Functorial Scaling of Ordinal Data

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Abstract

In this paper we investigate how methods of deriving conceptual structures from ordinal data can be justified structurally. For the investigation we choose a categorical approach which is elaborated for suitable functors from the category of all ordinal structures to the category of all closure structures. The main result yields that the best functors are those which correspond to contra-ordinal scaling, a method developed in Formal Concept Analysis [GW99].

1 Ordinal Structures and Ordinal Contexts

Mathematically, the ordinal nature of data is conceived best by *quasi-orders*, i.e., by reflexive transitive binary relations. Therefore, sets carrying a family of quasi-orders have been proposed as basic structures of a mathematical theory for ordinal data in [SW92]. In the present paper, we discuss how conceptual structures can be derived from those "ordinal structures" which may support meaningful interpretations of the represented data. A method of deriving conceptual structures has already been offered in [SW92], but it has not been fully justified as a "coherence-preserving" transformation between the two different structure theories. In the following, we use notions of category theory to approach a solution of the justification problem.

First we recall some basic definitions and results from [SW92].

Definition 1 *An* ordinal structure *is defined as a pair* $\underline{S} := (S, (\leq_n)_{n\in N})$ *where* S *is a set and* $(\leq_n)_{n\in N}$ *is a family of quasi-orders on* S.

Ordinal structures may be transformed, without loss of information, into so-called "ordinal contexts" which belong to the basic structures of Formal Concept Analysis [GW99].

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Definition 2 *An* ordinal context *is defined as a set structure*

$$\mathbb{K} := (G, M, (W_m, \sqsubseteq_m)_{m \in M}, I)$$

consisting of sets G and M, a family $(W_m, \sqsubseteq_m)_{m\in M}$ of ordered sets, and a ternary relation $I \subseteq G \times \bigcup_{m\in M}(\{m\} \times W_m)$ such that $(g, m, v) \in I$ and $(g, m, w) \in I$ always imply $v = w$. The elements of G, M, and W_m are called objects*,* attributes*, and* attribute values*, respectively.*

For each ordinal context $(G, M, (W_m, \sqsubseteq_m)_{m\in M}, I)$, partial maps $m : G \to W_m$ are defined by $m(g) = w :\iff (g, m, w) \in I$. An ordinal context is called *complete* if these maps are total. Every ordinal context can be completed by adding a greatest element "blank" to W_m if the map m is not total. In the following, ordinal contexts are supposed to be complete. An ordinal context can be represented by a data table where the rows and the columns are denoted by the objects and the attributes, respectively, and the value written in row g and column m is just $m(g)$.

Figure 1 shows an ordinal context in which the objects are sights, the attributes are guide-books, and the values are numbers classifying the sights according to the listed guide-books (see [GW99]).

Now, we describe the canonical correspondence between ordinal structures and ordinal contexts:

- Let $\mathbb{K} := (G, M, (W_m, \sqsubseteq_m)_{m\in M}, I)$ be an ordinal context.
  Then $\underline{S}(\mathbb{K}) := (G, (\leq_m)_{m\in M}))$ with $g \leq_m h :\iff m(g) \sqsubseteq_m m(h)$ is an ordinal structure.
- Let $\underline{S} := (S, (\leq_n)_{n\in N}))$ be an ordinal structure. We define
  $E_n := E_{\leq_n}$ to be the equivalence relation generated by the quasi-order $\leq_n$, $\sqsubseteq_n$ to be the order relation on S/E_n with $[x]E_n \sqsubseteq_n [y]E_n :\iff x \leq_n y$, and $I_{\underline{S}} := \{(x, n, [x]E_n) | x \in S, n \in N\}$.
  Then $\mathbb{K}(\underline{S}) := (S, N, (S/E_n, \sqsubseteq_n)_{n\in N}, I_{\underline{S}})$ is an ordinal context.
- For every ordinal structure $\underline{S}$, we obtain $\underline{S}(\mathbb{K}(\underline{S})) = \underline{S}$ and,
  for every ordinal context $\mathbb{K}$, we obtain $\mathbb{K}(\underline{S}(\mathbb{K})) \cong \mathbb{K}$ if and only if the attributes of which are surjective (as maps).

