Mathematical analysis of constitutive equations: Existence and collapse of solutions

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be the set of material points of a solid body. For many materials the history dependent deformation behavior of this body can be modelled by the equations

$$-\operatorname{div}_x T(x,t) = b(x,t) \tag{1.1}$$

$$T(x,t) = \mathcal{D}(\varepsilon(\nabla_x u(x,t)) - \varepsilon_p(x,t)) \tag{1.2}$$

$$\frac{\partial}{\partial t}\varepsilon_p(x,t) = \tilde{g}_1(T(x,t), -\tilde{z}(x,t)) \tag{1.3}$$

$$\frac{\partial}{\partial t}\tilde{z}(x,t) = \tilde{g}_2(T(x,t), -\tilde{z}(x,t)) \tag{1.4}$$

$$\varepsilon_p(x,0) = \varepsilon_p^{(0)}(x), \quad \tilde{z}(x,0) = \tilde{z}^{(0)}(x), \tag{1.5}$$

with the Dirichlet boundary condition

$$u(x,t) = \gamma_D(x,t), \quad (x,t) \in \partial\Omega \times (0,\infty) \tag{1.6}$$

or the Neumann boundary condition

$$T(x,t)n(x) = \gamma_N(x,t), \quad (x,t) \in \partial\Omega \times (0,\infty). \tag{1.7}$$

Here $u(x,t) : \Omega \times (0,\infty) \to \mathbb{R}^3$ denotes the displacement of the material point labeled x at time t. With the 3×3–matrix $\nabla_x u(x,t)$ of first order derivatives of u with respect to the components x_1, x_2, x_3 of x and with the transposed matrix $(\nabla_x u(x,t))^T$ the strain tensor is defined by

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\Big(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\Big) \in \mathcal{S}^3.$$

$\mathcal{S}^3$ is the set of symmetric 3×3–matrices. $T : \Omega \times (0,\infty) \to \mathcal{S}^3$ is the Cauchy stress tensor, $\varepsilon_p(x,t) \in \mathcal{S}^3$ is the plastic strain tensor, $\tilde{z}(x,t) \in \mathbb{R}^N$ is a vector of internal variables. Moreover, $\mathcal{D} : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping, the elasticity tensor, $b : \Omega \times (0,\infty) \to \mathbb{R}^3$ is a given volume force, $\gamma_D : \partial\Omega \times (0,\infty) \to \mathbb{R}^3$ is a given boundary displacement, $\gamma_N : \partial\Omega \times (0,\infty) \to \mathbb{R}^3$ is a given traction at the boundary, and $\varepsilon_p^{(0)}, \tilde{z}^{(0)}$ are given initial data. Finally, $n(x)$ in the Neumann boundary condition denotes the exterior unit normal to $\partial\Omega$ at x. The constitutive equations (1.3), (1.4) with given functions $\tilde{g}_1 : \mathcal{S}^3 \times \mathbb{R}^N \to \mathcal{S}^3$, $\tilde{g}_2 : \mathcal{S}^3 \times \mathbb{R}^N \to \mathbb{R}^N$ determine the inelastic behavior of the body.

A more general class of constitutive models consists of constitutive relations of monotone type. These relations have the form

$$z_t(x,t) \in g\Big(-\rho \nabla_z \, \psi(\varepsilon(\nabla_x \, u(x,t)), z(x,t))\Big)\,, \tag{1.8}$$

where $z = (\varepsilon_p, \tilde{z})$, and where $\rho > 0$ is the constant mass density. $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a given monotone mapping satisfying $0 \in g(0)$, and ψ is the free energy, which is assumed to be a positive definite or positive semi-definite quadratic form

$$\rho\psi(\varepsilon, z) = \frac{1}{2}[D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + (Lz) \cdot z.$$

L is a symmetric, positive definite or positive semi-definite matrix.

The class of constitutive equations of monotone type was introduced in [1]. It generalizes the class of constitutive equations of generalized standard materials introduced by Halphen and Nguyen Quoc Son in [6]. For a generalised standard material, g is the gradient of a convex function.

