Compact Groups with Large Abelian Subgroups

BY KARL H HOFMANN*
Fachbereich Mathematik, Technische Universität Darmstadt
Schlossgartenstr. 7, D-64289 Darmstadt, Germany
e-mail: hofmann@mathematik.tu-darmstadt.de

AND SIDNEY A. MORRIS**
School of Mathematics, University of South Australia
Mawson Blvd, Mawson Lakes S.A. 5095, Australia
e-mail: sid.morris@unisa.edu.au

1. Introduction: The abelian subgroup conjecture

In this paper we formulate a new conjecture and introduce methods to verify it in many cases.

CONJECTURE. (The Abelian Subgroup Conjecture) *Every infinite compact group G has an abelian subgroup A of weight $w(A) = w(G)$.*

As usual the weight $w(X)$ of a topological space X is defined by

$$w(X) = \min\{\operatorname{card}\mathcal{B} : \mathcal{B} \text{ is a basis of the topology of } X\}.$$

When G is an infinite metrizable compact group, that is $w(G) = \aleph_0$, the conjecture states that G contains an infinite abelian subgroup. It is a consequence of Wilson [14] that this is the case if every compact torsion p-group contains an infinite abelian subgroup; a purely group theoretic result by Hall and Kulatilak [3], and by Kargapolov [11] from the early sixties states that an infinite locally finite group has an infinite abelian subgroup. The finishing touch was applied by Zelmanov [16] who proved that every compact p-torsion group is locally finite. Thus our Abelian Subgroup Conjecture is settled for compact groups G with $w(G) = \aleph_0$. But this line of argument does not tell us whether a non-metrizable compact group G must contain a non-metrizable compact abelian subgroup A, that is, whether $w(G) > \aleph_0$ implies the existence of A such that $w(A) > \aleph_0$. If true, our conjecture would show this and more.

The following definition assists us in addressing our conjecture and what we know about it.

DEFINITION 1·1. (i) A subgroup A of a topological group G is said to be *large* if $w(A) = w(G)$ and is said to be *small* if $w(A) < w(G)$.

(ii) A topological group G is called an LAS-*group* if it has a large abelian subgroup. □

Thus the Abelian Subgroup Conjecture asserts that *every compact group is an* LAS-*group.*

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PROPOSITION A. *Let G be a locally compact group in which the identity component G_0 is large. Then G is an* LAS*-group.*

Proof. Firstly assume that G is compact. Then G contains a maximal compact connected abelian subgroup T (pro-torus in the language of [7]) such that $w(T) = w(G_0)$ (see [7, pp. 465, 466, Theorem 9·36(vi)]). Since G_0 is large we have $w(G_0) = w(G)$. Thus $w(T) = w(G)$ and G is an LAS-group.

Now if G is locally compact then for a maximal compact subgroup K of G_0 the group G_0 is homeomorphic to the product space $K \times \mathbb{R}^n$ for some n (see [10] or [5]). Let T be a maximal pro-torus of K. By the preceding, $w(T) = w(K)$, but clearly $w(K) = w(G_0)$ and since G_0 is large, $w(G_0) = w(G)$. □

COROLLARY B. *Every connected locally compact group is an* LAS*-group.* □

So the Abelian Subgroup Conjecture is true for connected compact groups.

Notice that any abelian subgroup of a discrete free group is countable, and thus there are discrete, hence locally compact, groups which are not LAS-groups.

We shall prove the following:

THEOREM C. (The Reduction Theorem) *Let G be an infinite compact group and $N \trianglelefteq G$ a closed normal small subgroup such that G/N is an* LAS*-group. Then G is an* LAS*-group.*

The next result is an easy consequence of the Reduction Theorem.

THEOREM D. (The Extension Theorem) *Let G be an infinite compact group and N a closed normal subgroup such that G/N and N are* LAS*-groups. Then G is an* LAS*-group.*

As a consequence of Proposition A and Theorem D we obtain at once a result which will show that proving or disproving the Abelian Subgroup Conjecture is a problem on profinite groups.

COROLLARY E. (The Reduction Corollary) *Let G be a compact group and assume that G/G_0 is an* LAS*-group. Then G is an* LAS*-group.*

In [8] we called a compact group *strictly reductive* if it is (isomorphic to) a cartesian product of compact simple groups, where we call a group *simple* if it has no more than two normal subgroups. Accordingly a compact simple group is either cyclic of prime order, or finite simple, or is a centre-free compact connected simple Lie group. It is important to point out that a compact connected Lie group is said to be a simple Lie group if its Lie algebra is simple. A simple Lie group such as SU(2) is not necessarily a simple group because it can have a non-trivial finite centre.

If $\{S_j : j \in J\}$ is a family of LAS-groups, then $G = \prod_{j\in J} S_j$ is either a finite group or an LAS-group: Indeed let A_j be a closed abelian subgroup of S_j such that $w(A_j) = w(S_j)$ and set $A \overset{\text{def}}{=} \prod_{j\in J} A_j$. For an infinite group G we have $w(G) = \sup(\{\operatorname{card} J\} \cup \{w(S_j) : j \in J\})$ (see e.g. [7, p. 764]). Thus $w(S) = w(A)$. As a consequence, the following lemma is quite elementary:

LEMMA F. *Every infinite strictly reductive group is an* LAS*-group.* □

The significance of the class of strictly reductive groups is clear from the next theorem which we proved in [8].

