Projective Limits of Finite Dimensional Lie Groups

by

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Abstract. For a topological group G we define $\mathcal{N}(G)$ to be the set of all normal subgroups N of G such that G/N is a finite dimensional Lie group. Then G is said to be a *pro-Lie group* if, firstly, G is complete, secondly, $\mathcal{N}(G)$ is a filter basis, and thirdly, every identity neighborhood of G contains some $N \in \mathcal{N}(G)$. It is easy to see that every pro-Lie group G is a projective limit $\lim_{N\in\mathcal{N}(G)} G/N$. The converse emerges as a difficult question, but it is shown here that any projective limit of finite dimensional Lie groups is a pro-Lie group. It is also shown that a closed subgroup H of a pro-Lie group G is a pro-Lie group, and that for any closed normal subgroup N of a pro-Lie group G, for any one parameter subgroup $Y:\mathbb{R} \to G/N$ there is a one parameter subgroup $X:\mathbb{R} \to G$ such that $X(t)N = Y(t)$ for $t \in \mathbb{R}$. It is proved that the category of all pro-Lie groups and continuous group homomorphisms between them is closed under the formation of *all* limits, and that the Lie algebra functor preserves limits and quotients.

1. Introduction

In the theory of compact and locally compact groups it has been customary to study and use "projective limits of Lie groups." By this one means usually that a topological group G is a projective limit of Lie groups if it has arbitrarily small **compact** normal subgroups N such that G/N is a finite dimensional Lie group. Such a group is necessarily locally compact; conversely if G is a locally compact group and U is a compact identity neighborhood, then any closed normal subgroup contained in U is trivially compact. At the root of this intuition of "projective limits of Lie groups" is,

— firstly, the theory of compact groups reaching back to the twenties of the last century (for a recent presentation see [4]),

— secondly, Iwasawa's fundamental paper of 1949 [7] giving decisive structural information on locally compact groups being projective limits of Lie groups *in this sense*, and,

— thirdly, Yamabe's article [10] in which he showed that every locally compact group G for which the factor group G/G_0 modulo the identity component is compact, is indeed a projective limit of Lie groups *in this sense.*

Groups for which G/G_0 is compact are called *almost connected.* All of this was made popular within the horizon of the theory of topological groups through the enormously influential book by Montgomery and Zippin [8].

We say that a topological group G is a *projective limit of Lie groups*, or, equivalently, is *Lie projective* if there is a projective system

$$\{f_{jk}: G_k \to G_j \mid j \le k,\ (j,k) \in J \times J\}$$

for a directed index set J and for finite dimensional Lie groups G_j and if

$$G = \lim_{j\in J} G_j = \{(g_j)_{j\in J} \in \prod_{j\in J} G_j : (\forall j \leq k)\, g_j = f_{jk}(g_k)\}$$

is the projective limit of this system.

We say that G is a *pro-Lie group* if G is a complete topological group and every identity neighborhood contains a normal subgroup N such that G/N is a finite dimensional Lie group, and that the intersection of every two such normal subgroups contains a third of the same type. Every pro-Lie group is a Lie projective group: Indeed let $\mathcal{N}(G)$ denote the filter basis of all N such that G/N is a finite dimensional Lie group. Then the natural quotient maps $G/N \to G/M$ for $M \supseteq N$ in $\mathcal{N}(G)$ form a projective system such that $G \cong \lim_{N\in\mathcal{N}(G)} G/N$. The converse, namely, that a Lie projective group is a pro-Lie group, is far from obvious, as one experiences as soon as one attempts a proof. But in this article, our first order of business is to prove that the two concepts are indeed equivalent.

Both of these concepts are vastly more general than the concept described in the beginning of the introduction. This is illustrated by any infinite power $\mathbb{R}^X$ or, for that matter, by any infinite product of a family of noncompact finite dimensional Lie groups which are pro-Lie groups but are not locally compact. We shall observe here (and give more details in a monograph under preparation [5]) that pro-Lie groups have an excellent Lie Theory in the sense that each pro-Lie group G has a generally infinite dimensional Lie algebra $\mathfrak{L}(G)$ with an exponential function $\exp\colon \mathfrak{L}(G) \to G$ whose image generates a dense subgroup of G_0; we illustrated the usefulness of this sort of Lie theory in our monograph on compact groups [4]. The additive group of $\mathfrak{L}(G)$ is itself a pro-Lie group. The category of all pro-Lie groups will be recognized as being complete and as being the smallest full subcategory of the category of all topological groups and continuous group morphisms (being closed under passing to isomorphic objects) such that it contains all finite dimensional Lie groups. It is relatively simple to prove that the category of Lie projective groups is complete; it seems prohibitively difficult to show *directly* that the category of pro-Lie groups is complete. Thus the category of Lie projective groups has good functorial properties while the category of pro-Lie groups has good structural properties, and it is therefore a great advantage to know that the two categories are indeed one and the same category. It is not easy at all to prove that a closed subgroup H of a pro-Lie group G is again a pro-Lie group, but we show this to be the case here; the stumbling blocks are firstly, that the continuous algebraic isomorphism $H/(H\cap N) \to HN/N$ is not an isomorphism of topological groups in general and, secondly, that an easy criterion is not available that says when a subgroup of a Lie group is an analytic group in the absence of closedness. If G is a pro-Lie group and N is a closed normal subgroup then G/N has arbitrarily small subgroups modulo which this quotient is a finite dimensional Lie group, but, unfortunately, in general it fails to be complete as we show elsewhere ([6] and [5]). Nevertheless we show here that this does not impair the Lie theory of pro-Lie groups in the following sense: If G is a pro-Lie group then the morphism $q\colon G \to G/N$ induces a *surjective* morphism of Lie algebras.

The bottom line: The category of pro-Lie groups is suitable in all respects in which any category of locally compact groups is defective:

- it is closed under *all* limits and contains all finite dimensional Lie groups;
- it has an excellent—albeit in general infinite dimensional—Lie theory;
- it is closed under passing to the additive groups of the Lie algebras.

And, in addition it still has the following property:

- it includes all almost connected locally compact groups and thus is the true background theory for any Lie theory of locally compact groups.

The classical example of a semidirect product $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \rtimes_\sigma \mathbb{Z}$ with the shift action of $\mathbb{Z}$ on the product is a locally compact group which is not a pro-Lie group; certain p-adic Lie groups such as $\mathrm{SL}(2, \mathbb{Q}_p)$ likewise are not pro-Lie groups in our sense.

The second major result in this article concerns the Lie algebra functor $\mathfrak{L}$ from the category of pro-Lie groups into the category of topological Lie algebras. It is not obvious whether or not a connected nondegenerate pro-Lie group G has nondegenerate one-parameter subgroups $\mathbb{R} \to G$ at all, that is, whether its Lie algebra $\mathfrak{L}(G)$ is nonzero. However, we shall show in this paper that for any quotient morphism $f: G \to H$ between pro-Lie groups, the induced morphism of topological Lie algebras $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$ is surjective, and since G has many nondegenerate quotients G/N which are finite dimensional Lie groups, this will answer the question in the affirmative.

It is a consequence of fairly general category theoretical considerations that $\mathfrak{L}$ preserves all limits and thus, notably, preserves kernels. The proof of the surjectivity of $\mathfrak{L}(f)$ for all quotient maps reduces comparatively quickly to the proof that $\mathfrak{L}(f)$ is surjective whenever f is a quotient morphism $G \to \mathbb{R}$. Thus we have to show that every quotient morphism $G \to \mathbb{R}$ splits. The proof of this fact is surprisingly complex, and, not surprisingly, it uses the Axiom of Choice.

For a recent thorough study of very general Lie algebra functors we refer to a forthcoming article by H. Glöckner [2] who discusses and strongly uses projective limits of finite dimensional Lie groups.

2. Projective limits

For a proof of the first background theorem on projective limits, see [1], [2], [3], or [5] 1.27 and 1.33.

Theorem 2.1. (Fundamental Theorem on Projective Limits) *Let $G = \lim_{j\in J} G_j$ be a projective limit of a projective system*

$$\mathcal{P} = \{f_{jk}: G_k \to G_j \mid (j,k) \in J \times J,\, j \le k\}$$

of topological groups with limit morphisms $f_j: G \to G_j$, and let $\mathcal{U}_j$ denote the filter of identity neighborhoods of G_j, $\mathcal{U}$ the filter of identity neighborhoods of G, and $\mathcal{N}$ the set $\{\ker f_j \mid j \in J\}$. Then

(i) *$\mathcal{U}$ has a basis of identity neighborhoods $\{f_k^{-1}(U) \mid k \in J, U \in \mathcal{U}_k\}$.*

(ii) *$\mathcal{N}$ is a filter basis of closed normal subgroups converging to 1.*

If $M \supseteq N$ in $\mathcal{N}$ and if $\nu_{MN}: G/N \to G/M$ is defined by $\nu_{MN}(gN) = gM$, then

$$\{\nu_{MN}: G/N \to G/M \mid (M, N) \in \mathcal{N} \times \mathcal{N},\ M \supseteq N\}$$

is a projective system of topological groups, and there is a unique isomorphism $\eta: \lim_{N\in\mathcal{N}} G/N \to G$ such that the following diagram commutes with $j \leq k$, $M = \ker f_j$, $N = \ker f_k$, and with the morphisms $f'_j: G/\ker f_j \to G_j$ induced by the limit map $f_j: G \to G_j$:

$$\begin{tikzcd}
\cdots & G/M \arrow[d, "f'_j"'] & G/N \arrow[l, "\nu_{MN}"'] \arrow[d, "f'_k"] & \lim_{P\in\mathcal{N}(G)} G/P \arrow[l, "\nu_N"'] \arrow[d, "\eta"] \\
\cdots & G_j & G_k \arrow[l, "f_{jk}"] & G. \arrow[l, "f_k"']
\end{tikzcd}$$

The limit maps ν_N are quotient morphisms.

(iii) *Assume that all bonding maps $f_{jk}: G_j \to G_k$ are quotient morphisms and that all limit maps f_j are surjective. Then the limit maps $f_j: G \to G_j$ are quotient morphisms.*

(iv) *Set $H_j = \overline{f_j(G)}$ for each $j \in J$ and let $f'_{jk}: H_k \to H_j$ be the morphisms defined by f_{jk} for $j \leq k$. Then*

$$\{f'_{jk}: H_k \to H_j \mid (j, k) \in J \times J,\ j \leq k\}$$

is a projective system of topological groups and $G = \lim_{j\in J} H_j$. The limit maps $f'_j: G \to H_j$ are corestrictions of the f_j and they have dense images.

(v) *Assume that all G_j are complete, then so is G.*

(vi) *Let G be a complete topological group and $\mathcal{N}$ a filter basis of closed normal subgroups converging to the identity. Then $\gamma_G: G \to G_{\mathcal{N}}$, $\gamma(g) = (gN)_{N\in\mathcal{N}(G)}$ is an isomorphism. That is, $G \cong \lim_{N\in\mathcal{N}} G/N$.* $\square$

Our next theorem implies, in particular, that a closed subgroup of a projective limit of finite dimensional Lie groups is a projective limit of finite dimensional Lie groups in a natural way. We remind the reader of the following terminology: A filter basis $\mathcal{F}$ in a topological group G is called a *Cauchy filter basis* if for each identity neighborhood U of G there is a member $F \in \mathcal{F}$ such that $FF^{-1} \subseteq U$. (See. e.g. [5], Theorem 1.30 and the paragraph preceding it.)

Theorem 2.2. (The Closed Subgroup Theorem for Projective Limits) *Assume that $\mathcal{N}$ is a filter basis of closed normal subgroups of the complete topological group G and assume that $\lim \mathcal{N} = 1$ and that all quotient groups G/N are complete for $N \in \mathcal{N}$. Let H be a closed subgroup of G. For $N \in \mathcal{N}$ set $H_N = \overline{HN}$. Then the following conclusions hold*

(i) *The isomorphism $\gamma_G: G \to \lim_{N\in\mathcal{N}} G/N$ maps H isomorphically onto $\lim_{N\in\mathcal{N}} H_N/N$.*

(ii) *Under the present hypotheses,*

$$H \cong \lim_{N\in\mathcal{N}} H/(H \cap N) \cong \lim_{N\in\mathcal{N}} HN/N \cong \lim_{N\in\mathcal{N}} \overline{HN}/N.$$

(iii) *The limit maps* $\mu_M: \lim_{N\in\mathcal{N}} HN/N \to HM/M$, $M \in \mathcal{N}$, *are quotient morphisms.*

(iv) *The standard morphisms* $H/(H\cap N) \to HN/N$ *are isomorphisms of topological groups.*

Proof. (i) We note

$$(a)\qquad H_N/N = \overline{H_N/N} \subseteq G/N,$$

and thus H_N/N, as a closed subgroup of a complete group is a complete group. Let $\mathcal{U}$ be the filter of identity neighborhoods of G; for $U \in \mathcal{U}$ find $V \in \mathcal{U}$ such that $VV \subseteq U$. Since $\lim \mathcal{N} = 1$ by hypothesis, there is an $N \in \mathcal{N}$ such that $N \subseteq V$. For any subset A of a topological group, the closure $\overline{A}$ is the intersection of the sets AW where W ranges through all identity neighborhoods. Thus $H_N = \overline{HN} \subseteq HNV \subseteq HVV \subseteq HU$ whence

$$(b)\qquad \bigcap_{N\in\mathcal{N}} H_N = \bigcap_{N\in\mathcal{N}} \overline{HN} \subseteq \bigcap_{U\in\mathcal{U}} HU = \overline{H} = H.$$