It turns out that for most materials the free energy is only positive semi-definite, but not positive definite. In this case, if we assume that the constitutive relation is rate dependent and under a minor additional assumption the constitutive equation (1.8) of monotone type can be transformed to the equations (1.3) and (1.4) with a monotone vector field

$$(\varepsilon_p, \tilde{z}) \to (\tilde{g}_1(\varepsilon_p, \tilde{z}), \tilde{g}_2(\varepsilon_p, \tilde{z})) : \mathcal{S}^3 \times \mathbb{R}^N \to \mathcal{S}^3 \times \mathbb{R}^N.$$

For initial-boundary value problem to constitutive equations of monotone type with positive definite free energy a satisfactory and general existence theory is available, cf. [7, 2, 3]. Moreover, the dynamic initial-boundary value problem to positive semi-definite free energy has been treated in a number of articles, cf. [5] for example, to mention just one. The aim of this note is to present existence results for the quasi-static initial-boundary value problems (1.1) – (1.7) with positive semi-definite free energy. For such initial-boundary value problems the existence theory is much less complete.

Proofs are not contained in this note. These are to be published in [3].

2 Existence of solutions for initial-boundary value problems with history functionals

To prove existence of solutions to the initial-boundary value problems (1.1) – (1.7) it is first shown that a more general initial-boundary value problem with constitutive relations given by a history functional can be solved. Subsequentially it is proved that the constitutive equations (1.3) – (1.5) define such a history functional satisfying the conditions needed in the existence proof. To present this approach, we need a few notations:

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with C^1-boundary. T_e denotes a positive number (time of existence), and for $0 \leq t \leq T_e$ we set

$$Z_t = \Omega \times [0,t], \quad Z = Z_{T_e}.$$

By $1 < p,\, q < \infty$ we denote numbers with $\frac{1}{p} + \frac{1}{q} = 1$. The norms on $L^p(\Omega, \mathcal{S}^3)$ and $L^p(\Omega, \mathbb{R}^N)$ are denoted by

$$\|u\|_{p,\Omega} = \Big[\int_\Omega |u(x)|^p \, dx \Big]^{1/p}.$$

The same notation is used for Z instead of Ω. For the scalar product of two symmetric matrices $\sigma, \tau \in \mathcal{S}^3$ we write

$$\sigma \cdot \tau = \sum_{i,j=1}^{3} \sigma_{ij} \tau_{ij} \,.$$

With this notation the canonical bilinear forms on the product spaces $L^p(\Omega, \mathcal{S}^3) \times L^q(\Omega, \mathcal{S}^3)$ and $L^p(Z, \mathcal{S}^3) \times L^q(Z, \mathcal{S}^3)$ are given by

$$(\sigma, \tau)_\Omega = \int_\Omega \sigma(x) \cdot \tau(x) \, dx \,, \quad (\sigma, \tau)_Z = \int_Z \sigma(x,t) \cdot \tau(x,t) \, d(x,t) \,.$$

A history functional is defined as follows:

Definition 2.1 *Let $F(Z, \mathcal{S}^3)$ be the set of all functions from Z to $\mathcal{S}^3$. A mapping $\mathcal{H} : \Delta(\mathcal{H}) \to F(Z, \mathcal{S}^3)$ with $\Delta(\mathcal{H}) \subseteq F(Z, \mathcal{S}^3)$ is called a history functional on $F(Z, \mathcal{S}^3)$, if it has the following property: For all $0 \leq t \leq T_e$ and all $\tau_1, \tau_2 \in \Delta(\mathcal{H})$, which satisfy $\tau_1\big|_{Z_t} = \tau_2\big|_{Z_t}$, it follows that $\mathcal{H}[\tau_1]\big|_{Z_t} = \mathcal{H}[\tau_2]\big|_{Z_t}$.*

With a suitable history functional $\mathcal{H}$ on $F(Z, \mathcal{S}^3)$ the equations (1.1) – (1.5) can now be written in the form

$$-\mathrm{div}_x T = b \tag{2.1}$$
$$T = \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) \tag{2.2}$$
$$\frac{\partial}{\partial t} \varepsilon_p = \mathcal{H}[T] \tag{2.3}$$
$$\varepsilon_p(x,0) = \varepsilon_p^{(0)}(x). \tag{2.4}$$

