Quasistatic problems in viscoplasticity theory

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Abstract

We study the existence theory to quasistatic initial-boundary value problems with internal variables, which model the viscoelastic or viscoplastic behavior of solids at small strain. In these problems a system of linear partial differential equations coupled with a nonlinear system of differential equations or differential inclusions must be solved. The solution theory is based on monotonicity properties of the differential equations or differential inclusions. The article gives an essentially complete account of the recent progress.

1 Introduction and statement of results

In this article we study existence and uniqueness of solutions to initial-boundary value problems, which model the viscoelastic or viscoplastic deformation behavior of solids at small strain. The initial-boundary value problems we study use differential equations or differential inclusions to model the dependence of the stress on the strain history.

To formulate the initial-boundary value problem let $\Omega \subseteq \mathbb{R}^3$ be an open bounded set, the set of material points of a solid body. $\mathcal{S}^3$ denotes the set of symmetric 3×3–matrices. Unknown are the displacement $u(x,t) \in \mathbb{R}^3$ of the material point labeled x at time t, the Cauchy stress $T(x,t) \in \mathcal{S}^3$ and the vector of internal variables $z(x,t) \in \mathbb{R}^N$. The model equations are

$$-\operatorname{div}_x T(x,t) = b(x,t), \tag{1.1}$$

$$T(x,t) = \mathcal{D}(\varepsilon(\nabla_x u(x,t)) - Bz(x,t)), \tag{1.2}$$

$$\frac{\partial}{\partial t} z(x,t) \in f(\varepsilon(\nabla_x u(x,t)), z(x,t)), \tag{1.3}$$

which must hold for $(x,t) \in \Omega \times [0,\infty)$. The unknowns must satisfy the initial condition

$$z(x,0) = z^{(0)}(x), \quad x \in \Omega, \tag{1.4}$$

and either the Dirichlet boundary condition

$$u(x,t) = \gamma_D(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.5}$$

or the Neumann boundary condition

$$T(x,t)n(x) = \gamma_N(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{1.6}$$

Here $\nabla_x u(x,t)$ denotes the 3×3–matrix of first order derivatives of u, the deformation gradient, $(\nabla_x u(x,t))^T$ denotes the transposed matrix, and

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\Big(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\Big) \in \mathcal{S}^3,$$

is the strain tensor. $B : \mathbb{R}^N \to \mathcal{S}^3$ is a linear mapping, which assigns to the vector $z(x,t)$ the plastic strain tensor $\varepsilon_p(x,t) = Bz(x,t)$. One often assumes that all six independent components of the symmetric matrix ε_p belong to the components of z, in which case B would be the projection to these six components. Moreover, $\mathcal{D} : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping, the elasticity tensor.

The given data of the problem are $b : \Omega \times [0,\infty) \to \mathbb{R}^3$, the volume force, $\gamma_D : \partial\Omega \times [0,\infty) \to \mathbb{R}^3$, the boundary displacement, $\gamma_N : \partial\Omega \times [0,\infty) \to \mathbb{R}^3$, the traction at the boundary, and $z^{(0)} : \Omega \to \mathbb{R}^N$, the initial data. $n(x)$ in the Neumann boundary condition denotes the exterior unit normal to $\partial\Omega$ at x.

The equation (1.2) and the differential inclusion (1.3) with a given function $f : \mathcal{S}^3 \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ together determine the dependence of the stress $T(x,t)$ on the strain history $s \to \varepsilon(\nabla_x u(x,s))$. They are the constitutive relations which model the inelastic behavior of the body. Clearly, these constitutive relations cannot be chosen arbitrarily. Instead, thermodynamical and mathematical requirements restrict the choice of f.

We first note that the formulation of equation (1.2) is based on the assumption that the deformation gradient only assumes small values, which makes it possible to additively decompose the strain tensor into an elastic and a plastic part:

$$\varepsilon(\nabla_x u) = (\varepsilon(\nabla_x u) - \varepsilon_p) + \varepsilon_p\,.$$

According to (1.2), the elastic part is the only source of stresses in the body.

The choice of the function f in equation (1.3) is restricted by the second law of thermodynamics, which requires that to f there exists a free energy function $\psi(\varepsilon,z)$ such that the relations

$$\begin{aligned} \rho\nabla_\varepsilon\psi(\varepsilon,z) &= T \\ \rho\nabla_z\psi(\varepsilon,z)\cdot\zeta &\le 0 \end{aligned} \tag{1.7}$$

hold for all $\zeta \in f(\varepsilon,z)$ and all $(\varepsilon,z) \in \mathcal{S}^3 \times \mathbb{R}^N$. Here $\rho > 0$ denotes the constant mass density. A class of functions, for which these relations are automatically satisfied and which has important mathematical properties consists of all f, for which a quadratic positive definite or positive semi-definite free energy

$$\rho\psi(\varepsilon,z) = \frac{1}{2}[\mathcal{D}(\varepsilon - Bz)]\cdot(\varepsilon - Bz) + \frac{1}{2}(Lz)\cdot z \tag{1.8}$$

and a monotone function $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ with $0 \in g(0)$ exist such that for all $(\varepsilon,z) \in \mathcal{S}^3 \times \mathbb{R}^N$

$$f(\varepsilon,z) = g\Big(-\rho\nabla_z\,\psi(\varepsilon(\nabla_x u(x,t)), z(x,t))\Big). \tag{1.9}$$

L in (1.8) is a symmetric positive definite or positive semi-definite $N\times N$–matrix. We call the constitutive relation (1.3) of monotone type if f is of the form (1.9). This is the class of constitutive relations we are interested in.

Thus, the aim of this article is to study the existence and uniqueness of solutions to the quasistatic problems (1.1) – (1.6) with constitutive equations of monotone type.

Using that (1.2) and (1.8) together yield $-\rho \nabla_z \psi(\varepsilon, z) = B^T T - Lz$, where $B^T : \mathcal{S}^3 \to \mathbb{R}^N$ is the mapping adjoint to B, these problems can be written in the transparent form

$$-\operatorname{div}_x T(x,t) = b(x,t), \tag{1.10}$$

$$T(x,t) = \mathcal{D}(\varepsilon(\nabla_x u(x,t)) - Bz(x,t)), \tag{1.11}$$

$$z_t(x,t) \in g(B^T T(x,t) - Lz(x,t)), \tag{1.12}$$

$$z(x,0) = z^{(0)}(x), \tag{1.13}$$

either with the Dirichlet boundary condition

$$u(x,t) = \gamma_D(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.14}$$

or with the Neumann boundary condition

$$T(x,t)n(x) = \gamma_N(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{1.15}$$

The class of constitutive relations of monotone type was introduced in [Alb98]. This class generalizes the class of constitutive relations of generalized standard materials defined by B. Halphen and Nguyen Quoc Son in [HS75]. For a generalized standard material the function g is the gradient or subdifferential of a convex function. [Alb98] contains a derivation of the dissipation inequality (1.7) from the second law of thermodynamics. Also, examples of constitutive equations from engineering are considered in this book and the problem is studied, whether these constitutive equations are of monotone type. Suffice it to say here that the class of constitutive relations of monotone type includes the classical constitutive models like the Norton-Hoff and the Prandtl-Reuss laws, but that it is too small to include most models from engineering for the inelastic behavior of metals. In fact, all these models can be written in the form (1.3) with f of the form (1.9), but with non-monotone g. In the majority of cases the free energy ψ is not positive definite, but only positive semi-definite. In particular, the Norton-Hoff law and the Prandtl-Reuss law are constitutive equations with positive semi-definite free energy, whereas models with linear hardening have positive definite free energy.

Nevertheless, it is an important mathematical goal to understand the existence theory of initial-boundary value problems to constitutive equations of monotone type. We give an incomplete survey of the literature to existence problems. More references can be found in [Alb98].

For the classical constitutive models the existence theory started with [Mor71] and [DL72]. Of the publications which appeared in the following years we only mention [Joh76, Grö78], in which elasto-plastic constitutive equations were studied, and [Joh78, Grö79], where the existence theory for constitutive equations with hardening was investigated. In particular, in [Grö79] it was noticed that the presence of linear hardening simplifies the existence proofs. For the elasto-plastic constitutive equation, the Prandtl-Reuss law, the existence of a function representing the displacement remained unclear in these publications. This question was solved in [Suq81], where it is shown that the displacement belongs to the space of bounded deformations, a space which was later studied in the book [Tem83] in connection with time independent problems of elasto-plasticity. We mention that the problem of existence of the displacement function is connected with the existence of the Helmholtz and Weyl projectors in general Banach spaces, which in the L^p-space we study in Section 2.

In [Tem86] it was clarified in what sense the solution satisfies the differential inclusions appearing in the Prandtl-Reuss law, a rate independent model. The regularity of the stress field to the Prandtl-Reuss law was studied in [BF93, BF96]; it turned out that the stress field belongs to the Sobolev space H_1^2 if the data of the problem are sufficiently smooth. Finally, the recent book [HR99] presents a treatment of the Prandtl-Reuss model and of some models, which in our terminology belong to the class of constitutive models with positive definite free energy.

These references are concerned with quasistatic problems. In the dynamic initial-boundary value problem the quasistatic equation (1.1) is replaced by

$$\rho u_{tt} - \operatorname{div}_x T(x,t) = b(x,t).$$

The existence theory of the dynamic problem to special models with linear hardening is considered in [LeT90], whereas the article [AL87] is devoted to the investigation of the dynamic problem to the Prandtl-Reuss law. The study of the existence theory for the dynamic problems to constitutive equations of monotone type with positive definite free energy was started in [Alb98]. In [Che97, Che01, CG00] this study is continued and extended to special classes of models of monotone type with positive semi-definite free energy. It is shown by these investigations that for the existence theory it is a principle difference, whether the free energy ψ is positive definite or only positive semi-definite.

That the same division also exists for quasistatic initial-boundary value problems to constitutive equations of monotone type is shown by the references [Alb01, Cheb, Chea, Ebe], which besides other investigations contain studies of quasistatic problems with positive definite free energy, and by the present article, which aims to develop a more general and complete theory: We give existence proofs for problems with positive definite free energy and, in particular, derive new existence results for the important and more difficult case of positive semi-definite free energy. As a by-product we complete the investigations in [Alb01], since several of the theorems stated and proved in the section on problems with positive definite free energy are announced in that reference and used without proof.

Our results on problems with positive definite free energy extend the results of the manuscripts [Chea, Ebe], which are to be published, only slightly. However, the method of proof used in Section 3, which we believe to be of principle interest, is new. Also, we includ these results here since they complete the investigations in [Alb01] and since it seems to be very desirable to have a unified treatment at hand which allows to compare this case with the case of positive semi-definite free energy. Our results are precisely discussed in the remainder of this introduction.

Statement of the main results. To state the results we need some notations and definitions.

We always assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded open set with C^1-boundary $\partial\Omega$. T_e denotes a positive number (time of existence), and for $0 \leq t \leq T_e$ we set

$$Z_t = \Omega \times [0,t], \quad Z = Z_{T_e}.$$

If w is a function defined on Z_t and if $0 \leq s \leq t$, we denote the function $x \mapsto w(x,s)$ by $w(s)$. For symmetric matrices $\sigma, \tau \in \mathcal{S}^3$ we write

$$\sigma \cdot \tau = \sum_{i,j=1}^{3} \sigma_{ij}\tau_{ij}\,, \quad |\sigma| = \sqrt{\sigma \cdot \sigma}.$$

By $1 < p,\ q < \infty$ we denote numbers with $\frac{1}{p} + \frac{1}{q} = 1$. The norms on $L^p(\Omega, \mathcal{S}^3)$ and

$L^p(\Omega,\mathbb{R}^N)$ are denoted by

$$\|u\|_{p,\Omega} = \Big[\int_\Omega |u(x)|^p \, dx \Big]^{1/p},$$

and for the canonical bilinear forms on the product spaces $L^p(\Omega,\mathcal{S}^3) \times L^q(\Omega,\mathcal{S}^3)$ and $L^p(Z,\mathcal{S}^3) \times L^q(Z,\mathcal{S}^3)$ we use the symbols

$$(\sigma,\tau)_\Omega = \int_\Omega \sigma(x)\cdot\tau(x)\,dx\,, \quad (\sigma,\tau)_Z = \int_Z \sigma(x,t)\cdot\tau(x,t)\,d(x,t)\,.$$

Since $\mathcal{D} : \mathcal{S}^3 \to \mathcal{S}^3$ is symmetric and positive definite, other bilinear forms on these product spaces are

$$[\sigma,\tau]_\Omega = (\mathcal{D}\sigma,\tau)_\Omega\,, \quad [\sigma,\tau]_Z = (\mathcal{D}\sigma,\tau)_Z\,.$$

$H_1^p(\Omega,\mathbb{R}^n)$ is the Banach space of functions in $L^p(\Omega,\mathbb{R}^n)$, for which the components have weak derivatives in $L^p(\Omega,\mathbb{R}^n)$, and $\overset{\circ}{H}{}_1^p(\Omega,\mathbb{R}^n)$ denotes the closure in $H_1^p(\Omega,\mathbb{R}^n)$ of the space $C_0^\infty(\Omega,\mathbb{R}^n)$ of all infinitely differentiable functions with compact support contained in Ω. The norm of $H_1^p(\Omega,\mathbb{R}^n)$ is $\|u\|_{1,p,\Omega}$. Clearly, for Z instead of Ω we use the same notations.

In our investigations solutions of the boundary value problems of linear elasticity theory in L^q–space with $1 < q < \infty$ play an important part. These problems are given by the equations

$$-\operatorname{div} T(x) = \hat{b}(x)\,, \quad x \in \Omega, \tag{1.16}$$
$$T(x) = \mathcal{D}\Big(\varepsilon(\nabla u(x)) - \hat{\varepsilon}_p(x)\Big), \quad x \in \Omega, \tag{1.17}$$

with the Dirichlet condition

$$u(x) = \hat{\gamma}_D(x), \quad x \in \partial\Omega, \tag{1.18}$$

or the Neumann condition

$$T(x)\,n(x) = \hat{\gamma}_N(x), \quad x \in \partial\Omega\,. \tag{1.19}$$

To define a weak solution of the Dirichlet problem, we insert (1.17) into (1.16), multiply the resulting equation with $v \in \overset{\circ}{H}{}_1^p(\Omega,\mathbb{R}^3)$ and integrate by parts. A weak solution is sought in the form $u = w + \hat{\gamma}_D$ with $\hat{\gamma}_D \in H_1^q(\Omega,\mathbb{R}^3)$ and $w \in \overset{\circ}{H}{}_1^q(\Omega,\mathbb{R}^3)$. Insertion of this representation in the resulting integral identity yields

$$\Big(\mathcal{D}(\varepsilon(\nabla w) - \hat{\varepsilon}_p), \varepsilon(\nabla v)\Big)_\Omega = (\hat{b}, v)_\Omega - \Big(\mathcal{D}(\varepsilon(\nabla\hat{\gamma}_D)), \varepsilon(\nabla v)\Big)_\Omega, \tag{1.20}$$

where we used that $\mathcal{D}((\varepsilon\nabla u(x)) - \hat{\varepsilon}_p(x))$ is a symmetric matrix.