For $M \supseteq N$, the bonding map $\nu_{MN}: G/N \to G/M$ induces a bonding map $\mu_{MN}: H_N/N \to H_M/M$ by restriction and corestriction, and

$$(c)\qquad \mathcal{P}_{\mathcal{N}} \stackrel{\text{def}}{=} \{\nu_{MN} : G/N \to G/M \mid (M,N) \in \mathcal{N}\times\mathcal{N}, M \supseteq N\},$$

$$(d)\qquad \mathcal{Q}_{\mathcal{N}} \stackrel{\text{def}}{=} \{\mu_{MN} : H_N/N \to H_M/M \mid (M,N) \in \mathcal{N}\times\mathcal{N}, M \supseteq N\}$$

are projective systems in which the bonding maps have dense image. (In the former system they are of course quotient morphisms. The projective limits are written $\lim_{N\in\mathcal{N}} G/N$ and $\lim_{N\in\mathcal{N}} H_N/N$, respectively. There is a unique morphism

$$\varepsilon: \lim_{N\in\mathcal{N}} H_N/N \to \lim_{N\in\mathcal{N}} G/N, \quad \varepsilon\big((g_N N)_{N\in\mathcal{N}}\big) = (g_N N)_{N\in\mathcal{N}}$$

such that the following diagram commutes:

$$(e)\qquad \begin{array}{ccccccc}
\cdots & H_M/M & \xleftarrow{\mu_{MN}} & H_N/N & \leftarrow & \cdots & \lim_{N\in\mathcal{N}} H_N/N \\
 & {\scriptstyle \mathrm{incl}_M}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{incl}_N} & & & \big\downarrow{\scriptstyle \varepsilon} \\
\cdots & G/M & \xleftarrow{\nu_{MN}} & G/N & \leftarrow & \cdots & \lim_{N\in\mathcal{N}} G/N.
\end{array}$$

Since G is complete, by Theorem 2.1, there is an isomorphism

$$\gamma_G: G \to \lim_{N\in\mathcal{N}} G/N,$$

and there is a a morphism $\delta_H: H \to \lim_{N\in\mathcal{N}} H_N/N$ defined by $\delta_H(h) = (hN)_{N\in\mathcal{N}}$ such that the following diagram commutes:

$$(f)\qquad \begin{array}{ccc}
H & \xrightarrow{\delta_H} & \lim_{N\in\mathcal{N}} H_N/N \\
{\scriptstyle \mathrm{incl}}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\
G & \xrightarrow[\gamma_G]{} & \lim_{N\in\mathcal{N}} G/N.
\end{array}$$

We claim that δ_H is an isomorphism. For this purpose we define a function $\sigma\colon \lim_{N\in\mathcal{N}} H_N/N \to H$ of which we shall show that it is a morphism of topological groups and inverts δ_H.

Let $(g_NN)_{N\in\mathcal{N}} \in \lim_{N\in\mathcal{N}} H_N/N$, that is, $g_N \in H_N$ and $M \supseteq N$ implies $g_M \in g_NM$, equivalently, $g_N \in g_MM$. Then $\mathcal{F} \stackrel{\rm def}{=} \{g_NN : N \in \mathcal{N}\}$ is a Cauchy filter basis in G, and $\mathcal{F}$ does not depend on the particular choice of the representatives g_N of the cosets g_NN, but only on the cosets. Since G is complete by hypothesis, $g = \lim \mathcal{F}$ exists. Note that g is also the limit of the net $(g_N)_{N\in\mathcal{N}}$, irrespective of the choice of the representatives g_N. We claim that $g \in H_N$ for all $N \in \mathcal{N}$. Fix $N \in \mathcal{N}$ and consider $N \supseteq P$ in $\mathcal{N}$. Then $g_PP \subseteq g_NN \subseteq H_N$, for all of these P and thus $g \in \overline{H_N} = H_N$ for all $N \in \mathcal{N}$; this proves the claim. Therefore $g \in \bigcap_{N\in\mathcal{N}} H_N = H$ by (b). We thus define a function $\sigma\colon \lim_{N\in\mathcal{N}} H_N/N \to H$ by setting

$$\sigma\big((g_NN)_{N\in\mathcal{N}}\big) = \lim\{g_NN : N \in \mathcal{N}\}. \tag{g}$$

From this definition it follows that

$$\begin{aligned}\sigma\big((g_NN)_{N\in\mathcal{N}}(g'_NN)_{N\in\mathcal{N}}\big) = \sigma\big((g_Ng'_NN)_{N\in\mathcal{N}}\big) = \lim g_Ng'_N = \\ \lim g_N \lim g'_N = \sigma\big((g_NN)_{N\in\mathcal{N}}\big)\sigma\big((g'_NN)_{N\in\mathcal{N}}\big).\end{aligned}$$

Thus σ is a morphism of groups. Next we show that σ is continuous at the identity. Let $V \in \mathcal{U}$; pick a $U \in \mathcal{U}$ such that $UU \subseteq V$; by Theorem 2.1(i) we may assume that $U = UN = NU$ for some $N \in \mathcal{N}$. Now we define $U_M \subseteq H_M/M$ by $U_M = H_M/M$ for $M \neq N$ and by $U_N = U/N$ and set $\widetilde{U} = (\prod_{M\in\mathcal{N}} U_M) \cap \lim_{M\in\mathcal{N}} H_M/M$. Now let $g = (g_MM)_{M\in\mathcal{N}} \in \widetilde{U}$. Then $g_NN \in U_N = U/N$. Hence for $N \supseteq P$ we have $g_P \in g_NN \subseteq U$. Thus $\sigma(g) = \lim_{M\in\mathcal{N}} g_M \in \overline{U} \subseteq UU \subseteq V$. This concludes the proof of the claim that $\sigma\colon \lim_{N\in\mathcal{N}} H_N/N \to H$ is a morphism of topological groups. For $h \in H$ we have $\sigma\big(\delta_H(h)\big) = \sigma\big((hN)_{N\in\mathcal{N}}\big) = \lim_{N\in\mathcal{N}} h = h$. Now let $g = (g_NN)_{N\in\mathcal{N}}$, then $\delta_H\big(\sigma(g)\big) = \delta_H\big(\lim_{N\in\mathcal{N}} g_N\big) = (hN)_{N\in\mathcal{N}}$ with $h = \lim_{N\in\mathcal{N}} g_N$. If now $N \in \mathcal{N}$ then $N \supseteq P$ implies $g_P \in g_NN$ whence $h = \lim_{P\in\mathcal{N}} g_P \in g_NN$, and thus $hN = g_NN$ for all $N \in \mathcal{N}$. We conclude $\delta_H\big(\sigma(g)\big) = g$. Therefore σ and δ_H are inverses of each other. We have shown that $H \cong \lim_{N\in\mathcal{N}} H_N/N$ where H_N/N is a closed subgroup of G/N for each N from the filter basis $\mathcal{N}$.

(ii) The filter basis $\{H \cap N : N \in \mathcal{N}\}$ in H converges to 1. We know that $\gamma_H\colon H \to \lim_{N\in\mathcal{N}} H/(H\cap N)$, $\gamma_H(h) = \big(h(H\cap N)\big)_{N\in\mathcal{N}}$ is an isomorphism by Theorem 2.1. The bijective morphisms of topological groups $H/(H\cap N) \to HN/N$, $N \in \mathcal{N}$, induce a bijective morphism j in the following sequence of morphisms

$$H \xrightarrow{\gamma_H} \lim_{N\in\mathcal{N}} H/(H\cap N) \xrightarrow{j} \lim_{N\in\mathcal{N}} HN/N \xrightarrow{\rm incl} \lim_{N\in\mathcal{N}} H_N/N \xrightarrow{\sigma} H$$

whose composition is the identity, i.e. $\sigma\circ{\rm incl}\circ j\circ\gamma_H = {\rm id}$, so that ${\rm incl}\circ(j\circ\gamma_H\circ\sigma) = {\rm id}$. Hence the inclusion morphism incl is an isomorphism.

(iii) We must show that the limit morphisms $\mu_M\colon \lim_{N\in\mathcal{N}} HN/N \to HM/M$ are quotient morphisms. Indeed, let U be an identity neighborhood of the limit; since $\lim\mathcal{N} = 1$ by hypothesis, we may assume that there is an $N \subseteq M$ such that $U\ker\mu_N = U$. Then $\mu_N(U)$ is an identity neighborhood of HN/N. Since $\mu_{MN}: HN/N \to HM/M$ is a quotient morphism and $\mu_M = \mu_{MN}\circ\mu_N$ we conclude that $\mu_M(U)$ is open which establishes the claim.

(iv) We must show that

$$\eta\big(h(H\cap N)\big) = hN, \eta: H/(H\cap N) \to HN/N,\ \eta\big(h(H\cap N)\big) = hN,$$

is an isomorphism. In the proof of (ii) we saw that $\delta = j\circ\gamma_H: H \to \lim_{N\in\mathcal{N}} HN/N$ is an isomorphism of topological groups. By that which we just saw, for each $M\in\mathcal{N}$, the morphism $\mu_M\circ j\circ\gamma_H: H \to HM/M$ is a quotient morphism. Its kernel, however, is $H\cap M$. Hence in the canonical decomposition

$$\begin{array}{ccc} H & \xrightarrow{\mu_M\circ j\circ\gamma_H} & HM/M \\ {\scriptstyle\mathrm{quot}}\big\downarrow & & \big\uparrow{\scriptstyle\mathrm{id}_{HM/M}} \\ H/(H\cap M) & \xrightarrow[\eta]{} & G/M, \end{array}$$

the morphism η is an isomorphism. $\square$

Corollary 2.3. *Every closed subgroup H of a pro-Lie group G is Lie projective.*

Proof. We continue the notation of Theorem 2.2. As a closed subgroup of the finite dimensional Lie group G/N, the group $\overline{HN}/N$ is a finite dimensional Lie group. By 2.2(ii) we have $H \cong \lim_{N\in\mathcal{N}(H)} \overline{H}N/N$, and thus H is a projective limit of finite dimensional Lie groups. $\square$

A topological group G is said to be a *proto-Lie group* if the the set $\mathcal{N}(G)$ of all closed normal subgroups N of G such that G/N is a finite dimensional Lie group, is a filter basis converging to 1. Note that it is a pro-Lie group if it is, in addition, complete. A proto-Lie group is densely embedded into a pro-Lie group via $\gamma_G: G \to \lim_{N\in\mathcal{N}(G)} G/N$, $\gamma_G(g) = (gN)_{N\in\mathcal{N}(G)}$. For easy reference we quote the following characterisation of of pro-Lie groups from [5]:

Proposition 2.4. *For a topological group G, the following two conditions are equivalent:*
(i) *G is a proto-Lie group.*
(ii) *There is a filter basis $\mathcal{M}$ of closed normal subgroups converging to 1 such that G/M is a finite-dimensional Lie group for each $M\in\mathcal{M}$.*
If these conditions hold, then $\mathcal{M}$ is cofinal in $\mathcal{N}(G)$. Moreover, if G is complete, then these conditions are equivalent to
(iii) *G is a pro-Lie group.*
If (iii) *holds then $G \cong \lim_{M\in\mathcal{M}} G/M$.*

Proof. Since (i)⇒(ii) is trivial by the definition of a proto-Lie group, we prove (ii)⇒(i). Clearly, $\mathcal{M}\subseteq\mathcal{N}(G)$. We claim that

$$(*)\qquad \big(\forall N\in\mathcal{N}(G)\big)(\exists M\in\mathcal{M})\ N\supseteq M.$$

Firstly, condition (8) implies that $\mathcal{N}(G)$ is closed under finite intersections and hence is a filter basis: Let $N_1, N_2 \in \mathcal{N}(G)$, then by $(*)$ there are subgroups $M_1, M_2 \in \mathcal{M}$ with $N_j \supseteq M_j$ for $j = 1, 2$. Since $\mathcal{M}$ is a filter basis, there is an $M \in \mathcal{M}$ such that $M_1 \cap M_2 \supseteq M$. Hence $N_1 \cap N_2 \supseteq M$. Therefore $G/(N_1 \cap N_2)$ is a quotient group of the finite dimensional Lie group G/M and is therefore itself a finite dimensional Lie group. Hence $N_1 \cap N_2 \in \mathcal{N}(G)$. Secondly, since $\mathcal{M} \subseteq \mathcal{N}(G)$, and since $\mathcal{M}$ converges to 1, the filter basis $\mathcal{N}(G)$ converges to 1 as well. And finally, by $(*)$, $\mathcal{M}$ is cofinal in $\mathcal{N}(G)$, whence $G \cong \lim_{M\in\mathcal{M}} G/M = \lim_{N\in\mathcal{N}(G)} G/N$ by cofinality (see [5], Cofinality Lemma 1.21). Thus it remains to prove $(*)$. So let $N \in \mathcal{N}(G)$ be given. Let $U = UN$ be an open identity neighborhood of G such that UN/N is an identity neighborhood of the finite dimensional Lie group G/N which contains no subgroups other than the singleton one. If $p: G \to G/N$ is the quotient map, then the image filter basis $p(\mathcal{M})$ converges to the identity in G/N. Hence there is an M such that $p(M) \subseteq UN/N$. Then the subgroup $p(M)$ is singleton, that is $M \subseteq N$, which is what we had to show.

If G is complete, then (i) shows that G is a pro-Lie group and by Theorem 2.1(vi) we then know that $G \cong \lim_{M\in\mathcal{M}} G/M$. □

3. Weakly complete vector spaces and Lie algebras.

For the concept of weakly complete vector spaces see [4], p. 319ff. Here is one way of saying what a weakly complete vector space is: A topological vector space is *weakly complete* if there is an isomorphism of topological vector spaces to some product vector space $\mathbb{R}^X$.