COUNTABLE LAYER THEOREM. *Any compact group G has a canonical countable descending sequence $G = \Omega_0(G) \supseteq \cdots \supseteq \Omega_n(G) \supseteq \cdots$ of closed characteristic subgroups of G with the two properties, that their intersection $\bigcap_{n=1}^{\infty} \Omega_n(G)$ is $Z_0(G_0)$, the identity* *component of the centre of the identity component* G_0 *of* G, *and that each quotient group* $\Lambda_n(G) \stackrel{\text{def}}{=} \Omega_{n-1}(G)/\Omega_n(G)$ *is a strictly reductive group.* □

In [7] there are numerous pieces of information on the structure of compact groups which express the intuition that large compact groups are "broad and wide" but not too "deep." The Countable Layer Theorem confirms this impression. We shall use it in this paper to prove:

THEOREM G. (The Dominant Layer Theorem for Profinite Groups) *Assume that* G *is an infinite profinite group for which there is a natural number* n *such that* $w\big(\Omega_n(G)\big) < w(G)$. *Then* G *is an* LAS*-group.*

The proofs of our results require some tools in addition to the Countable Layer Theorem, and we shall first provide these.

2. The Automorphism group of a Strictly Reductive Group.

For a locally compact group the topology on the automorphism group is a refinement of the compact open topology (cf. [7, p. 257]). If $\mathcal{B}_j$, $j = 1, 2$ are bases for two topologies $\mathcal{O}_j$ on a set X, then $\{U \cap V : U \in \mathcal{B}_1,\, V \in \mathcal{B}_2\}$ is a basis for $\mathcal{O}_1 \vee \mathcal{O}_2$. Let $X_j = (X, \mathcal{O}_j)$ and $X = (X, \mathcal{O}_1 \vee \mathcal{O}_2)$. Since clearly $w(X) \le w(X_j)$, we deduce

$$(*) \qquad w(X) = \max\{w(X_1), w(X_2)\}$$

if at least one of the topologies is infinite.

Now we recall that for a locally compact group G, the topology $\mathcal{O}$ of the automorphism group $\operatorname{Aut} G$ is $\mathcal{CO} \vee \mathcal{CO}^{-1}$ where $\mathcal{OC}$ is the compact open topology induced from that of $\operatorname{Hom}(G, G)$ (cf. [7, p. 257]). In [7, p. 361, Corollary 7·75], it is shown that for two locally compact *abelian* groups A and B one has

$$(**) \qquad w\big(\operatorname{Hom}(A, B)\big) \le \max\{w(A), w(B)\}.$$

The proof of Claim (b) of the required Proposition 7·74 of [7] is readily modified so that the last line of that proof remains true for non-abelian groups; this is the only place where commutativity is used. Thus Proposition 7.75 of [7] is available for locally compact groups which are not necessarily abelian. We shall write $\operatorname{End}(G)$ instead of $\operatorname{Hom}(G, G)$.

LEMMA 2·1. *Let* G *be an infinite locally compact group. Then*

$$w\big(\operatorname{Aut}(G)\big) \le w\big(\operatorname{End}(G)\big) \le w(G).$$

Proof. By the preceding remarks, we have $w\big(\operatorname{End}(G)\big) \le w(G)$ from $(**)$. Then by the definition of the topology of $\operatorname{Aut}(G)$, in view of $(*)$ above, we obtain $w\big(\operatorname{Aut}(G)\big) \le w\big(\operatorname{End}(G)\big)$. □

Now we return to strictly reductive compact groups. Let us recall some notation from [8].

NOTATION. Let $\mathcal{S}$ denote a set of representatives for the set of isomorphism classes of the class of all compact simple groups. For a compact group G and $S \in \mathcal{S}$, the smallest closed subgroup G_S of G containing all closed normal subgroups isomorphic to S is called the S*-socle* of G.

We showed in [8], 2·3 that for a strictly reductive compact group G and the sequence $(G_S)_{S\in\mathcal{S}}$ of S-socles of G, there is a sequence of cardinals $\big(J(G,S)\big)_{S\in\mathcal{S}}$ such that

$$G \cong \prod_{S\in\mathcal{S}} G_S, \quad G_S \cong S^{J(G,S)}.$$

Proposition 2·2. *Let* $G = \prod_{S\in\mathcal{S}} S^{J(G,S)}$ *be strictly reductive. Then every automorphism* f *of* G *preserves* G_S*, and the morphism*

$$(f_S)_{S\in\mathcal{S}} \mapsto \big((s_S)_{S\in\mathcal{S}} \mapsto (f_S(s_S))_{S\in\mathcal{S}}\big) : \prod_{S\in\mathcal{S}} \mathrm{Aut}(S^{J(G,S)}) \to \mathrm{Aut}(G)$$

is an isomorphism.

Proof. This is a straightforward exercise. □

Proposition 2·2 reduces the structure theory of $\mathrm{Aut}(G)$ to the determination of the automorphism group of S^X for a set X and $S \in \mathcal{S}$. The situation will be different according as S is abelian or non-abelian.

Firstly we deal with the abelian case

Proposition 2·3. *Assume that* $S \in \mathcal{S}$ *is abelian, say,* $S = \mathbb{Z}(p)$*, then*
(i) $\mathrm{Aut}\, S^X \cong \mathrm{GL}\big(\mathbb{Z}(p)^{(X)}\big)$.
(ii) *If* X *is infinite, then* $w(\mathrm{Aut}\, S^X) = \mathrm{card}\, X$.