To see this, let $\tilde{z}^{(0)} : \Omega \to \mathbb{R}^N,\ T : Z \to \mathcal{S}^3$ be given and let $(\tilde{z}, h) : Z \mapsto \mathbb{R}^N \times \mathcal{S}^3$ be a solution of the initial value problem

$$\begin{aligned} h(x,t) &= \tilde{g}_1(T(x,t), -\tilde{z}(x,t)) && (2.5)\\ \frac{\partial}{\partial t}\tilde{z}(x,t) &= \tilde{g}_2(T(x,t), -\tilde{z}(x,t)) && (2.6)\\ \tilde{z}(x,0) &= \tilde{z}^{(0)}(x), && (2.7) \end{aligned}$$

for $x \in \Omega$ and $t \in [0, T_e]$. Then a history functional $\mathcal{H}_{\tilde{z}^{(0)}}$ on $F(Z, \mathcal{S}^3)$ is defined by

$$\mathcal{H}_{\tilde{z}^{(0)}}[T] = h. \qquad (2.8)$$

Insertion of $\mathcal{H}_{\tilde{z}^{(0)}}$ for $\mathcal{H}$ into (2.3) reduces (2.1) – (2.4) to the equations (1.1) – (1.5).

Next we state the existence result for the Dirichlet or Neumann initial-boundary value problems to the equations (2.1) – (2.4) containing an abstract history functional. We need the following assumptions:

For numbers $2 \le p < \infty$ and $1 < q \le 2$ with $\frac{1}{p} + \frac{1}{q} = 1$ let the history functional be a mapping $\mathcal{H} : L^p(Z, \mathcal{S}^3) \to L^q(Z, \mathcal{S}^3)$, which satisfies four conditons:

(H1) There is a constant C such that for all $T \in L^p(Z, \mathcal{S}^3)$

$$\|\mathcal{H}(T)\|_{q,Z} \le C\Big(\|T\|_{p,Z}^{\frac{p}{q}} + 1\Big).$$

(H2) $\mathcal{H}$ is hemicontinuous and monontone with respect to the bilinear form $(\sigma, T)_Z$.

(H3) $\mathcal{H}$ satisfies a first coercivity condition:

$$\frac{\Big(T, \mathcal{H}[T]\Big)_Z}{\|\mathcal{H}[T]\|_{q,Z}} \to \infty \quad \text{for} \quad \|T\|_{p,Z} \to \infty\,.$$

(H4) $\mathcal{H}$ satisfies a second coercivity condition:

$$\frac{\Big(T, \mathcal{H}[T]\Big)_Z}{\|T\|_{p,Z}} \to \infty \quad \text{for} \quad \|T\|_{p,Z} \to \infty\,.$$

Theorem 2.2 *Let $\mathcal{H}$ satisfy the conditons* (H1) – (H4)*, and let $b \subseteq L^p(Z, \mathbb{R}^3)$, $\varepsilon_p^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ be given functions. Then the following assertions hold:*

(i) *For $\gamma_D \in L^p(0, T_e; H_1^p(\Omega, \mathbb{R}^3))$ there is a unique solution*

$$(u, T, \varepsilon_p) \in L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^p(Z, \mathcal{S}^3) \times H_1^q([0, T_e], L^q(\Omega, \mathcal{S}^3))$$

of the Dirichlet problem (2.1) – (2.4), (1.6).

(ii) *Assume that* $\gamma_N \in L^p(\partial\Omega \times [0,T_e], \mathbb{R}^3)$ *and that for all infinitesimal rigid motions* $a + \omega \times x$ *and for almost every* $t \in [0,T_e]$ *the equation*

$$\int_\Omega b(x,t) \cdot (a + \omega \times x)dx + \int_{\partial\Omega} \gamma_N(x,t) \cdot (a + \omega \times x)dS_x = 0 \qquad (2.9)$$

holds. Then there exists a solution

$$(u_0, T, \varepsilon_p) \in L^q(0,T_e; H_1^q(\Omega,\mathbb{R}^3)) \times L^p(Z,\mathcal{S}^3) \times H_1^q([0,T_e], L^q(\Omega,\mathcal{S}^3))$$

of the Neumann problem (2.1) – (2.4), (1.7). *All solutions of this problem are obtained in the form* $(u,T,\varepsilon_p) = (u_0,T,\varepsilon_p) + (w,0,0)$, *where* $w(x,t) = a(t) + \omega(t) \times x$ *with* $a, \omega \in L^q([0,T_e],\mathbb{R}^3)$ *is an infinitesimal rigid motion.*