Lemma 3.1. *Let $f: V \to W$ be a morphism of weakly complete vector spaces. Then $f(V)$ is a closed vector subspace of W.*

Proof. We have a canonical decomposition

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle j} \\ V/\ker f & \xrightarrow[f']{} & \overline{f(V)}, \end{array}$$

where $q(v) = v + \ker f$, $j(w) = w$, $f'(v + \ker f) = f(v)$. After replacing f by f' we may assume without loss of generality that f is injective and has a dense image. Then f is both a monic and an epic in the category of weakly complete vector spaces since it has a zero cokernel. By the Duality Theorem for Real Vector Spaces (see [4], p. 325, Theorem 7.30) the dual $\widehat{f}: \widehat{W} \to \widehat{V}$ is a linear map between real vector spaces which is both a monic and an epic. But then it is bijective, i.e. is an isomorphism. By duality again, $f = \widehat{\widehat{f}}$ is an isomorphism and thus has an inverse in the category of weakly complete vector spaces. Hence it is bijective. In particular, it is surjective and thus the lemma is proved.

Lemma 3.2. *Let* $\mathfrak{g} = \lim_{k\in J} \mathfrak{g}_k$ *be a projective limit of a projective system*

$$\{\gamma_{jk}: \mathfrak{g}_k \to \mathfrak{g}_j \mid j \le k,\ (j,k) \in J \times J\}$$

of finite dimensional real vector spaces in the category of weakly complete vector spaces. Let $\gamma_j: \mathfrak{g} \to \mathfrak{g}_j$ *denote the limit maps. Then for each* $j \in J$ *there is an index* $k_j \ge j$ *such that* $\gamma_{jk_j}(\mathfrak{g}_k) \subseteq \gamma_j(\mathfrak{g})$.

Proof. By Lemma 3.1 above, $\gamma_j(\mathfrak{g})$ is a closed vector subspace of $\mathfrak{g}_j$. By the Duality Theorem for Real Vector Spaces (see [4], p. 324, Theorem 7.30, statement (ii) is equivalent to the following assertion

(∗) Let $E = \operatorname{colim}_{k\in J} E_k$ be the direct limit of a direct system

$$\{\eta_{jk}: E_j \to E_k \mid j \le k,\ (j,k) \in J \times J\}$$

of finite dimensional vector spaces. Fix an index $j \in J$. Then there is an index $k_j \ge j$ such that η_{jk_j} vanishes on $\ker \eta_j$.

Now E is the directed union of the images $\eta_k(E_k)$. If $x \in E_j$ is such that $\eta_{jk}(x) \ne 0$ for all k, then $\eta_j(x) \ne 0$. Thus for each $x \in \ker \eta_j$ there is a $k_x \ge j$ such that $\eta_{jk_x}(x) = 0$. Since $\dim \ker \eta_j \le \dim E_j$ is finite, $\ker \eta_j$ is finitely generated. Statement (∗) follows. □

We record that for a topological group G, a *one parameter subgroup* is a continuous group morphism $f: \mathbb{R} \to G$.

We shall deal with topological groups that have a Lie algebra. The space $\operatorname{Hom}(\mathbb{R}, G)$ of all one parameter subgroups $X: \mathbb{R} \to G$ endowed with the topology of uniform convergence on compact sets is denoted $\mathfrak{L}(G)$. Accordingly $\mathfrak{L}$ is a limit preserving functor from the category of topological groups to the category of pointed topological spaces. For suitably good specimen of topological groups, the assignment $\mathfrak{L}$ has much better properties, as we shall outline in the following definition. For a real number r we set $\square(r) = r^2$.

Definitions 3.3. Let G be a topological group. Then it is said that G *has a Lie algebra* or, equivalently, that G is a *topological group with a Lie algebra* if the following conditions hold:

(i) For all $X, Y \in \mathfrak{L}(G)$, the following limits exist pointwise:

$$X + Y \stackrel{\text{def}}{=} \lim_{n\to\infty} \left(\left(\frac{1}{n}\cdot X\right)\left(\frac{1}{n}\cdot Y\right)\right)^n, \tag{1}$$

$$[X, Y] \circ \square \stackrel{\text{def}}{=} \lim_{n\to\infty} \operatorname{comm}\left(\frac{1}{n}\cdot X, \frac{1}{n}\cdot Y\right)^{n^2} \tag{2}$$

and $X + Y$, $[X, Y] \in \mathfrak{L}(G)$.

(ii) *Addition* $(X, Y) \mapsto X + Y : \mathfrak{L}(G) \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ and *bracket multiplication* $(X, Y) \mapsto [X, Y] : \mathfrak{L}(G) \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ are continuous.

(iii) With respect to scalar multiplication $\cdot$, addition $+$ and bracket multiplication $[\cdot,\cdot]$ the set $\mathfrak{L}(G)$ is a real Lie algebra. □

In particular, if G has a Lie algebra, then $\mathfrak{L}(G)$ is a *topological Lie algebra*. Note that a topological group G has a Lie algebra if and only if the connected component G_0 of the identity has a Lie algebra.

A Lie algebra is said to be *profinite dimensional* if it is a projective limit of finite dimensional real Lie algebras. The underlying vector space of a profinite dimensional Lie algebra is a weakly complete vector space.

Using the continuity of the functor $\mathfrak{L}$, it is not hard to see that all Lie projective groups have a Lie algebra, and indeed a profinite dimensional one.

We shall have to deal with topological groups G for which we make some standard assumptions:

Notation 3.4. For G there is a projective system

$$\{f_{jk}\colon G_k \to G_j \mid (j,k) \in J \times J,\, j \le k\}$$

of finite dimensional Lie groups G_j such that $G = \lim_{j\in J} G_j$. The limit maps are denoted $f_j\colon G \to G_j$, the kernels $\ker f_j$ of the limit maps will be abreviated by K_j. The finite dimensional Lie algebras $\mathfrak{L}(G_j)$ will be written $\mathfrak{g}_j$. Let us write $\mathfrak{f}_{jk} \stackrel{\text{def}}{=} \mathfrak{L}(f_{jk})$ and $\mathfrak{f}_j \stackrel{\text{def}}{=} \mathfrak{L}(f_j)$.

Proposition 3.5. *There is a projective system*

$$\{\mathfrak{f}_{jk}\colon \mathfrak{g}_k \to \mathfrak{g}_j \mid (j,k) \in J \times J,\, j \le k\}$$

of finite dimensional real Lie algebras and Lie algebra morphisms such that

$$\mathfrak{L}(G) = \lim_{j\in J} \mathfrak{g}_j$$

and that the continuous Lie algebra morphisms $\mathfrak{f}_j\colon \mathfrak{L}(G) \to \mathfrak{g}_j$ *are the limit morphisms.*

Proof. By Theorem [5] 2.25(ii), the functor $\mathfrak{L}$ from the category of all topological groups having a Lie algebra and continuous group morphisms between them to the category of topological Lie algebras is continuous and thus, in particular preserves projective limits. Hence $\mathfrak{L}(G) \cong \lim_{j\in J} \mathfrak{L}(G_j)$, and we may identify the two profinite dimensional Lie algebras. □

We set $\mathfrak{a}_j = \mathfrak{f}_j\big(\mathfrak{L}(G)\big) \subseteq \mathfrak{g}_j$ for each $j \in J$, and let $\alpha_{jk}\colon \mathfrak{a}_k \to \mathfrak{a}_j$ be the morphism of finite dimensional Lie algebras induced by $\mathfrak{f}_{jk}$ for $j \le k$.

Lemma 3.6. *The system*

$$\mathcal{L}' \stackrel{\text{def}}{=} \{\alpha_{jk}\colon \mathfrak{a}_k \to \mathfrak{a}_j \mid (j,k) \in J \times J,\, j \le k\}$$

is a projective system of finite dimensional Lie algebras and surjective bonding maps. We have

$$\mathfrak{L}(G) = \lim_{j\in J} \mathfrak{a}_j.$$

The limit maps $\alpha_j\colon \mathfrak{L}(G) \to \mathfrak{a}_j$ are quotient morphisms.

Proof. We apply the Fundamental Theorem on Projective Limits 2.1(iv) to the system $\mathcal{L}$ and conclude that $\lim \mathcal{L}' = \lim \mathcal{L}$. The limit maps $\alpha_j\colon \mathfrak{L}(G) \to \mathfrak{a}_j$ are surjective and thus are quotient maps ([4], p. 3.26, Theorem 7.30). It also follows that the bonding maps $\alpha_{jk}\colon \mathfrak{a}_k \to \mathfrak{a}_j$ are surjective. □

The following diagram illustrates the situation:

$$\begin{array}{ccccccc}
\mathfrak{a}_j & \xleftarrow[\alpha_{jk}]{} & \mathfrak{a}_k & \cdots & \leftarrow \mathfrak{L}(G) & = & \lim_{j\in J}\mathfrak{a}_j \\
{\scriptstyle\mathrm{incl}_j}\big\downarrow & & {\scriptstyle\mathrm{incl}_k}\big\downarrow & \cdots & \big\downarrow{\scriptstyle\mathrm{id}_{\mathfrak{L}(G)}} & & \\
\mathfrak{g}_j & \xleftarrow[\mathfrak{f}_{jk}]{} & \mathfrak{g}_k & \cdots & \leftarrow \mathfrak{L}(G) & = & \lim_{j\in J}\mathfrak{g}_j.
\end{array}$$

Now we apply Lemma 3.2 and obtain

Lemma 3.7. *Under our general assumptions for $G = \lim_{j\in J} G_j$ we have*

$$(\forall j \in J)(\exists k_j \geq j,\ k_j \in J)\ f_{jk_j}\big((G_{k_j})_0\big) \subseteq A_j.$$

Proof. From 3.2 we have

$$(\forall j \in J)(\exists k_j \geq j,\ k_j \in J)\ \mathfrak{f}_{jk_j}(\mathfrak{g}_{k_j}) \subseteq \mathfrak{a}_j.$$

The assertion now follows from the fact that as a finite dimensional connected Lie group, $G_{k_j})_0$ is algebraically generated by $\exp \mathfrak{g}_{k_j}$ and that A_j is algebraically generated by $\mathfrak{a}_j$. Thus $f_{jk_j}\big((G_{k_j})_0\big) = f_{jk_j}\big(\langle \exp \mathfrak{g}_{k_j}\rangle\big) = \langle \exp \mathfrak{L}(f_{jk_j})(\mathfrak{g}_{k_j})\rangle \subseteq \langle \exp \mathfrak{a}_j\rangle = A_j$. □

Now for each $j \in J$ the subgroup $A_j \stackrel{\text{def}}{=} \langle \exp_{G_j} \mathfrak{L}(G_j)\rangle$ is an analytic subgroup of G_j such that $\mathfrak{L}(A_j) = \mathfrak{a}_j$. (For linear Lie groups a reference is [4], p. 155, Theorem 5.52. The proof there does not depend of the assumption that G is a *linear* Lie group.) The morphisms $f_{jk}\colon G_k \to G_j$ induce morphisms $\psi_{jk} \stackrel{\text{def}}{=} f_{jk}|A_k\colon A_k \to A_j$ with $\mathfrak{L}(\psi_{jk}) = \alpha_{jk}$ and $f^0_{jk}\colon (G_k)_0 \to (G_j)_0$. Then

$$\{\psi_{jk}\colon A_k \to A_j \mid (j,k) \in J \times J,\ j \leq k\}$$

is a projective system of analytic groups; let $A \stackrel{\text{def}}{=} \lim_{j\in J} A_j$ be its limit. Each analytic group carries a topology which is in general finer than the induced topology, making the subgroup A_j into a connected Lie group H_j such that $\mathfrak{L}(H_j) = \mathfrak{L}(A_j) = \mathfrak{a}_j$ and that the morphisms $\psi_{jk}\colon A_j \to A_k$ induce morphisms of Lie groups $\varphi_{jk}\colon H_k \to H_j$ such that $\mathfrak{L}(\varphi_{jk}) = \alpha_{jk}$. We have injective morphisms

$$H_j \xrightarrow{\varepsilon_j} A_j \xrightarrow{\mathrm{incl}_{A_j}} (G_j)_0 \xrightarrow{\mathrm{incl}_{(G_j)_0}} G_j$$

where ε_j is the bijective morphism of topological groups given by $\varepsilon_j(h) = h$ and incl denotes the respective inclusion morphisms.

We consider the projective system

$$\mathcal{H} \stackrel{\text{def}}{=} \{\varphi_{jk}: H_k \to H_j \mid (j,k) \in J \times J,\, j \le k\}$$

of finite dimensional Lie groups and let $H = \lim_{j\in J} H_j$ denote its limit; we note that due to the continuity of the functor $\mathfrak{L}$ we have

$$(\mathfrak{L}) \qquad \mathfrak{L}(H) = \lim_{j\in J} \mathfrak{L}(H_j) = \lim_{j\in J} \mathfrak{a}_j = \mathfrak{L}(G).$$

It is not at all clear at this time that a pro-Lie group is connected if its finite dimensional Lie group quotients are connected. However, we observe the following lemma which we shall presently apply to $H = \lim_{j\in J} H_j$.

Lemma 3.8. *Assume that H is a projective limit $\lim_{j\in J} H_j$ of finite dimensional Lie groups satisfying the following two hypotheses:*
(i) *For all $j \in J$ the Lie group H_j is connected, and*
(ii) *the limit maps $\varphi_j: H \to H_j$, $j \in J$ induce surjective morphisms $\mathfrak{L}(\varphi_j)$: $\mathfrak{L}(H) \to \mathfrak{L}(H_j)$.*
Then H is connected.