Proof. (i) The compact abelian group $G \overset{\text{def}}{=} S^X$ has character group $\widehat{G}$ which can be identified with $\mathbb{Z}(p)^{(X)}$. Now $\phi \mapsto \widehat{\phi} : \mathrm{Aut}\, G \to \mathrm{Aut}\, \widehat{G}$ is an anti-isomorphism and $\mathrm{Aut}\, \widehat{G} \cong \mathrm{GL}\big(\mathbb{Z}(p)^{(X)}\big)$. Every group has the anti-automorphism $x \mapsto x^{-1}$.

(ii) Since linear self-maps of $\mathbb{Z}(p)^{(X)}$ are determined by their action on a basis we have an isomorphism $\mathrm{GL}(\mathbb{Z}(p)^{(X)}) \cong (\mathbb{Z}(p)^{(X)})^X$ and $w\big((\mathbb{Z}(p)^{(X)})^X\big) = w\big((\mathbb{Z}(p)^{(X)})^X\big) = \max\{\mathrm{card}\, X, w(\mathbb{Z}(p)^{(X)})\}$ (cf. [7, p. 763, 764, EA4·3]), and this cardinal equals $\mathrm{card}\, X$. □

Next we turn to the non-abelian case

If G is a compact group, let $\mathrm{Inn}\, G \le \mathrm{Aut}\, G$ denote the normal subgroup of inner automorphisms and define $\mathrm{Out}\, G \overset{\text{def}}{=} (\mathrm{Aut}\, S)/(\mathrm{Inn}\, S)$, called the *outer automorphism group.*

We acknowledge the fact that the outer automorphism group is not a group of automorphisms, but the terminology is entrenched in the literature.

Lemma 2·4. *If* $S \in \mathcal{S}$*, then* $\mathrm{Out}\, G$ *is a finite soluble group.*

Proof. If S is connected, then $\mathrm{Out}\, G$ is isomorphic to the symmetry group of the Dynkin diagram which is a finite soluble group. (Indeed it is abelian with one exception, D_4 whose automorphism group is S_3.)

If S is a finite simple group, then the Schreier Conjecture applies; it asserts that for a finite simple group the outer automorphism group $\mathrm{Out}(S)$ is a (finite) soluble group, and it is verified by the classification of finite simple groups. □

Let X be a set. If F is a finite field, such as $\mathbb{Z}(p)$, then any F-vector space of dimension $\mathrm{card}\, X$ is isomorphic to the direct sum $V = F^{(X)}$ and $\mathrm{End}\, V \cong M_{X\times X}(F)$ the ring of column finite matrices. These form a subset of the compact space $F^{X\times X}$ with the Tychonoff topology. If $\mathrm{End}\, V$ is equipped with the topology of pointwise convergence then the identification $\mathrm{End}\, V \to M_{X\times X}(F)$ is a homeomorphism. The group of units of $\mathrm{End}\, V$ is the automorphism group $\mathrm{Aut}\, V = \mathrm{GL}(V)$ and $w(\mathrm{Aut}\, V) \le w(V) = \mathrm{card}\, X$ if X is infinite.

Now define $P(X) \subseteq X^X$ to be the group of all bijections with the topology introduced in [7, p. 506]. For each finite set $E \subseteq X$ set $W_{\mathrm{id}}(E) = \{f \in P(X) : (\forall x \in E)\, f(x) = x\}$; then the set of all $W_{\mathrm{id}}(E)$ as E ranges through the set of finite subsets of X is a basis for the identity neighbourhoods of a group topology for $P(X)$.

The group $P(X)$ operates on $\mathbb{Z}(2)^{(X)}$ by $\sigma\cdot(r_x)_{x\in X}=(r_{\sigma^{-1}(x)})_{x\in X}$. Thus we obtain a faithful representation

$$\pi\colon P(X)\to \mathrm{GL}\,\big(X,\mathbb{Z}(2)\big),\quad \pi(\sigma)\big((r_x)_{x\in X}\big)=(r_{\sigma^{-1}(x)})_{x\in X}.$$

LEMMA 2·5. *The representation* $\pi\colon P(X)\to \mathrm{GL}\,\big(X,\mathbb{Z}(2)\big)$ *is a topological embedding.*

Proof. An element $\sigma\in P(X)$ is in $W_{\mathrm{id}}(E)$ for some finite subset E if and only if $\pi(\sigma)$ fixes the basis vectors $(\delta_{xe})_{x\in X}$, $e\in E$ for the Kronecker

$$\delta_{xy}=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$$

The proof of Lemma 9.83 of [7, p. 508] which is expressed for a compact connected simple group S applies also to any non-abelian finite simple group and yields the following.

THEOREM 2·6. (The Automorphism Group of Strictly Reductive Groups) *Let* $G=\prod_{S\in\mathcal{S}}S^{J(G,S)}$ *be a strictly reductive group. Then* $\mathrm{Aut}\,G\cong\prod_{S\in\mathcal{S}}\mathrm{Aut}(S^{J(G,S)})$, *and the groups* $\mathrm{Aut}\,S^{J(G,S)}$ *are determined as follows:*

Assume that S *is a compact simple group and* X *is an arbitrary set.*

Case A. *Assume that* $S=\mathbb{Z}(p)$. *Then* $\mathrm{Aut}\,S^X\cong \mathrm{GL}(\mathbb{Z}(p)^{(X)})$ *and* $w(\mathrm{Aut}\,S^X)=\mathrm{card}\,X$.