To define a weak solution of the Neumann problem, we use (1.19) in the process of partial integration and obtain for $v \in H_1^p(\Omega,\mathbb{R}^3)$

$$\Big(\mathcal{D}(\varepsilon(\nabla u) - \hat{\varepsilon}_p), \varepsilon(\nabla v)\Big)_\Omega = (\hat{b}, v)_\Omega - \int_{\partial\Omega} \hat{\gamma}_N \cdot v\,dS. \tag{1.21}$$

Therefore we define weak solutions as follows:

Definition 1.1 *Let $\hat{b} \in L^q(\Omega, \mathbb{R}^3)$ and $\hat{\varepsilon}_p \in L^q(\Omega, \mathcal{S}^3)$ be given.*
(i) *Let $\hat{\gamma}_D \in H_1^q(\Omega, \mathcal{S}^3)$. A function $(u, T) \in H_1^q(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3)$ is a weak solution of the Dirichlet problem* (1.16) – (1.18), *if* (1.17) *holds and if $w \in \overset{\circ}{H}{}_1^q(\Omega, \mathbb{R}^3)$ exists with $u = w + \hat{\gamma}_D$, such that* (1.20) *is satisfied for all $v \in \overset{\circ}{H}{}_1^p(\Omega, \mathbb{R}^3)$.*
(ii) *Let $\hat{\gamma}_N \in L^q(\partial\Omega, \mathbb{R}^3)$. A function $(u, T) \in H_1^q(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3)$ is a weak solution of the Neumann problem* (1.16), (1.17), (1.19), *if* (1.17) *and the equation* (1.21) *hold for all $v \in H_1^p(\Omega, \mathbb{R}^3)$.*

It is well known that for the case $q = 2$ the Dirichlet problem has a unique weak solution, and that the Neumann problem has a weak solution u_0 if $\hat{b}$ and $\hat{\gamma}_N$ satisfy the equation

$$\int_\Omega \hat{b}(x) \cdot (a + \omega \times x)dx = \int_{\partial\Omega} \hat{\gamma}_N(x) \cdot (a + \omega \times x)dS$$

for all $a, \omega \in \mathbb{R}^3$. The function $a + \omega \times x$ is an infinitesimal rigid motion. All solutions of the Neumann problem are obtained in the form $u_0(x) + a + \omega \times x$ with $a, \omega \in \mathbb{R}^3$. The proof of these assertions is based on Korn's inequality stated later in Lemma 2.1.

It is generally believed, we cite [Val88] for this, that the same assertions also hold for $q \neq 2$, and that the methods used for example in [Mey63, GT83, Sim72, SS92, SS96] to prove results for boundary value problem to scalar partial differential equations in L^q can be transfered to the system of linear elasticity theory to prove these assertions. However, although important partial results are available even for systems with variable coefficients, cf. [Gia93, Gre91, Kos62], to the best of our knowledge a general proof has not been published. Therefore our results for positive semi-definite free energy, which are based on the existence of solutions to the boundary value problems of linear elasticity theory for $q \neq 2$, are proved under the assumption that the above assertions hold for all $1 < q < \infty$.

Next we give the definition of weak and strong solutions of the initial-boundary value problems (1.10) – (1.15).

Definition 1.2 *Assume that $1 < q \leq 2 \leq p < \infty$, $b \in L^1(0, T_e; L^p(\Omega, \mathbb{R}^3))$, $z^{(0)} \in L^2(\Omega, \mathbb{R}^N)$, $\gamma_D \in L^1(0, T_e; H_1^p(\Omega, \mathbb{R}^3))$ and $\gamma_N \in L^1(0, T_e; L^p(\partial\Omega, \mathbb{R}^3))$.*

A function $(u, T, z) \in L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^p(Z, \mathcal{S}^3) \times C([0, T_e], L^q(\Omega, \mathbb{R}^N))$ is a strong solution of the Dirichlet problem (1.10) – (1.14) *or the Neumann problem* (1.10) – (1.13), (1.15), *respectively, if*

(i) *for almost every $t \in [0, T_e]$ the function $(u(t), T(t))$ is a weak solution of the Dirichlet problem* (1.16) – (1.18) *or of the Neumann problem* (1.16), (1.17), (1.19), *respectively, with $\hat{\varepsilon}_p = Bz(t)$, $\hat{b} = b(t)$, $\hat{\gamma}_D = \gamma_D(t)$ and $\hat{\gamma}_N = \gamma_N(t)$.*

(ii) *for almost every $t \in [0, T_e]$ the derivative $\frac{d}{dt} z(t)$ exists, belongs to $L^q(\Omega, \mathbb{R}^N)$, and satisfies* (1.12).

(iii) $z(0) = z^{(0)}$.

The function (u, T, z) is a weak solution of the Dirichlet initial-boundary value problem (1.10) – (1.14), *or of the Neumann initial-boundary value problem* (1.10) – (1.13), (1.15), *respectively, if there exists a sequence $\{k_n\}_{n=1}^\infty$ of functions $k_n \in L^1(0, T_e; L^q(\Omega, \mathbb{R}^N))$, which converges in $L^1(0, T_e; L^q(\Omega, \mathbb{R}^N))$ to 0, and if there exists a sequence $\{(u_n, T_n, z_n)\}_{n=1}^\infty$ of strong solutions of the Dirichlet or Neumann problem with* (1.12) *replaced by*

$$\frac{\partial}{\partial t} z_n(x, t) \in g\Big(B^T T_n(x, t) - L z_n(x, t)\Big) + k_n(x, t)\,, \tag{1.22}$$

which converges in $L^q(0,T_e;H_1^q(\Omega,\mathbb{R}^3))\times L^p(Z,\mathbb{R}^3)\times C([0,T_e],L^q(\Omega,\mathbb{R}^N))$ *to* (u,T,z).

Positive definite free energy. Now we can state our main results for constitutive equations with positve definite free engergy ψ, which are proved in Sections 3 and 6. Note that ψ is positive definite if and only if the symmetric $N\times N$–matrix L in (1.8) is positive definite.

Theorem 1.3 *Assume that*

$$\rho\psi(\varepsilon,z)=\frac{1}{2}\,[\mathcal{D}(\varepsilon-Bz)]\cdot(\varepsilon-Bz)+\frac{1}{2}\,(Lz)\cdot z$$

is a positive definite quadratic form and that $g:\mathbb{R}^N\to 2^{\mathbb{R}^N}$ *is a maximal monotone function satisfying* $0\in g(0)$.

For $i=1$ *or* $i=2$ *suppose that* $b\in H_i^1(0,T_e;L^2(\Omega,\mathbb{R}^3))$, $\gamma_D\in H_i^1(0,T_e;H_1^2(\Omega,\mathbb{R}^3))$ *and* $\gamma_N\in H_i^1(0,T_e;L^2(\partial\Omega,\mathbb{R}^3))$. *Moreover, for the Neumann problem assume that* b *and* γ_N *satisfy*

$$\int_\Omega b(x,t)\cdot(a+\omega\times x)dx=\int_{\partial\Omega}\gamma_N(x,t)\cdot(a+\omega\times x)dS_x \tag{1.23}$$

to all vectors $a,\omega\in\mathbb{R}^3$ *and almost all* t *in* $[0,T_e]$. *Finally, assume that* $z^{(0)}\in L^2(\Omega,\mathbb{R}^n)$ *and that there is* $\zeta\in L^2(\Omega,\mathbb{R}^N)$ *such that*

$$\zeta(x)\in g(B^TT^{(0)}(x)-Lz^{(0)}(x))\quad\text{a.e. in }\Omega\,, \tag{1.24}$$

where $(u^{(0)},T^{(0)})$ *is a weak solution of the Dirichlet or Neumann problem* (1.16) – (1.19), *respectively, to the data* $\hat{b}=b(0)$, $\hat{\varepsilon}_p=Bz^{(0)}$, $\hat{\gamma}_D=\gamma_D(0)$, $\hat{\gamma}_N=\gamma_N(0)$.

Then, for $i=1$ *there is a unique weak solution and for* $i=2$ *a unique strong solution of the Dirichlet initial-boundary value problem* (1.10) – (1.14). *Furthermore, for* $i=1$ *there is a weak solution and for* $i=2$ *a strong solution of the Neumann initial-boundary value problem* (1.10) – (1.13), (1.15). *If* (u_0,T,z) *is a weak or a strong solution of the Neumann problem, then all solutions are obtained in the form* $(u_0+a+\omega\times x,T,z)$ *with* $a,\omega\in L^2([0,T_e],\mathbb{R}^3)$.

This theorem is proved in Sections 3. For the proof we reduce the initial-boundary value problem to an evolution equation in the Hilbert space L^2 with a maximal monotone evolution operator. Theorem 1.3 then follows from well known results for such abstract evolution equations. In the reduction we use projection operators to tensor fields, which are symmetric gradients, and projection operators to tensor fields, for which a generalized divergence vanishes. The projectiors of the latter type generalize the classical Helmholtz and Weyl projections. We study these projectors in Section 2.

In Section 6 we present another proof of the existence and the uniqueness of global in time solutions to the problem (1.10) – (1.13) with a positive definite free energy function ψ. In that section we consider our problem with another boundary condition, namely the condition of mixed type

$$u(x,t)=\gamma_D(x,t)\text{ for }x\in\Gamma_1\,,\quad T(x,t)n(x)=\gamma_N(x,t)\text{ for }x\in\Gamma_2\,,\ t>0, \tag{1.25}$$

where $\Gamma_1\,,\Gamma_2\subseteq\partial\Omega$ are relatively open subsets of $\partial\Omega$ satisfying $\Gamma_1\cap\Gamma_2=\emptyset$, $\partial\Omega=\overline{\Gamma}_1\cup\overline{\Gamma}_2$ and $|\Gamma_1|>0$. Here $|\Gamma_1|$ denotes the 2-dimensional boundary measure of Γ_1. The method used for the proof is based on the partial Yosida approximation. The main result of Section 6 is stated in the following theorem:

Theorem 1.4 *Let us suppose that the boundary data γ_D, γ_N and the external force b possess the regularity*

$$b \in H_2^\infty(0, T_e; L^2(\Omega; \mathbb{R}^3))$$

$$\gamma_D \in H_3^\infty(0, T_e; H^2_{1/2}(\Gamma_1; \mathbb{R}^3))\,, \quad \gamma_N \in H_2^\infty(0, T_e; H^2_{-1/2}(\Gamma_2; \mathbb{R}^3))\,,$$

for all $T_e > 0$, and that to the initial data $z^0 \in L^2(\Omega; \mathbb{R}^N)$ there is $\zeta \in L^2(\Omega, \mathbb{R}^N)$ such that

$$\zeta(x) \in g(B^T T^{(0)}(x) - Lz^{(0)}(x)) \quad \text{a.e. in } \Omega\,,$$

where $(u^{(0)}, T^{(0)})$ is a weak solution of the problem formed by (1.16), (1.17), (1.25) *with $t = 0$, $\hat{b} = b(0)$ and $\hat{\varepsilon}_p = Bz^{(0)}$. If the considered model is of monotone type with the maximal monotone constitutive function $g : \Delta(g) \subset \mathbb{R}^N \to 2^{\mathbb{R}^N}$, which satisfies $0 \in g(0)$, and with a positive definite matrix L, then the system* (1.10) – (1.13) *with the boundary condition* (1.25) *possesses a global in time, unique solution*

$$(u, T, z) \in H_1^\infty(0, T_e; H_1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathcal{S}^3 \times \mathbb{R}^N)) \ \textit{for all } T_e > 0\,.$$

Positive semi-definite free energy. We remarked earlier that there is a principle difference between initial-boundary value problems with positive definite and positive semi-definite free energy. Accordingly, to study problems with positive semi-definite free energy we follow a different line of attack, which we introduce next. Since we can only treat the case where the function g in (1.12) is single-valued, we assume in the following that g is a function with values in $\mathbb{R}^N$.

If the free energy

$$\rho\psi(\varepsilon, z) = \frac{1}{2}\,[\mathcal{D}(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2}\,(Lz) \cdot z$$

is not positive definite, it follows that $\ker L \neq \{0\}$. For constitutive equations of monotone type it is required that the $N \times N$–matrix $M = B^T \mathcal{D} B + L$ is positive definite, cf. [Alb98, p. 25], since otherwise the vector z of internal variables contains unnecessary components, which do not contribute to the strain-stress relation $\varepsilon(\nabla_x u) \to T$, and which can be eliminated. This requirement implies $\ker B \cap \ker L = \{0\}$. Therefore $\ker B + \ker L$ is a subspace of $\mathbb{R}^N$, whose dimension is strictly greater than the dimension of $\ker B$. For technical reasons we assume in our investigation of constitutive models with positive semi-definite free energy, that this subspace is equal to $\mathbb{R}^N$ and that $\dim(\ker B) = N - 6$, $\dim(\ker L) = 6$. For all models from engineering this assumption is satisfied. In the appendix it is shown that in this case the initial-boundary value problems (1.10) – (1.15) can be written in the equivalent form

$$-\mathrm{div}_x\, T(x,t) = b(x,t) \tag{1.26}$$

$$T(x,t) = \mathcal{D}(\varepsilon(\nabla_x u(x,t)) - \varepsilon_p(x,t)) \tag{1.27}$$

$$\frac{\partial}{\partial t}\,\varepsilon_p(x,t) = \tilde{g}_1(T(x,t), -\tilde{z}(x,t)) \tag{1.28}$$

$$\frac{\partial}{\partial t}\,\tilde{z}(x,t) = \tilde{g}_2(T(x,t), -\tilde{z}(x,t)) \tag{1.29}$$

$$\varepsilon_p(x,0) = \varepsilon_p^{(0)}(x), \quad \tilde{z}(x,0) = \tilde{z}^{(0)}(x), \tag{1.30}$$

with the Dirichlet boundary condition

$$u(x,t) = \gamma_D(x,t), \quad (x,t) \in \partial\Omega \times [0, \infty) \tag{1.31}$$

or the Neumann boundary condition

$$T(x,t)n(x) = \gamma_N(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{1.32}$$

Here $\varepsilon_p(x,t) \in \mathcal{S}^3$ is the plastic strain tensor, $\tilde{z}(x,t) \in \mathbb{R}^{N-6}$ is a vector of internal variables, and $\tilde{g}_1 : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathcal{S}^3$, $\tilde{g}_2 : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathbb{R}^{N-6}$ are given functions such that

$$(T,y) \to (\tilde{g}_1(T,y), \tilde{g}_2(T,y)) : \mathbb{R}^N \to \mathbb{R}^N$$

is a monotone vector field.

To prove existence of solutions to this initial-boundary value problem, we put it into a more general setting:

Definition 1.5 *Let $F(Z,\mathcal{S}^3)$ be the set of all functions from Z to $\mathcal{S}^3$. A mapping $\mathcal{H} : \Delta(\mathcal{H}) \to F(Z,\mathcal{S}^3)$ with $\Delta(\mathcal{H}) \subseteq F(Z,\mathcal{S}^3)$ is called a history functional on $F(Z,\mathcal{S}^3)$, if it has the following property: For all $0 \le t \le T_e$ and all $\tau_1, \tau_2 \in \Delta(\mathcal{H})$, which satisfy $\tau_1|_{Z_t} = \tau_2|_{Z_t}$, it follows that $\mathcal{H}[\tau_1]|_{Z_t} = \mathcal{H}[\tau_2]|_{Z_t}$.*

With a suitable history functional $\mathcal{H}$ on $F(Z,\mathcal{S}^3)$ the equations (1.26) – (1.30) can now be written in the form

$$-\mathrm{div}_x T = b \tag{1.33}$$

$$T = \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) \tag{1.34}$$

$$\frac{\partial}{\partial t}\varepsilon_p = \mathcal{H}[T] \tag{1.35}$$

$$\varepsilon_p(x,0) = \varepsilon_p^{(0)}(x). \tag{1.36}$$

To see this, let $\tilde{z}^{(0)} : \Omega \to \mathbb{R}^{N-6}$ and $T : Z \to \mathcal{S}^3$ be given and let $(h,\tilde{z}) : Z \mapsto \mathcal{S}^3 \times \mathbb{R}^{N-6}$ be a solution of the initial value problem

$$h(x,t) = \tilde{g}_1(T(x,t), -\tilde{z}(x,t)) \tag{1.37}$$

$$\frac{\partial}{\partial t}\tilde{z}(x,t) = \tilde{g}_2(T(x,t), -\tilde{z}(x,t)) \tag{1.38}$$

$$\tilde{z}(x,0) = \tilde{z}^{(0)}(x), \tag{1.39}$$

for $x \in \Omega$ and $t \in [0,T_e]$. Then a history functional $\mathcal{H}_{\tilde{z}^{(0)}}$ on $F(Z,\mathcal{S}^3)$ is defined by

$$\mathcal{H}_{\tilde{z}^{(0)}}[T] = h. \tag{1.40}$$

Insertion of $\mathcal{H}_{\tilde{z}^{(0)}}$ for $\mathcal{H}$ into (1.35) reduces (1.33) – (1.36) to the equations (1.26) – (1.30).