The proof of this theorem is based on the reduction of the Dirichlet initial-boundary value problem (2.1) – (2.4), (1.6) to an initial value problem for an evolution equation in the space

$$H^q_{\mathrm{sol},D} = \{\sigma \in L^q(\Omega,\mathcal{S}^3) \mid \operatorname{div}(\mathcal{D}\sigma) = 0\},$$

the Neumann initial-boundary value problem is reduced to an initial value problem in the space

$$H^q_{\mathrm{sol},N} = \{\sigma \in L^q(\Omega,\mathcal{S}^3) \mid \operatorname{div}(\mathcal{D}\sigma) = 0\,,\ (\mathcal{D}\sigma)\big|_{\partial\Omega} n = 0\}\,.$$

This initial value problem is

$$\frac{\partial}{\partial t}\tau(t) = Q\mathcal{H}[-\mathcal{D}\tau + \hat{\sigma}](t), \qquad 0 \le t \le T_e \qquad (2.10)$$

$$\tau(0) = Q\varepsilon_p^{(0)}, \qquad (2.11)$$

where $Q : L^q(\Omega,\mathcal{S}^3) \to L^q(\Omega,\mathcal{S}^3)$ is the projection onto the closed subspace $H^q_{\mathrm{sol},D}$ with $\ker Q = \{\varepsilon(\nabla u) \mid u \in \mathring{H}_1^p(\Omega,\mathbb{R}^3)\}$ in the Dirichlet case, and onto the closed subspace $H^q_{\mathrm{sol},N}$ with $\ker Q = \{\varepsilon(\nabla u) \mid u \in H_1^p(\Omega,\mathbb{R}^3)\}$ in the Neumann case. The solution τ satisfies $\tau = Q\varepsilon_p$, and for almost every t the function $x \mapsto \hat{\sigma}(x,t)$ is the Cauchy stress in the solution of the linear boundary value problem, which consists of the equations (2.1), (2.2) with $\varepsilon_p = 0$, and the Dirichlet or Neumann boundary condition. Using arguments from the theory of evolution equations to monotone operators, cf. [4], it can be shown that the initial value problem (2.10), (2.11) has a unique solution.

3 Existence of solutions for two examples of constitutive models

We apply the preceding result to study the existence of solutions to the initial-boundary value problems (1.1) – (1.7) for two examples of constitutive models.

The first example is the Norton-Hoff law:

$$
\begin{aligned}
-\mathrm{div}_x T &= b && (3.1)\\
T &= \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) && (3.2)\\
\frac{\partial}{\partial t}\varepsilon_p &= c|T|^r \frac{T}{|T|} && (3.3)\\
\varepsilon_p(0) &= \varepsilon_p^{(0)}, && (3.4)
\end{aligned}
$$

with constants $c > 0, r > 1$. The second example incorporates kinematic hardening:

$$
\begin{aligned}
-\mathrm{div}_x T &= b && (3.5)\\
T &= \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) && (3.6)\\
\frac{\partial}{\partial t}\varepsilon_p &= c_1|T - k(\varepsilon_p - \varepsilon_n)|^r \frac{T - k(\varepsilon_p - \varepsilon_n)}{|T - k(\varepsilon_p - \varepsilon_n)|} && (3.7)\\
\frac{\partial}{\partial t}\varepsilon_n &= c_2(k|\varepsilon_p - \varepsilon_n|)^\gamma \frac{\varepsilon_p - \varepsilon_n}{|\varepsilon_p - \varepsilon_n|} && (3.8)\\
\varepsilon_p(0) &= \varepsilon_p^{(0)}, \qquad \varepsilon_n(0) = \varepsilon_n^{(0)}, && (3.9)
\end{aligned}
$$

where $c_1, c_2, k > 0$ and $r, \gamma > 1$ are constants, and where the internal variable $\varepsilon_n(x,t) \in \mathcal{S}^3$ is of the type of a strain tensor.

For both models it can be shown that the history functional defined by the constitutive equations satisfies the conditions (H1) – (H4). This yields the following results:

Theorem 3.1 (Norton-Hoff law) *Let $c > 0$ and $r > 1$ be constants, and let $p = 1 + r$, $q = 1 + \frac{1}{r}$. Then under the regularity assumptions for b, $\varepsilon_p^{(0)}$, γ_D and γ_N from Theorem 2.2 the assertions of that theorem also hold for the Dirichlet and Neumann initial-boundary value problems to the equations* (3.1) - (3.4).