Case B. *Assume that* S *is non-abelian. Then*

(i) $\mathrm{Aut}(S^X)\cong\mathrm{Aut}(S)^X\rtimes_\Sigma P(X)$ *for a suitable automorphic action* $\Sigma\colon P(X)\to\mathrm{Aut}\,\big([\mathrm{Aut}(S)]^X\big)$.

(ii) *If* X *is infinite then* $w(\mathrm{Aut}\,S^X)=w(S^X)$. *The group* $\mathrm{Inn}\,S$ *is isomorphic to* S, *and* $\mathrm{Out}\,S=\mathrm{Aut}\,S/\,\mathrm{Inn}\,S$ *is finite and soluble.*

(iii) *If* X *is infinite,*

$$w\big(P(X)\big)\le w\big(\,\mathrm{GL}(X,\mathbb{Z}(2))\big)\le \mathrm{card}\,X.$$

Proof. The first assertion follows from 2·2. Case A follows from 2·3. Case B: Assertion (i) is taken, mutatis mutandis, from the proof of Lemma 9.83 of [7, p. 508].

Proof of (ii): $\mathrm{Aut}\,S$ is an extension of the normal subgroup $\mathrm{Inn}\,S\cong S$ of inner automorphisms by the finite soluble group $\mathrm{Out}\,S$. Thus we know that $w\big((\mathrm{Aut}\,S)^X\big)=\max\{\aleph_0,\mathrm{card}\,X\}$.

(iii) follows from Lemma 2·5. □

The significance of Theorem 2·6 is that for any compact group G with a normal subgroup N which is strictly reductive, we have a representation $\phi\colon G\to\mathrm{Aut}\,N$ given by $\phi(g)(n)=gng^{-1}$; the structure and size of $\mathrm{Aut}\,N$ has just been determined in Theorem 2·6.

In this paper we do not need the full power of Theorem 2·6. For most of our applications it suffices to know the size of the automorphism group measured by its weight. But Theorem 2·6 is a viable result in its own right and is likely to be useful in future applications of the Countable Layer Theorem.

3. Abelian subgroups of a Compact Group

Recall that the centralizer of a closed subgroup is closed, and that it is normal if the subgroup is normal: indeed if $g\in G$, then $x\in Z(N,G)$ implies

$$(gxg^{-1})n(gxg^{-1})=g\big(x(g^{-1}ng)x^{-1}\big)g^{-1}=g1g^{-1}=1.$$

For a subset N of G let $Z(N,G)$ be the centralizer of N in G.

Proposition 3·1. *Let G be an infinite compact group and $N \trianglelefteq G$ a closed normal subgroup. Then*

$$w\big(G/Z(N,G)\big)\begin{cases}\le w(N) & \text{if } N \text{ is infinite,}\\ <\infty & \text{if } N \text{ is finite.}\end{cases}$$

In particular, if N is small then $Z(N,G)$ is very large.

Proof. (i) The function $I{:}\,G \to \operatorname{Aut} N$, $I(x)(n) = xnx^{-1}$ is a morphism of topological groups. Clearly $\ker I = Z(N,G)$. Since G is compact, so is $G/\ker I$ and thus this group is embedded into $\operatorname{Aut} N$. Therefore, $w(G/Z(N,G)) \le w(\operatorname{Aut} N)$. If N is infinite, by Lemma 2·1 we have $w(\operatorname{Aut} N) \le w(N)$ and thus

$$w\big(G/Z(N,G)\big) \le w(N).$$

If N is finite, then $\operatorname{Aut} N$ is finite and thus

$$\operatorname{card}\big(G/Z(N,G)\big) < \infty.$$

In the second case, as $w(G)$ is infinite, $Z(N,G)$ is very large in G. In the first case this is true if $w(N) < w(G)$. □

Remark 3·2. Let N be a closed normal subgroup of a compact group G.
(i) If N is abelian, then $N \subseteq Z(G,N)$, and
(ii) if N is centre-free, then $N \cap Z(N,G) = \{1\}$ and the product $NZ(N,G)$ is a direct product of compact groups.

Proof. (i) Every abelian subgroup is contained in its centralizer. (ii) Since $Z(N)$, the centre of N, is $N \cap Z(N,G)$ and since all groups in sight are compact, the assertion follows. □

The following lemma pertains to the multilinear algebra of topological groups in general. In this lemma and its proof we shall write all groups additively. Let V be a compact abelian group and let $V\otimes V$ be the tensor product of compact abelian groups [6] which, together with the bilinear continuous function $\otimes : V\times V \to V\otimes V$ classifies continuous $\mathbb{Z}$-bilinear maps $b{:}\,V\times V \to W$ into a topological group by providing a unique morphism $b'{:}\,V\otimes V \to W$ such that $b(v,v') = b'(v\otimes b')$.