Proof. Let $h \in H$ be an arbitrary element of G. We shall show that arbitrarily close to g there are elements from the arc component of the identity of H; thus the arc component of the identity is dense in H and thus H is indeed connected. For a proof let U be any identity neighborhood of H. By 1.27(i) we may assume that such that $U = \varphi_j^{-1}V)$ for some identity neighborhood V of H_j. Since H_j is connected by hypothesis (i), and since any connected finite dimensional Lie group is algebraically generated by the image of its exponential function, there are elements $X_1, \dots, X_n \in \mathfrak{L}(H_j)$ such that $\varphi_j(h) = \exp X_1 \cdots \exp X_n$. By hypothesis (ii) the morphism $\mathfrak{L}(\varphi_j): \mathfrak{L}(H) \to \mathfrak{L}(H_j)$ is surjective, and thus we find elements $Y_m \in \mathfrak{L}(H)$, such that $X_m = \mathfrak{L}(\varphi_j)(Y_m)$ for all $m = 1, \dots, n$. Now $\varphi_j(h) = \exp_{H_j} X_m = \exp_{H_j} \mathfrak{L}(\varphi_j)(Y_m) = \varphi_j(\exp_H Y_m)$in H_j for all m. Let $\alpha: [0,1] \to H$ denote the arc given by $\alpha(t) = \exp_H(t{\cdot}Y_1) \cdots \exp_H(t{\cdot}Y_n)$. Then $\alpha(0) = 1$ and $\alpha(1) = \exp_H Y_1 \cdots \exp_H Y_n \in \varphi_j^{-1}\big(\varphi_j(h)\big) \subseteq hU$. This proves our claim and thus finishes the proof of the lemma. □

Lemma 3.9. *The system*

$$\mathcal{H} \stackrel{\text{def}}{=} \{\varphi_{jk}: H_k \to H_j \mid (j,k) \in J \times J,\, j \le k\}$$

is a projective system of quotient morphisms between finite dimensional connected Lie groups and its limit $H = \lim_{j\in J} H_j$ is a connected pro-Lie group. The limit maps $\varphi_j: H \to H_j$ are quotient morphisms.

Proof. Since all $\mathfrak{L}(\varphi_{jk}) = \alpha_{jk}$ are surjective, the morphisms φ_{jk} are surjective, and since H_k as a connected finite dimensional Lie group is σ-compact and locally compact and H_j is locally compact, by the Open Mapping Theorem (see e.g. [4],

p. 650) the morphisms φ_{jk} are quotient morphisms. Therefore, the limit maps $\varphi_j\colon H \to H_j$ are quotient morphisms by Theorem 2.1(iii). It follows that H is a pro-Lie group and that $\mathcal{B} \stackrel{\text{def}}{=} \{\ker\varphi_j : j \in J\}$ is a cofinite filter basis of $\mathcal{N}(H)$. Now the preceding Lemma 3.8 applies to show that H is connected. ◻

We illustrate the situation in the following diagram showing the limits of the various projective systems we consider:

$$\begin{tikzcd}
H_j \arrow[d, "\varepsilon_j"'] & H_k \arrow[l, "\varphi_{jk}"] \arrow[d, "\varepsilon_k"'] & \cdots & H = \lim_{j\in J} H_j \arrow[d, "\varepsilon"] \\
A_j \arrow[d, "\mathrm{incl}_{A_j}"'] & A_k \arrow[l, "f'_{jk}"'] \arrow[d, "\mathrm{incl}_{A_k}"'] & \cdots & A = \lim_{j\in J} A_j \arrow[d, "\mathrm{incl}_A"] \\
(G_j)_0 \arrow[d, "\mathrm{incl}_{(G_j)_0}"'] & (G_k)_0 \arrow[l, "f^0_{jk}"'] \arrow[d, "\mathrm{incl}_{(G_k)_0}"'] & \cdots & G_0 = \big(\lim_{j\in J}(G_j)_0\big)_0 \arrow[d, "\mathrm{incl}_{G_0}"] \\
G_j & G_k \arrow[l, "f_{jk}"'] & \cdots & G = \lim_{j\in J} G_j.
\end{tikzcd}$$

The universal property of the limit G gives us the morphisms $\varepsilon\colon H \to A$ and the various inclusion morphisms incl filling in diagram (1). Notice that $\mathfrak{L}(H) = \mathfrak{L}(A) = \mathfrak{L}(B) = \mathfrak{L}(G)$ and we may identify $\mathfrak{L}(\varepsilon)$ and the various maps $\mathfrak{L}(\mathrm{incl})$ with $\mathrm{id}_{\mathfrak{L}(G)}$. By the concrete construction of the limits
we have

$$G = \{(g_j)_{j\in J} \in \prod_{j\in J} G_j \mid (\forall j \le k \text{ in } J)\, f_{jk}(g_k) = g_j\},$$

$$A = \{(a_j)_{j\in J} \in \prod_{j\in J} A_j \mid (\forall j \le k \text{ in } J)\, f_{jk}(a_k) = a_j\},$$

$$H = \{(h_j)_{j\in J} \in \prod_{j\in J} H_j \mid (\forall j \le k \text{ in } J)\, f_{jk}(h_k) = h_j\}.$$

Thus A is a subgroup of G and we may identify H with A except that its topology may be finer than the topology induced from G on A.

The situation is again illustrated by the following diagram:

$$\begin{tikzcd}
\mathfrak{L}(H) \arrow[r, "="] \arrow[d, "\exp_H"'] & \mathfrak{L}(A) \arrow[r, "="] \arrow[d, "\exp_A"'] & \mathfrak{L}(G) \arrow[d, "\exp_G"] \\
H \arrow[r, "\varepsilon"'] \arrow[d, "\varphi_j"'] & A \arrow[r, "\mathrm{incl}_A"'] \arrow[d, "\psi_j"'] & G \arrow[d, "f_j"] \\
H_j \arrow[r, "\varepsilon_j"'] & A_j \arrow[r, "\mathrm{incl}_{A_j}"'] & G_j,
\end{tikzcd}$$

where ε and all ε_j are bijective and all incl are embeddings.

For a given Lie projective group $G = \lim_{j\in J} G_j$ a connected pro-Lie group H emerged almost out of nowhere and it is mapped under the bijective morphism ε onto the subgroup A of G. Clearly we must identify this subgroup of G_0.

Lemma 3.10. $H = G_0$.

Proof. By Lemma 3.7,

$$(\forall j \in J)(\exists j \leq k_j \in J)\, f_{jk_j}\big((G_{k_j})_0\big) \subseteq A_j.$$

Now we notice that $(G_{k_j})_0$ is locally arcwise connected and H_j is A_j equipped with the arc component topology (cf. [4], p. 156, Theorem 5.52(iv) and pp. 760 ff.). Hence the restriction and corestriction $f_{jk_j}|(G_{j_k})_0\colon (G_{j_k})_0 \to A_j$ factors through $\varepsilon_j\colon H_j \to A_j$ for a morphism $\overline{f}_{jk_j}\colon (G_{k_j})_0 \to H_j$ such that

$$f^0_{jk_j} \stackrel{\rm def}{=} \mathrm{incl}_{A_j} \circ \varepsilon_j \circ \overline{f}_{jk_j}\colon (G_{k_j})_0 \to (G_j)_0.$$

Temporarily, set

$$(*) \qquad G^0 \stackrel{\rm def}{=} \lim_{j\in J} (G_j)_0 \supseteq G_0$$

in the category of topological groups and continuous morphisms. Thus for each $j \in J$ there is a $k_j \geq j$ and a commutative diagram

$$\begin{tikzcd}
(G_j)_0 & (G_{k_j})_0 \arrow[l, "f^0_{jk_j}"'] \arrow[dl, "\overline{f}_{jk_j}"'] & \cdots & G^0 \arrow[d, "\beta"] \\
H_j \arrow[d, "\mathrm{incl}_j \circ \varepsilon_j"'] & H_{k_j} \arrow[l, "\varphi_{jk_j}"] \arrow[d, "\mathrm{incl}_{k_j} \circ \varepsilon_{k_j}"'] & \cdots & H \arrow[d, "\mathrm{incl}_A \circ \varepsilon"] \\
(G_j)_0 & (G_{k_j})_0 \arrow[l, "f^0_{jk_j}"] & \cdots & G^0.
\end{tikzcd}$$

It follows that there is a morphism $\beta_j \stackrel{\rm def}{=} \overline{f}_{jk_j}|(G_{k_j})_0 \circ f_{k_j}|G^0 : G^0 \to H_j$ which is independent of the choice of k_j in as much as it agrees with $\overline{f}_{jk_j} \circ f_{k_j k} \circ f_k|G^0$ for $k \geq k_j$. We notice that for $j \leq j'$ we get $\beta_j = \varphi_{jj'} \circ \beta'_j\colon G^0 \to H_j$. Thus the universal property of $H = \lim_{j\in J} H_k$ implies the existence of a unique morphism $\beta\colon H \to G^0$ such that $\beta_j = \varphi_j \circ \beta$.

From $\mathrm{incl}_A \circ \varepsilon \circ \overline{f}_{jk} = f_{jk_j}$ we conclude that

$$\mathrm{incl}_A \circ \varepsilon \circ \beta = \mathrm{id}_{G^0}\,.$$

Thus $\mathrm{incl}_A \circ \varepsilon\colon H \to G^0$ is a retraction, and since it is injective, it is an isomorphism. As it is also an inclusion map (except for continuity), we now see that it is an isomorphism. This shows $G^0 = H$. Thus G^0 is connected and so

$$(**) \qquad H = G^0 \subseteq G_0.$$

Now (8) and $(**)$ imply $H = G_0$ □

4. Are Lie Projective Groups Pro-Lie Groups?

For easy reference we recall the definition of $\mathcal{N}(G)$ and complement it in a way that will be useful to us:

Definition 4.1. For a topological group G let

$$\mathcal{N}(G) \stackrel{\text{def}}{=} \{N \trianglelefteq G : G/N \text{ is a Lie group}\},$$
$$\mathcal{N}_0(G) \stackrel{\text{def}}{=} \{N \trianglelefteq G_0 : G_0/N \text{ is a Lie group}\} = \mathcal{N}(G_0). \qquad \square$$

In a pro-Lie group, $\mathcal{N}(G)$ is a filter basis which converges to 1.

We work in the setting of Notation 3.4. Recall that $K_j \stackrel{\text{def}}{=} \ker f_j$ and that for each $j \in J$ we have an injective morphism $G/K_j \to G_j$.

Main Lemma 4.2. (The First Fundamental Lemma) *Let G be projective limit $\lim_{j\in J} G_j$ of finite dimensional Lie groups. Then the following conclusions hold:*
(i) *The identity component G_0 is a pro-Lie group and thus*

$$G_0 \cong \lim_{M\in\mathcal{N}_0(G)} G_0/M.$$

(ii) *Set $\mathcal{M} \stackrel{\text{def}}{=} \{G_0 \cap K_j : j \in J\}$. Then $\mathcal{M}$ is a cofinal subset of $\mathcal{N}_0(G)$; that is, for each $M \in \mathcal{N}_0(G)$ there is a $j \in J$ such that $G_0 \cap K_j \subseteq M$.*
(iii) *For each $j \in J$, the natural map $\big(G_0/(G_0 \cap K_j)\big) \to (G_0K_j)/K_j$ is an isomorphism, the group G_0K_j/K_j is a Lie group and a closed subgroup of G/K_j, and*

$$G_0 = \lim_{j\in J}(G_0K_j)/K_j.$$

Proof. (i) Since G_0 is a pro-Lie group by Lemmas 3.8 and 3.9, we have $G \cong \lim_{M\in\mathcal{N}_0(G)} G_0/M$ by Theorem 2.1(vi).

(ii) Since $\lim \mathcal{N}_0(G) = 1$ and $f_j\colon G \to G_j$ is continuous for each $j \in J$, we have $\lim f_j(\mathcal{N}_0(G)) = 1$. But G_j is a Lie group and thus has no small subgroups. Hence there is an $M \in \mathcal{N}_0(G)$ such that $f_j(M) = \{1\}$, that is, $M \subseteq K_j$. Thus we have a quotient morphism $G_0/M \to G_0/(G_0 \cap K_j)$. Since quotients of finite dimensional Lie groups are Lie groups, $G_0/(G_0 \cap K_j)$ is a Lie group whence $G_0 \cap K_j \in \mathcal{N}_0(G)$ by Definition 4.1. Hence $\mathcal{M} \subseteq \mathcal{N}_0(G)$.

By Theorem 2.1(i) we know that $\lim_{j\in J} K_j = 1$. Then $\lim_{j\in J} G_0 \cap K_j = 1$. Let $M \in \mathcal{N}_0(G)$. Then G_0/M is a Lie group, and thus there is an open identity neighborhood U of G_0 such that $UM = U$ and U/M has no nonsingleton subgroups. Then there is a $j \in J$ such that $G_0 \cap K_j \subseteq U$. Since $(G_0 \cap K_j)M/M$ is a subgroup of G_0/M contained in U/M we have $G_0 \cap K_j \subseteq M$.

(iii) By (ii) above, $G_0/(G_0 \cap K_j)$ is a finite dimensional Lie group. We set $\mathcal{N} \stackrel{\text{def}}{=} \{K_j : j \in J\}$. By 2.1 we know that $\lim \mathcal{N} = 1$. So we can apply Theorem 2.2 with $H = G_0$. In particular, 2.2(iv) yields the assertions of (iii). $\square$

Note that we have shown, in particular, that every connected Lie projective group is a pro-Lie group.