2 Formal Contexts and Closure Structures

Formal Concept Analysis, as presented in [GW99], offers a mathematical theory of conceptual structures which have been proven successful to support meaningful interpretations of data. In particular, it allows to turn ordinal contexts into informative conceptual structures, the so-called "concept lat-

Forum Romanum	Baedecker	Les Guides Bleus	Michelin	Polyglott
1 Arch of Septimus Severus	1	1	2	1
2 Arch of Titus	1	2	2	0
3 Basilica Julia	0	0	1	0
4 Basilica of Maxentius	1	0	0	0
5 Column of Phocas	0	1	2	0
6 Curia	0	0	0	1
7 House of Vestals	0	0	1	0
8 Porticus of the twelve gods	0	1	1	1
9 Temple of Antonius and Faustina	1	1	3	1
10 Temple of Castor and Pollux	1	2	3	1
11 Temple of Romulus	0	1	0	0
12 Temple of Saturn	0	0	2	1
13 Temple of Vespasian	0	0	2	0
14 Temple of Vesta	0	2	2	1

Fig. 1. Ordinal context about sights of the "Forum Romanum"

tices". In this section we recall some basic notions and results from Formal Concept Analysis and show how they can be related to ordinal structures.

Definition 3 *A* formal context *is defined as a set structure* $\mathbb{K} := (G, M, I)$ *consisting of sets* G *and* M *and a binary relation* $I \subseteq G \times M$*. The elements of* G *and* M *are called* objects *and* attributes*, respectively, and the relationship* gIm *is read: the object* g has *the attribute* m*.*

A formal context can be represented by a cross-table where the rows and columns are denoted by the objects and attributes, respectively, and a cross in row g and column m indicates the relationship gIm. Each formal context can be understood as an ordinal context $(G, M, (W_m, \sqsubseteq_m)_{m \in M}, I)$ where W_m is a one-element set for every $m \in M$. The following assignments

$$
\begin{aligned}
A \mapsto A' &:= \{m \in M : gIm \text{ for every } g \in A\} \qquad \text{for } A \subseteq G, \\
B \mapsto B' &:= \{g \in G : gIm \text{ for every } m \in B\} \qquad \text{for } B \subseteq M
\end{aligned}
$$

establish a Galois connection between the power sets of G and M. A *(formal) concept* of the context (G, M, I) is defined as a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$. The set A is called the *extent* and B the *intent* of the concept (A, B). An order relation between the concepts of a context (G, M, I), modelling the *subconcept-superconcept-relation* is defined by

$$(A_1, B_1) \leq (A_2, B_2) :\Longleftrightarrow A_1 \subseteq A_2 \quad (\Longleftrightarrow B_2 \subseteq B_1).$$

The ordered set $\underline{\mathfrak{B}}(G, M, I)$ of all concepts of the context (G, M, I) together with the defined order relation is always a complete lattice, called the *concept lattice* of (G, M, I). The set $\mathfrak{U}(\mathbb{K})$ of all (concept) extents of $\mathbb{K} := (G, M, I)$ ordered by set-theoretical inclusion is denoted by $\underline{\mathfrak{U}}(\mathbb{K})$. Since a concept (A, B) is uniquely determined by its extent A, in particular $A = \bigcap_{m \in B}\{m\}'$, the lattices $\underline{\mathfrak{B}}(G, M, I)$ and $\underline{\mathfrak{U}}(\mathbb{K})$ are naturally isomorphic ($\{m\}'$ is the extent of the concept $(\{m\}', \{m\}'')$ and called the *attribute extent* of m). The fact that, in (G, M, I), intersections of extents are always extents again, yields that $(G, \mathfrak{U}(G, M, I))$ is a "closure structure" within the meaning of the following definition:

Definition 4 *A* closure structure *is defined as a pair* $\underline{\mathfrak{H}} := (S, \mathfrak{H})$ *where* S *is a set and* $\mathfrak{H}$ *is a closure system on* S*, i.e., a set of subsets of* S *containing* S *itself and each intersection of its members. The corresponding* closure operator $H_{\underline{\mathfrak{H}}}$ *is defined by*

$$H_{\underline{\mathfrak{H}}}(A) := \bigcap\{B \in \mathfrak{H} \mid A \subseteq B\} \qquad \text{for } A \subseteq S.$$

Now, we are able to describe a canonical correspondence between closure structures and formal contexts:

- Let $\mathbb{K} := (G, M, I)$ be a formal context.
  Then $\underline{\mathfrak{H}}(\mathbb{K}) := (G, \mathfrak{U}(G, M, I))$ is a closure structure.
- Let $\underline{\mathfrak{H}} := (S, \mathfrak{H})$ be a closure structure.
  Then $\mathbb{K}(\underline{\mathfrak{H}}) := (S, \mathfrak{H}, \in)$ is a formal context.
- For every closure structure $\underline{\mathfrak{H}}$, we obtain $\underline{\mathfrak{H}}(\mathbb{K}(\underline{\mathfrak{H}})) = \underline{\mathfrak{H}}$ and,
  for every formal context $\mathbb{K}$, we obtain $\mathbb{K}(\underline{\mathfrak{H}}(\mathbb{K})) \cong \mathbb{K}$ if and only if, in $\mathbb{K}$, the set of all extents equals the set of all attribute extents (i.e. $\mathfrak{U}(\mathbb{K}) = \{\{m\}' | m \in M\}$).

Because of the correspondence between ordinal structures and ordinal contexts described in Section 1, we are able to establish connections between ordinal structures and formal contexts (together with their closure structures of concept extents) by using methods of *conceptual scaling* offered in [GW89]

(s. also [GW99]). Two frequently used methods of conceptual scaling are "ordinal scaling" and "contra-ordinal scaling":

- By *ordinal scaling*, an ordinal context $\mathbb{K} := (G, M, (W_m, \sqsubseteq_m)_{m \in M}, I)$ is transformed into the formal context $\mathbb{K}_o := (G, M_o, I_o)$ where $M_o := \bigcup_{m \in M} \{m\} \times W_m$ and $gI_o(m, w_m) :\Longleftrightarrow m(g) \sqsubseteq_m w_m$.
- By *contra-ordinal scaling*, an ordinal context $\mathbb{K} := (G, M, (W_m, \sqsubseteq_m)_{m \in M}, I)$ is transformed into the formal context $\mathbb{K}_{co} := (G, M_{co}, I_{co})$ where $M_{co} := \bigtimes_{m \in M} W_m$ and $gI_{co}(w_m)_{m \in M} :\Longleftrightarrow \exists n \in M : n(g) \not\sqsupseteq_n w_n$.

3 Ordinal Closure Functors

In the previous sections, correspondences between ordinal structures and ordinal contexts as well as between closure structures and formal contexts have been described. In addition, by methods of turning ordinal contexts into formal contexts, it was made explicit how we may derive from ordinal contexts conceptual structures, namely the closure structures of concept extents of the resulting contexts (and, even more informative, their concept lattices). All together yields procedures turning ordinal structures into closure structures which allow the interpretation as conceptual hierarchies. Now, we want to investigate how far those procedures can be justified as appropriate transformations of the theory of ordinal structures to the theory of closure structures. For this, we use an understanding of "theories" which is given by suitably chosen categories. Then the considered transformations can be represented by covariant functors which should keep the given information as much as possible. Figure 2 visualizes the transformations which are of interest.

$$\begin{tikzcd}
\begin{array}{l}\text{ordinal structure}\\ \underline{S} := (S, (\leq_n)_{n \in N})\end{array} \arrow[r, shift left] \arrow[d] & \begin{array}{l}\text{ordinal context}\\ \mathbb{K} := (G, M, (W_m, \sqsubseteq_m)_{m \in M}, I)\end{array} \arrow[l, shift left] \arrow[d] \\
\begin{array}{l}\text{closure structure}\\ \underline{\mathfrak{H}} := (S, \mathfrak{H})\end{array} \arrow[r, shift left] & \begin{array}{l}\text{formal context}\\ \mathbb{K} := (G, M, I)\end{array} \arrow[l, shift left]
\end{tikzcd}$$

Fig. 2. Functorial transformations

The full description of Figure 2 as a categorical diagram is given in [PW02]. Here we restrict our considerations to covariant functors from the category OS of all ordinal structures to the category CS of all closure structures. First we have to define the morphism of the categories OS and CS:

Definition 5 *An OS-*morphism *from an ordinal structure $\underline{S} := (S, (\leq_n)_{n \in N}))$ to an ordinal structure $\underline{T} := (T, (\leq_p)_{p \in P}))$ is defined as a pair (φ, ψ) of maps $\varphi : S \to T$ and $\psi : N \to P$ where ψ is surjective and*

$$x \leq_n y \Longrightarrow \varphi(x) \leq_{\psi(n)} \varphi(y) \text{ for all } x, y \in S.$$

Definition 6 *A CS-*morphism *from a closure structure $\underline{\mathfrak{H}} := (S, \mathfrak{H})$ to a closure structure $\underline{\mathfrak{K}} := (T, \mathfrak{K})$ is defined as a map $\varphi : S \to T$ satisfying*

$$x \in H_{\underline{\mathfrak{H}}}(A) \Longrightarrow \varphi(x) \in H_{\underline{\mathfrak{K}}}(\varphi(A)) \text{ for all } x \in S \text{ and } A \subseteq S.$$

By replacing the arrows $\Longrightarrow$ by double arrows $\Longleftrightarrow$ in the definitions above, we obtain the definitions of the so-called OS- and CS-*quasi-embeddings* which are used to specify "information preserving" covariant functors as the so-called "strong ordinal closure functors".

Definition 7 *A covariant functor Γ from the category OS to the category CS is called an* ordinal closure functor *if $\Gamma(\underline{S}) = (S, \mathfrak{H}_{\Gamma(\underline{S})})$ for every ordinal structure $\underline{S}$ and if $\Gamma(\varphi, \psi) = \varphi$ for every OS-morphism (φ, ψ).*
An ordinal closure functor Γ from OS to CS is called strong *if, for every OS-quasi-embedding (φ, ψ) between ordinal structures $\underline{S}$ and $\underline{T}$, φ is a CS-quasi-embedding between the closure structures $\Gamma(\underline{S})$ and $\Gamma(\underline{T})$.*

The strong ordinal closure functors are models for the desired transformations of the theory of ordinal structures to the theory of closure structures. In order to describe all strong ordinal closure functors, we regard their effects to special ordinal structures, namely quasi-ordered sets. Surprisingly, already the three two-element quasi-ordered sets $\underline{S_i} := (S_i, \leq_i)$ (i=0,1,2) with $S_0 = S_1 = S_2 = \{s, t\}$ and

$$\leq_0 := \{(s,s),(t,t)\},\ \leq_1 := \{(s,s),(t,t),(s,t)\},\ \leq_2 := \{(s,s),(t,t),(s,t),(t,s)\}$$

are essential for our investigations. The following theorem allows us to restrict our further investigations to strong ordinal closure functors Γ with

$$(\star) \qquad \Gamma(\underline{S_0}) = (S_2, \mathfrak{P}(S_2)) \text{ and } \Gamma(\underline{S_1}) = (S_2, \{\emptyset, \{s\}, S_2\}).$$

(For the proof of this theorem and the following theorems and propositions we must refer to [PW02] because of space restrictions.)

Theorem 8 *According to their effect to two-element quasi-ordered sets there are four classes of strong ordinal closure functors Γ from OS to CS:*

(1) $\Gamma(\underline{S_0}) = \Gamma(\underline{S_1}) = (S_2, \{\emptyset, S_2\})$,

(2) $\Gamma(\underline{S}_0) = \Gamma(\underline{S}_1) = (S_2, \mathfrak{P}(S_2))$,
(3) $\Gamma(\underline{S}_0) = (S_2, \mathfrak{P}(S_2))$ *and* $\Gamma(\underline{S}_1) = (S_2, \{\emptyset, \{s\}, S_2\})$,
(4) $\Gamma(\underline{S}_0) = (S_2, \mathfrak{P}(S_2))$ *and* $\Gamma(\underline{S}_1) = (S_2, \{\emptyset, \{t\}, S_2\})$ *(dual to (3))*.

For all strong ordinal closure functors, we have $\Gamma(\underline{S}_2) = (S_2, \{\emptyset, S_2\})$.

There is, up to duality, only one non-trivial case, namely the case in which Γ satisfies the condition $(\star)$; let us call Γ in this case a *non-trivial strong ordinal closure functor*. For those functors we can prove the following structure theorem:

Theorem 9 *A non-trivial strong ordinal closure functor* Γ *maps each quasi-ordered set* $\underline{S} := (S, \leq)$ *to the closure system isomorphic to* $(S, \mathfrak{I}(S))$ *where* $\mathfrak{I}(S)$ *is the set of all order ideals of* $\underline{S} := (S, \leq)$.