Lemma 3·3. *For a compact abelian group V*
(i) *the tensor product $V\otimes V$ is totally disconnected, and*
(ii) *if V/V_0 is infinite, then $w(V\otimes V) = w(V/V_0)$.*

Proof. We have $\operatorname{Hom}(V\otimes V,\mathbb{T}) \cong \operatorname{Hom}(V,\operatorname{Hom}(V,\mathbb{T})) = \operatorname{Hom}(V,\widehat{V})$. Since $\widehat{V}$ is discrete each morphism $\phi{:}\,V\to\widehat{V}$ annihilates the identity component V_0 and $\phi(V)$ is finite. Hence $V_0 \in \ker\phi$ and $\phi(V) \subseteq \operatorname{tor}\widehat{V}$. Thus $V\otimes V \cong (V/V_0)\otimes(V/V_0)$. Write $W = V/V_0$. Then W is totally disconnected. For a prime p let W_p denote the p-Sylow subgroup of V (cf. [7, p. 370]). So $\widehat{W_p}$ is the p-Sylow subgroup of $\widehat{V}$. Then $(V\otimes V)\,\widehat{}\, \cong \operatorname{Hom}(V,\widehat{V}) = \operatorname{Hom}(V,\sum_p \widehat{W_p}) \cong \sum_p \operatorname{Hom}(V,\widehat{W_p}) = \sum_p \operatorname{Hom}(W_p,\widehat{W_p})$. Thus

$$w(V\otimes V) = \operatorname{card}\big((V\otimes V)\,\widehat{}\, = \operatorname{card}\Big(\sum_p \operatorname{Hom}(W_p,\widehat{W_p})\Big).$$

Thus we have to determine the cardinality of $\sum_p \operatorname{Hom}(W_p,\widehat{W_p})$. If W is infinite,

$$\operatorname{card}\Big(\sum_p \operatorname{Hom}(W_p,\widehat{W_p})\Big) = \max\big\{\aleph_0, \sup_p \operatorname{card}\big(\operatorname{Hom}(W_p,\widehat{W_p})\big)\big\}.$$

If W_p is infinite, then the divisible hull of $\widehat{W_p}$ has the same cardinality as $\widehat{W_p}$ itself and is therefore of the form $\mathbb{Z}(p^\infty)^{(X)}$ with $\operatorname{card} X = w(W_p)$. Then $\operatorname{Hom}(W_p, \widehat{W_p})$ is isomorphic to a subgroup of $\operatorname{Hom}(W_p, \mathbb{Z}(p^\infty)^{(X)}) \cong \operatorname{Hom}\big(W_p, \mathbb{Z}(p^\infty)\big)^{(X)} = \widehat{W_p}^{(X)}$ and the cardinal of this group is $w(W_p)$. Thus

$$\operatorname{card}\big(\sum_p \operatorname{Hom}(W_p, \widehat{W_p})\big) \leq \max\{\aleph_0, \sup_p w(W_p)\} = w(W).$$

Thus for infinite $W = V/V_0$ we have $w(V \otimes V) \leq w(W)$.

On the other hand, let S be the p-socle of $\widehat{W_p}$, then $S \cong \mathbb{Z}(p)^{(X)}$, and $\operatorname{Hom}(W_p, \widehat{W_p})$ contains a subgroup isomorphic to a subgroup of

$$\operatorname{Hom}(W_p, \mathbb{Z}(p)^{(X)}) \cong \operatorname{Hom}(W_p, Z(p))^{(X)};$$

since W_p has epimorphisms onto $\mathbb{Z}(p)$, this group has cardinality at least $\operatorname{card} X = w(W_p)$. This implies $w(V \otimes V) \geq w(W)$. This completes the proof. □

We are interested in *symplectic maps*, that is, continuous bilinear functions $\sigma: V \times V \to H$, i.e. those satisfying $\sigma(v,v) = 0$ for all $v \in V$; because of the polarisation trick $0 = \sigma(v+w, v+w)\sigma(v,v) + \sigma(v,w) + \sigma(w,v) + \sigma(w,w) = \sigma(v,w) + \sigma(w,v)$, they satisfy $\sigma(w,v) = -\sigma(v,w)$.

Let $\beta: V \times V \to V \otimes V$ be the bilinear map given by $\beta(v_1, v_2) = v_1 \otimes v_2$. For a compact abelian group V, let $\bigwedge^2 V = (V \otimes V)/\overline{\langle \beta(v,v) : v \in V\rangle}$ and let $p: V \otimes V \to \bigwedge^2 V$ be the quotient map. For $v, w \in V$ set $v \wedge w = p\big(\beta(v,w)\big)$. Then for any symplectic map $\sigma: V \times V \to H$ there is a unique morphism of topological groups $f_\sigma: \bigwedge^2 V \to H$ such that $\sigma(v,w) = f_\sigma(v \wedge w)$.

Lemma 3·4. *Let V be an infinite compact abelian group and let K be a closed subgroup of $\bigwedge^2 V$.*

(i) *Assume that $(\bigwedge^2 V)/K$ is infinite. Then there is a closed subgroup A of V such that $A \wedge A \subseteq K$ and $w(V/A) \leq w\big((\bigwedge^2 V)/K\big)$.*

(ii) *Assume that $(\bigwedge^2 V)/K$ is finite. then there is a closed subgroup A of finite index in V such that $A \wedge A \subseteq K$.*

Proof. (i) Set $D \overset{\text{def}}{=} (\bigwedge^2 V)/K$; then D is totally disconnected by Lemma 3·3(i). Let $F: V \times V \to D$ be the unique bilinear map such that $F(v, v') = (v \wedge v') + K$.