A topological group G is said to be *protodiscrete* if the filter basis of open normal subgroups converges to 1. If G is in addition complete, it is called *prodiscrete.*

Proposition 4.3. (a) *For a Lie projective group G, the following statements are equivalent:*
(i) *G is prodiscrete.*
(ii) *G is zero dimensional.*
(iii) *G is totally disconnected.*
(iv) $\mathfrak{L}(G) = \{0\}$.

(b) *A quotient of a protodiscrete group is protodiscrete.*

Proof. First we prove (a):
(i)$\Rightarrow$(ii): By (i) G is a closed subgroup of a product of discrete groups and therefore the filter of its identity neighborhoods has a basis of open subgroups.

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv): This is clear.

(iv)$\Rightarrow$(i): Let $G = \lim_{j\in J} G_j$ with a projective system as in 3.4 and assume that the limit maps $f_j\colon G \to G_j$ have dense images. Let D_j be the discrete group $G_j/(G_j)_0$ and let

$$\mathcal{D} = \{F_{jk}\colon D_k \to D_j \mid j \le k,\ (j,k) \in J \times J\}$$

be the projective system induced by $\mathcal{P}$ and let $D = \lim_{j\in J} D_j$. Then each quotient $D/\ker F_j$ for the limit maps $F_j\colon D \to D_j$ is discrete, and $D = \lim_{j\in J} D/\ker F_j$ by Theorem 2.1(ii). Hence D is a prodiscrete group. Now by hypothesis (v) we have $\{0\} = \mathfrak{L}(G) = \lim_{j\in J} \mathfrak{L}(G_j)$. Then by Lemma 3.6, for each $j \in J$, there is a $k_j \ge j$ such that $\mathfrak{f}_{jk_j}(\mathfrak{g}_{k_j}) = \{0\}$, i.e. $f_{jk}\big((G_{k_j})_0\big) = \{1\}$. Thus f_{jk_j} factors through a morphism $\overline{F}_{jk_j}\colon D_{jk_j} \to G_j$. We have a diagram

$$\begin{tikzcd}
G_j \arrow[d, "q_j"'] & G_{k_j} \arrow[l, "f_{jk_j}"'] \arrow[d, "q_k"'] & \cdots & G \arrow[d, "q"] \\
D_j & D_{k_j} \arrow[l, "F_{jk_j}"'] \arrow[dl, "\overline{F}_{jk_j}"'] & \cdots & D \arrow[d, "p"] \\
G_j & G_{k_j} \arrow[l, "f_{jk}"] & \cdots & G.
\end{tikzcd}$$

By an argument entirely analogous to that in the proof of 3.9 regarding the diagram in that proof we conclude the existence of a morphism $p\colon D \to G$ such that $p \circ q = \mathrm{id}_G$. Thus G is a retract of D. But retracts of prodiscrete groups are easily seen to be prodiscrete. This completes the proof.

Proof of (b): If G is protodiscrete, $\mathcal{N}(G)$ is a filter basis of open normal subgroups which converges to 1. Now let N be a closed normal subgroup of G. Define $\mathcal{U} = \{NU/U : U \in \mathcal{N}(G)\}$. Each NU is an open and hence closed normal subgroup of G and thus the NU/U are open-closed subgroups of G/N, and we claim that $\mathcal{U}$ converges to the identity of G/N. Let W be an open identity neighborhood of G/N and V its full inverse image in G/N. Then V is an open identity neighborhood of G such that $NV = V$. Since $\mathcal{N}(G)$ converges to 1, there is a $U \in \mathcal{N}(G)$ such that $U \subseteq V$. Then $NU \subseteq NV = V$ and thus $NU/N \subseteq W$. This proves the claim and proves (b) in view of 2.4. □

Main Lemma 4.4. (The Second Fundamental Lemma) *For any Lie projective group G, the component factor group G/G_0 is protodiscrete; if it is complete, then it is prodiscrete.*

Proof. We retain the notation of the proof of Proposition 4.3 and consider the commutative diagram

$$\begin{array}{ccccccc}
(G_j)_0 & \xleftarrow{f^0_{jk}} & (G_k)_0 & \cdots & G_0 & = & \lim_{j\in J}(G_j)_0 \\
\downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} & & \\
G_j & \xleftarrow{f_{jk}} & G_k & \cdots & G & = & \lim_{j\in J} G_j \\
\downarrow{\scriptstyle \mathrm{quot}} & & \downarrow{\scriptstyle \mathrm{quot}} & & \downarrow{\scriptstyle \rho} & & \\
D_j & \xleftarrow{F_{jk}} & D_k & \cdots & D & = & \lim_{j\in J} D_j.
\end{array}$$

The morphism $\rho\colon G \to D$ is the fill-in map given by the universal property of the limit in the last row. Since the composition

$$(G_j)_0 \xrightarrow{\mathrm{incl}} G_j \xrightarrow{quot} D_j$$

is constant, so is the composition

$$G_0 \xrightarrow{\mathrm{incl}} G \xrightarrow{\rho} D.$$

Hence we have a unique morphism $p\colon G/G_0 \to D$, $p(gG_0) = \rho(g)$. Assume that $g = (g_j)_{j\in J} \in G$ is such that $p\big(gG_0\big) = 1$, i.e., $(g_j(G_j)_0)_{j\in J} = \rho(g) = 1$ in $\lim_{j\in J} D_j$; thus $g \in \bigcap_{j\in J} f_j^{-1}\big((G_j)_0\big) = \lim_{j\in J}(G_j)_0 = G_0$. This shows that p is injective. The sets $F_j^{-1}(1)$ are basic identity neighborhoods of D by 2.1(i). As $p^{-1}F_j^{-1}(1) = f_j^{-1}\big((G_j)_0\big)/G_0$ and this is an open-closed subgroup we see that p is an embedding. Therefore G/G_0 may be identified with the subgroup $S \stackrel{\mathrm{def}}{=} \operatorname{im}\rho = \{(g_j(G_j)_0)_{j\in J} : (g_j)_{j\in J} \in G\}$ of D.

Let $N_j = F_j^{-1}(1)$. Then N_j is an open-closed normal subgroup of D and $S \cap N_j$ is an open-closed normal subgroup of S. Since $\lim_{j\in J} N_j = 1$ we have $\lim_{j\in J} S \cap N_j = 1$. Hence $G/G_0 \cong S$ is a protodiscrete group and $\overline{S} = \bigcap_{j\in J} SN_j$ is prodiscrete. If G/G_0 is complete, then $G/G_0 \cong \overline{S}$ and G/G_0 is prodiscrete. □

Before we continue, we record an independent elementary lemma:

Lemma 4.5. *Let $f: A \to B$ be a quotient morphism of topological groups with discrete kernel. Then there is an open symmetric identity neighborhood V of A and an open symmetric identity neighborhood W of B such that $f|V): V \to W$ is a homeomorphism, and for every subgroup K of B contained in W there is a subgroup S of A contained in V such that $f(S) = K$.*

Proof. Let U be a symmetric open identity neighborhood of A such that $U^2 \cap \ker f = \{1\}$. Then $f(U)$ is an open symmetric identity neighborhood of B. Then $f|U: U \to f(U)$ is continuous, open and surjective; if $u_1, u-2 \in U$ and $f(u_1) = f(u_2)$, then $u_1u_2^{-1} \in (\ker f) \cap U^2$. Thus $f|U$ is a homeomorphism. Now let V be an open symmetric identity neighborhood in A such that $V^2 \subseteq U$, and set $W \stackrel{\text{def}}{=} f(V)$. Then $f|V: V \to W$ is a homeomorphism onto an open identity neighborhood of B. Define $\varphi: W \to V$ to be its inverse and take $w_1, w_2 \in V$ such that $w_1w_2 \in W$. Set $v_j = \varphi(w_j)$, $j = 1, 2$ and $v = \varphi(w_1w_2)$. Then $(f|U)(v) = (f|V)\varphi(w_1w_2) = w_1w_2$. Further $v_1v_2 \in V^2 \subseteq U$. Then $(f|U)(v_1v_2) = f(v_1)f(v_2) = (f|V)\varphi(w_1)(f|V)\varphi(w_2) = w_1w_2 = (f|U)(v)$. Since $(f|U)$ is injective, we conclude $v = v_1v_2$, that is, $\varphi: W \to V$ is a homeomoprhism such that

$$(*) \qquad (\forall w_1, w_2 \in W)\,(w_1w_2 \in W) \Rightarrow (\varphi(w_1w_2) = \varphi(w_1)\varphi(w_2).$$

In particular, if $w \in W$ then $w^{-1} \in W$ and $ww^{-1} = 1 \in W$ and thus $\varphi(w)\varphi(w^{-1}) = \varphi(1) = 1$ and thus $\varphi(w^{-1}) = \varphi(w)^{-1}$. Now let K be a subgroup of B contained in W. Let $g_1, g_2 \in \varphi(K)$ Then there are elements $w_1, w_2 \in K \subseteq W$ such that $g_j = \varphi(k_j)$, $j = 1, 2$ and $k_1k_2^{-1} \in K \subseteq W$. Hence $g_1g_2^{-1} = \varphi(w_1)\varphi(w_2)^{-1} = \varphi(w_1)\varphi(w_2^{-1}) = \varphi(w_1w_2^{-1}) \in \varphi(K)$. It follows that $\varphi(K) \subseteq V$ is a subgroup of A. ∎

Main Lemma 4.6. (The Third Fundamental Lemma) *Let G be a topological group such that G_0 is a finite dimensional Lie group and assume that $f: G \to L$ is an injection into a finite dimensional Lie group. If G/G_0 is a protodiscrete group, then G is a finite dimensional Lie group.*

Proof. We must show that G_0 is open in G. First we make some reductions. Since $f^{-1}(L_0)$ is open in G there is no loss in assuming that $L = L_0$, i.e. that L is connected.

Let $M = \overline{f(G_0)}$, then M is a closed normal subgroup of L and f induces an injective map $G/f^{-1}(M) \to L/M$. Now $G/f^{-1}(M)$, being a quotient of the protodiscrete group G/G_0 is protodiscrete by 4.3(b) and is, at the same time, without small subgroups. Hence it is discrete, that is, $f^{-1}(M)$ is open. We may therefore assume that $G = f^{-1}(M)$, i.e. that $M = L$. Thus we may assume that $f(G_0)$ is dense in L.

Now we consider the universal covering $q: \widetilde{L} \to L$ and form the pullback

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d, "Q"'] & \widetilde{L} \arrow[d, "q"] \\ G \arrow[r, "f"'] & L. \end{tikzcd}$$

In terms of elements, we have $p = \{(g, \widetilde{\ell}) \in G \times \widetilde{L} : f(g) = q(\widetilde{\ell})\}$. If $p = (g, \widetilde{\ell}) \in P$ and $F(p) = 1$, then $\widetilde{\ell} = 1$, whence $f(g) = q(\widetilde{\ell}) = 1$ and thus $g = 1$ as f is injective. Thus $p = (1,1)$ and this shows that F is injective.

Next F maps $\ker Q$ isomorphically onto $\ker q$. Indeed let $p = (g, \widetilde{\ell}) \in \ker Q$. Then $1 = Q(p) = g$ and then $qF(p) = q(\ell) = f(g) = 1$, that is, $F(p) \in \ker q$. Conversely, if $\widetilde{\ell}q \in \ker q$, then $1 = q(\widetilde{\ell}) = f(1)$, whence $p \stackrel{\text{def}}{=} (1, q) \in P$, and $Q(p) = 1$, i.e. $p \in \ker Q$ and $F(p) = q$. Now let V is an identity neighborhood of $\widetilde{L}$ such that $V \cap \ker q = \{1\}$ and assume that $p = (g, \widetilde{\ell}) \in \ker Q \cap G \times V$; then $g = Q(p) = 1$ and $1 = f(g) = q(\widetilde{\ell})$; thus $\widetilde{\ell} \in V \cap \ker q = \{1\}$. Thus $p = 1$. Theferefore $\ker Q$ is discrete in P. If $(U \times V) \cap P$ is an identity neighborhood then $p = (u, v) \in P$ implies $Q(p) = u$ and $f(u) = q(v)$, whence $Q\big((U \times V) \cap P\big) U \cap f^{-1}q(V)$, and this is an identity neighborhood. Thus the morphism Q is open and thus, since its kernel is open and hence implements a local isomorphism.

Therefore G is a Lie group if and only if P is a Lie group. Thus we must show that P is a Lie group, that is, that P_0 is open.

Now $F(P_0)$ is a normal analytic subgroup of $\widetilde{L}$, and normal analytic subgroups in simply connected Lie groups are closed. The full inverse of $f(G_0)$ in $\widetilde{L}$ is $F(P_0) \ker q$, and thus this group is dense, and $F(P_0) \ker q / F(P_0)$ is dense in $\widetilde{L}/F(P_0)$. Since $\ker q$ is central in $\widetilde{L}$, the group $\widetilde{L}/F(P_0)$ is abelian and simply connected, hence is isomorphic to a vector group $\mathbb{R}^n$. Thus F induces an injective morphism of P/P_0 into the vector group $\widetilde{L}/F(P_0)$ and thus has no small subgroups. The quotient morphism

$$P \xrightarrow{Q} G \xrightarrow{\mathrm{quot}_G} G/G_0$$

vanishes on P_0 and therefore factors through P/P_0:

$$\mathrm{quot}_G \circ Q = (P \xrightarrow{\mathrm{quot}_P} P/P_0 \xrightarrow{Q^*} G/G_0).$$

We have $\ker Q^* = P_1/P_0$ where $P_1 = Q^{-1}G_0$. The following is a diagram of abelian topological groups

$$\begin{array}{ccc} Q/Q_0 & \xrightarrow{\iota} & \mathbb{R}^n \\ \Big\downarrow{\scriptstyle Q^*} & & \\ G/G_0, & & \end{array} \qquad \text{where } \iota \text{ is an injective morphism.}$$

The morphism $Q' \stackrel{\text{def}}{=} Q|P_1 : P_1 \to G_0$ is a covering morphism of the Lie group G_0 with kernel $\ker Q \cong \ker q$ and thus is a Lie group containing the closed normal subgroup $P_0 = (P_1)_0$. Then $\ker Q^* = P_1/P_0$ is a totally disconnected Lie group and is therefore discrete. Since G/G_0 has arbitrarily small open subgroups by the hypothesis of protodiscreteness, Lemma 4.5 applies to Q^* and shows that P/P_0 has arbitrarily small open subgroups (that is, P/P_0 is a protodiscrete group). But ι injects P/P_0 into $\mathbb{R}^n$, and thus P/P_0 has an identity neighborhood in which the singleton group $\{P_0\}$ is the only subgroup; this subgroup, therefore, is open and thus P_0 is open which is what we had to show. □

Now we are ready for the principal result of the first part of the article.