The structure theorem yields that a non-trivial strong ordinal closure functor is, up to isomorphism, uniquely-defined on all quasi-ordered sets. For lifting this uniqueness result to arbitrary ordinal structures, the following propositions are useful:

Proposition 10 *For a non-trivial strong ordinal closure functor* Γ*, an* OS*-morphism* (φ, ψ) *between quasi-ordered sets* $\underline{S}$ *and* $\underline{T}$ *is an* OS*-quasi-embedding if and only if* φ *is a* CS*-quasi-embedding between* $\Gamma(\underline{S})$ *and* $\Gamma(\underline{T})$.

Proposition 11 *Let* $\underline{S} := (S, (\leq_n)_{n\in N})$ *be an ordinal structure and let* E_n *be the equivalence relation on* S *generated by* $\leq_n$*. Then*

$$(\varphi, \psi) : \underline{S} \to (\bigtimes_{n\in N} (S/E_n), (\sqsubseteq_p)_{p\in N})$$

with $\varphi(x) := ([x]E_n)_{n\in N}$*,* $\psi(p) := p$ *and* $(x_n)_{n\in N} \sqsubseteq_p (y_n)_{n\in N} :\iff x_p \leq_p y_p$ $(p \in N)$ *is an* OS*-quasi-embedding.*

Now, we can summarize our investigations by the following theorem, for which we need one more definition, namely that of the *direct product* of formal contexts: $\bigtimes_{t\in T} (G_t, M_t, I_t) := (\bigtimes_{t\in T} G_t, \bigtimes_{t\in T} M_t, \nabla)$ *where* $(\mathrm{g}_t)_{t\in T} \nabla (m_t)_{t\in T} :\iff \exists s \in T : g_s I_s m_s$.

Theorem 12 (**Main Theorem**) *Let* Γ *be a non-trivial strong ordinal closure functor. Then, for all quasi-ordered sets* $(S, \leq)$ *and* $(S_n, \leq_n)$ $(n \in N)$*, we have*

$$\Gamma(S, \leq) = (S, \mathfrak{U}(S, S, \not\geq)) \quad \text{and}$$

$$\Gamma(\bigtimes_{n\in N} (S_n, \leq_n)) = (\bigtimes_{n\in N} S_n, \mathfrak{U}(\bigtimes_{n\in N} (S_n, S_n, \not\geq_n))).$$

In general, $\Gamma_{co}(\underline{S}) := (S, \mathfrak{U}(\mathbb{K}_{co}))$ *with* $\mathbb{K} := \mathbb{K}(\underline{S})$ *for ordinal structures* $\underline{S}$ *and*

$\Gamma_{co}(\varphi, \psi) := \varphi$ *for OS-morphisms* (φ, ψ) *define a non-trivial strong ordinal closure functor* Γ_{co}.

Our investigations have shown that, from the viewpoint of categorical structure theory, contra-ordinal scaling is the best method for deriving conceptual structures from ordinal data. Of course, contents and purposes might nevertheless suggest other methods.

Finally we want to demonstrate our results by the example in Section 1. There is given an ordinal context describing the classification of sights by different guide-books. The corresponding ordinal structure consists of a set S of objects (i.e. sights) and of four quasi-orders $\leq_i$ $(i = 1, 2, 3, 4)$ on the values of the four attributes (i.e. the guide-books). The defined non-trivial strong ordinal closure functor Γ_{co} maps this ordinal structure to the closure structure represented in Figure 3 by a line diagram of the corresponding lattice of concept extents. A little circle in the diagram represents the extent consisting of all the sights the labels of which are attached to a circle on some descending path starting from the considered circle. According to the Main Theorem, the presented lattice is isomorhic to a sublattice of the direct product of the chains $(\{0,1\}, \leq)$, $(\{0,1,2\}, \leq)$, $(\{0,1,2,3\}, \leq)$, and $(\{0,1\}, \leq)$. In this sublattice, a sight is represented by the quadruple of those numbers which are in the row of the table of Figure 1 headed by the name of the sight. Each $\vee$-irreducible quadruple is also an "attribute label" of a $\wedge$-irreducible element, namely the largest element of the sublattice not above the $\vee$-irreducible quadruple. Via those "attribute labels", the object quadruples obtain their intensional meaning.