The family $\mathcal{N}(D)$ of compact open subgroups of D has cardinality $w(D)$. Let $f: \bigwedge^2 V \to D$ be the unique morphism such that $F(v,w) = f(v \wedge w)$. For each $U \in \mathcal{N}(D)$ let $W(U) = f^{-1}(U)$. Then by the surjectivity of f we have $\operatorname{card}\{W(U) : U \in \mathcal{N}(D) = \operatorname{card} \mathcal{N}(D)$. Moreover $\bigcap_U W(U) = C$. There is some open subgroup $A(U) \leq V$ such that $(A(U) \wedge A(U) \subseteq W(U)$. Set $A = \bigcap_{U \in \mathcal{N}(D)} A(U)$. Then $A \wedge A \subseteq W(U)$ for all $U \in \mathcal{N}$. Hence $A \times A \subseteq C$. Then the filter basis $\mathcal{F}$ consisting of the finite intersections of the set $\{A(U)/A : U \in \mathcal{N}(D)$ intersects in the singleton set $\{A\}$, and thus by the compactness of V/A converges to A. Since $\mathcal{F} \subseteq \mathcal{N}(V/A)$ we know that $\mathcal{F}$ is cofinal in $\mathcal{N}(V/A)$ and thus $\operatorname{card} \mathcal{F} = \operatorname{card} \mathcal{N}(V/A) = w(V/A)$. But $\operatorname{card} \mathcal{F} = \operatorname{card}\{A(U) : U \in \mathcal{N}(D)\} \leq \operatorname{card} \mathcal{N}(D) = w(D)$. Hence $w(V/A) \leq w(D)$. This completes the proof of (i).

(ii) If $(\bigwedge^2 V)/K$ is finite, the K is open in $\bigwedge^2 V$. Also $V_0 \wedge V = \{0\} \subseteq K$. Hence by the compactness of V there is an open subgroup A of V such that $A \wedge A \subseteq A \wedge V \subseteq K$. Since V is compact and A open, V/A is discrete and compact, hence finite. □

Lemma 3·5. *Let V be an infinite compact abelian group and $\sigma: V \times V \to H$ a symplectic map into a topological group H and let C be the smallest closed subgroup of H containing $\sigma(V \times V)$. Then*

(i) *there is a closed subgroup A of V such that $\sigma(A \times A) = \{0\}$ and $w(V/A) \le w(H)$, and*

(ii) *C is totally disconnected and compact, and $w(C) \le w(V)$.*

Proof. (i) There is a morphism $f: \bigwedge^2 V \to H$ such that $\sigma(v, w) = f(v \wedge w)$. Let $K = \ker f$. Then $w\big((\bigwedge^2 V)/K\big) = w(\operatorname{im} f) \le w(H)$. Then by Lemma 3·4 there is a closed subgroup A of V such that $w(V/A) \le w\big(\bigwedge^2 V)/K\big) \le w(H)$ and $A \wedge A \subseteq K$, i.e. $\sigma(A \times A) = \{1\}$

(ii) Recall that $V \wedge V$ generates a dense subgroup of $\bigwedge^2 V$ whence $C = f(\bigwedge^2 V)$. Then by Lemma 3·3, the homomorphic image C of a totally disconnected compact group is compact totally disconnected. Thus we have $w(C) = w\big(f(\bigwedge^2 V)\big) \le w(V \wedge V) \le w(V)$. □

Let G' be the closure of the commutator subgroup of a topological group G.

Lemma 3·6. *Let N be a closed central subgroup of a compact group G such that G/N is abelian. Then G contains a closed abelian subgroup $A \supseteq N$ such that $w(G/A) \le w(G') \le w(N)$ if N is infinite and G/A is finite if G' is finite.*

Proof. Since N is central in G, the group G is nilpotent of class at most two. Then $(x,y) \mapsto [x,y] = xyx^{-1}y^{-1}: G \times G \to G' \subseteq N$ is bilinear. We note that $\beta: G/N \times G/N \to N$, $\sigma(xN, yN) = [x,y]$ is a symplectic map $(G/N) \times (G/N) \to G'$. Then by Lemma 3·4 if G' is infinite, there is a closed subgroup A of G containing N such that $[A,A] = \beta(A/N \times A/N) = \{0\}$, that is, A is abelian and $w(V/A) \le w(G') \le w(N)$. If G' is finite, then A can be found to have finite index in G. □

In Definition 1.1 we called a subgroup H of a topological group G large if $w(H) = w(G)$ and we called G an LAS-group if it has large abelian subgroups. In order to facilitate the formulation of some technical results we complement these definitions as follows.

Definition 3·7. Let G be a topological group. A subgroup H of G is called *very large* if $w(G/H) < w(G)$. A topological group is called a VLAS-*group* if it has very large abelian subgroups.

The set of cardinals $\{w(G/H) : H \le G \text{ and } H \text{ is an abelian subgroup of } G\}$ has a smallest element since every set of cardinals is well-ordered. This smallest element is called the *abelian index* $\operatorname{abind}(G)$. □

In [8] we established a result of which the following is a corollary:

Fact 3·8. *Let G be an infinite compact group and H a closed subgroup. Then*

$$w(G) = \max\{w(H), w(G/H)\}.$$

□

These concepts are intended for infinite groups G. The following remarks are immediate:

Remarks 3·9. Let G be an infinite compact group. For a small subgroup H we have $w(G) = w(G/H)$. If H is a very large subgroup of G, it is large, and thus a VLAS–group is an LAS-group. The group G is a VLAS-group if an only if $\operatorname{abind}(G) < w(G)$. □

All abelian topological groups are VLAS-groups. A finite non-abelian group is never an LAS-group unless it is singleton. In Corollary B of the Introduction it was shown that every *connected* compact group is an LAS-group. Since for a maximal pro-torus

T of a connected compact group G, the factor space G/T has weight $w(G)$, a similar statement is true for all other maximal abelian subgroups T, a compact connected group is *not* a VLAS-group.

