphic to

$$\mathrm{Herm}_2(H) \times \mathrm{U}_2(H) \times \mathrm{U}(H) \cong \mathrm{Herm}_2(H) \times N \times \mathrm{U}(H),$$

showing that N is a submanifold of S. Hence S/N carries a natural Lie group structure such that it is diffeomorphic to $\mathrm{U}(H) \times \mathrm{Herm}_2(H)$ (cf. Remark IV.4).■

Some general results on homotopy groups

Lemma A.7. *If X is a Hausdorff space which is carries the direct limit topology with respect to the subspaces X_n, $n \in \mathbb{N}$, with $X_n \subseteq X_{n+1}$, then*

$$\pi_k(X) = \lim_{\longrightarrow} \pi_k(X_n)$$

for every $k \in \mathbb{N}_0$.

Proof. We claim that each compact subset $K \subseteq X$ is contained in some X_n. If this is not so, then for each $n \in \mathbb{N}$ we pick $x_n \in K \setminus X_n$. The set $M := \{x_n \colon n \in \mathbb{N}\}$ satisfies $M \cap X_m \subseteq \{x_1, \ldots, x_{m-1}\}$. Therefore $M \cap X_m$ is closed for each m, so that M is a closed subset of X. Thus $M \subseteq K$ implies that M is compact. The same argument applies to the subsets $M_m := \{x_m, x_{m+1}, \ldots\} \subseteq M$. Now $\bigcap_{m\in\mathbb{N}} M_m \neq \emptyset$ follows from the compactness of the sets M_m. On the other hand $M_{m+1} \cap X_m = \emptyset$ implies that

$$\bigcap_{m\in\mathbb{N}} M_m \subseteq X \setminus \bigcup_{m\in\mathbb{N}} X_m = \emptyset.$$

This contradiction shows that there exists an $n \in \mathbb{N}$ with $K \subseteq X_n$.

Now let $\gamma\colon \mathbb{S}^k \to X$ be a continuous map. Then $\gamma(\mathbb{S}^k)$ is a compact subset of X, hence contained in some X_n, and since $X_n \hookrightarrow X$ is an embedding, the corestriction $\gamma\colon \mathbb{S}^k \to X_n$ is continuous. Therefore the natural map

$$\varinjlim \pi_k(X_n) \to \pi_k(X)$$

is surjective. To see that it is injective, we apply the same argument to the range of a homotopy of two continuous maps $\gamma_1\colon \mathbb{S}^k \to X_{n_1}$ and $\gamma_2\colon \mathbb{S}^k \to X_{n_2}$. We find that there exists $n_3 > \max(n_1, n_2)$ such that $\pi_k(\varphi_{n_3,n_1})([\gamma_1]) = \pi_k(\varphi_{n_3,n_2})([\gamma_2])$, where $\varphi_{n_3,n_1}\colon X_1 \to X_3$ and $\varphi_{n_3,n_2}\colon X_2 \to X_3$ are the embeddings. ■

The following theorem is quite useful to calculate homotopy groups:

Theorem A.8. *Let V_1 and V_2 be locally convex topological vector spaces and $f\colon V_1 \to V_2$ a continuous linear map with dense range. Let $U \subseteq V_2$ be an open subset and put $\widetilde{U} := f^{-1}(U)$ and $\widetilde{f} := f|_{\widetilde{U}}$. Assume that V_1 and V_2 are metrizable or, more generally, that $\widetilde{U}$ and U are paracompact. Then $\widetilde{f}\colon \widetilde{U} \to U$ is a homotopy equivalence.*

Proof. This is Theorem 16 in [Pa66]. A quite direct proof of the corresponding result for Banach spaces can be found in [At67, p.164]. ■

The following theorem is particularly useful for separable spaces:

Theorem A.9. *Let V be a locally convex space and $(E_n)_{n\in\mathbb{N}}$ an increasing sequence of finite-dimensional subspaces of V such that their union is dense in V. Given an open subset $U \subseteq V$, let $U_n := U \cap E_n$ and consider the direct limit topological space $U_\infty := \varinjlim U_n$. Then if V is metrizable or, more generally, if U is paracompact, then the inclusion map $U_\infty \to U$ is a homotopy equivalence.*

Proof. This is the corollary to Theorem 17 in [Pa66]. ■

Theorem A.10. *Let H be an infinite-dimensional Hilbert space over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and $p \in [1, \infty]$. Then for every $k \in \mathbb{N}_0$ we have*

$$\pi_k(\mathrm{GL}_p(H)) \cong \varinjlim \pi_k(\mathrm{GL}(n, \mathbb{K})).$$

Proof. Let $H_s \subseteq H$ be a separable closed subspace. We write operators on H as (2×2)-block matrices with respect to the decomposition $H = H_s \oplus H_s^\perp$. First we show that the natural inclusion map

$$\mathrm{GL}_p(H_s) \hookrightarrow \mathrm{GL}_p(H), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

induces isomorphism for all homotopy groups.