To obtain existence results for the initial-boundary value problems (1.26) – (1.32) we now proceed as follows: In Section 4 we prove existence of solutions to the problems (1.31) – (1.36) containing an abstract history functional. This history functional has to satisfy monotonicity, boundedness and coercivity conditions. In Section 5 we consider several examples of constitutive equations with positive semi-definite free energy, and verify that the history functionals defined by these constitutive equations satisfy these conditions.

The main result of Section 4 is stated in the following theorem: For numbers $2 \le p < \infty$ and $1 < q \le 2$ with $\frac{1}{p} + \frac{1}{q} = 1$ let $\mathcal{H} : L^p(Z,\mathcal{S}^3) \to L^q(Z,\mathcal{S}^3)$ be a history functional, which satisfies the following four conditons:

(H1) There is a constant C such that for all $T \in L^p(Z, \mathcal{S}^3)$

$$\|\mathcal{H}(T)\|_{q,Z} \leq C\Big(\|T\|_{p,Z}^{\frac{p}{q}} + 1\Big).$$

(H2) $\mathcal{H}$ is hemicontinuous and monontone with respect to the bilinear form $(\sigma, \tau)_Z$.

(H3) $\mathcal{H}$ satisfies a first coercivity condition:

$$\frac{\Big(T, \mathcal{H}[T]\Big)_Z}{1 + \|\mathcal{H}[T]\|_{q,Z}} \to \infty \quad \text{for} \quad \|T\|_{p,Z} \to \infty\,.$$

(H4) $\mathcal{H}$ satisfies a second coercivity condition:

$$\frac{\Big(T, \mathcal{H}[T]\Big)_Z}{\|T\|_{p,Z}} \to \infty \quad \text{for} \quad \|T\|_{p,Z} \to \infty\,.$$

Theorem 1.6 *Let $\mathcal{H}$ satisfy the conditons* (H1) – (H4)*, and let $b \subseteq L^p(Z, \mathbb{R}^3)$, $\varepsilon_p^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ be given functions. Then the following assertions hold:*

(i) *For $\gamma_D \in L^p(0, T_e; H_1^p(\Omega, \mathbb{R}^3))$ there is a unique strong solution*

$$(u, T, \varepsilon_p) \in L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^p(Z, \mathcal{S}^3) \times H_1^q([0, T_e], L^q(\Omega, \mathcal{S}^3))$$

of the Dirichlet problem (1.33) – (1.36)*,* (1.31)*.*

(ii) *Assume that $\gamma_N \in L^p(\partial\Omega \times [0, T_e], \mathbb{R}^3)$ and that for all infinitesimal rigid motions $a + \omega \times x$ and for almost every $t \in [0, T_e]$ the equation*

$$\int_\Omega b(x,t) \cdot (a + \omega \times x)dx = \int_{\partial\Omega} \gamma_N(x,t) \cdot (a + \omega \times x)dS_x \tag{1.41}$$

holds. Then there exists a strong solution

$$(u_0, T, \varepsilon_p) \in L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^p(Z, \mathcal{S}^3) \times H_1^q([0, T_e], L^q(\Omega, \mathcal{S}^3))$$

of the Neumann problem (1.33) – (1.36)*,* (1.32)*. All strong solutions of this problem are abtained in the form $(u, T, \varepsilon_p) = (u_0, T, \varepsilon_p) + (w, 0, 0)$, where $w(x,t) = a(t) + \omega(t) \times x$ with $a, \omega \in L^q([0, T_e], \mathbb{R}^3)$ is an infinitesimal rigid motion.*

To prove this theorem we define and study in Section 2 the projection operators not only on the Hilbert space L^2, but also on the Banach space L^q. The proof of Theorem 1.6 is given in Section 4, where we use these L^q–projections to reduce the initial-boundary value problem to an evolution equation in the space of all L^q–tensor fields with vanishing generalized divergence. Arguments from the theory of monotone operators can then be used to prove existence of a solution.

In Section 5 we apply Theorem 1.6 to study two examples of constitutive models. The first example is the Norton-Hoff law:

$$-\mathrm{div}_x T = b \tag{1.42}$$

$$T = \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) \tag{1.43}$$

$$\frac{\partial}{\partial t}\varepsilon_p = c|T|^r \frac{T}{|T|} \tag{1.44}$$

$$\varepsilon_p(0) = \varepsilon_p^{(0)}, \tag{1.45}$$

with constants $c > 0, r > 1$. The second example incorporates kinematic hardening:

$$
\begin{aligned}
-\mathrm{div}_x\, T &= b && (1.46)\\
T &= \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) && (1.47)\\
\frac{\partial}{\partial t}\varepsilon_p &= c_1|T - k(\varepsilon_p - \varepsilon_n)|^r \frac{T - k(\varepsilon_p - \varepsilon_n)}{|T - k(\varepsilon_p - \varepsilon_n)|} && (1.48)\\
\frac{\partial}{\partial t}\varepsilon_n &= c_2(k|\varepsilon_p - \varepsilon_n|)^\gamma \frac{\varepsilon_p - \varepsilon_n}{|\varepsilon_p - \varepsilon_n|} && (1.49)\\
\varepsilon_p(0) &= \varepsilon_p^{(0)}, \qquad \varepsilon_n(0) = \varepsilon_n^{(0)}, && (1.50)
\end{aligned}
$$

where $c_1, c_2, k > 0$ and $r, \gamma > 1$ are constants, and where the internal variable $\varepsilon_n(x,t) \in \mathcal{S}^3$ is of the type of a strain tensor.

For both models we show that the conditions (H1) – (H4) are satisfied. Theorem 1.6 thus yields the following results:

Theorem 1.7 (Norton-Hoff law) *Let $c > 0$ and $r > 1$ be constants, and let $p = 1 + r$, $q = 1 + \frac{1}{r}$. Then under the regularity assumptions for b, $\varepsilon_p^{(0)}$, γ_D and γ_N from Theorem 1.6 the assertions of that theorem also hold for the Dirichlet and Neumann initial-boundary value problems to the equations* (1.42) – (1.45).

Theorem 1.8 (Kinematic hardening) *Let c_1, c_2, k be positive constants and let the constants r and γ satisfy $\gamma > r > 1$. Set $p = 1 + r$, $q = 1 + \frac{1}{r}$, $\hat{p} = 1 + \gamma$, $\hat{q} = 1 + \frac{1}{\gamma}$. Suppose that $b \in L^p(Z, \mathbb{R}^3)$ and $\varepsilon_p^{(0)}, \varepsilon_n^{(0)} \in L^2(\Omega, \mathcal{S}^3)$. Then the following assertions hold:*
(i) *For $\gamma_D \in L^p(0, T_e; H_1^p(\Omega, \mathbb{R}^3))$ there is a unique strong solution*

$$
\begin{aligned}
(u, T, \varepsilon_p, \varepsilon_n) \;\in\; & L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^p(Z, \mathcal{S}^3)\\
& \times H_1^q([0, T_e], L^q(\Omega, \mathcal{S}^3)) \times H_1^{\hat{q}}([0, T_e], L^{\hat{q}}(\Omega, \mathcal{S}^3))
\end{aligned}
$$

of the Dirichlet problem to the equations (1.46) – (1.50) *with $\varepsilon_p - \varepsilon_n \in L^{\hat{p}}(Z, \mathcal{S}^3)$.*
(ii) *Assume that $\gamma_N \in L^p(\partial\Omega \times [0, T_e], \mathbb{R}^3)$ and that b and γ_N satisfy* (1.41). *Then there exists a strong solution*

$$
\begin{aligned}
(u_0, T, \varepsilon_p, \varepsilon_n) \;\in\; & L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^p(Z, \mathcal{S}^3)\\
& \times H_1^q([0, T_e], L^q(\Omega, \mathcal{S}^3)) \times H_1^{\hat{q}}([0, T_e]; L^{\hat{q}}(\Omega, \mathcal{S}^3))
\end{aligned}
$$

of the Neumann problem to the equations (1.46) – (1.50) *with $\varepsilon_p - \varepsilon_n \in L^{\hat{p}}(Z, \mathcal{S}^3)$. All strong solutions of this problem are obtained in the form $(u, T, \varepsilon_p, \varepsilon_n) = (u_0, T, \varepsilon_p, \varepsilon_n) + (w, 0, 0, 0)$, where $w(x,t) = a(t) + \omega(t) \times x$ with $a, \omega \in L^q([0, T_e], \mathbb{R}^3)$.*

Remarks. *1.)* Existence results for the Norton-Hoff law are known since long, cf. [Tem86, LeT90]. These authors use different methods to treat the problem and must impose regularity assumptions on the time derivatives $\frac{\partial b}{\partial t}$, $\frac{\partial \gamma_D}{\partial t}$, $\frac{\partial \gamma_N}{\partial t}$ of the data, in contrast to our result in Theorem 1.7, which holds without any assumption for these time derivatives.
2.) The existence and regularity result from Theorem 1.8 is new. We surmise that the restriction $\gamma > r$ is not necessary to obtain solutions. However, for $\gamma < r$ the Cauchy stress T will be less regular and belong to the space $L^{1+\gamma}$, and not to the smaller space $L^p = L^{1+r}$. Since our method of proof automatically yields solutions in the space L^p, it

cannot be applied to find solutions in the larger space $L^{1+\gamma}$. It is thus an open problem to prove existence of solutions for $\gamma \le r$.
3.) A related problem is connected with the stress deviator. Instead of T, constitutive models usually contain the stress deviator $T - \frac{1}{3}\,\mathrm{trace}(T)I$. Therefore it is important to study the systems (1.42) – (1.45) and (1.46) – (1.50) with T in (1.44) and (1.48) replaced by the stress deviator. Again, our method of proof fails, we surmise for the same reason: the coercivity condition (H4) cannot be verified in L^p, hence the stress T will not lie in L^p, but in a larger space, and our method cannot yield the solution either. To prove existence of solutions is an open problem also in this case.
4.) More generally, for the case of positive semi-definite free energy an important open problem is to prove existence of solutions for constitutive relations with right hand sides, which grow faster than a power or which are multivalued. Multivalued right hand sides are needed to formulate rate independent constitutive relations. We already mentioned that constitutive models used in engineering are rarely monotone. Both for positive definite and positive semi-definite free energy the final goal is therefore to prove existence of solutions for a class of non-monotone constitutive equations large enough to include most models used in practice.

2 Helmholtz and Weyl projections in L^p

In the proofs of Theorems 1.3 and 1.6 we need projection operators in L^p to spaces of tensor fields, which are symmetric gradients and to spaces of tensor fields, for which a generalized divergence vanishes. The projection operators of the second type are generalizations of the classical Helmholtz or Weyl projections in spaces of vector fields. Though the topic is completely classical, for $p \neq 2$ these projections have not been considered before; we refer to the remark after Definition 1.1. In this section we introduce and study these projections.

To this end we need two results, which we state without proof. As always we assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded open set with C^1–boundary.

Lemma 2.1 *Let*

$$\mathcal{R}_0 = \{u \in H_1^p(\Omega, \mathbb{R}^3) \mid \varepsilon(\nabla u) = 0 \text{ on } \Omega\}.$$

(i) *The linear space $\mathcal{R}_0$ consists of all infinitesimal rigid motions, i.e.*

$$\mathcal{R}_0 = \{u : \Omega \to \mathbb{R}^3 \mid u(x) = a + \omega \times x;\ a, \omega \in \mathbb{R}^3\}.$$

In particular, $\mathcal{R}_0$ does not depend on p. Let

$$\mathcal{R}^p = \left\{H_1^p(\Omega, \mathbb{R}^3) \mid (u, v)_\Omega + (\nabla u, \nabla v)_\Omega = 0 \quad \text{for all } v \in \mathcal{R}_0\right\}.$$

$\mathcal{R}^p$ is a closed subspace and $H_1^p(\Omega, \mathbb{R}^3) = \mathcal{R}^p + \mathcal{R}_0$.

(ii) *(Korn's first inequality.) There is a constant c_D such that*

$$\|u\|_{p,1,\Omega} \le c_D \|\varepsilon(\nabla u)\|_{p,\Omega} \tag{2.1}$$

for all $u \in \overset{\circ}{H}{}_1^p(\Omega, \mathbb{R}^3)$.

(iii) *(Korn's second inequality.) There is a constant c_N such that*

$$\|u\|_{p,1,\Omega} \le c_N \|\varepsilon(\nabla u)\|_{p,\Omega} \tag{2.2}$$

for all $u \in \mathcal{R}^p$.

A **proof** of this lemma can be found for example in [KO88] or in [Par92].

The second result concerns existence of solutions in L^p of the Dirichlet or Neumann boundary value problems (1.16) – (1.19) with partly homogeneous data: To a given function $\hat{\varepsilon}_p : \Omega \to \mathcal{S}^3$ we seek $u : \Omega \to \mathbb{R}^3$ and $T : \Omega \to \mathcal{S}^3$ satisfying the partial differential equations

$$-\operatorname{div} T(x) = 0, \qquad x \in \Omega, \tag{2.3}$$

$$T(x) = \mathcal{D}(\varepsilon(\nabla u(x)) - \hat{\varepsilon}_p(x)), \quad x \in \Omega, \tag{2.4}$$

and the homogeneous Dirichlet boundary condition

$$u(x) = 0, \quad x \in \partial\Omega, \tag{2.5}$$

or the homogeneous Neumann boundary condition

$$T(x)n(x) = 0, \quad x \in \partial\Omega. \tag{2.6}$$

By Definition 1.1, to a given function $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ the pair $(u, T) \in \overset{\circ}{H}{}^p_1(\Omega, \mathbb{R}^3) \times L^p(\Omega, \mathcal{S}^3)$ is a weak solution of (2.3)–(2.5) if (2.4) holds and if

$$(\mathcal{D}(\varepsilon(\nabla u) - \hat{\varepsilon}_p), \varepsilon(\nabla v))_\Omega = 0 \tag{2.7}$$

for all $v \in \overset{\circ}{H}{}^q_1(\Omega, \mathbb{R}^3)$. Also, $(u, T) \in H^p_1(\Omega, \mathbb{R}^3) \times L^p(\Omega, \mathcal{S}^3)$ is a weak solution of (2.3), (2.4) and (2.6) if (2.4) holds and if (2.7) is satisfied for all $v \in H^q_1(\Omega, \mathbb{R}^3)$.

The following existence theorem for this problem is stated under the assumption discussed after Definition 1.1:

Theorem 2.2 (i) *To every* $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ *there exists a unique weak solution* $(u, T) \in \overset{\circ}{H}{}^p_1(\Omega, \mathbb{R}^3) \times L^p(\Omega, \mathcal{S}^3)$ *of the Dirichlet problem* (2.3) - (2.5). *This solution satisfies*

$$||\varepsilon(\nabla u)||_{p,\Omega} \leq c_D ||\hat{\varepsilon}_p||_{p,\Omega} \tag{2.8}$$

with a constant c_D *independent of* $\hat{\varepsilon}_p$.
(ii) *Also, to every* $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ *there exists a unique weak solution* $(u, T) \in \mathcal{R}^p \times L^p(\Omega, \mathcal{S}^3)$ *of the Neumann problem* (2.3), (2.4), (2.6). *This solution satisfies*

$$||\varepsilon(\nabla u)||_{p,\Omega} \leq c_N ||\hat{\varepsilon}_p||_{p,\Omega} \tag{2.9}$$

with a constant c_N *independent of* $\hat{\varepsilon}_p$. *All solutions of the Neumann problem are given by*

$$u + a + \omega \times x$$

with arbitrary vectors $a, \omega \in \mathbb{R}^3$.

With these results we can construct the projections. First we define subspaces $\mathcal{G}^p_D$ and $\mathcal{G}^p_N$ of $L^p(\Omega, \mathcal{S}^3)$ by

$$\mathcal{G}^p_D = \{\varepsilon(\nabla u) \mid u \in \overset{\circ}{H}{}^p_1(\Omega, \mathbb{R}^3)\}$$

$$\mathcal{G}^p_N = \{\varepsilon(\nabla u) \mid u \in H^p_1(\Omega, \mathbb{R}^3)\} = \{\varepsilon(\nabla u) \mid u \in \mathcal{R}^p\}.$$

$(\mathcal{G}^p_D, \|\cdot\|_{p,\Omega})$ and $(\mathcal{G}^p_N, \|\cdot\|_{p,\Omega})$ are normed spaces, and from Korn's inequalities we obtain

Lemma 2.3 *The bounded linear mappings*

$$u \mapsto \varepsilon(\nabla u) : \overset{\circ}{H}{}^p_1(\Omega, \mathbb{R}^3) \to \mathcal{G}^p_D\,, \quad u \mapsto \varepsilon(\nabla u) : \mathcal{R}^p \to \mathcal{G}^p_N$$

have bounded inverses $V^p_D : \mathcal{G}^p_D \to \overset{\circ}{H}{}^p_1(\Omega, \mathbb{R}^3)$, $V^p_N : \mathcal{G}^p_N \to \mathcal{R}^p$.