Theorem 4.7. (The Pro-Lie Group Theorem) *Every Lie projective group is a pro-Lie group.*

Proof. By the First Fundamental Lemma 4.2, a Lie projective group G has a filter basis $\mathcal{M}$ of closed normal subgroups M converging to 1 such that G_0M/M is a connected Lie subgroup of G/M, and that there is an injective morphism of G/M into a finite dimensional Lie group. By the Second Fundamental Lemma 4.4, G/G_0 has a basis $\mathcal{O}$ of open normal subgroups converging to the identity. It follows that for each $M \in \mathcal{M}$, the factor group G/M has a filter basis of open normal subgroups U/M such that every open set V containing G_0M/M contains one of the $U/M, U \in \mathcal{O}$. Thus every G/M, $M \in \mathcal{M}$, satisfies the hypotheses of the Third Fundamental Lemma 4.6. As a consequence of 4.6, G/M is a finite dimensional Lie group. Then by Proposition 2.4, it follows that G is a pro-Lie group. □

Corollary 4.8. (The Closed Subgroup Theorem for pro-Lie Groups) *A closed subgroup of a pro-Lie group is a pro-Lie group.*

Proof. This immediate from Corollary 2.3 and Theorem 4.6. □

The Lie algebra $\mathfrak{L}(G)$ of a pro-Lie group is $\lim_{N \in \mathcal{N}(G)} \mathfrak{L}(G/N)$ with finite dimensional Lie algebras $\mathfrak{L}(G/N)$ since $\mathfrak{L}$ preserve limits. So the additive group of $\mathfrak{L}(G)$ is a Lie projective group. Hence it is a pro-Lie group by 4.6 and we may conclude what is also observed in [5]:

Corollary 4.9. *The underlying topological vector space of the Lie algebra of a pro-Lie group is a pro-Lie group in its own right and is a weakly complete topological vector space.* □

5. The category of pro-Lie groups is complete.

We shall henceforth denote by pro$\mathbb{LIEGR}$ the full subcategory of the category $\mathbb{TOPGR}$ of all topological groups and continuous group homomorphisms between them whose objects are pro-Lie groups. After the Pro-Lie Group Theorem 4.7, pro$\mathbb{LIEGR}$ can also be described as the full subcategory of $\mathbb{TOPGR}$ of all projective limits of finite dimensional Lie groups.

We begin with a basic lemma on limits in categories. Recall that a category is said to be *complete* if it has all limits.

Lemma 5.1. (The Limit Existence Theorem)

(i) *If a category has arbitrary products and equalizers, then it is complete.*

(ii) *If a category has arbitrary products and has intersections of retracts, then it is complete.*

(iii) *If a full subcategory $\mathcal{A}$ of a complete category $\mathcal{C}$ is closed in $\mathcal{C}$ under the formation of products and passing to intersections of retracts, then it is closed under the formation of all limits and is therefore complete.*

Proof. We refer to any source on category theory or to [4], Appendix 3, or to [5] Theorem 1.10. ◻

Theorem 5.2. (Completeness Theorem for pro-Lie Groups)

(i) *The category* pro$\mathbb{LIEGR}$ *of pro-Lie groups is closed in* $\mathbb{TOPGR}$ *under all limits and is therefore complete.*

(ii) pro$\mathbb{LIEGR}$ *is the smallest full subcategory of* $\mathbb{TOPGR}$ *that contains all finite dimensional Lie groups and is closed under the formation of all limits.*

Proof. (i) We shall invoke Lemma 5.1(iii) and show that pro$\mathbb{LIEGR}$ is closed in $\mathbb{TOPGR}$ under the formation of products and the passing to closed subgroups; since any retract of a topological group in $\mathbb{TOPGR}$ is a closed subgroup, this will settle the claim. But by Corollary 4.9, the category of pro-Lie groups is closed under the passage to closed subgroups so it remains to show that pro$\mathbb{LIEGR}$ is closed in $\mathbb{TOPGR}$ under the formation of arbitrary products.

So let $\{G_\alpha : \alpha \in A\}$ be a family of Lie projective groups. We must show that $G \overset{\text{def}}{=} \prod_{\alpha\in A} G_\alpha$ is a Lie projective group. Since every G_α is a projective limit of finite dimensional Lie groups, it is a closed subgroup of a product $\prod_{j\in J^\alpha} L_j^\alpha$ of finite dimensional Lie groups. Thus G is isomorphic to a closed subgroup of a product $P = \prod_{\alpha\in A, j\in J^\alpha} L_j^\alpha$ of finite dimensional Lie groups. Then P is is the projective limit of the projective system of all finite partial products and the corresponding projections. Hence P is Lie projective and thus is a pro-Lie group by the Pro-Lie-Group Theorem 4.7. Since G a a closed subgroup of P it is a pro-Lie group by 4.8. Thus (i) is proved.

(ii) Let $\mathcal{C}$ be any full subcategory of $\mathbb{TOPGR}$ which contains all finite dimensional Lie groups and is closed in $\mathbb{TOPGR}$ under the formation of all limits. Let G be Lie projective. Then $G = \lim_{j\in J} G_j$ for a projective system of finite dimensional Lie groups G_j. Then all G_j are contained in $\mathcal{C}$ and since $\mathcal{C}$ is closed under the formation of all limits, G is in $\mathcal{C}$. Thus pro$\mathbb{LIEGR} \subseteq \mathcal{C}$. ◻

6. The One Parameter Subgroup Lifting Theorem

Many categories of topological groups are stable under the passage to quotient groups; the category of pro-Lie groups, regrettably, is not, as we see now.

Proposition 6.1. (The Quotient Theorem for Pro-Lie Groups) *A quotient group of a pro-Lie group is a proto-Lie group and thus is isomorphic as a topological group to a dense subgroup of a pro-Lie group. If the quotient group is complete, then it is a pro-Lie group.*

Proof. (i) Let G be a pro-Lie group and K a closed normal subgroup. Define $f: G \to H \overset{\text{def}}{=} G/K$ to be the open quotient morphism. For $N \in \mathcal{N}(G)$ the set $\overline{NK}$ is a closed subgroup of G containing K, and since f is a quotient map and $\overline{NK}$ is K-saturated, the set $N^* \overset{\text{def}}{=} f(\overline{NK}) \subseteq H$ is closed and agrees with $\overline{f(N)}$.

Then N^* is a closed normal subgroup of H, and since f is open, we have that $H/N^* \cong G/f^{-1}(N^*) = G/\overline{NK} \cong (G/N)/(\overline{NK}/N)$ is a finite dimensional Lie group as a quotient of a finite dimensional Lie group. Let $\mathcal{M} = \{N^* \mid N \in \mathcal{N}(G)\}$. Then $\mathcal{M}$ is a filter basis of closed normal subgroups of H such that all factor groups H/M, $M \in \mathcal{M}$ are finite dimensional Lie groups. Since $\mathcal{N}(G)$ converges to 1 as G is a pro-Lie group, from the continuity of f we conclude that $f(\mathcal{N}(G)) = \{f(N) \mid N \in \mathcal{N}(G)\}$ converges to 1 in H. But since H is regular, i.e. the filter of identity neighborhoods has a basis of closed sets, $\mathcal{M}$ converges to 1 in H. Thus H is a proto-Lie group and we have a natural dense embedding morphism $\gamma_H: H \to H_{\mathcal{M}}$ into the pro-Lie group $H_{\mathcal{M}} \stackrel{\text{def}}{=} \lim_{N \in \mathcal{N}(H)} H/N$. It follows by definition that the group H is a pro-Lie group if an only if it is complete. □

The pro-Lie group $\mathbb{R}^{\mathbb{R}}$ has an incomplete quotient group modulo a totally disconnected and algebraically free subgroup (see [6]); hence 6.1 cannot be improved.

The lifting of one parameter subgroups deals with the following situation: Assume that $f: G \to H$ is a quotient morphism and $Y \in \mathfrak{L}(H)$; under which circumstances is there an $X \in \mathfrak{L}(G)$ such that $\mathfrak{L}(f)(X) = Y$?

Lemma 6.2. *Assume that*

(1)
$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "Y"] \\ G \arrow[r, "f"'] & H. \end{tikzcd}$$

is a pullback of topological groups. Set $K \stackrel{\text{def}}{=} \ker \varphi$. Then the following conditions are equivalent:

(i) *K is a semidirect factor.*
(ii) *φ is a retraction.*
(ii′) *$\varphi|P_0: P_0 \to \mathbb{R}$ is a retraction, where P_0 is the identity component of P.*
(iii) *There is an $X \in \mathfrak{L}(G)$ such that $\mathfrak{L}(f)(X) = Y$.*
(iv) *There is a subgroup R of P such that $KR = P$ and $K \cap R = \{1\}$, and further that $\varphi|R: R \to \mathbb{R}$ is open.*

These conditions imply
(v) *There is a closed subgroup R of P such that $KR = P$ and $K \cap R = \{1\}$.*

Proof. (i)⇔(ii): The equivalence of (i) and (ii) is a standard exercise in topological group theory (see e.g. [5], E1.5).
(ii)⇒(ii′): If a morphism $\sigma: \mathbb{R} \to P$ satisfies $\varphi \circ \sigma = \mathrm{id}_{\mathbb{R}}$, then $\sigma(\mathbb{R}) \subseteq P_0$ as $\mathbb{R}$ is connected, and thus its corestriction $\underline{\sigma}: \mathbb{R} \to P_0$ satisfies $\varphi \circ \underline{\sigma} = \mathrm{id}_{\mathbb{R}}$.
(ii′)⇒(ii): Conversely, if $\sigma: \mathbb{R} \to P_0$ satisfies $\varphi \circ \sigma = \mathrm{id}_{\mathbb{R}}$, then its coextension $\overline{\sigma}: \mathbb{R} \to P$ satisfies $\varphi \circ \overline{\sigma} = \mathrm{id}_{\mathbb{R}}$.
(ii)⇒(iii): If $X': \mathbb{R} \to P$ is a one parameter subgroup satisfying $\varphi \circ X' = \mathrm{id}_{\mathbb{R}}$ then $X \stackrel{\text{def}}{=} \pi \circ X': \mathbb{R} \to G$ is a one parameter subgroup of G such that $\mathfrak{L}(f)(X) = f \circ X = f \circ \pi \circ X' = Y \circ \varphi \circ X' = Y \circ \mathrm{id}_{\mathbb{R}} = Y$.

(iii)⇒(ii): Assume $Y = \mathfrak{L}(f)(X) = f \circ X$. Then for all $r \in \mathbb{R}$ we have $f\big(X(r)\big) = Y(r)$. Now the explicit form of the pullback is $P = \{(g, r) \in G \times \mathbb{R} \mid f(g) = Y(r)\}$ and $\varphi(g, r) = r$ (see e.g. [5], Theorem 1.5). Hence $(X(r), r) \in P$ for all $r \in \mathbb{R}$ and if we set $X'(r) = (X(r), r)$, then $X' : \mathbb{R} \to P$ is a morphism satisfying $\varphi\big(X'(r)\big) = r$ for all r.

(i)⇒(iv)⇒(v) is trivial.

(iv)⇒(ii): The function $\varphi|R : R \to \mathbb{R}$ is continuous and open; and since $K \cap R = \{0\}$ it is injective, and since $KR = P$ it is surjective. Hence it is an isomorphism of topological groups and thus is invertible; the coextension $\sigma : \mathbb{R} \to P$ of $(\varphi|R)^{-1} : \mathbb{R} \to R$ satisfies $\varphi \circ \sigma = \mathrm{id}_{\mathbb{R}}$. □

Lemma 6.3. *If f in the pullback* (1) *is surjective, then φ is surjective. If f is open, then φ is open. If f is a quotient morphism so is φ.*

Proof. Surjectivity: if $r \in \mathbb{R}$ then, since f is surjective, there is a $g \in G$ such that $f(g) = Y(r)$.
Openness: The filter of identity neighborhoods of P has a basis of open sets of the form $W = P \cap (U \times I)$, where U is an open identity neighborhood of G and I an open interval around 0 in $\mathbb{R}$. Then $\varphi(W) = \{r \in I \mid (\exists g \in U)\, f(g) = Y(r)\} = I \cap Y^{-1}\big(f(U)\big)$. Since f is an open map, $f(U)$ is an open subset of H and thus by the continuity of Y, the set $\varphi(W)$ is open.
Quotients: This assertion follows from the combination of the preceding two. □

Lemma 6.4. *In the pullback* (1)*, assume that the morphism f is a quotient morphism and that G is a pro-Lie group. Then P is a pro-Lie group.*

Proof. By Proposition 6.1, H is a proto-Lie group. Let $\gamma_H : H \to H_{\mathcal{N}(H)}$ be the natural completion morphism. Then we obtain a diagram

$$(1') \qquad \begin{array}{ccc} P & \xrightarrow{\varphi} & \mathbb{R} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_H \circ Y} \\ G & \xrightarrow[\gamma_H \circ f]{} & H_{\mathcal{N}(H)}. \end{array}$$

We claim that $(1')$ is a pullback in $\mathbb{TOPGR}$: Thus we let T be a topological group and let $\alpha_G : T \to G$ as well as $\alpha_{\mathbb{R}} : T \to \mathbb{R}$ be morphisms of topological groups such that $\gamma_H \circ f \circ \alpha_G = \gamma_H \circ Y \circ \alpha_{\mathbb{R}}$. Since γ_H is injective, $f \circ \alpha_H = Y \circ \alpha_{\mathbb{R}}$. Since (1) is a pullback in $\mathbb{TOPGR}$, there is a unique $\xi : T \to P$ such that $\alpha_G = \pi \circ \xi$ and $\alpha_{\mathbb{R}} = \varphi \circ \xi$. This shows that $(1')$ is a pullback as well.