Let X be a separable compact space and $\gamma\colon X \to \mathrm{GL}_p(H)$ be a continuous map. Then for every $x \in X$ the the range of $\gamma(x) - \mathbf{1}$ is a separable subspace because the operator $\gamma(x)-\mathbf{1}$ is compact. Since X is separable, the closure H_e of the space spanned by all the separable subspaces $\mathrm{im}(\gamma(x)-\mathbf{1})$ and $\mathrm{im}(\gamma(x)^*-\mathbf{1})$, $x \in X$, is separable. Let $u \in \mathrm{U}(H)$ with $u.H_e = H_s$. Since $\mathrm{U}(H)$ is connected (an easy consequence of the Spectral Theorem for Unitary Operators), there exists a continuous curve $\alpha\colon [0,1] \to \mathrm{U}(H)$ with $\alpha(0) = \mathbf{1}$ and $\alpha(1) = u$. Then $h(t,x) := \alpha(t)\gamma(x)$ is a homotopy of γ to a map whose range is contained in H_s. Applying this to $X = \mathbb{S}^k$ and $X = \mathbb{S}^k \times [0,1]$, we conclude that the inclusion map $\mathrm{GL}_p(H_s) \hookrightarrow \mathrm{GL}_p(H)$ induces isomorphism of all homotopy groups. Therefore we may assume that $H = H_s \cong l^2(\mathbb{N}, \mathbb{K})$.

Let e_n, $n \in \mathbb{N}$, be the canonical orthonormal basis and consider the corresponding subspaces $E_n := \mathrm{span}\{e_1, \ldots, e_n\}$. Then the affine subspaces

$$\mathbf{1} + B(E_n) \subseteq \mathbf{1} + B_p(H)$$

form an ascending chain of finite-dimensional subspaces whose union is dense (Exercise!). Now Theorem A.9 implies that the inclusion map

$$\varinjlim \mathrm{GL}(E_n) = \varinjlim (\mathbf{1} + B(E_n)) \cap \mathrm{GL}(H) \to \mathrm{GL}_p(H) = (\mathbf{1} + B_p(H)) \cap \mathrm{GL}(H)$$

is a homotopy equivalence. Hence the assertion follows from Lemma A.7. ∎

The main point in Theorem A.10 is that it permits to describe the homotopy groups of all the groups $\mathrm{GL}_p(H)$ explicitly by the Bott Periodicity Theorem.

Theorem A.11. (Bott Periodicity Theorem) *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $d := \dim_{\mathbb{R}} \mathbb{K}$ and*

$$\mathrm{GL}(\infty, \mathbb{K}) := \varinjlim \mathrm{GL}(n, \mathbb{K}).$$

Then for $k \leq d(n+1) - 3$ and $q \in \mathbb{N}$ the maps

$$\pi_k(\mathrm{GL}(n, \mathbb{K})) \to \pi_k(\mathrm{GL}(n+q, \mathbb{K}))$$

are isomorphism, so that

$$\pi_k(\mathrm{GL}(\infty, \mathbb{K})) \cong \pi_k(\mathrm{GL}(n, \mathbb{K})).$$

Moreover, we have the periodicity relations

$$\pi_{n+2}(\mathrm{GL}(\infty, \mathbb{C})) \cong \pi_n(\mathrm{GL}(\infty, \mathbb{C})), \quad \pi_{n+4}(\mathrm{GL}(\infty, \mathbb{R})) \cong \pi_n(\mathrm{GL}(\infty, \mathbb{H}))$$

and

$$\pi_{n+4}(\mathrm{GL}(\infty, \mathbb{H})) \cong \pi_n(\mathrm{GL}(\infty, \mathbb{R})).$$

Therefore the homotopy groups of $\mathrm{GL}(\infty, \mathbb{K})$ *are determined by the following table:*

	$\mathrm{GL}(\infty, \mathbb{R})$	$\mathrm{GL}(\infty, \mathbb{C})$	$\mathrm{GL}(\infty, \mathbb{H})$
π_0	$\mathbb{Z}_2$	$\{\mathbf{1}\}$	$\{\mathbf{1}\}$
π_1	$\mathbb{Z}_2$	$\mathbb{Z}$	$\{\mathbf{1}\}$
π_2	$\{\mathbf{1}\}$	$\{\mathbf{1}\}$	$\{\mathbf{1}\}$
π_3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

Proof. The first part is [Hu94, Th. 8.4.1] and the second part [Hu94, Cor. 9.5.2 and Rem. 9.5.4]. ■

The preceding theorem implies in particular that for a complex Hilbert space $\pi_1(\mathrm{GL}_p(H)) \cong \mathbb{Z}$, so that it is a natural question how to describe the universal covering group $\widetilde{\mathrm{GL}}_p(H)$. Below we will see how this can be done for $p \in \mathbb{N}$. Here the case $p = 1$ is quite special.