Theorem 3.2 (Kinematic hardening) *Let c_1, c_2, k be positive constants and let the constants r and γ satisfy $\gamma > r > 1$. Set $p = 1 + r$, $q = 1 + \frac{1}{r}$, $\hat{p} = 1 + \gamma$, $\hat{q} = 1 + \frac{1}{\gamma}$. Suppose that $b \in L^p(Z, \mathbb{R}^3)$ and $\varepsilon_p^{(0)}, \varepsilon_n^{(0)} \in L^2(\Omega, \mathcal{S}^3)$. Then the following assertions hold:*

(i) *For $\gamma_D \in L^p(0, T_e; H_1^p(\Omega, \mathbb{R}^3))$ there is a unique solution*

$$
\begin{aligned}
(u, T, \varepsilon_p, \varepsilon_n) &\in L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^p(Z, \mathcal{S}^3)\\
&\times H_1^q([0, T_e], L^q(\Omega, \mathcal{S}^3)) \times H_1^{\hat{q}}([0, T_e], L^{\hat{q}}(\Omega, \mathcal{S}^3))
\end{aligned}
$$

of the Dirichlet problem to the equations (3.5) – (3.9).

(ii) *Assume that $\gamma_N \in L^p(\partial\Omega \times [0, T_e], \mathbb{R}^3)$ and that b and γ_N satisfy* (2.9). *Then there exists a solution*

$$
\begin{aligned}
(u_0, T, \varepsilon_p, \varepsilon_n) &\in L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^p(Z, \mathcal{S}^3)\\
&\times H_1^q([0, T_e], L^q(\Omega, \mathcal{S}^3)) \times H_1^{\hat{q}}([0, T_e]; L^{\hat{q}}(\Omega, \mathcal{S}^3))
\end{aligned}
$$

of the Neumann problem to the equations (3.5) – (3.9)*. All solutions of this problem are obtained in the form* $(u, T, \varepsilon_p, \varepsilon_n) = (u_0, T, \varepsilon_p, \varepsilon_n) + (w, 0, 0)$*, where* $w(x,t) = a(t) + \omega(t) \times x$ *with* $a, \omega \in L^q([0, T_e], \mathbb{R}^3)$*.*

4 The coercivity conditions

Here we discuss the meaning of the coercivity conditions (H3) and (H4). For simplicity we only consider the Neumann problem with homogeneous boundary data. Assume that (u, T, ε_p) is a solution of the Neumann initial-boundary value problem to the equations (2.1) – (2.4). For the positive semi-definite free energy

$$\rho\psi(\varepsilon\nabla_x u), \varepsilon_p) = \frac{1}{2}[\mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p)] \cdot (\varepsilon(\nabla_x u) - \varepsilon_p)$$

we then obtain by formal differentiation and partial integration that

$$\frac{d}{dt}\int_\Omega \rho\psi(\varepsilon(\nabla_x u), \varepsilon_p)\, dx = \int_\Omega T \cdot (\varepsilon(\nabla_x u_t) - \varepsilon_{pt})\, dx$$
$$= \int_\Omega (-\mathrm{div}_x T) \cdot u_t - T \cdot \mathcal{H}[T]\, dx = \int_\Omega b \cdot u_t dx - (T, \mathcal{H}[T])_\Omega\,.$$

It follows that $(T, \mathcal{H}[T])_Z$ is the energy dissipated during the time interval $[0, T_e]$ due to plastic deformation.

For the Norton-Hoff law the definition (2.5) – (2.8) of the history functional reduces to the equation

$$\mathcal{H}[T] = f(T)$$

with $f(T) = c|T|^r \frac{T}{|T|}$. Assume now that there is a volume force $b : \Omega \to \mathbb{R}^3$ with the following property: To the stress field $\hat{\sigma}$ in the solution $(\hat{u}, \hat{\sigma})$ of the (time independent) Neumann boundary value problem to the equations (3.1), (3.2) with data b and with $\varepsilon_p = 0$ there is a stress function $(x \to \sigma(x)) \in \mathcal{D}H^p_{\mathrm{sol},N}$ such that $f(\sigma + \hat{\sigma})$ is a gradient field, i.e.