The group $\mathbb{R}$ is a Lie group, hence trivially a pro-Lie group. Thus $\mathbb{R}, G, H_{\mathcal{N}(H)}$ are pro-Lie groups. By 5.2(i), the category pro$\mathbb{LIEGR}$ is closed under the formation of pullbacks. We apply this to $(1')$ and conclude that P is a pro-Lie group. □

We now are ready for a proof of the lifting of one parameter subgroups. This is not easy because in the absence of countability assumptions, this requires the Axiom of Choice, and the absence of compactness in the present situation forces us to rely on completeness and the convergence of Cauchy filters. The proof will require from the reader a certain facility handling "multivalued morphisms" as a special type of binary relations; but most of what is required will be self-explanatory in the proof.

Lemma 6.5. (The One Parameter Subgroup Lifting Lemma) *Let $f: G \to H$ be a quotient morphism of topological groups and assume that G is a pro-Lie group. Then every one parameter subgroup $Y: \mathbb{R} \to H$ lifts to one of G, that is, there is a one parameter subgroup σ of G such that $Y = f \circ \sigma$.*

Proof. We for the pullback

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & \mathbb{R} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle Y} \\ G & \xrightarrow[f]{} & H \end{array} \tag{1}$$

in the category of topological groups. Since f is assumed to be a quotient morphism, by Lemma 3, the morphism φ is a quotient morphism, and by Lemma 6.4, the pullback group P is a pro-Lie group. If we can show that φ is a homorphic retraction, then by Lemma 6.2 we have an $X \in \mathfrak{L}(G)$ such that $\mathfrak{L}(f)(X) = Y$. This reduces our task to shoing that φ is a homomorphic retraction. Thus, in order to simplify notation we may assume that $H = \mathbb{R}$ and that we have to show that f is a retraction.

Let $K = \ker f$. Since $\{N^* = \overline{f(N)} \mid N \in \mathcal{N}(G)\}$ converges to 1 in $\mathbb{R}$, and since there are no subgroups in $]-1,1[$ other than $\{0\}$ there is an $\underline{N} \in \mathcal{N}(G)$ such that $\overline{f(\underline{N})} = \underline{N}^* = \{0\}$, and thus $\underline{N} \subseteq K$. Then for all $N \in \mathcal{N}(G)$, $\underline{N} \supseteq N$, the morphism f induces a quotient morphism $f_N: G/N \to \mathbb{R}$, $f_N(gN) = f(g)$, and $f_N(gN) = 0$ iff $f(g) = 0$ iff $g \in K$, that is, $\ker f_N = K/N$. If we let $p_N: K \to K/N$ and $q_N: G \to G/N$ denote the quotient morphisms, then we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & N & \xrightarrow{\mathrm{id}_N} & N & & & & \\ & & {\scriptstyle \mathrm{incl}}\downarrow & & \downarrow{\scriptstyle \mathrm{incl}} & & & & \\ 1 & \to & K & \xrightarrow{\mathrm{incl}} & G & \xrightarrow{f} & \mathbb{R} & \to & 0 \\ & & {\scriptstyle p_N}\downarrow & & \downarrow{\scriptstyle q_N} & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{R}}} & & \\ 1 & \to & K/N & \xrightarrow{\mathrm{incl}} & G/N & \xrightarrow{f_N} & \mathbb{R} & \to & 0 \end{array} \tag{2}$$

with exact rows and columns. Due to the fact that the exponential map of a Lie group is a local homeomorphism at 0, an open morphism $\psi: L_1 \to L_2$ between Lie groups induces an open morphism $\mathfrak{L}(\psi)$ between their Lie algebras:

$$\begin{array}{ccc} \mathfrak{L}(L_1) & \xrightarrow{\mathfrak{L}(\psi)} & \mathfrak{L}(L_2) \\ {\scriptstyle \exp_{L_1}}\downarrow & & \downarrow{\scriptstyle \exp_{L_2}} \\ L_1 & \xrightarrow[\psi]{} & L_2. \end{array}$$

An open morphism between topological real vector spaces is automatically surjective, and thus $\mathfrak{L}(\psi)$ is surjective. Hence there is a morphism $\sigma\colon \mathbb{R} \to G/N$ such that $f_N \circ \sigma = \mathrm{id}_{\mathbb{R}}$. The binary relation $\Sigma \stackrel{\mathrm{def}}{=} q_N^{-1} \circ \sigma\colon \mathbb{R} \to G$ satisfies the following conditions

(i) $\Sigma(0) = N$ and every $\Sigma(r) \subseteq G$ is a coset mod N.

(ii) The graph of Σ is a closed subgroup of $\mathbb{R} \times G$.

(iii) We have a commutative diagram of binary relations of which all but Σ are functions:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\mathrm{id}_{\mathbb{R}}} & \mathbb{R} \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle\mathrm{id}_{\mathbb{R}}} \\ G & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle q_N}\downarrow & & \downarrow{\scriptstyle\mathrm{id}_{\mathbb{R}}} \\ G/N & \xrightarrow[f_N]{} & \mathbb{R}. \end{array} \tag{3}$$

A binary relation $\Sigma\colon \mathbb{R} \to G$ satisfying (i), (ii) and (iii) will be called a *multivalued morphism associated with* N. The set $\mathcal{S}$ of all multivalued morphisms $\Sigma\colon \mathbb{R} \to G$ associated with some $N \in \mathcal{N}(G)$ is partially ordered under containment $\subseteq$. By Zorn's Lemma we find a maximal filter $\mathcal{F} \subseteq \mathcal{S}$. It is our goal to show that $\mathcal{M} \stackrel{\mathrm{def}}{=} \{\Sigma(0) \mid \Sigma \in \mathcal{F}\}$ is cofinal in $\mathcal{N}(G)$. Assuming that this is proved, we note that for each $r \in \mathbb{R}$ and $\Sigma \in \mathcal{F}$ the subset $\Sigma(r)$ is a coset $Nx = xN$ with $N = \Sigma(0) \in \mathcal{N}(G)$, and thus $\Sigma(r)\Sigma(r)^{-1} = Nx(Nx)^{-1} = N$; since $\mathcal{M}$ converges to 1, we conclude that $\{\Sigma(r) \mid \Sigma \in \mathcal{F}\}$ is a Cauchy filter basis. Since G is complete, it converges to an element $\sigma(r) \in G$, giving us a function $\sigma\colon \mathbb{R} \to G$. As each $\Sigma(r)$, being a coset mod $N = \Sigma(0) \in \mathcal{M}$, is closed, we have $\sigma \in \Sigma(r)$ for all $\Sigma \in \mathcal{F}$. Consequently, since (3) is commutative for each $\Sigma \in \mathcal{F}$ for $N = \Sigma(0)$ we have the following commutative diagram for all $N \in \mathcal{M}$:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\mathrm{id}_{\mathbb{R}}} & \mathbb{R} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\mathrm{id}_{\mathbb{R}}} \\ G & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle q_N}\downarrow & & \downarrow{\scriptstyle\mathrm{id}_{\mathbb{R}}} \\ G/N & \xrightarrow[f_N]{} & \mathbb{R}. \end{array} \tag{4}$$

The upper rectangle shows that $f \circ \sigma = \mathrm{id}_{\mathbb{R}}$, and the fact that each $q_N \circ \Sigma\colon \mathbb{R} \to G$ is a morphism of topological groups shows that $q_N \circ \sigma\colon \mathbb{R} \to G/N$ is continuous. Theorem 2.1(i) shows that G has arbitrarily small open identity neighborhoods U satisfying $UN = U$ for some $N \in \mathcal{M}$. Then if V is a zero neighborhood of $\mathbb{R}$ such that $q_N\big(\sigma(V)\big) \subseteq U/N$, then $\sigma(V) \in q_N^{-1}(U/N) = U$. This shows that σ is continuous. Hence σ is the required coretraction for f.

Thus the remainder of the proof will show that $\mathcal{M}$ is cofinal in $\mathcal{N}(G)$. Suppose that this is not the case. Then there exists an $N \in \mathcal{N}(G)$, $\underline{N} \supseteq N$ such that $M \not\subseteq N$ for all $M \in \mathcal{M} \subseteq \mathcal{N}(G)$. Hence for all $\Sigma \in \mathcal{F}$ we have $q_N(\Sigma) \cap q_N(K) = MN/N$ for $M = \Sigma(0)$. Let us temporarily fix M; then $M \cap N \in \mathcal{N}(G)$, and thus

$G^\dagger \stackrel{\rm def}{=} G/(M\cap N)$ is a Lie group:

$$\left.\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash] & \\
 & MN \arrow[dl, dash] \arrow[dr, dash] & \\
M \arrow[dr, dash] & & N \arrow[dl, dash] \\
 & M\cap N &
\end{tikzcd}\right\} G^\dagger.$$

This shows that for fixed M everything takes place in the Lie group $G^\dagger$ in which $M^\dagger \stackrel{\rm def}{=} M/(M\cap N)$ and $N^\dagger \stackrel{\rm def}{=} N/(M\cap N)$ are closed normal Lie subgroups with $M^\dagger\cap N^\dagger=\{1\}$. Thus $\mu\colon M^\dagger\to G/N$, $\mu\big(m(M\cap N)\big)=mN$ is a morphism of Lie groups mapping $M^\dagger$ bijectively onto MN/N and inducing an isomorphism of Lie algebras $\mathfrak{L}(M^\dagger)\to\mathfrak{L}(MN/N)\subseteq\mathfrak{L}(G/N)$. Now $M^\dagger$ is a closed normal subgroup of the Lie group $G^\dagger$ and thus $M^\dagger/M_0^\dagger$ is a discrete normal subgroup of the Lie group $\mathbb{G}^\dagger/M_0^\dagger$. We let $M^\ddagger$ be the open subgroup of $M^\dagger$ containing $(M^\dagger)_0$ and being such that $M^\ddagger/M_0^\dagger=(M^\dagger/M_0^\dagger)\cap(G^\dagger)_0/M_0^\dagger$. Hence $M^\ddagger/M_0^\dagger$ is a discrete normal subgroup of a connected Lie group. Hence it is finitely generated and thus countable. Thus $M^\ddagger$ has countably many components and therefore $\mu(M_0^\dagger)$ is an analytic subgroup $M_{\rm an}\subseteq G/N$ agreeing with $(MN/N)_0$ and having Lie algebra $\mathfrak{L}(M_{\rm an})=\mathfrak{L}(MN/N)=\mathfrak{L}\big((MN/N)_0\big)$. (See [4], p. 155, 156, 157.) Accordingly,

$$\{\mathfrak{L}(MN/N)\mid M\in\mathcal{M}\}$$

is a filter basis of finite dimensional vector subspaces of $\mathfrak{L}(G/N)$. Hence there is a smallest element $\mathfrak{m}=\mathfrak{L}(M_\#N/N)$ in it such that for all $M\le M_\#$ in $\mathcal{M}$ we have $\mathfrak{L}(MN/N)=\mathfrak{m}$. Let us abbreviate $q_{(M_\#\cap N)}\colon G\to G/(M_\#\cap N)$ by $q^\#\colon G\to G^\#$, further $f_{(M_\#\cap N)}\colon G^\#\to\mathbb{R}$ by $f^\#$, and $M_\#/(M_\#\cap N)$ by $M^\#$. Since $\mathfrak{L}(\mu)\colon\mathfrak{L}(M^\#)\to\mathfrak{L}(M^\#N/N)=\mathfrak{m}$ is an isomorphism we have

$$q^\#\left(\left(\frac{M}{M^\#\cap N}\right)_0\right)=q^\#(M_0^\#)\text{ for }M\subseteq M_\#\text{ in }\mathcal{M}. \tag{5}$$

There is a $\Xi\in\mathcal{F}$ such that $M^\#=\Sigma^\#(0)$. Then for all $\Sigma\in\mathcal{F}$ contained in $\Sigma^\#$, the subgroup $q^\#(\Sigma)$ of the Lie group $G^\#$ is contained in $q^\#(\Sigma^\#)$, satisfies $q^\#(\Sigma(0))=M_0^\#$ and $(f^\#\circ q^\#\circ\Sigma)(\mathbb{R})=\mathbb{R}$. Thus for all $r\in\mathbb{R}$ we have $q^\#\big(\Sigma^\#(r)\big)=q^\#\big(\Sigma(r)\big)$ since the right side is contained in the left and both are cosets mod $M^\#$. In the