Proof. The existence and boundedness of V^p_D is an immediate consequence of (2.1), whereas (2.2) implies the existence and boundedness of V^p_N.

As a consequence of this lemma, $\mathcal{G}^p_D$ and $\mathcal{G}^p_N$ are closed subspaces of $L^p(\Omega, \mathcal{S}^3)$, hence $(\mathcal{G}^p_D, \|\cdot\|_{p,\Omega})$ and $(\mathcal{G}^p_N, \|\cdot\|_{p,\Omega})$ are Banach spaces. Clearly, in the Hilbert space case $p = 2$ this implies that there exist bounded projection operators $P_D : L^2(\Omega, \mathcal{S}^3) \to L^2(\Omega, \mathcal{S}^3)$ onto $\mathcal{G}^2_D$ and $P_N : L^2(\Omega, \mathcal{S}^3) \to L^2(\Omega, \mathcal{S}^3)$ onto $\mathcal{G}^2_N$, which are orthogonal with respect to the scalar product $[\sigma, \tau]_\Omega$. This means that these projections are selfadjoint with respect to this scalar product. From the preceding existence theorem it follows that projections with analogous properties exist for all $1 < p < \infty$:

Definition 2.4 *Let the linear operators* $P_D = P^p_D : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$ *and* $P_N = P^p_N : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$ *be defined by*

$$P_D \hat{\varepsilon}_p = \varepsilon(\nabla u_D)\,, \qquad P_N \hat{\varepsilon}_p = \varepsilon(\nabla u_N)\,,$$

for every $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$, *where* $u_D \in \overset{\circ}{H}{}^p_1(\Omega, \mathbb{R}^3)$ *is the weak solution of the boundary value problem* (2.3) – (2.5) *and* $u_N \in \mathcal{R}^p$ *is the weak solution of the boundary value problem* (2.3), (2.4), (2.6) *to the given function* $\hat{\varepsilon}_p$.

The solutions u_D and u_N exist and are unique by Theorem 2.2.

Lemma 2.5 *For every* $1 < p < \infty$ *the mapping* $P_D = P^p_D$ *is a bounded projection operator onto the subspace* $\mathcal{G}^p_D$ *of* $L^p(\Omega, \mathcal{S}^3)$ *and* $P_N = P^p_N$ *is a bounded projection operator onto the subspace* $\mathcal{G}^p_N$. *The projections* $(P^p_D)^*$ *and* $(P^p_N)^*$ *adjoint with respect to the bilinear form* $[\sigma, \tau]_\Omega$ *on* $L^p(\Omega, \mathcal{S}^3) \times L^q(\Omega, \mathcal{S}^3)$ *satisfy*

$$(P^p_D)^* = P^q_D\,, \quad (P^p_N)^* = P^q_N\,,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. *This implies* $\ker(P^p_D) = H^p_{\mathrm{sol},D}$ *and* $\ker(P^p_N) = H^p_{\mathrm{sol},N}$, *where*

$$H^p_{\mathrm{sol},D} = \{\sigma \in L^p(\Omega, \mathcal{S}^3) \mid [\sigma, \tau]_\Omega = 0 \quad \textit{for all } \tau \in \mathcal{G}^q_D\}\,, \tag{2.10}$$

$$H^p_{\mathrm{sol},N} = \{\sigma \in L^p(\Omega, \mathcal{S}^3) \mid [\sigma, \tau]_\Omega = 0 \quad \textit{for all } \tau \in \mathcal{G}^q_N\}\,. \tag{2.11}$$

Remark. *1.)* In the following we drop the indices p and q if they are understood from the context. Thus for P^p_D and P^q_D we simply write P_D, for P^p_N and P^q_N we write P_N. Also, if a statement is valid for P_D and P_N, then we drop D and N and write P instead. Similarly, we write $\mathcal{G}^p$ if a statement is valid for $\mathcal{G}^p_D$ and $\mathcal{G}^p_N$.
2.) We note that $[\sigma, \tau]_\Omega = 0$ holds for all $\tau \in \mathcal{G}^q$ if and only if

$$(\mathcal{D}\sigma, \nabla v)_\Omega = (\mathcal{D}\sigma, \varepsilon(\nabla v))_\Omega = [\sigma, \varepsilon(\nabla v)]_\Omega = 0$$

for all $v \in \overset{\circ}{H}{}^q_1(\Omega, \mathbb{R}^3)$ if $\mathcal{G}^q = \mathcal{G}^q_D$ and for all $v \in H^q_1(\Omega, \mathbb{R}^3)$ if $\mathcal{G}^q = \mathcal{G}^q_N$. Here we used that $\mathcal{D}\sigma(x)$ is a symmetric matrix for all $x \in \Omega$. Using this observation, we formally obtain

$$H^p_{\mathrm{sol},D} = \{\sigma \in L^p(\Omega, \mathcal{S}^3) \mid \operatorname{div}(\mathcal{D}\sigma) = 0\}$$

$$H^p_{\mathrm{sol},N} = \{\sigma \in L^p(\Omega, \mathcal{S}^3) \mid \operatorname{div}(\mathcal{D}\sigma) = 0\,,\ (\mathcal{D}\sigma)_{|\partial\Omega} n = 0\}\,,$$

where $n(x)$ is the exterior unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Therefore the projection operators

$$Q_D^p = (I - P_D^p)\,, Q_N^p = (I - P_N^p) : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$$

with $Q_D^p(L^p(\Omega, \mathcal{S}^3)) = H^p_{\text{sol},D}$ and $Q_N^p(L^p(\Omega, \mathcal{S}^3)) = H^p_{\text{sol},N}$ are generalizations of the classical *Helmholtz* and *Weyl* projections.

Proof of the lemma: Obviously, the range of P_D belongs to $\mathcal{G}_D^p$ and the range of P_N belongs to $\mathcal{G}_N^p$, by definition of these spaces. Moreover, the estimates (2.8) and (2.9) yield

$$||P_D\hat{\varepsilon}_p||_{p,\Omega} \le c_D||\hat{\varepsilon}_p||_{p,\Omega}, \quad ||P_N\hat{\varepsilon}_p||_{p,\Omega} \le c_N||\hat{\varepsilon}_p||_{p,\Omega},$$

hence P_D and P_N are bounded linear mappings.

To prove that P_D is a projection onto $\mathcal{G}_D^p$ let $\hat{\varepsilon}_p \in \mathcal{G}_D^p$. Then there is $u \in \overset{\circ}{H}{}_1^p(\Omega, \mathbb{R}^3)$ such that $\hat{\varepsilon}_p = \varepsilon(\nabla u)$. Thus, by definition of P_D we have $P_D\varepsilon(\nabla u) = P_D\hat{\varepsilon}_p = \varepsilon(\nabla u_D)$, where u_D is the unique function in $\overset{\circ}{H}{}_1^p(\Omega, \mathbb{R}^3)$ which satisfies (2.7) for all $v \in \overset{\circ}{H}{}_1^q(\Omega, \mathbb{R}^3)$. Clearly u satisfies (2.7), hence $u_D = u$ and $P_D\hat{\varepsilon}_p = \varepsilon(\nabla u) = \hat{\varepsilon}_p$, which shows that P_D projects onto $\mathcal{G}_D^p$. In the same way it follows that P_N projects onto $\mathcal{G}_N^p$.

We now drop the indices D and N. Since $P^p\sigma \in \mathcal{G}^p$ for all $\sigma \in L^p(\Omega, \mathcal{S}^3)$, the definition of P^q implies for all $\tau \in L^q(\Omega, \mathcal{S}^3)$

$$[P^p\sigma, \tau - P^q\tau]_\Omega = 0.$$

Similarly, since $P^q\tau \in \mathcal{G}^q$, the definition of P^p implies for all $\sigma \in L^p(\Omega, \mathcal{S}^3)$

$$[\sigma - P^p\sigma, P^q\tau]_\Omega = 0.$$

Thus, for all $\sigma \in L^p(\Omega, \mathcal{S}^3)$, $\tau \in L^q(\Omega, \mathcal{S}^3)$

$$[\sigma, (P^p)^*\tau - P^q\tau]_\Omega = [P^p\sigma, \tau]_\Omega - [\sigma, P^q\tau]_\Omega = [P^p\sigma, P^q\tau]_\Omega - [P^p\sigma, P^q\tau]_\Omega = 0\,,$$

whence $(P^p)^* = P^q$.

Finally,

$$\ker P^p = (R((P^p)^*))^\perp = (R(P^q))^\perp = (\mathcal{G}^q)^\perp = H^p_{\text{sol}}\,.$$

The proof of the lemma is complete.

Corollary 2.6 *Let $(B^T\mathcal{D}P^pB)^T$ be the operator adjoint to $B^T\mathcal{D}P^pB : L^p(\Omega, \mathbb{R}^N) \to L^p(\Omega, \mathbb{R}^N)$ with respect to the bilinear form $(z, \hat{z})_\Omega$ on the product space $L^p(\Omega, \mathbb{R}^N) \times L^q(\Omega, \mathbb{R}^N)$. Then $(B^T\mathcal{D}P^pB)^T = B^T\mathcal{D}P^qB : L^q(\Omega, \mathbb{R}^N) \to L^q(\Omega, \mathbb{R}^N)$.*

Proof For $z \in L^p(\Omega, \mathbb{R}^N)$ and $\hat{z} \in L^q(\Omega, \mathbb{R}^N)$ we have

$$\begin{aligned}(B^T\mathcal{D}P^pBz, \hat{z})_\Omega &= (\mathcal{D}P^pBz, B\hat{z})_\Omega = [P^pBz, B\hat{z}]_\Omega = [Bz, P^qB\hat{z}]_\Omega \\ &= (\mathcal{D}Bz, P^qB\hat{z})_\Omega = (Bz, \mathcal{D}P^qB\hat{z})_\Omega = (z, B^T\mathcal{D}P^qB\hat{z})_\Omega\,,\end{aligned}$$

which implies $(B^T\mathcal{D}P^pB)^T = B^T\mathcal{D}P^qB$.

Corollary 2.7 *H^p_{sol} is a reflexive Banach space with dual space H^q_{sol}.*

Proof: $Q^p = I - P^p$ is a bounded projection on the reflexive Banach space $L^p(\Omega, \mathcal{S}^3)$ with adjoint projection $(Q^p)^* = I - (P^p)^* = I - P^q = Q^q$. Thus, by an easy to prove result from functional analysis the range $H^p_{\text{sol}} = Q^p(L^p(\Omega, \mathcal{S}^3))$ of the projection Q^p is a reflexive Banach space with dual given by the range $H^q_{\text{sol}} = Q^q(L^q(\Omega, \mathcal{S}^3))$ of the adjoint projection.

3 Positive definite free energy

In this section we prove Theorem 1.3. First we reduce the initial-boundary value problem (1.10) – (1.15) to an evolution equation. To this end we note that (1.11) yields

$$B^T T - Lz = B^T \mathcal{D}(\varepsilon - Bz) - Lz = B^T \mathcal{D}\varepsilon - Mz\,,$$

with the symmetric, positive semi-definite $N \times N$-matrix $M = B^T \mathcal{D} B + L$. Therefore (1.8) can be written as

$$z_t \in g(B^T \mathcal{D}\varepsilon(\nabla_x u) - Mz)\,. \tag{3.1}$$

Assume now that (u, T, z) is a solution of the Dirichlet or Neumann initial-boundary value problem (1.10) – (1.15). If $z(t)$ is known, then (1.10), (1.11), (1.14) or (1.10), (1.11), (1.15) form boundary value problems for the components $u(t)$ and $T(t)$ of the solution. These functions are obtained in the form

$$(u(t), T(t)) = (\tilde{u}(t), \tilde{T}(t)) + (\hat{u}(t), \hat{\sigma}(t)),$$

where $(\hat{u}(t), \hat{\sigma}(t))$ is a solution of the Dirichlet or Neumann boundary value problem (1.16) – (1.19) to the data $\hat{b} = b(t)$, $\hat{\varepsilon}_p = 0$, $\hat{\gamma}_D = \gamma_D(t)$, $\hat{\gamma}_N = \gamma_N(t)$, and $(\tilde{u}(t), \tilde{T}(t))$ is a solution of the boundary value problem (2.3) – (2.6) with data $\hat{\varepsilon}_p(t) = Bz(t)$. The definition of the projector P in Definition 2.4 thus yields

$$\varepsilon(\nabla_x u(t)) = \varepsilon(\nabla_x \tilde{u}(t)) + \varepsilon(\nabla_x \hat{u}(t)) = PBz(t) + \varepsilon(\nabla_x \hat{u}(t)). \tag{3.2}$$

Using that $\hat{\sigma}(t) = \mathcal{D}\varepsilon(\nabla_x \hat{u}(t))$, we obtain by insertion of this equation into (3.1) that

$$z_t(t) \in g\Big((B^T \mathcal{D} P B - M) z(t) + B^T \hat{\sigma}(t)\Big)\,. \tag{3.3}$$

Since $\hat{\sigma}$ is computed from the data b, γ_D, γ_N, and thus is known, (3.3) is an evolution equation for z. With the evolution operator $A(t)$ defined by

$$A(t)z = -g\Big((B^T \mathcal{D} P B - M) z + B^T \hat{\sigma}(t)\Big),$$

this evolution equation can be written as

$$z_t(t) + A(t) z(t) \ni 0\,.$$

This is a non-autonomous evolution equation. In the following we show that in the case of positive definite free energy it is possible to transform this equation into an autonomous evolution equation. Moreover, we prove that the resulting time independent evolution operator is maximal monotone in the Hilbert space $L^2(\Omega, \mathbb{R}^N)$. This permits to apply the strong existence results known for such evolution equations. This good behavior of the initial-boundary value problem with positive definite free energy is a consequence of the following property of the linear mapping $B^T \mathcal{D} P B - M$ in the evolution equation (3.3):

Lemma 3.1 *Assume that the free energy ψ defined in* (1.9) *is positive definite. With the symmetric matrix L from* (1.8) *let $M = B^T \mathcal{D} B + L$, and let $P = P_D^2$ or $P = P_N^2$ be the projection in $L^2(\Omega, \mathcal{S}^3)$ onto $\mathcal{G}_D^2$ or $\mathcal{G}_N^2$. Then the linear operator $M - B^T \mathcal{D} P B$: $L^2(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$ is bounded, symmetric and positive definite with respect to the scalar product $(z, \hat{z})_\Omega$.*

Proof: $M - B^T\mathcal{D}PB$ is obviously bounded. With $p = q = 2$ we obtain from Corollary Corollary 2.6 that $B^T\mathcal{D}PB$ is symmetric. Since M is symmetric, we conclude that also $M - B^T\mathcal{D}PB$ is symmetric. To see that this mapping is positive definite, note that ψ is positive definite if and only if the matrix L is positive definite. Thus, for $z \in L^2(\Omega, \mathbb{R}^N)$,

$$\begin{aligned}
&((M - B^T\mathcal{D}PB)z, z)_\Omega = (Lz, z)_\Omega + (\mathcal{D}(I - P)Bz, Bz)_\Omega \\
&= \ (Lz, z)_\Omega + \Big[(I - P)Bz, Bz\Big]_\Omega = (Lz, z)_\Omega + \Big[(I - P)Bz, (I - P)Bz\Big]_\Omega \\
&\geq \ (Lz, z)_\Omega \geq \mu \|z\|^2_\Omega,
\end{aligned}$$

with a positive constant μ. Here we used that P is orthogonal with respect to the scalar product $[\sigma, \tau]_\Omega$ on $L^2(\Omega, \mathcal{S}^3)$ and that L is positive definite. This proves the lemma.