For our example, inspite of a missing categorical justification in general, ordinal scaling yields a meaningful conceptual structure too which is represented as a concept lattice in [GW99], p. 45.

References

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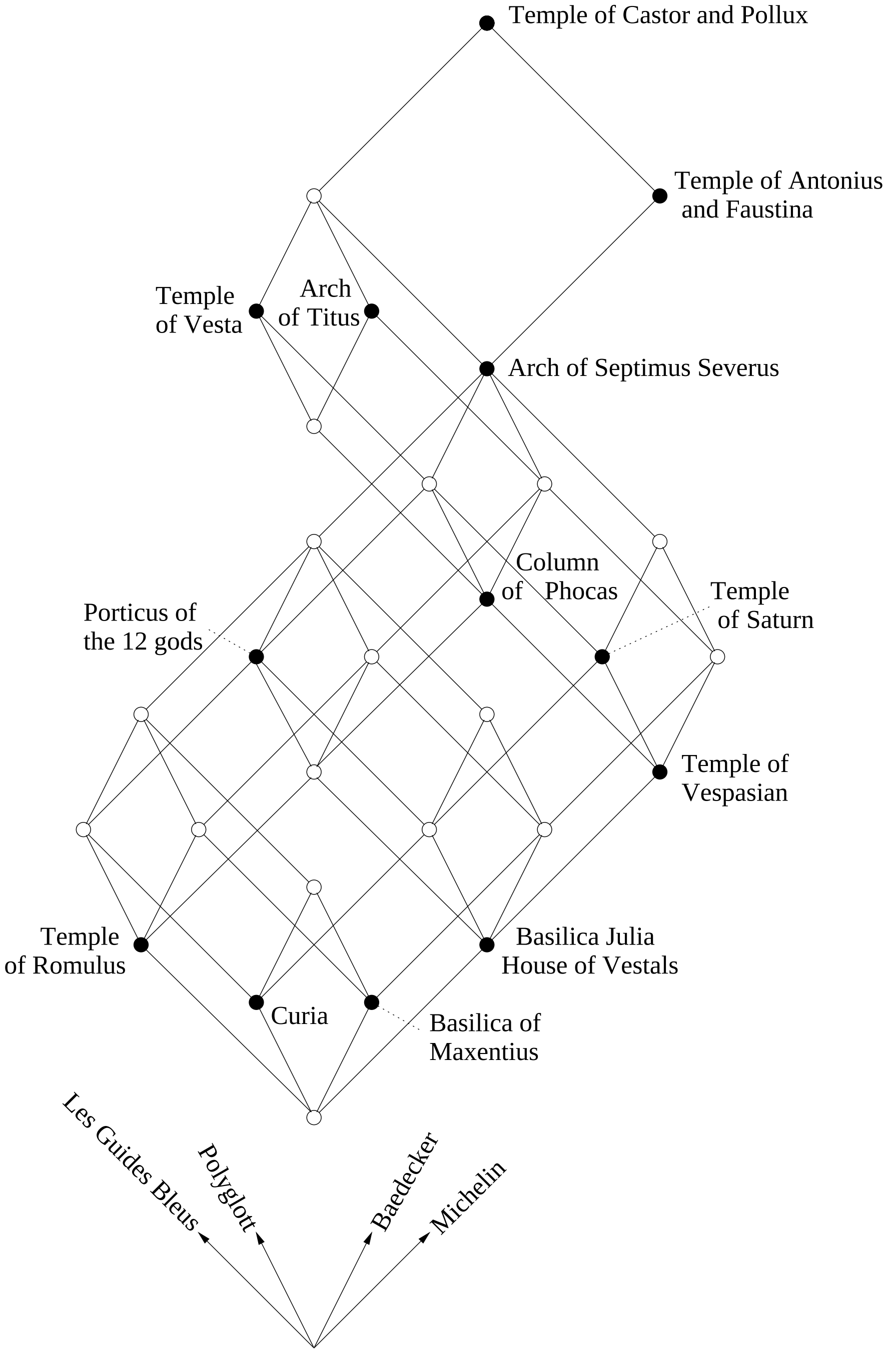

Fig. 3. Closure structure of concept extents of the ordinal data in Figure 1