$$f(\sigma + \hat{\sigma}) \in \{\varepsilon(\nabla_x v) \mid v \in H^q_1(\Omega, \mathbb{R}^3)\}\,. \tag{4.1}$$

Thus, to these $\hat{\sigma}$ and σ there is a function v with $\varepsilon(\nabla_x v) = f(\sigma + \hat{\sigma})$. We set

$$\varepsilon^{(0)}_p = \varepsilon(\nabla\hat{u}) - \mathcal{D}^{-1}\sigma.$$

Then (u, T, ε_p) with

$$u(x,t) = \hat{u}(x) + tv(x),\ T(x,t) = \sigma(x) + \hat{\sigma}(x),\ \varepsilon_p(x,t) = tf(\sigma(x) + \hat{\sigma}(x)) + \varepsilon^{(0)}_p(x)$$

is a solution of the homogeneous Neumann initial-boundary value problem to the equations (3.1) – (3.4) with volume force $b(x,t) = b(x)$. Though the volume force

b is constant in time, this solution shows indefinite plastic deformation. Condition (H3) excludes that a sequence $\sigma_n \in H^p_{\text{sol},N}$ exists with $\|\sigma_n\|_{p,\Omega} \to \infty$ for $n \to \infty$, such that $f(\sigma_n + \hat{\sigma}) \in \{\varepsilon(\nabla_x v) \mid v \in H^q_1(\Omega, \mathbb{R}^3)\}$. For, this would imply

$$(\sigma_n + \hat{\sigma}, \mathcal{H}[\sigma_n + \hat{\sigma}])_Z = (\sigma_n + \hat{\sigma}, f(\sigma_n + \hat{\sigma})_Z$$
$$= (\sigma_n + \hat{\sigma}, \varepsilon(\nabla_x v))_Z = (\hat{\sigma}, \varepsilon(\nabla_x v))_Z = (\hat{\sigma}, \mathcal{H}[\sigma_n + \hat{\sigma}])_Z\,,$$

which immediately shows that (H3) cannot be satisfied. Hence, (H3) excludes that solutions of the type just constructed exist for large stress fields $\sigma_n + \hat{\sigma}$.

These observations are connected to the collapse of solutions for elasto-plastic constitutive relations: To see this, replace (3.3) by a rate independent constitutive relation, for example by the Prandtl-Reuss law,

$$\frac{\partial}{\partial t}\varepsilon_p \in \partial\chi(T), \tag{4.2}$$

with the subdifferential $\partial\chi$ of the characteristic function $\chi : \mathcal{S}^3 \to [0, \infty]$ of a closed convex set $K \in \mathcal{S}^3$,

$$\chi(T) = \begin{cases} 0 \quad , & T \in K \\ \infty \quad , & T \in \mathcal{S}^3 \backslash K, \end{cases}$$

cf. [1][p. 31]. Let $b, \hat{u}, \hat{\sigma}, \sigma, \varepsilon_p^{(0)}$ be as above, and instead of (4.1) suppose that there is a nonvanishing function $v \in H^q_1(\Omega, \mathbb{R}^3)$ such that

$$\varepsilon(\nabla_x v(x)) \in \partial\chi(\sigma(x) + \hat{\sigma}(x)) \tag{4.3}$$

for almost all $x \in \Omega$. Then for every function $\kappa : [0, T_e] \to [0, \infty)$ with $\kappa(0) = 0$ the function (u, T, ε_p) defined by

$$u(x,t) = \hat{u}(x) + \kappa(t)v(x),\ T(x,t) = \sigma(x) + \hat{\sigma}(x),\ \varepsilon_p(x,t) = \kappa(t)\varepsilon(\nabla_x v(x)) + \varepsilon_p^{(0)}(x)$$

solves the homogeneous Neumann initial-boundary value problem to the equations (3.1), (3.2), (4.2), (3.4). In particular, for every $T_e > 0$ we can choose κ such that $\kappa(t) \to \infty$ for $t \to T_e$. Thus, if (4.3) holds we can construct solutions which blow up (or collapse, in another terminology) in an arbitrarily short time. Therefore in any existence theory for the Prandtl-Reuss law the possibility that (4.3) can hold must be excluded by a save-load condition slightly stronger than the coercivity condition (H3). It is known that such a safe-load condition restricts the choice of the volume force and the boundary data, cf. [8].

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