If H is a large subgroup of G and H is an LAS-group then G is an LAS-group, and an analogous statement is true for VLAS-groups. By the Reduction Theorem (Theorem C in Section 1) which we shall prove in 3·10 below, if N is a small closed normal subgroup and G/N is an LAS-group, then G is an LAS-group.

The following theorem is the enabling result for proofs by induction.

THEOREM 3·10. (The Reduction Theorem) *Let G be an infinite compact group and $N \trianglelefteq G$ a small closed normal subgroup.*

(i) *If G/N is an* LAS-*group then G is an* LAS-*group. Indeed, if B is an abelian subgroup of N there is a closed abelian subgroup A of G containing B such that $w(A) = w(G)$.*

(ii) *If G/N is a* VLAS-*group, then G is a* VLAS-*group. More specifically,*

$$\operatorname{abind}(G) \leq \max\{w(N), \operatorname{abind} G/N\} \text{ or } \operatorname{abind}(G) \text{ is finite.}$$

Proof. Since G/N is an LAS- [VLAS-] group, G contains a large subgroup H such that $N \trianglelefteq H$ and H/N is a large [very large] abelian subgroup of G/N. Thus

$$w(H) = \max\{w(N), w(H/N)\} = \max\{w(N), w(G/N)\} = w(G), \text{ respectively,}$$
$$w(G/H) = w\big((G/N)/(H/N)\big) < w(G/N) \leq w(G);$$

in the second case we may assume that $w(G/H) = \operatorname{abind}(G/N)$.

By Proposition 3·1 we have $w\big(H/Z(N,H)\big) \leq w(N)$ if N is infinite and know that $H/Z(N,H)$ is finite otherwise. Then $Z(N) = Z(N,H) \cap N$ is central in $Z(N,H)$ and $Z(N,H)/Z(N) \cong H/N$ is abelian. Now Lemma 3·6 applies and shows that $Z(N,H)$ contains an abelian subgroup A_1 containing $Z(N)$ such that

$$w(Z(N,H)/A_1) = w\big((Z(N,H)/Z(N))/(A_1/Z(N))\big) \leq w(Z(N,H)') \leq w\big(Z(N)\big)$$

provided $Z(N,H)'$ is infinite. Otherwise A_1 can be found so as to have finite index in $Z(N,H)$. Thus $w(H/A_1) \leq w(N)$ or H/A_1 is finite. If B is any abelian subgroup of N, then $A_1 \subseteq Z(N,G)$ and B commute elementwise. Hence $A = \overline{A_1 B}$ is a closed abelian subgroup of G containing B and satisfying $w(H/A) \leq w(N)$ or $|H/A| < \infty$. Thus if $w(G/H) = \operatorname{abind}(G/N)$, then $\operatorname{abind}(G) \leq w(G/A) \leq \max\{w(N), \operatorname{abind}(G/N)\}$. □

We observe that in case (ii), the group A may likewise be chosen so as to include any given abelian subgroup of N.

Sufficient conditions for G/N to be an LAS-group are not hard to find:

(i) G/N is abelian.
(ii) G/N is strictly reductive.
(iii) G/N is connected.

In those cases, if N is a small closed normal subgroup of the infinite compact group G, then G is an LAS-group.

THEOREM 3·11. (The Extension Theorem) *Let G be an infinite compact group and N a closed normal subgroup. If both G/N and N are* LAS-*groups then G is an* LAS-*group.*

Proof. Either N is large, that is $w(N) = w(G)$, or N is small, that is $w(N) < w(G)$. In the first case, since N is an LAS-group, so is G. In the second case, the Reduction Theorem 3·10 applies. □

Corollary 3·12. (The Reduction Corollary) *Let G be a compact group and assume that G/G_0 is an* LAS-*group. Then G is an* LAS-*group.*

Proof. By Proposition A of the Introduction, G_0 is an LAS-group. Hence the corollary is a direct consequence of the Extension Theorem 3·11. □

Another immediate consequence of the Extension Theorem is

Corollary 3·13. (Reduction mod $Z_0(G_0)$) *Let G be a compact group. If $G/Z_0(G_0)$ is an* LAS-*group, then G is an* LAS-*group.* □

By this corollary, whenever we wish to use the Countable Layer Theorem for proving that a compact group G is an LAS-group we may assume that $Z_0(G_0) = \{1\}$.

A simple induction argument yields the following consequence of the Extension Theorem 3·11:

Corollary 3·14. *Let G be an infinite compact group and let $N_1 \supseteq \cdots \supseteq N_k$ be a finite sequence of closed normal subgroups of G such that G/N_1 and N_{j-1}/N_j are LAS-groups for $j = 2, \ldots, k$. Then G is an LAS-group.* □

Recall the concept of layers from the Countable Layer Theorem and the paragraph following it in the Introduction. The next result is an immediate consequence:

Proposition 3·15. *If an infinite compact group has only a finite number of layers, then it is an* LAS-*group.*

Proof. Apply Corollary 3·14 with $N_j = \Omega_j(G)$ and notice that all of G/N_1 and N_{j-1}/N_j are either strictly reductive or connected abelian. □

4. Applying the Countable Layer Theorem

Let G be a compact group and recall the sequence of characteristic subgroups $\Omega_n(G)$ of the Countable Layer Theorem in the Introduction.