Lie group $G^{\#}$ we have the configuration

$$\left.\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash] & \\
 & M^{\#}N \arrow[dl, dash] \arrow[dr, dash] & \\
M^{\#} \arrow[dr, dash] & & N \arrow[dl, dash] \\
 & M^{\#} \cap N &
\end{tikzcd}\right\} G^{\#},$$

and thus obtain $q^{\#}(\Sigma) = q^{\#}(\Sigma^{\#})$ for all $\Sigma \subseteq \Sigma^{\#}$ in $\mathcal{F}$. The definition of $\Sigma^{\#}$ as $q_{M^{\#}}^{-1} \circ \sigma^{\#}$ applies and gives a commutative diagram of binary relations

$$(3^{\#}) \qquad \begin{tikzcd}
\mathbb{R} \arrow[r, "\mathrm{id}_{\mathbb{R}}"] \arrow[d, "\Sigma^{\#}"'] & \mathbb{R} \arrow[d, "\mathrm{id}_{\mathbb{R}}"] \\
G \arrow[r, "f"] \arrow[d, "q_{M^{\#}}"'] & \mathbb{R} \arrow[d, "\mathrm{id}_{\mathbb{R}}"] \\
G/M^{\#} \arrow[r, "f_{M^{\#}}"'] & \mathbb{R}
\end{tikzcd}$$

with $\sigma^{\#} = q_{M^{\#}} \circ \Sigma^{\#}$. We conclude that $S \stackrel{\text{def}}{=} q^{\#}\big(\Sigma^{\#}(\mathbb{R})\big) = \Sigma^{\#}(\mathbb{R})/(M^{\#} \cap N)$ is a closed subgroup of $G^{\#}$ whose Lie algebra $\mathfrak{L}(S)$ cannot be contained in $K^{\#} = K/(M^{\#} \cap N) = \ker f^{\#}$. From $\dim G^{\#}/K^{\#} = 1$ we conclude $\mathfrak{L}(G^{\#}) = \mathfrak{L}(K^{\#}) + \mathfrak{L}(S)$ and $\mathfrak{L}(S) = \mathfrak{L}(S) \cap \mathfrak{L}(K^{\#}) + \mathbb{R}\cdot X$ for a suitable element $X \in \mathfrak{L}(G^{\#})$ satisfying $f^{\#}(\exp_{G^{\#}} X) = 1$. Setting $\tau\colon \mathbb{R} \to S$, $\tau(r) = \exp_{G^{\#}} r\cdot X$ we obtain a coretraction for $f^{\#}\colon G^{\#} \to \mathbb{R}$. The binary relation $\underline{\Sigma} \stackrel{\text{def}}{=} (q^{\#})^{-1} \circ \tau\colon \mathbb{R} \to G$ is a member of $\mathcal{S}$. Moreover, for all $\Sigma \in \mathcal{F}$ we have $q^{\#}(\Sigma)(r) \supseteq \tau(r)$ for all $r \in \mathbb{R}$. Hence $\Sigma \cap \underline{\Sigma}$ is a member of $\mathcal{S}$. But now the maximality of $\mathcal{F}$ shows that $\Sigma^{\#} \in \mathcal{F}$. But this implies that $M^{\#} \cap N = \underline{\Sigma}(0) \in \mathcal{M}$ and that is a contradiction to our supposition allowing us a choice of an N such that $M \cap N \neq M$ for all $M \in \mathcal{M}$. This contradiction finally completes the proof. □

There are some subtleties here which we should point out. In [4], p. 157 we have seen the additive group $\mathfrak{h}$ of a Banach space mapped surjectively onto an abelian Lie group G (which itself is quotient of a Banach space modulo a discrete subgroup) such that G has a one parameter subgroup which does **not** lift to $\mathfrak{h}$. This cannot happen if the domain is separable, but it does happen in the category of not necessarily finite dimensional Lie groups. While being surjective, the morphism in question is not open and the Open Mapping Theorem fails.

We have seen that the functor $\mathfrak{L}$ preserves all limits and thus, in particular, all kernels (since $\ker f$ for a morphism f of topological groups is nothing but the equalizer of f and the constant morphism). We shall say that a functor $\mathfrak{F}\colon \mathcal{A} \to \mathcal{B}$ between categories of topological groups is *strictly exact* if it preserves kernels and quotients. As a corollary of the One Parameter Subgroup Lifting Lemma we obtain the following Theorem

Theorem 6.6. (The Strict Exactness Theorem for $\mathfrak{L}$) *The functor* $\mathfrak{L}\colon \mathrm{pro}\mathbb{LIEGR} \to \mathrm{pro}\mathbb{LIEALG}$ *is strictly exact.*

Proof. We observed that $\mathfrak{L}$ preserves kernels because kernels are limits. We have to show that $\mathfrak{L}$ preserves quotients.

Let $f\colon G \to H$ be a quotient morphism between pro-Lie groups. The morphism $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ is surjective by the One Parameter Subgroup Lifting Lemma 6.5. Now any surjective morphism of topological vector spaces between weakly complete vector spaces splits (see e.g. [4], p. 326, Theorem 7.30(iv)) and thus is in particular a quotient morphism. Since the underlying topological vector spaces of $\mathfrak{L}(G)$ and $\mathfrak{L}(H)$ are weakly complete by 4.10, this applies to the morphism $\mathfrak{L}(f)$ and shows that it is a quotient morphism. $\square$

Corollary 6.7. (i) *If N is a closed normal subgroup of a pro-Lie group G, then the quotient morphism $q\colon G \to G/N$ induces a map $\mathfrak{L}(q)\colon \mathfrak{L}(G) \to \mathfrak{L}(G/N)$ which is a quotient morphism with kernel $\mathfrak{L}(N)$. Accordingly there is a natural isomorphism $X + \mathfrak{L}(N) \mapsto \mathfrak{L}(f)(X) : \mathfrak{L}(G)/\mathfrak{L}(N) \to \mathfrak{L}(G/N)$.*

(ii) *Let G be a pro-Lie group. Then $\{\mathfrak{L}(N) \mid N \in \mathcal{N}(G)\}$ converges to zero and is cofinal in the filter $\mathcal{I}(\mathfrak{L})(G)$ of all ideals $\mathfrak{i}$ such that $\mathfrak{L}(G)/\mathfrak{i}$ is finite dimensional.*

Furthermore, $\mathfrak{L}(G)$ is the projective limit $\lim_{N\in\mathcal{N}(G)} \mathfrak{L}(G)/\mathfrak{L}(N)$ of a projective system of bonding morphisms and limit maps all of which are quotient morphisms, and there is a commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(\gamma_G)} & \mathfrak{L}(G_{\mathcal{N}(G)}) & = & \mathfrak{L}(\lim_{N\in\mathcal{N}(G)} \frac{G}{N}) & \cong & \lim_{n\in\mathcal{N}(G)} \frac{\mathfrak{L}(G)}{\mathfrak{L}(N)} \\
{\scriptstyle \exp_G}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathfrak{L}(\lim_{N\in\mathcal{N}(G)} \exp_{G/N})} & & \\
G & \xrightarrow[\gamma_G]{} & G_{\mathcal{N}(G)} & = & \lim_{N\in\mathcal{N}(G)} G/N. & &
\end{array}$$

Proof. (i) is an immediate consequence of the Strict Exactness Theorem 6.6.

(ii) We know that $\mathfrak{L}$ presevers limits. Thus

$$\mathfrak{L}(\gamma_G)\colon \mathfrak{L}(G) \to \mathfrak{L}(G_{\mathcal{N}(G)})$$

is an isomorphism. By (i) above, $\mathfrak{L}(G/N) \cong \mathfrak{L}(G)/L(N)$ and thus

$$\mathfrak{L}(G) \cong \lim_{N\in\mathcal{N}(G)} \mathfrak{L}(G)/\mathfrak{L}(N).$$

Thus by 2.1(ii), the filter basis $\{\mathfrak{L}(N) \mid N \in \mathcal{N}(G)\}$ of the kernels of the limit maps converges to 0 and the projective system of the $\mathfrak{L}(G)/\mathfrak{L}(N)$ has the natural quotient morphisms as bonding maps; by 2.1(ii) it follows that the limit maps are quotient morphsms as well. It then follows that this filter basis is cofinal in $\mathcal{N}(\mathfrak{L})(G)$. (Compare [5], 1.40.) $\square$

For a topological group G let $\mathbb{E}(G)$ denote the subgroup $\langle \exp_G G\rangle$ generated by the (images of the) one parameter subgroups and set $E(G) = \overline{\mathbb{E}(G)}$.

Corollary 6.8. (i) *For a pro-Lie group G, the subgroup $\mathbb{E}(G)$ is dense in G_0, i.e. $E(G) = G_0$. In particular, a connected nonsingleton pro-Lie group has nontrivial one parameter subgroups.*

(ii) *For a pro-Lie group G the following statements are equivalent:*

(a) *G is totally disconnected.*

(b) $\mathfrak{L}(G) = \{0\}$.

(iii) *If a morphism $f\colon G \to H$ of pro-Lie groups is a quotient morphism then the induced morphism $\mathbb{E}(f)\colon \mathbb{E}(G) \to \mathbb{E}(H)$ is surjective, that is $\mathbb{E}(H) = f\big(\mathbb{E}(G)\big)$. As a consequence $H_0 = \overline{f(G_0)}$.*

(iv) *Let G be a pro-Lie group and assume that for all N from a basis of $\mathcal{N}(G)$ the quotient G/N is connected. Then G is connected.*

Proof. (i) First we show that nonsingleton connected pro-Lie groups have nontrivial one parameter subgroups, Let G be a nonsingleton connected pro-Lie group. There is a $g \in G$, $g \neq 1$. Since $\lim \mathcal{N}(G) = 1$ there is an $N \in \mathcal{N}(G)$ such that $g \notin N$. Then G/N is a nonsingleton connected Lie group. Thus $\mathfrak{L}(G/N) \neq \{0\}$. Then $\mathfrak{L}(G) \neq \{0\}$ by 6.7(i).

Next we let G be an arbitrary pro-Lie group. The closed subgroup $E(G) = \overline{\langle \exp_G \mathfrak{L}(G)\rangle}$ is fully characteristic, hence normal. By the One Parameter Subgroup Lifting Lemma 6.5, every one parameter subgroup of $G/E(G)$ lifts to one in G which is contained in $E(G)$ by the definition of $E(G)$. Hence $\mathfrak{L}\big(G/E(G)\big) = \{0\}$. Thus $G/E(G)$ is totally disconnected by what we just proved, and thus $G_0 \subseteq E(G) \subseteq G_0$.

(ii) (a)$\Rightarrow$(b): If $G_0 = \{1\}$ then $E(G) = \{1\}$ and thus $\mathfrak{L}(G) = \{0\}$. (b)$\Rightarrow$(a): Assume $\mathfrak{L}(G) = \{0\}$, then $G_0 = \{1\}$ by (i).

(iii) By 6.6, $\mathfrak{L}(H) = \mathfrak{L}(f)\big(\mathfrak{L}(G)\big)$, and thus $\exp_H \mathfrak{L}(H) = \exp_H \mathfrak{L}(f)\big(\mathfrak{L}(G)\big) = f\big(\exp_G \mathfrak{L}(G)\big)$, and consequently $\mathbb{E}(H) = \langle \exp_H \mathfrak{L}(H)\rangle = \big\langle f\big(\exp_G \mathfrak{L}(G)\big)\big\rangle = f\langle \exp_G \mathfrak{L}(G)\rangle = f\big(\mathbb{E}(G)\big)$.

Thus $H_0 = \overline{\mathbb{E}(H)} = \overline{f\big(\mathbb{E}(G)\big)} \subseteq \overline{f}(G_0) \subseteq \overline{H_0} = H_0$, and this shows $\overline{f(G_0)} = H_0$.

(iv) Let $q_N\colon G \to G/N$ denote the quotient morphism. By (iii) we have $G/N = \mathbb{E}(G/N) = q_N\big(\mathbb{E}(G)\big)$. Thus $G = \mathbb{E}(G)N$ for all $N \in \mathcal{N}(()G)$ and thus $G_0 = E(G) = \overline{\mathbb{E}(G)} = G$. □

The relation $H_0 = \overline{f(G_0)}$ for a quotient morphism f cannot be improved as the example of the following quotient morphism of locally compact abelian groups shows: Let $G = \mathbb{R} \times \mathbb{Z}_p$ for the group of p-adic integers $\mathbb{Z}_p$, let $H = G/\{(n, -n) \mid n \in \mathbb{Z}\} \cong \mathbb{S}_p$ and let f be the corresponding quotient morphism. Note that H is compact and connected. (Cf. [4], p. 19, Exercise E1.11. We consider $\mathbb{Z}$ as a subgroup of $\mathbb{S}_p$ as well.) Then $G_0 = \mathbb{R} \times \{0\}$, and $f(G_0) = H_a \neq H = H_0$.

Corollary 6.9. *An open morphism $f\colon G \to H$ of pro-Lie groups induces a quotient (hence surjective) morphism $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$.*

Proof. Let $f: G \to H$ be an open morphism of topological groups. Then $f(G)$ is an open hence closed subgroup of H and thus a pro-Lie group by Corollary 4.8. The open and surjective corestriction $G \to f(G)$ (inducing an isomorphism of topological groups $G/\ker f \to f(G)$) is a quotiemn morphism between pro-Lie groups and thus induces a quotient morphism $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}\big(f(G)\big)$ by the Strict Exactness Theorem 6.6. Since $f(G)$ is open in H, then inclusion $j: f(G) \to H$ induces an isomorphism $\mathfrak{L}(j): \mathfrak{L}\big(f(G)\big) \to \mathfrak{L}(H)$ of topological Lie algebras. Thus $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$ is a quotient morphism. □

This Corollary remains intact if G and H are merely topological groups that have open subgroups being pro-Lie groups. This applies, for instance, to all locally compact groups G and H.

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