In particular, this lemma implies that $M - B^T\mathcal{D}PB$ has a bounded, symmetric, positive definite inverse.

To transform (3.3) to an autonomous equation we insert

$$h = (B^T\mathcal{D}PB - M)z + B^T\hat{\sigma}$$

into (3.3) and obtain

$$h_t \in (B^T\mathcal{D}PB - M)g(h) + B^T\hat{\sigma}_t\,. \tag{3.4}$$

Definition 3.2 *Let the evolution operator* $\mathcal{C} : L^2(\Omega, \mathbb{R}^n) \to 2^{L^2(\Omega,\mathbb{R}^N)}$ *be defined as follows: For* $h \in L^2(\Omega, \mathbb{R}^N)$ *set*

$$\mathcal{C}h = \Big\{(M - B^T\mathcal{D}PB)\zeta \ \Big|\ \zeta \in L^2(\Omega, \mathbb{R}^N),\ \zeta(x) \in g(h(x)) \ \textit{a.e. in } \Omega\Big\}.$$

The domain of $\mathcal{C}$ *is*

$$\Delta(\mathcal{C}) = \{h \in L^2(\Omega, \mathbb{R}^N) \,|\, \mathcal{C}h \neq \emptyset\}.$$

With this operator $\mathcal{C}$ the evolution equation (3.4) on $L^2(\Omega, \mathbb{R}^N)$ can be written as

$$h_t(t) + \mathcal{C}h(t) \ni B^T\hat{\sigma}_t(t). \tag{3.5}$$

We prove now that the operator $\mathcal{C}$ is maximal monotone if the vector field g is maximal monotone. The standard theory for evolution equations to such operators implies then that (3.5) has a unique solution for suitable functions $\hat{\sigma}_t$. To prove this, we need that by Lemma 3.1

$$\langle z, \hat{z}\rangle_\Omega = \Big((M - B^T\mathcal{D}PB)^{-1}z, \hat{z}\Big)_\Omega$$

is a scalar product on $L^2(\Omega, \mathbb{R}^N)$. Well known considerations show hat

$$\|(M - B^T\mathcal{D}PB)^{-1}\|^{-1}\langle z, z\rangle_\Omega \leq \|z\|^2_{2,\Omega} \leq \|M - B^T\mathcal{D}PB\|\langle z, z\rangle_\Omega.$$

Whence the associated norm

$$|z|_\Omega = \langle z, z\rangle^{1/2}_\Omega$$

is equivalent to $\|z\|_{2,\Omega}$.

Theorem 3.3 (i) *Let the mapping* $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ *be monotone. Then the operator* $\mathcal{C}$ *is monotone with respect the scalar product* $\langle z, \hat{z}\rangle_\Omega$.
(ii) *If* g *is maximal monotone and satisfies* $0 \in g(0)$, *then* $\mathcal{C}$ *is maximal monotone with respect to this scalar product.*

Proof: Let $z_1, z_2 \in \Delta(\mathcal{C})$. Then for $\zeta_1 \in \mathcal{C}z_1$, $\zeta_2 \in \mathcal{C}z_2$ the functions $\xi_i = (M - B^T\mathcal{D}PB)^{-1}\zeta_i$ satisfy $\xi_i(x) \in g(z_i(x))$ a.e. in Ω, for $i = 1, 2$. Thus,

$$\begin{aligned} \langle \zeta_1 - \zeta_2, z_1 - z_2 \rangle_\Omega &= \Big((M - B^T\mathcal{D}PB)^{-1}(\zeta_1 - \zeta_2), z_1 - z_2 \Big)_\Omega \\ = (\xi_1 - \xi_2, z_1 - z_2)_\Omega &= \int_\Omega \Big(\xi_1(x) - \xi_2(x)\Big) \cdot \Big(z_1(x) - z_2(x)\Big) dx \geq 0 \end{aligned}$$

since the monotonicity of g implies $(\xi_1(x) - \xi_2(x)) \cdot (z_1(x) - z_2(x)) \geq 0$. Therefore $\mathcal{C}$ is monotone.

(ii) We first show that the maximal monotone mapping $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ with $g(0) = 0$ defines a mapping $G : L^2(\Omega, \mathbb{R}^N) \to 2^{L^2(\Omega,\mathbb{R}^N)}$, which is maximal monotone with respect to the scalar product $(z, \hat{z})_\Omega$. For $z \in L^2(\Omega, \mathbb{R}^N)$ we set $Gz = \{\zeta \in L^2(\Omega, \mathbb{R}^N) \,|\, \zeta(x) \in g(z(x))$ a.e. in $\Omega\}$. The domain is $\Delta(G) = \{z \in L^2(\Omega, \mathbb{R}^N) \,|\, Gz \neq \emptyset\}$. Monotonicity of G is seen similarly as in (i). The mapping G is maximal monotone if and only if $I + G$ is surjective. To prove surjectivity, it must be shown that to every $h \in L^2(\Omega, \mathbb{R}^N)$ the equation

$$h \in z + Gz \tag{3.6}$$

has a solution $z \in L^2(\Omega, \mathbb{R}^N)$. Since g is maximal monotone, $(I + g) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ has an inverse $j : \mathbb{R}^N \to \mathbb{R}^N$, which satisfies $|j(\zeta) - j(\zeta')| \leq |\zeta - \zeta'|$ for all $\zeta, \zeta' \in \mathbb{R}^N$. Thus j is Lipschitz continuous. From $0 \in g(0)$ it follows that $j(0) = 0$, whence

$$|j(\zeta)| = |j(\zeta) - j(0)| \leq |\zeta - 0| = |\zeta|. \tag{3.7}$$

For $h \in L^2(\Omega, \mathbb{R}^N)$ we now define $z(x) = j(h(x))$ for all $x \in \Omega$. Clearly, z solves (3.6) if $z \in L^2(\Omega, \mathbb{R}^N)$. To see that z belongs to this space, note that z is measurable, since h is measurable and j is Lipschitz continuous. Moreover, (3.7) implies $|z(x)| \leq |h(x)|$, hence $z \in L^2(\Omega, \mathbb{R}^N)$. Consequently, z solves (3.6) and we conclude that $I + G$ is surjective. Hence G is maximal monotone.

With this result we can show that $I + \mathcal{C}$ is surjective. Since $\mathcal{C} = (M - B^T\mathcal{D}PB)G$, it must be shown that to every $h \in L^2(\Omega, \mathbb{R}^N)$ there is a solution $z \in L^2(\Omega, \mathbb{R}^N)$ of

$$h \in z + (M - B^T\mathcal{D}PB)Gz. \tag{3.8}$$

We apply $(M - B^T\mathcal{D}PB)^{-1}$ to (3.8) and obtain the equivalent equation

$$(M - B^T\mathcal{D}PB)^{-1}h \in (M - B^T\mathcal{D}PB)^{-1}z + Gz. \tag{3.9}$$

To prove that (3.9) has a solution z we note that since $(M - B^T\mathcal{D}PB)^{-1}$ is positive definite, there is $\mu > 0$ such that

$$(M - B^T\mathcal{D}PB)^{-1} - \mu I$$

is positive definite on $L^2(\Omega, \mathbb{R}^N)$. As a bounded mapping, it is Lipschitz continuous. Thus, $G + ((M - B^T\mathcal{D}PB)^{-1} - \mu I)$ is maximal montone as sum of the maximal monotone mapping G and a Lipschitz continuous monotone mapping, cf. [Bré73, p. 34]. This implies that

$$G + ((M - B^T\mathcal{D}PB)^{-1} - \mu I) + \mu I = G + (M - B^T\mathcal{D}PB)^{-1}$$

is surjective. Therefore (3.9) and as a consequence also (3.8) has a solution $z \in L^2(\Omega, \mathbb{R}^N)$. This implies that $I + (M - B^T\mathcal{D}PB)G = I + \mathcal{C}$ is surjective, whence $\mathcal{C}$

is maximal monotone. The proof of Theorem 3.3 is complete.

With this result we can apply standard existence results to the evolution equation (3.5). To state two of these results we need the following definitions (cf. [Bré73, p. 64], [Bar76, p. 123, 134]):

Definition 3.4 *Let $k \in L^1(0,T_e;L^2(\Omega,\mathbb{R}^N))$. A function $h \in C([0,T_e],L^2(\Omega,\mathbb{R}^N))$ is called a strong solution of the equation*

$$\frac{dh}{dt} + \mathcal{C}h \ni k, \tag{3.10}$$

on the interval $(0,T_e)$, if it satisfies

(i) $t \mapsto h(t)$ *is absolutely continuous on every compact subset of* $(0,T_e)$,

(ii) $h(t) \in \Delta(\mathcal{C})$, *a.e. on* $(0,T_e)$,

(iii) $\frac{dh}{dt}(t) + \mathcal{C}h(t) \ni k(t)$, *a.e. on* $(0,T_e)$.

We remark that if X is a Hilbert space (more generally, if it is a reflexive Banach space), then every absolutely continuous function $t \mapsto h(t) : (0,T_e) \to X$ is almost everywhere differentiable, cf. [Bré73, p. 145].

Definition 3.5 *Let $k \in L^1(0,T_e;L^2(\Omega,\mathbb{R}^N))$. A function $h \in C([0,T_e],L^2(\Omega,\mathbb{R}^N))$ is called a weak solution of the equation (3.10), it there exists a sequence $\{k_n\}_{n=1}^\infty$ of functions $k_n \in L^1(0,T_e;L^2(\Omega,\mathbb{R}^N))$, which converges in $L^1(0,T_e;L^2(\Omega,\mathbb{R}^N))$ to k, and if there exists a sequence $\{h_n\}_{n=1}^\infty$ of strong solutions $h_n \in C([0,T_e],L^2(\Omega,\mathbb{R}^n))$ of $\frac{dh_n}{dt} + \mathcal{C}h_n \ni k_n$, which converges to h in $C([0,T_e],L^2(\Omega,\mathbb{R}^N))$.*

Theorem 3.6 *If $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone function satisfying $0 \in g(0)$, then the following assertions hold:*

(i) *Let $\hat{\sigma} \in H_1^1(0,T_e;L^2(\Omega,\mathcal{S}^3))$ and let $h^{(0)} \in \Delta(\mathcal{C}) \subseteq L^2(\Omega,\mathbb{R}^N)$. Then the evolution equation*

$$h_t + \mathcal{C}h \ni B^T\hat{\sigma}_t$$

has a unique weak solution on $[0,T_e]$ satisfying

$$h(0) = h^{(0)}.$$

(ii) *Let $\hat{\sigma}_1, \hat{\sigma}_2 \in H_1^1(0,T_e;L^2(\Omega,\mathcal{S}^3))$, and let h_1, h_2 be weak solutions of*

$$h_{it} + \mathcal{C}h_i \ni B^T\hat{\sigma}_{it}, \quad i = 1,2.$$

Then

$$\|h_1(t) - h_2(t)\|_{2,\Omega} \le \|h_1(s) - h_2(s)\|_{2,\Omega} + \int_s^t \|B^T(\hat{\sigma}_{1t}(\eta) - \hat{\sigma}_{2t}(\eta))\|_{2,\Omega}\, d\eta\,,$$

for all $0 \le s \le t \le T_e$.

Proof: Theorem 3.3 shows that $\mathcal{C}$ is maximal monotone under the assumptions of this theorem. Since for $\hat{\sigma} \in H_1^1(0,T_e;L^2(\Omega,\mathcal{S}^3))$ the function $B^T\hat{\sigma}_t$ belongs to $L^1(0,T_e;L^2(\Omega,\mathbb{R}^N))$, this theorem is therefore an immediate consequence of [Bré73, Lemma 3.1 and Theorem 3.4, p. 64, 65].

Theorem 3.7 *Suppose that $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is maximal monotone and satisfies $0 \in g(0)$. Also, let $\hat{\sigma} \in H_2^1(0, T_e; L^2(\Omega, \mathcal{S}^3))$ and let $h^{(0)} \in \Delta(\mathcal{C})$. Then the evolution equation*

$$h_t + \mathcal{C}h \ni B^T \hat{\sigma}_t \tag{3.11}$$

has a unique strong solution on $[0, T_e]$ with

$$h(0) = h^{(0)}. \tag{3.12}$$

Moreover, h belongs to the space $H_1^\infty(0, T_e; L^2(\Omega, \mathbb{R}^N))$ and satisfies

$$\|h_t(t)\|_{2,\Omega} \le |||\mathcal{C}h^{(0)} + B^T \hat{\sigma}_t(0)||| + \int_0^t \|B^T \hat{\sigma}_{tt}(s)\|_{2,\Omega}\, ds \quad a.e.,$$

where $|||\mathcal{C}h^{(0)} + B^T \hat{\sigma}_t(0)||| = \inf\{\|\zeta\|_{2,\Omega} \mid \zeta \in \mathcal{C}h^{(0)} + B^T \hat{\sigma}_t(0)\}$.

We remark that for a Hilbert space X (more generally, for a reflexive Banach space X), the space $H_1^\infty(0, T_e; X)$ coincides with the space of Lipschitz continuous functions on $[0, T_e]$ with values in X, cf. [Bré73, p. 143-145]. Therefore the solution in this theorem is Lipschitz continuous.

Proof: Since $\mathcal{C}$ is maximal monotone und since for $\hat{\sigma} \in H_2^1(0, T_e; L^2(\Omega, \mathcal{S}^3))$ the function $B^T \hat{\sigma}_t$ belongs to $H_1^1(0, T_e; L^2(\Omega, \mathbb{R}^n))$, this theorem is an immediate consequence of [Bar76, Theorem 2.2, p. 131].

Proof of Theorem 1.3: We first prove that strong solutions of the Dirichlet problem are unique and strong solutions of the Neumann problem are of the multiplicity stated in Theorem 1.3. Thus, assume that (u_i, T_i, z_i) for $i = 1, 2$ are two strong solutions of the Dirichlet problem or of the Neumann problem to the data b, γ_D, γ_N and $z^{(0)}$. We set $(\overline{u}, \overline{T}, \overline{z}) = (u_1 - u_2, T_1 - T_2, z_1 - z_2)$.

By Definition 1.2 of strong solutions, $(\overline{u}(t), \overline{T}(t))$ is a weak solution of the Dirichlet or Neumann problem (2.3) – (2.6) to the data $\hat{\varepsilon}_p = B\overline{z}(t)$ with $\overline{u} \in L^2(0, T_e; \overset{\circ}{H}{}_1^2(\Omega, \mathbb{R}^3))$ in the case of the Dirichlet problem. In the case of the Neumann problem we can assume that $\overline{u} \in L^2(0, T_e; \mathcal{R}^2)$ with the space $\mathcal{R}^2$ introduced in Lemma 2.1. Otherwise we add the function $a(t) + \omega(t) \times x$ with suitable coefficients a and ω to achieve this. Since, again by Definition 1.2, $\overline{z}_t(t)$ exists for almost all t and belongs to $L^2(\Omega, \mathbb{R}^N)$, we conclude from the regularity properties of the linear elliptic boundary value problems (2.3) – (2.6) that $\overline{u}_t(t)$ exists for almost all t with $\overline{u}_t(t) \in \overset{\circ}{H}{}_1^2(\Omega, \mathbb{R}^3)$ or $\overline{u}_t(t) \in \mathcal{R}^2$. Using that weak solutions satisfy (2.7), we thus obtain

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \rho\psi(\varepsilon(\nabla_x \overline{u}(t)), \overline{z}(t))\, dx \\ &= \int_\Omega (\mathcal{D}(\varepsilon(\nabla_x \overline{u}(t)) - B\overline{z}(t))) \cdot \varepsilon(\nabla_x \overline{u}_t(t)) + \nabla_z\, \psi(\varepsilon(\nabla_x \overline{u}(t)), \overline{z}(t)) \cdot \overline{z}_t(t)\, dx \\ &= -\int_\Omega \Big[B^T \mathcal{D}(\varepsilon(\nabla_x \overline{u}(t)) - B\overline{z}(t)) - L\overline{z}(t)\Big] \cdot \overline{z}_t(t)\, dx \\ &= -\int_\Omega (\overline{B^T T(t) - Lz(t)}) \cdot \overline{z}_t(t)\, dx \le 0. \end{aligned} \tag{3.13}$$

Here we used that the relation $z_{it}(x,t) \in g(B^T T_i(x,t) - Lz_i(x,t))$ a. e. and the monotonicity of g together imply $(\overline{B^T T(x,t) - Lz(x,t)}) \cdot \overline{z_t(x,t)} \ge 0$ a. e. Since

$\overline{u}(0) = \overline{z}(0) = 0$, it follows that $\int_\Omega \rho\psi(\varepsilon(\nabla_x u(0)), z(0))\,dx = 0$. Hence (3.13) implies $\psi(\varepsilon(\nabla_x \overline{u}(x,t)), \overline{z}(x,t)) = 0$ almost everywhere, which yields $\varepsilon(\nabla_x \overline{u}) = \overline{z} = 0$, since ψ is positive definite. From Lemma 2.1 we now conclude that $\overline{u} = 0$ in the Dirichlet case and that $\overline{u} = a(t) + \omega(t) \times x$ with $a, \omega \in L^2([0,T_e], \mathbb{R}^3)$ in the Neumann case. This proves the uniqueness statements for strong solutions in Theorem 1.3.