Theorem 4·1. (The Dominant Layer Theorem) *Let G be a compact group. If there is a natural number n such that $w\big(\Omega_n(G)/Z_0(G_0)\big) < w\big(G/Z_0(G_0)\big)$ then G is an* LAS-*group.*

Proof. By Proposition 3·15, $G/\Omega_n(G)$ is an LAS-group. Therefore

$$\big(G/Z_0(G_0)\big)/\big(\Omega_n(G)/Z_0(G_0)\big)$$

is an LAS-group. By hypothesis, $\Omega_n(G)/Z_0(G_0)$ is a small subgroup of $G/Z_0(G_0)$. Then by the Reduction Theorem 3·10, the group $G/Z_0(G_0)$ is an LAS-group and thus by 3·14, G is an LAS-group. □

The name of the theorem will become more obvious when we formulate and prove Theorem 4·4 below.

Corollary 4·2. *Let G be a profinite group. If $w(\Omega_n(G)) < w(G)$ for some n, then G is an* LAS-*group.* □

Accordingly, if the Abelian Subgroup Conjecture is false, then there exists a counterexample G where G is a profinite group such that $w\big(\Omega_n(G)\big) = w(G)$ for all $n = 1, 2 \ldots$.

In view of 3·14, for further comments, we may just as well assume that $Z_0(G_0) = \{1\}$. Then let us finally express the condition that $w\big(\Omega_n(G)\big) < w(G)$ in terms of the layers of the Countable Layer Theorem.

LEMMA 4·3. *Let $G \supseteq N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be a descending sequence of infinite normal subgroups with $\bigcap_{n=1}^{\infty} N_n = \{1\}$.*

Proof. Then the following conditions are equivalent:

(i) $w(N_1) < w(G)$.

(ii) There is a cardinal $\aleph < w(G)$ such that $w(N_{n-1}/N_n) \leq \aleph$ for $n = 2, 3, \ldots$.

Proof. (i) implies (ii): $w(N_{n-1}/N_n) \leq w(N_{n-1}) \leq w(N_1)$. Thus let $\aleph = w(N_1)$.

(ii)$\Rightarrow$(i): Repeated application of Fact 3·8 yields

$$w(N_1/N_n) = \max\{w(N_1/N_2), \ldots, w(N_{n-1}/N_n)\}.$$

Thus $w(N_1/N_n) \leq \aleph$ by (ii). From $\bigcap_n N_n = \{1\}$ and the compactness of N_1 we conclude that $N_1 \cong \lim_n N_1/N_n$ and thus $w(N_1) = \sup_n w(N_1/N_n) \leq \aleph < w(G)$. □

Let us now return to the Countable Layer Theorem in the Introduction. The factor groups $\Lambda_n(G)$, $n = 1, 2, \ldots$ are called the *layers.* We shall say that a layer $\Lambda_n(G)$ *is dominant* if there is a cardinal $\aleph$ such that $w(\Lambda_m(G)) \leq \aleph < w(\Lambda_n(G))$ for $m \neq n$.

THEOREM 4·4. (The Dominant Layer Theorem, Second Version) *Any infinite compact group with a dominant layer is an* LAS*-group.*

Proof. We may assume that $Z_0(G_0) = \{1\}$. Let $\Lambda_n(G) = \Omega_{n-1}(G)/\Omega_n(G)$ be a dominant layer with a minimal n. We apply Lemma 4·3 with $N_1 = \Omega_n(G)$, $N_2 = \Omega_{n+1}(G)$ etc. and obtain $w(\Omega_n(G)) < w(G/\Omega_n(G)) \leq w(G)$. Then Theorem 4·1 proves the claim. □

Let us comment finally that under the hypotheses of Theorems 4·1 and 4·4 we have an additional piece of information.

PROPOSITION 4·5. *Let G be an infinite profinite group such that $\Omega_n(G)$ is small for some $n \in \{1, 2, \ldots\}$. Then G has a large subgroup S containing $\Omega_n(G)$ which contains an abelian subgroup A such that $w(S/A) < w(S)$; that is, S is a* VLAS*-group.*

Proof. We obtain S directly from 3·15 such that $S/\Omega_n(G)$ is abelian and $w(S) = w(G)$. Now we apply 3·10(ii) with S in place of G and $\Omega_n(G)$ in place of N and find $\operatorname{abind}(S) \leq \max\{w(\Omega(G)), \operatorname{abind} S/\Omega_n(G)\} = w(\Omega_n(G))$ since $S/\Omega_n(G)$ is abelian and thus has abelian index zero. But $w(\Omega_n(G)) < w(G) = w(S)$. Thus $\operatorname{abind}(S) < w(S)$. □

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Added in proof. In the meantime, WOLFGANG HERFORT pointed out (W. Herfort, The Abelian Subgroup Conjecture: A Counter Example, J. of Lie Theory **12** (2002), 305–308) that the free profinite p-group $F_p(X)$ on any infinite set X converging to 1 has weight $\mathrm{card}(X)$ and has the property that all of its closed subgroups are free (see [12] or [15]). Thus the only nondegenerate abelian closed subgroups are isomorphic to the additive group $\mathbb{Z}_p$ of p-adic integers, and thus all abelian subgroups have a countable weight. Hence $F_p(X)$ is a counterexample to the Abelian Subgroup Conjecture for any uncountable set X. (See our comment preceding Theorem C for the discrete case.) In contrast with the hypothesis of the Dominant Layer Theorem 4·4, the layers of $F_p(X)$ are all isomorphic to $\mathbb{Z}(p)^X$ and thus have weight $\mathrm{card}(X)$. The results of this paper remain valid and are a challenge to find further sufficient, perhaps even necessary and sufficient conditions for a profinite group to be an LAS-group.