Higher order determinants

Definition A.12. (a) Let H be a Hilbert space and $X \in B_2(H)$. Then $(\mathbf{1} + X)e^{-X} - \mathbf{1} \in B_1(H)$ follows from $\mathbf{1} + X - e^X = X^2(\cdots)$. Hence the *generalized determinant*

$$\det_2(\mathbf{1} + X) := \det\big((\mathbf{1} + X)e^{-X}\big)$$

makes sense for $X \in B_2(H)$ (cf. [Mi89, Prop. 6.2.3]). This means that for $g \in \mathrm{GL}_2(H)$ we have

$$\det_2(g) = \det(ge^{\mathbf{1}-g}).$$

For $g \in \mathrm{GL}_1(H)$ this simplifies to $\det_2(g) = \det(g)e^{\mathrm{tr}(\mathbf{1}-g)}$.
(b) The construction in (a) can be generalized to all $p \in \mathbb{N}$ as follows. For $X \in B_p(H)$ we define

$$\det_p(\mathbf{1} + X) := \det(\mathbf{1} + R_p(X)),$$

where

$$R_p(X) = -\mathbf{1} + (\mathbf{1} + X)\exp\Big(\sum_{j=1}^{p-1}(-1)^j\frac{X^j}{j}\Big).$$

The function $\det_p$ is called the *Carleman–Fredholm determinant of order* p. For $p = 1$ it is simply called the *Fredholm determinant* and for $p = 2$ the *Hilbert-Carleman determinant* ([GGK00, Section IX.1]). Then $R_p(X) \in B_1(H)$ for every $X \in B_p(H)$ because R_p is defined by an everywhere convergent power series which starts with a term of the form $\lambda_p X^p + \lambda_{p+1} X^{p+1} + \ldots \in B_1(H)$. This is most easily seen by observing that

$$\sum_{j=1}^{p-1} (-1)^j \frac{z^j}{j} = -\log(1+z) - \sum_{j=p}^{\infty} (-1)^j \frac{z^j}{j},$$

so that

$$g(z) := -1 + (1+z)\exp\Big(\sum_{j=1}^{p-1} (-1)^j \frac{z^j}{j}\Big) = -1 + \exp\big(-\sum_{j=p}^{\infty} (-1)^j \frac{z^j}{j}\big)$$

for $|z| < 1$. We conclude that $g(z) = z^p f(z)$ for some holomorphic function on the open unit disc in $\mathbb{C}$. Since g is entire, the function f is entire with $g(z) = z^p f(z)$ for all $z \in \mathbb{C}$ which implies that

$$X \mapsto R_p(X) = g(X) \in X^p B(H) \subseteq B_1(H)$$

is a holomorphic function. Therefore

$$\det_p : \mathbf{1} + B_p(H) \to \mathbb{C}$$

is a holomorphic function. ■

Remark A.13. Let $\gamma : \mathbb{C}^\times \to \mathrm{GL}_p(H)$ be the holomorphic group homomorphism from Proposition IV.21. Then

$$\begin{aligned}(\det_p \circ \gamma)(z) &= \det(\gamma(z)) \det\Bigg(\exp\Big(\sum_{j=1}^{p-1} (-1)^j \frac{(\mathbf{1} - \gamma(z))^j}{j}\Big)\Bigg) \\ &= z e^{\mathrm{tr}\left(\sum_{j=1}^{p-1} (-1)^j \frac{(\mathbf{1}-\gamma(z))^j}{j}\right)} = z e^{\sum_{j=1}^{p-1} (-1)^j \frac{(1-z)^j}{j}} = z e^{f(z)}\end{aligned}$$

for some polynomial function $f : \mathbb{C} \to \mathbb{C}$. We conclude that the winding number of the function $\det_p \circ \gamma : \mathbb{C}^\times \to \mathbb{C}^\times$ is 1, and hence that

$$\det_p : \pi_1(\mathrm{GL}_p(H)) \to \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$$

is an isomorphism. This provides a natural construction of the universal covering space by a pullback construction

$$\widetilde{\mathrm{GL}}_p(H) := \{(g, z) \in \mathrm{GL}_p(H) \times \mathbb{C} : \det_p(g) = e^z\}.$$

For $p = 1$ this leads immediately to the group

$$\widetilde{\mathrm{GL}}_1(H) \cong \mathrm{SL}(H) \rtimes \mathbb{C}$$

(cf. Proposition IV.21). ■

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