To prove existence of strong solutions, let b, γ_D, γ_N and $z^{(0)}$ satisfy the assumptions of Theorem 1.3 with $i = 2$, and let $(\hat{u}(t), \hat{\sigma}(t)) \in \overset{\circ}{H}{}_1^2(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)$ or $(\hat{u}(t), \hat{\sigma}(t)) \in \mathcal{R}^2 \times L^2(\Omega, \mathcal{S}^3)$, respectively, be a solution of the Dirichlet problem (1.16) – (1.18) or of the Neumann problem (1.16), (1.17), (1.19), respectively, to the data $\hat{b} = b(t), \hat{\gamma}_D = \gamma_D(t), \hat{\gamma}_N = \gamma_N(t)$ and $\hat{\varepsilon}_p = 0$, for almost every t. From the regularity properties of these elliptic boundary value problems we conclude under the assumptions for b, γ_D and γ_N of Theorem 1.3 that $\hat{\sigma} \in H_2^1(0,T_e; L^2(\Omega, \mathcal{S}^3))$. With a solution $(\tilde{u}(0), \tilde{T}(0))$ of the Dirichlet or Neumann problem (2.3) – (2.6) to the data $\hat{\varepsilon}_p = Bz^{(0)}$ we obtain a solution $(u^{(0)}, T^{(0)}) = (\tilde{u}(0) + \hat{u}(0), \tilde{T}(0) + \hat{\sigma}(0))$ of the equations (1.10), (1.11), (1.14) or (1.10), (1.11), (1.15) for $t = 0$, and the definition of the projector P in Definition 2.4 yields

$$T^{(0)} = \mathcal{D}(\varepsilon(\nabla_x \tilde{u}(0)) - Bz^{(0)}) + \mathcal{D}\varepsilon(\nabla_x \hat{u}(0)) = \mathcal{D}(P - I)Bz^{(0)} + \hat{\sigma}(0).$$

With $M = B^T\mathcal{D}B + L$ we thus conclude that $h^{(0)}$ defined by

$$h^{(0)} = (B^T\mathcal{D}PB - M)z^{(0)} + B^T\hat{\sigma}(0)$$

satisfies $h^{(0)} = B^T\mathcal{D}(P - I)Bz^{(0)} - Lz^{(0)} + B^T\hat{\sigma}(0) = B^T T^{(0)} - Lz^{(0)}$, and thus belongs to $\Delta(\mathcal{C})$, by (1.24). Whence the assumptions of Theorem 3.7 are satisfied, and consequently there is a unique strong solution h of (3.11), (3.12). Set $z = (B^T\mathcal{D}PB - M)^{-1}(h - B^T\hat{\sigma})$, and for almost every t let $(\tilde{u}(t), \tilde{T}(t))$ be a weak solution of the Dirichlet or Neumann boundary value problems (2.3) – (2.6) to the data $\hat{\varepsilon}_p = Bz(t)$. We leave it to the reader to verify that $(u, T, z) = (\tilde{u} + \hat{u}, \tilde{T} + \hat{\sigma}, z)$ is a strong solution of the Dirichlet or Neumann problems (1.10) – (1.15). This concludes the proof of the statements of Theorem 1.3 for strong solutions.

To prove the statements for weak solutions, we note that from the preceding uniqueness and existence proof it follows in particular that (u, T, z) is a strong solution of the initial-boundary value problems (1.10) – (1.15) if and only if $h = (B^T\mathcal{D}PB - M)z + B^T\hat{\sigma}$ is a strong solution of (3.11), (3.12). For $k_n \in L^1(0, T_e; L^2(\Omega, \mathbb{R}^N))$ it can be shown in exactly the same way that (u_n, T_n, z_n) is a strong solution of the problems (1.10) – (1.15) with (1.12) replaced by (1.22) if and only if $h_n = (B^T\mathcal{D}PB - M)z_n + B^T\hat{\sigma}$ is a strong solution of $\frac{d}{dt}h_n + \mathcal{C}h_n \ni B^T\hat{\sigma}_t + (B^T\mathcal{D}PB - M)k_n$. Using this result it can be verified in a straightforward way that (u, T, z) is a weak solution of the problems (1.10) – (1.15) if and only if $h = (B^T\mathcal{D}PB - M)z + B^T\hat{\sigma}$ is a weak solution of $h_t + \mathcal{C}h \ni B^T\hat{\sigma}_t$, $h(0) = h^{(0)}$. The statements about weak solutions in Theorem 1.3 now follow from Theorem 3.6.

4 Constitutive equations with history functionals

In this section we prove Theorem 1.6. As in the preceding section, the proof is based on the reduction of the initial-boundary value problems (1.31) – (1.36) to an evolution equation.

To derive this evolution equation, let (u, T, ε_p) be a solution of the equations (1.33) – (1.36) to the Dirichlet condition (1.31) or to the Neumann condition (1.32). Following the reasoning used to derive equation (3.2), we obtain for the strain tensor

$$\varepsilon(\nabla_x u(t)) = P\varepsilon_p(t) + \varepsilon(\nabla_x \hat{u}(t)),$$

where $(\hat{u}(t), \hat{\sigma}(t))$ is a solution of the Dirichlet or Neumann boundary value problem (1.16) – (1.19) to the data $\hat{b} = b(t)$, $\hat{\varepsilon}_p = 0$, $\hat{\gamma}_D = \gamma_D(t)$, $\hat{\gamma}_N = \gamma_N(t)$. Since $\hat{\sigma}(t) = \mathcal{D}\varepsilon(\nabla_x \hat{u}(t))$, we thus obtain from (1.34) that

$$T(t) = \mathcal{D}(P - I)\varepsilon_p(t) + \mathcal{D}\varepsilon(\nabla_x \hat{u}(t)) = -\mathcal{D}Q\varepsilon_p(t) + \hat{\sigma}(t),$$

where $Q = I - P$ is the projection onto the kernel $H_{\rm sol}$ of P, which was introduced after Lemma 2.5. Insertion of this equation into (1.35) yields

$$\frac{\partial}{\partial t}\varepsilon_p(t) = \mathcal{H}[-\mathcal{D}Q\varepsilon_p + \hat{\sigma}](t).$$

We apply Q to the left of this equation and obtain for the component $\tau = Q\varepsilon_p$ in $H_{\rm sol}$ of the plastic strain ε_p the evolution problem

$$\frac{\partial}{\partial t}\tau(t) = Q\mathcal{H}[-\mathcal{D}\tau + \hat{\sigma}](t), \qquad 0 \le t \le T_e \tag{4.1}$$

$$\tau(0) = Q\varepsilon_p^{(0)}. \tag{4.2}$$

Next we show that this initial value problem has a unique solution if $\mathcal{H}$ satisfies the conditions (H1) – (H4) stated before Theorem 1.6. As preparation we need the following

Lemma 4.1 *Let the history functional $\mathcal{H}$ satisfy the conditions* (H1) – (H4)*, let $Q = Q^q : L^q(\Omega) \to H^q_{\rm sol}$ be the projection operator in L^q, and suppose that $\hat{\sigma} \in L^p(Z, \mathcal{S}^3)$. Then the history functional $\mathcal{H}' : L^p(0, T_e; H^p_{\rm sol}) \to L^q(0, T_e; H^q_{\rm sol})$ defined by*

$$\mathcal{H}'[\tau] = -Q\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]$$

has the following properties:

(H1') *$\mathcal{H}'$ is hemicontinuous and monotone with respect to the bilinear form $[\sigma, \tau]_Z$ on the product space $H^p_{\rm sol} \times H^q_{\rm sol}$.*

(H2') *$\mathcal{H}'$ is bounded: There is a constant C_1 such that for all $\tau \in L^p(0, T_e; H^p_{\rm sol})$*

$$\|\mathcal{H}'[\tau]\|_{q,Z} \le C_1(\|\tau\|_{p,Z}^{\frac{p}{q}} + 1). \tag{4.3}$$

(H3') *$\mathcal{H}'$ is coercive:*

$$\frac{[\tau, \mathcal{H}'[\tau]\,]_Z}{\|\tau\|_{p,Z}} \to \infty \quad \text{for} \quad \|\tau\|_{p,Z} \to \infty. \tag{4.4}$$

Proof: (H1'): The mapping $Q : L^q(Z, \mathcal{S}^3) \to L^q(0, T_e; H^q_{\rm sol}) \subseteq L^q(Z, \mathcal{S}^3)$ is bounded. Hence it is continuous with respect to the weak topology. Since by condition (H2) the history functional $\mathcal{H} : L^p(Z, \mathcal{S}^3) \to L^q(Z, \mathcal{S}^3)$ is hemicontinuous, it follows that the composition $\mathcal{H}' = -Q\mathcal{H}[\hat{\sigma} - \mathcal{D}\,\cdot]$ is hemicontinuous. To prove monotonicity, note that by Lemma 2.5 the projection adjoint to Q^q with respect to the bilinear form $[\sigma, \hat{\sigma}]_Z$ is Q^p, which projects to $L^p(0, T_e; H^p_{\text{sol}})$. Hence, for $\tau, \hat{\tau} \in L^p(0, T_e; H^p_{\text{sol}})$ we obtain

$$\begin{aligned}
[\,\tau - \hat{\tau}\,,\, \mathcal{H}'[\tau] - \mathcal{H}'[\hat{\tau}]\,] &= [\,\tau - \hat{\tau}\,,\, -Q^q\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau] + Q^q\mathcal{H}[\hat{\sigma} - \mathcal{D}\hat{\tau}]\,]_Z \\
&= [\,Q^p(\hat{\tau} - \tau)\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau] - \mathcal{H}[\hat{\sigma} - \mathcal{D}\hat{\tau}]\,]_Z \\
&= \Big(\mathcal{D}(\hat{\tau} - \tau)\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau] - \mathcal{H}[\hat{\sigma} - \mathcal{D}\hat{\tau}]\Big)_Z \\
&= \Big((\hat{\sigma} - \mathcal{D}\tau) - (\hat{\sigma} - \mathcal{D}\hat{\tau}))\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau] - \mathcal{H}[\hat{\sigma} - \mathcal{D}\hat{\tau}]\Big)_Z \\
&\geq 0,
\end{aligned}$$

since by condition (H2) the functional $\mathcal{H}$ is monotone with respect to the bilinear form $(\sigma, \tau)_Z$. This proves that $\mathcal{H}'$ is monotone.
(H2'): To prove the estimate (4.3), we use that $\frac{p}{q} > 1$. Consequently $r \mapsto r^{\frac{p}{q}}$ is a convex function on $[0, \infty)$, which yields that $((a+b)/2)^{\frac{p}{q}} \leq (a^{\frac{p}{q}} + b^{\frac{p}{q}})/2$ for all $a, b \geq 0$. The boundedness of Q and the condition (H1) thus imply for $\tau \in L^p(0, T_e; H^p_{\text{sol}})$

$$\begin{aligned}
\|\mathcal{H}'[\tau]\|_{q,Z} &= \| - Q\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\|_{q,Z} \leq C_1\|\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\|_{q,Z} \leq C_1C(\|\mathcal{D}\tau - \hat{\sigma}\|_{p,Z}^{\frac{p}{q}} + 1) \\
&\leq C_1C\Big(2^{\frac{p}{q}-1}(\|\mathcal{D}\tau\|_{p,Z}^{\frac{p}{q}} + \|\hat{\sigma}\|_{p,Z}^{\frac{p}{q}}) + 1\Big) \leq C_2(\|\tau\|_{p,Z}^{\frac{p}{q}} + 1),
\end{aligned}$$

with a suitable constant C_2. This is (4.3).
(H3'): To verify (4.4), let $\tau \in L^p(0, T_e; H^p_{\text{sol}})$. As in the proof of the monotonicity we use that $(Q^q)^*\,\tau = Q^p\tau = \tau$. Together with Hölder's inequality we thus obtain

$$\begin{aligned}
[\,\tau\,,\, \mathcal{H}'[\tau]\,]_Z &= [\,\tau\,,\, -Q\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,]_Z = [\,-\tau, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,]_Z = (\,-\mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z \\
&= (\,\hat{\sigma} - \mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z - (\,\hat{\sigma}\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z \\
&\geq (\,\hat{\sigma} - \mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z - \|\hat{\sigma}\|_{p,Z}\|\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\|_{q,Z}\,. \qquad (4.5)
\end{aligned}$$

Condition (H3) implies that there is $C > 0$ with

$$\frac{(\,\hat{\sigma} - \mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z}{1 + \|\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\|_{q,Z}} \geq 2\|\hat{\sigma}\|_{p,Z}$$

for all $\|\tau\|_{p,Z} \geq C$, hence

$$\begin{aligned}
\frac{1}{2}(\,\hat{\sigma} - \mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z &= \frac{1}{2}\frac{(\,\hat{\sigma} - \mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z}{1 + \|\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\|_{q,Z}}\Big(1 + \|\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\|_{q,Z}\Big) \\
&\geq \|\hat{\sigma}\|_{p,Z}\|\mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\|_{q,Z}
\end{aligned}$$

for $\|\tau\|_{p,Z} \geq C$. From this estimate and from (4.5) we obtain for $\|\tau\|_{p,Z} \geq C$

$$[\,\tau\,,\, \mathcal{H}'[\tau]\,]_Z \geq \frac{1}{2}(\,\hat{\sigma} - \mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z\,. \qquad (4.6)$$

Condition (H4) implies

$$\frac{(\,\hat{\sigma} - \mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z}{\|\tau\|_{p,Z}} = \frac{(\,\hat{\sigma} - \mathcal{D}\tau\,,\, \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau]\,)_Z}{\|\hat{\sigma} - \mathcal{D}\tau\|_{p,Z}}\,\frac{\|\hat{\sigma} - \mathcal{D}\tau\|_{p,Z}}{\|\tau\|_{p,Z}} \to \infty$$

for $\|\tau\|_{p,Z} \to \infty$. This relation and (4.6) yield (4.4). The proof of Lemma 4.1 is complete.

Theorem 4.2 *Suppose that the history functional $\mathcal{H}$ satisfies the conditions* (H1) – (H4). *Then to every $\hat{\sigma} \in L^p(Z, \mathcal{S}^3)$ and to every $\tau^{(0)} \in H^2_{\mathrm{sol}}$ there is a unique function*

$$\tau \in L^p(0, T_e; H^p_{\mathrm{sol}}) \cap C(0, T_e; H^2_{\mathrm{sol}})$$

with $\tau_t \in L^q(0, T_e; H^q_{\mathrm{sol}})$ such that

$$\tau_t(t) - Q\mathcal{H}[-\mathcal{D}\tau + \hat{\sigma}](t) = 0 \quad a.e.\ on\ [0, T_e] \tag{4.7}$$

$$\tau(0) = \tau^{(0)}. \tag{4.8}$$

Proof: To prove this theorem we modify the proof of Theorem 4.2 in [Bar76, p. 166]. In the proof we only need that the functional $\mathcal{H}'$ has the properties (H1') – (H3') verified in Lemma 4.1.

By Corollary 2.7, H^p_{sol} is a reflexive Banach space with dual space H^q_{sol}. Therefore $L^p(0, T_e; H^p_{\mathrm{sol}})$ is a reflexive Banach space with dual $L^q(0, T_e; H^q_{\mathrm{sol}})$. Also, $L^2(0, T_e; H^2_{\mathrm{sol}})$ is a Hilbert space, which satisfies

$$L^p(0, T_e; H^p_{\mathrm{sol}}) \subseteq L^2(0, T_e; H^2_{\mathrm{sol}}) \subseteq L^q(0, T_e; H^q_{\mathrm{sol}}),$$

since $q \leq 2 \leq p$ and since Ω is bounded. As bilinear form on the product space $L^p(0, T_e; H^p_{\mathrm{sol}}) \times L^q(0, T_e; H^q_{\mathrm{sol}})$ we use $[\sigma, \tau]_Z$. Define the operator $\mathcal{B} : L^p(0, T_e; H^p_{\mathrm{sol}}) \to L^q(0, T_e; H^q_{\mathrm{sol}})$ by

$$\mathcal{B}\tau = \tau_t\,,$$

for τ from the domain

$$\Delta(\mathcal{B}) = \{\sigma \in L^p(0, T_e; H^p_{\mathrm{sol}}) \mid \sigma_t \in L^q(0, T_e; H^q_{\mathrm{sol}}),\ \sigma(0) = \tau^{(0)}\}.$$

We note that $\Delta(\mathcal{B})$ is well defined, since the Sobolev imbedding theorem implies that every function $\sigma \in L^p(0, T_e; H^p_{\mathrm{sol}})$ with derivative $\sigma_t \in L^q(0, T_e; H^q_{\mathrm{sol}})$ belongs to $C(0, T_e; H^2_{\mathrm{sol}})$ and thus has traces $\sigma(t) \in H^2_{\mathrm{sol}}$. The operator $\mathcal{B}$ is maximal monotone with respect to the bilinear form $[\sigma, \tau]_Z$, cf. [Bar76, Lemma 4.1, p. 167]. By the properties (H1') and (H2') the operator $\mathcal{H}'$ is everywhere defined on $L^p(0, T_e; H^p_{\mathrm{sol}})$, bounded, hemicontinuous and monotone. These properties of $\mathcal{B}$ and $\mathcal{H}'$ imply that the sum $\mathcal{B} + \mathcal{H}'$ is maximal monotone, cf. [Bar76, Corollary 1.1, p.39]. Furthermore, $\mathcal{B} + \mathcal{H}'$ is coercive. To see this, note that $\tau \in \Delta(\mathcal{B})$ satisfies $\tau(0) = \tau^{(0)}$, by definition of the domain, whence

$$[\tau, \mathcal{B}\tau]_Z = \int_Z (\mathcal{D}\tau(x,t)) \cdot \tau_t(x,t)\, d(x,t) = [\tau(t), \tau(t)]_\Omega - [\tau^{(0)}, \tau^{(0)}]_\Omega \geq -[\tau^{(0)}, \tau^{(0)}]_\Omega\,.$$

From the property (H3') we thus conclude

$$\frac{[\tau\,,\ \mathcal{B}\tau + \mathcal{H}'[\tau]\,]_Z}{\|\tau\|_{p,Z}} = \frac{[\tau, \mathcal{B}\tau]_Z + [\tau, \mathcal{H}'[\tau]\,]_Z}{\|\tau\|_{p,Z}} \geq \frac{[\tau\,,\ \mathcal{H}'[\tau]\,]_Z - [\tau^{(0)}, \tau^{(0)}]_\Omega}{\|\tau\|_{p,Z}} \to \infty,$$

for $\|\tau\|_{p,Z} \to \infty$. Thus, $\mathcal{B} + \mathcal{H}'$ is coercive, which implies that the inverse of $\mathcal{B} + \mathcal{H}'$ is bounded, whence it follows that $\mathcal{B} + \mathcal{H}'$ is surjective, cf. [Bar76, Theorem 1.6, p. 45]. The surjectivity and the monotonicity imply that the equation

$$\mathcal{B}\tau + \mathcal{H}'[\tau] = 0$$

has a unique solution $\tau \in \Delta(\mathcal{B} + \mathcal{H}') = \Delta(\mathcal{B})$. This proves the theorem.

Proof of Theorem 1.6: We first prove the uniqueness part of the theorem. Thus, assume that $(u_i, T_i, \varepsilon_{pi}) \in L^q(0,T_e; H_1^q(\Omega,\mathbb{R}^3)) \times L^p(Z,\mathcal{S}^3) \times H_1^q([0,T_e], L^q(\Omega,\mathcal{S}^3))$, $i = 1,2$, are two solutions of the Dirichlet or the Neumann initial-boundary value problem. Then

$$(u, T, \hat\varepsilon_p) = (u_1 - u_2, T_1 - T_2, \varepsilon_{p1} - \varepsilon_{p2})$$

is a solution of the Dirichlet or Neumann boundary value problem (2.3)–(2.6) for almost every $t \in [0,T_e]$. Since $\varepsilon(\nabla_x(a(t) + \omega(t)\times x)) = 0$, the definition of the projector P in Definition 2.4 yields $\varepsilon(\nabla_x u) = P^q\hat\varepsilon_p$, hence

$$T = \mathcal{D}(\varepsilon(\nabla_x u) - \hat\varepsilon_p) = \mathcal{D}(P^q - I)\hat\varepsilon_p = -\mathcal{D}Q^q\hat\varepsilon_p\,. \tag{4.9}$$

Since $T \in L^p(Z,\mathcal{S}^3),\ \hat\varepsilon_p \in H_1^q([0,T_e], L^q(\Omega,\mathcal{S}^3))$, this equation implies for $Q = Q^q$ that

$$Q\hat\varepsilon_p \in L^p(Z,\mathcal{S}^3),\ T_t \in L^q(Z,\mathcal{S}^3).$$

Also, for $\varepsilon(\nabla_x u) - \hat\varepsilon_p = \mathcal{D}^{-1}T$ we obtain

$$\varepsilon(\nabla_x u) - \hat\varepsilon_p \in L^p(Z,\mathcal{S}^3), \quad (\varepsilon(\nabla_x u) - \hat\varepsilon_p)_t \in L^q(Z,\mathcal{S}^3).$$

The Sobolev imbedding theorem now yields that T and $\varepsilon(\nabla_x u) - \hat\varepsilon_p$ have traces $T(t), (\varepsilon(\nabla_x u) - \hat\varepsilon_p)(t) \in L^2(\Omega,\mathcal{S}^3)$. Since $\hat\varepsilon_p(0) = \varepsilon_p^{(0)} - \varepsilon_p^{(0)} = 0$, for the traces at $t = 0$ we have $(\varepsilon(\nabla_x u) - \hat\varepsilon_p)(0) = T(0) = 0$. Thus, for $0 \le s \le T_e$,

$$\begin{aligned}
0 &\le \Big(\mathcal{D}(\varepsilon(\nabla_x u) - \hat\varepsilon_p)(s)\,,\,(\varepsilon(\nabla_x u) - \hat\varepsilon_p)(s)\Big)_\Omega \\
&= \Big(T_t\,,\,\varepsilon(\nabla_x u) - \hat\varepsilon_p\Big)_{Z_s} + \Big(T\,,\,(\varepsilon(\nabla_x u) - \hat\varepsilon_p)_t\Big)_{Z_s} \\
&= (\mathcal{D}Q\hat\varepsilon_{p,t}\,,\,Q\hat\varepsilon_p)_{Z_s} + (\mathcal{D}Q\hat\varepsilon_p\,,\,Q\hat\varepsilon_{p,t})_{Z_s} = (\mathcal{D}\hat\varepsilon_{p,t}\,,\,Q^pQ^q\hat\varepsilon_p)_{Z_s} + (\mathcal{D}Q^pQ^q\hat\varepsilon_p\,,\,\hat\varepsilon_{p,t})_{Z_s} \\
&= 2(\hat\varepsilon_{p,t}\,,\,\mathcal{D}Q^q\hat\varepsilon_p)_{Z_s} = -2(\mathcal{H}[T_1] - \mathcal{H}[T_2]\,,\,T_1 - T_2)_{Z_s} \le 0,
\end{aligned}$$

where in the last step we used the differential equation (1.35) and the fact that $\mathcal{H}$ is monotone with respect to the bilinear form $(\sigma,\tau)_Z$, by condition (H2). Clearly, this computation implies $T_1 = T_2$ and $\varepsilon(\nabla_x u_1) - \varepsilon_{p1} = \varepsilon(\nabla_x u_2) - \varepsilon_{p2}$. From (1.35) and (1.36) we then conclude that $\varepsilon_{p1} = \varepsilon_{p2}$, and so $\varepsilon(\nabla_x u_1) = \varepsilon(\nabla_x u_2)$. For the Dirichlet problem Lemma 2.3 now yields $u_1 = u_2$, whereas for the Neumann problem Lemma 2.3 together with Lemma 2.1 implies that $u_2(x,t) = u_1(x,t) + a(t) + \omega(t)\times x$. This proves the uniqueness part of Theorem 1.6, and it remains to prove the existence of solutions.

We construct a solution as follows: With the given initial data $\varepsilon_p^{(0)}$ let $\tau^{(0)} = Q\varepsilon_p^{(0)}$, and to almost every $t \in [0,T_e]$ let $(\hat u(t), \hat\sigma(t)) \in H_1^p(\Omega,\mathbb{R}^3) \times L^p(\Omega,\mathcal{S}^3)$ be a solution of the Dirichlet or Neumann boundary value problem (1.16) – (1.19) with the data $\hat b = b(t),\ \hat\gamma_D = \gamma_D(t),\ \hat\gamma_N = \gamma_N(t),\ \hat\varepsilon_p = 0$. This defines a function $\hat\sigma \in L^p(Z,\mathcal{S}^3)$. Let $\tau \in L^p(0,T_e; H^p_{\mathrm{sol}})$ be the unique solution of the initial value problem (4.7), (4.8) to the functions $\tau^{(0)}$ and $\hat\sigma$ thus defined. With this function τ let $\varepsilon_p \in H_1^q(0,T_e; L^q(\Omega,\mathcal{S}^3))$ be the solution of

$$\frac{\partial}{\partial t}\varepsilon_p(t) = \mathcal{H}[\hat\sigma - \mathcal{D}\tau](t), \qquad \text{a.e. in } [0,T_e], \tag{4.10}$$

$$\varepsilon_p(0) = \varepsilon_p^{(0)}. \tag{4.11}$$

Finally, for almost every $t \in [0, T_e]$ let $(\tilde{u}(t), \tilde{T}(t)) \in H_1^q(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3)$ be a solution of the Dirichlet or Neumann boundary value problem (2.3) – (2.6) to the data $\hat{\varepsilon}_p = \varepsilon_p(t)$. Then

$$(u, T, \varepsilon_p) = (\tilde{u} + \hat{u}, \tilde{T} + \hat{\sigma}, \varepsilon_p) \in L^q(0, T_e; H_1^q(\Omega, \mathbb{R}^3)) \times L^q(Z, \mathcal{S}^3) \times H_1^q(0, T_e; L^q(\Omega, \mathcal{S}^3))$$

is a solution of the Dirichlet or Neumann initial-boundary value problem (1.31) – (1.36).

To see this, note that by construction $(u(t), T(t), \varepsilon_p(t))$ solves the Dirichlet or Neumann boundary value problem formed by the equations (1.31) – (1.34), almost everywhere in $[0, T_e]$. To see that (1.35) also holds, note that application of Q^q to (4.10) and (4.11) yields

$$(Q^q \varepsilon_p)_t = Q^q \mathcal{H}[\hat{\sigma} - \mathcal{D}\tau] = \tau_t, \qquad Q\varepsilon_p(0) = Q\varepsilon_p^{(0)} = \tau^{(0)} = \tau(0),$$

whence $Q^q \varepsilon_p = \tau \in L^p(0, T_e; H^p_{\text{sol}})$. The reasoning which leads to (4.9) is valid also in the present situation, hence $\tilde{T} = -\mathcal{D}Q^q\varepsilon_p = -\mathcal{D}\tau$, and therefore

$$T = \hat{\sigma} + \tilde{T} = \hat{\sigma} - \mathcal{D}\tau \in L^p(Z, \mathcal{S}^3).$$

Insertion of this equation into (4.10) shows that also the equations (1.35) and (1.36) are satisfied, hence (u, T, ε_p) satisfies the initial-boundary value problem and has the regularity properties stated in Theorem 1.6. The proof is complete.

5 Positive semi-definite free energy

In this section we study examples for initial-boundary value problems to constitutive equations with positive semi-definite free energy, and prove Theorems 1.7 and 1.8. In the proofs we need the following lemma, which shows that history functionals defined by monotone constitutive equations are itself monotone.

To state the lemma we need some definitions: Let $\tilde{g}_1 : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathcal{S}^3$, $\tilde{g}_2 : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathbb{R}^{N-6}$ and $\tilde{z}^{(0)} \in L^2(\Omega, \mathbb{R}^{N-6})$ be given, assume that $1 < \hat{q} \le \hat{p}$ are numbers with $\frac{1}{\hat{p}} + \frac{1}{\hat{q}} = 1$, and suppose that for every $T \in L^p(Z, \mathcal{S}^3)$ the system of equations

$$h(x,t) = \tilde{g}_1(T(x,t), -\tilde{z}(x,t)) \tag{5.1}$$

$$\frac{\partial}{\partial t}\tilde{z}(x,t) = \tilde{g}_2(T(x,t), -\tilde{z}(x,t)) \tag{5.2}$$

$$\tilde{z}(x,0) = \tilde{z}^{(0)}(x) \tag{5.3}$$

has a unique solution $(\tilde{z}, h) \in L^{\hat{p}}(Z, \mathbb{R}^{N-6}) \times L^q(Z, \mathcal{S}^3)$ with $\frac{\partial}{\partial t}\tilde{z} \in L^{\hat{q}}(Z, \mathbb{R}^{N-6})$. Then

$$\mathcal{H}_{\tilde{z}^{(0)}}[T] = h \tag{5.4}$$

defines a history functional $\mathcal{H}_{\tilde{z}^{(0)}} : L^p(Z, \mathcal{S}^3) \to L^q(Z, \mathcal{S}^3)$.

Lemma 5.1 *Assume that the vector field $(T, \tilde{z}) \to (\tilde{g}_1(T, \tilde{z}), \tilde{g}_2(T, \tilde{z}))$ is monotone. Then the history functional $\mathcal{H}_{\tilde{z}^{(0)}}$ is monotone with respect to the bilinear form $(\sigma, \tau)_Z$ on the product space $L^q(Z, \mathcal{S}^3) \times L^p(Z, \mathcal{S}^3)$.*

Proof: Let $T_1, T_2 \in L^p(Z, \mathcal{S}^3)$ and let $\tilde{z}_1, \tilde{z}_2 \in L^{\hat{p}}(Z, \mathbb{R}^{N-6})$ be the solutions of the initial value problem (5.2), (5.3) with T_1 or T_2 inserted for T. Then the monotonicity of $(\tilde{g}_1, \tilde{g}_2)$ and the equations (5.1) – (5.4) yield

$$
\begin{aligned}
&(\mathcal{H}_{\tilde{z}^{(0)}}[T_1] - \mathcal{H}_{\tilde{z}^{(0)}}[T_2]\,,\, T_1 - T_2)_Z \\
&= (\tilde{g}_1(T_1, -\tilde{z}_1) - \tilde{g}_1(T_2, -\tilde{z}_2)\,,\, T_1 - T_2)_Z + (\tilde{g}_2(T_1, -\tilde{z}_1) - \tilde{g}_2(T_2, -\tilde{z}_2)\,,\, -\tilde{z}_1 + \tilde{z}_2)_Z \\
&\qquad\qquad - (\tilde{g}_2(T_1, -\tilde{z}_1) - \tilde{g}_2(T_2, -\tilde{z}_2)\,,\, -\tilde{z}_1 + \tilde{z}_2)_Z \\
&\geq (\frac{\partial}{\partial t}\tilde{z}_1 - \frac{\partial}{\partial t}\tilde{z}_2\,,\, \tilde{z}_1 - \tilde{z}_2)_Z = \frac{1}{2}|\tilde{z}_1(T_e) - \tilde{z}_2(T_e)|^2_{2,\Omega} - \frac{1}{2}|\tilde{z}_1(0) - \tilde{z}_2(0)|^2_{2,\Omega} \geq 0\,,
\end{aligned}
$$

since $\tilde{z}_1(0) - \tilde{z}_2(0) = \tilde{z}^{(0)} - \tilde{z}^{(0)} = 0$. Note that $\tilde{z}_i$ has traces $\tilde{z}_i(t) \in L^2(\Omega)$, since $\tilde{z}_i \in L^{\hat{p}}(Z)$ and $\frac{\partial}{\partial t}\tilde{z}_i \in L^{\hat{q}}(Z)$.

5.1 Example 1 (Norton-Hoff law)

We consider the equations

$$
\begin{aligned}
-\operatorname{div}_x T &= b && (5.5)\\
T &= \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) && (5.6)\\
\frac{\partial}{\partial t}\varepsilon_p &= c\,|T|^r\,\frac{T}{|T|} && (5.7)\\
\varepsilon_p(0) &= \varepsilon_p^{(0)}, && (5.8)
\end{aligned}
$$

with constants $c > 0\,,\, r > 1\,.$ Equation (5.7) is the Norton-Hoff law. These equations can be written in the form (1.26) – (1.30) if we define $\tilde{g}_1 : \mathcal{S}^3 \to \mathcal{S}^3$ by

$$\tilde{g}_1(T) = c\,|T|^r\,\frac{T}{|T|}\,. \tag{5.9}$$

The variable $\tilde{z}$ and the equation (1.29) are not needed in this example. Consequently, the definition (5.1) – (5.4) of the history functional $\mathcal{H}_{\tilde{z}^{(0)}} = \mathcal{H}$ reduces to the equation

$$\mathcal{H}[T] = \tilde{g}_1(T) = c|T|^r\frac{T}{|T|}\,. \tag{5.10}$$

We set $p = 1 + r$ and $q = 1 + \frac{1}{r}$. Then $\frac{1}{p} + \frac{1}{q} = 1$, and we obtain the following

Lemma 5.2 *The history functional defined by* (5.10) *is a mapping* $\mathcal{H} : L^p(Z, \mathcal{S}^3) \to L^q(Z, \mathcal{S}^3)$*, which satisfies the conditions* (H1) – (H4)*.*

Proof: We remind the reader that the conditions (H1) – (H4) are stated before Theorem 1.6.
(H1) For $T \in L^p(Z,\, \mathcal{S}^3)$ we have

$$\|\mathcal{H}[T]\|^q_{q,Z} = \|c\,|T|^r\,\frac{T}{|T|}\,\|^q_{q,Z} = c^q\int_Z |T|^{1+r}\,d(x,t) = c^q\,\|T\|^p_{p,Z}\,,$$

hence $\mathcal{H}$ is everywhere defined on $L^p(Z, \mathcal{S}^3)$ as a mapping with values in $L^q(Z, \mathcal{S}^3)$ and satisfies

$$\|\mathcal{H}[T]\|_{q,Z} \leq c\,\|T\|^{\frac{p}{q}}_{p,Z}\,.$$

(H2) To show that $\mathcal{H}$ is monotone, define the convex function $\varphi : \mathcal{S}^3 \to \mathbb{R}$ by $\varphi(T) = \frac{c}{r+1}|T|^{r+1}$. Then $\tilde{g}_1$ given by (5.9) satisfies

$$\tilde{g}_1(T) = c\,|T|^r \frac{T}{|T|} = \nabla\varphi(T)\,,$$

hence $\tilde{g}_1$ is monotone as gradient of a convex function. From Lemma 5.1 we now conclude that $\mathcal{H}$ is monotone.

To prove that $\mathcal{H}$ is hemicontinuous, it suffices to show that $\mathcal{H}$ is demicontinuous, i.e. that $T_n \to T$ in $L^p(Z,\mathcal{S}^3)$ implies $\mathcal{H}[T_n] \rightharpoonup \mathcal{H}[T]$ weakly in $L^q(Z,\mathcal{S}^3)$. Thus, let $T, T_n \in L^p(Z,\mathcal{S}^3)$ with $\|T - T_n\|_{p,Z} \to 0$ for $n \to \infty$. Then (H1) shows that $\{\mathcal{H}[T_n]\}_{n=1}^\infty$ is a bounded sequence in $L^q(Z,\mathcal{S}^3)$. Therefore we can choose a subsequence $\{T'_n\}_{n=1}^\infty$, which converges pointwise almost everywhere, and for which $\{\mathcal{H}[T'_n]\}_{n=1}^\infty$ converges weakly. Since $\{\mathcal{H}[T'_n]\}_{n=1}^\infty$ converges pointwise almost everywhere to $\mathcal{H}[T]$, it follows that $\{\mathcal{H}[T'_n]\}_{n=1}^\infty$ converges weakly to $\mathcal{H}[T]$. The same construction shows that every subsequence of $\{T_n\}_{n=1}^\infty$ has a subsequence $\{T''_n\}_{n=1}^\infty$, for which $\{\mathcal{H}[T''_n]\}_{n=1}^\infty$ converges weakly to $\mathcal{H}[T]$, hence the sequence $\{\mathcal{H}[T_n]\}_{n=1}^\infty$ converges weakly to $\mathcal{H}[T]$. This proves that $\mathcal{H}$ is demicontinuous.

(H3) For $T \in L^p(Z,\, \mathcal{S}^3)$ we have

$$(T, \mathcal{H}[T])_Z = \int_Z c\,|T(x,t)|^{r+1}\, d(x,t) = c\,\|T\|_{p,Z}^p\,, \tag{5.11}$$

and

$$\|\mathcal{H}[T]\|_{q,Z}^q = \int_Z \left(c\,|T(x,t)|^r\right)^{1+\frac{1}{r}} d(x,t) = c^q \|T\|_{p,Z}^p\,,$$

hence

$$(T, \mathcal{H}[T])_Z = c\,\|T\|_{p,Z}^p = \beta_0\,\|\mathcal{H}[T]\|_{q,Z}^q\,,$$

with $\beta_0 = c^{1-q} > 0$. This yields (H3), since $q > 1$.

(H4) Equation (5.11) implies

$$\frac{(T, \mathcal{H}[T])_Z}{\|T\|_{p,Z}} = c\,\|T\|_{p,Z}^{p-1} \to \infty$$

for $\|T\|_{p,Z} \to \infty$, since $p - 1 = r > 1$. This completes the proof of Lemma 5.2.

Proof of Theorem 1.7: From Lemma 5.2 we see that the history functional $\mathcal{H}$ defined by the constitutive equation (1.44) satisfies the assumptions of Theorem 1.6. That theorem immediately yields Theorem 1.7.

5.2 Example 2 (Kinematic hardening)

We study the equations

$$\begin{aligned}
-\operatorname{div}_x T &= b && (5.12)\\
T &= \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) && (5.13)\\
\frac{\partial}{\partial t}\varepsilon_p &= c_1|T - k(\varepsilon_p - \varepsilon_n)|^r \frac{T - k(\varepsilon_p - \varepsilon_n)}{|T - k(\varepsilon_p - \varepsilon_n)|} && (5.14)\\
\frac{\partial}{\partial t}\varepsilon_n &= c_2(k|\varepsilon_p - \varepsilon_n|)^\gamma \frac{\varepsilon_p - \varepsilon_n}{|\varepsilon_p - \varepsilon_n|}\,, && (5.15)\\
\varepsilon_p(0) &= \varepsilon_p^{(0)}, \quad \varepsilon_n(0) = \varepsilon_n^{(0)}, && (5.16)
\end{aligned}$$

where $c_1, c_2, k > 0$ and $r, \gamma > 1$ are constants, and where $\varepsilon_p(x,\tau), \varepsilon_n(x,\tau) \in \mathcal{S}^3$ are inelastic strain tensors. This is a model, which incorporates kinematic hardening. For other constitutive models, which are developed for the application in engineering and which incorporate kinematic and isotropic hardening we refer to [Alb98].

With the definition $\tilde{z} = k^{1/2}(\varepsilon_p - \varepsilon_n)$ we obtain the equivalent equations

$$-\mathrm{div}_x T = b \tag{5.17}$$

$$T = \mathcal{D}(\varepsilon(\nabla_x u) - \varepsilon_p) \tag{5.18}$$

$$\frac{\partial}{\partial t}\varepsilon_p = c_1 |T - k^{\frac{1}{2}}\tilde{z}|^r \frac{T - k^{\frac{1}{2}}\tilde{z}}{|T - k^{\frac{1}{2}}\tilde{z}|} \tag{5.19}$$

$$\frac{\partial}{\partial t}\tilde{z} = k^{\frac{1}{2}} c_1 |T - k^{\frac{1}{2}}\tilde{z}|^r \frac{T - k^{\frac{1}{2}}\tilde{z}}{|T - k^{\frac{1}{2}}\tilde{z}|} - c_2 k^{\frac{1}{2}} |k^{\frac{1}{2}}\tilde{z}|^\gamma \frac{\tilde{z}}{|\tilde{z}|}, \tag{5.20}$$

$$\varepsilon_p(0) = \varepsilon_p^{(0)}, \quad \tilde{z}(0) = k^{1/2}(\varepsilon_p^{(0)} - \varepsilon_n^{(0)}). \tag{5.21}$$

These equations can be written in the form (1.26) – (1.30) if we choose $N = 12$, identify $\mathbb{R}^6$ with $\mathcal{S}^3$, and define $\tilde{g} = (\tilde{g}_1, \tilde{g}_2) : \mathcal{S}^3 \times \mathcal{S}^3 \to \mathcal{S}^3 \times \mathcal{S}^3$ by

$$\tilde{g}_1(T, \tilde{z}) = c_1 |T + k^{\frac{1}{2}}\tilde{z}|^r \frac{T + k^{\frac{1}{2}}\tilde{z}}{|T + k^{\frac{1}{2}}\tilde{z}|}, \tag{5.22}$$

$$\tilde{g}_2(T, \tilde{z}) = k^{\frac{1}{2}} c_1 |T + k^{\frac{1}{2}}\tilde{z}|^r \frac{T + k^{\frac{1}{2}}\tilde{z}}{|T + k^{\frac{1}{2}}\tilde{z}|} + k^{\frac{1}{2}} c_2 |k^{\frac{1}{2}}\tilde{z}|^\gamma \frac{\tilde{z}}{|\tilde{z}|}. \tag{5.23}$$

For the convex function $\varphi : \mathcal{S}^3 \times \mathcal{S}^3 \to \mathbb{R}$ given by

$$\varphi(T, \tilde{z}) = \frac{c_1}{r+1}|T + k^{1/2}\tilde{z}|^{r+1} + \frac{c_2}{\gamma+1}|k^{1/2}\tilde{z}|^{\gamma+1}$$

we have

$$\tilde{g}(T, \tilde{z}) = \nabla\varphi(T, \tilde{z}),$$

whence $\tilde{g}$ is monotone as gradient of a convex function.

The history functional $\mathcal{H}_{\tilde{z}^{(0)}}$ generated by the constitutive equations (5.14), (5.15), or equivalently by (5.19), (5.20), is defined by (5.1) – (5.4) with the functions $\tilde{g}_1$ and $\tilde{g}_2$ from (5.22) and (5.23) inserted. We study now this history functional and always assume that $\tilde{g}_1$ and $\tilde{g}_2$ are equal to these functions.

We set $p = 1 + r,\ q = 1 + \frac{1}{r},\ \hat{p} = \max(1+r, 1+\gamma),\ \hat{q} = \min(1 + \frac{1}{r}, 1 + \frac{1}{\gamma})$. Then $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{\hat{p}} + \frac{1}{\hat{q}} = 1$.

Lemma 5.3 *Let* $\tilde{z}^{(0)} \in L^2(\Omega, \mathcal{S}^3)$.
(i) *To every* $T \in L^p(Z, \mathcal{S}^3)$ *the initial value problem* (5.2), (5.3) *has a unique solution* $\tilde{z} \in L^{\hat{p}}(Z, \mathcal{S}^3)$ *with* $\frac{\partial}{\partial t}\tilde{z} \in L^{\hat{q}}(Z, \mathcal{S}^3)$.

(ii) *The equations* (5.1) – (5.4) *define a history functional* $\mathcal{H}_{\tilde{z}^{(0)}} : L^p(Z, \mathcal{S}^3) \to L^q(Z, \mathcal{S}^3)$, *which satisfies the conditions* (H1) and (H2). *If* $\gamma > r$, *then also the coercivity conditions* (H3) *and* (H4) *are satisfied.*

Proof: For notational convenience we set $y = \varepsilon_p - \varepsilon_n = k^{-1/2}\tilde{z}$ and $y^{(0)} = \varepsilon_p^{(0)} - \varepsilon_n^{(0)}$. From (5.1) – (5.4) and (5.22), (5.23) we see that y satisfies the equations

$$\mathcal{H}_{\tilde{z}^{(0)}}[T] = c_1|T-ky|^r \frac{T-ky}{|T-ky|}\,, \tag{5.24}$$

$$y_t = c_1|T-ky|^r \frac{T-ky}{|T-ky|} - c_2|ky|^\gamma \frac{y}{|y|}\,, \tag{5.25}$$

$$y(0) = y^{(0)}\,. \tag{5.26}$$

Proof of (i): We can use the same results from the theory of monotone operators as in the proof of Theorem 4.2 to show that for given functions $y^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ and $T \in L^p(Z, \mathcal{S}^3)$ the initial value problem (5.25), (5.26) has a unique solution $y \in L^{\hat{p}}(Z, \mathcal{S}^3)$ with $y_t \in L^{\hat{q}}(Z, \mathcal{S}^3)$. In fact, for the proof it suffices to show that this initial value problem satisfies conditions analogous to the conditions (H1') – (H3') stated in Lemma 4.1. These conditions can be verified directly by a computation, which we leave to the reader.

Proof of (ii): To show that the conditions (H1) – (H4) are satisfied, we derive estimates for the solution y of the initial value problem (5.25), (5.26). Thus, let $y^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ and $T \in L^p(Z, \mathcal{S}^3)$. For brevity we set

$$\eta = c_1|T-ky|^r \frac{T-ky}{|T-ky|}\,, \qquad \kappa = c_2|ky|^\gamma \frac{y}{|y|}\,.$$

Note first that (5.25) implies for $0 \le t \le T_e$ that

$$\begin{aligned}
\partial_t \frac{k}{2}|y(x,t)|^2 &= ky\cdot y_t = \eta\cdot ky - \kappa\cdot ky = -\eta\cdot(T-ky) - \kappa\cdot ky + \eta\cdot T \\
&\le -c_1|T-ky|^{r+1} - c_2|ky|^{\gamma+1} + |\eta|\,|T| \qquad (5.27)\\
&\le -c_1|T-ky|^{r+1} - c_2|ky|^{\gamma+1} + \frac{\delta^q}{q}\,|\eta|^q + \frac{1}{p\delta^p}\,|T|^p \\
&= -c_1|T-ky|^{r+1} + \frac{\delta^q}{q}(c_1|T-ky|^r)^{1+\frac{1}{r}} - c_2|ky|^{\gamma+1} + \frac{1}{p\delta^p}\,|T|^p\,,
\end{aligned}$$

for a constant $\delta > 0$. Here we used Young's inequality. Now choose δ such that $\frac{\delta^q c_1^q}{q} = \frac{c_1}{2}$. Since $r(1+\frac{1}{r}) = 1 + r = p$, integration of this inequality over the domain Z yields

$$\int_\Omega \frac{k}{2}|y(T_e)|^2\,dx + \frac{c_1}{2}\,\|T-ky\|_{p,Z}^p + c_2\|ky\|_{\gamma+1,Z}^{\gamma+1} \le \int_\Omega \frac{k}{2}\,|y^{(0)}|^2 dx + C_1\|T\|_{p,Z}^p\,, \tag{5.28}$$

with $C_1 = \frac{1}{p\delta^p} > 0$. This inequality and the triangle inequality together yield

$$\|ky\|_{p,Z} \le \|T\|_{p,Z} + \|T-ky\|_{p,Z} \le \Big[\frac{2}{c_1}\big(\frac{k}{2}\|y^{(0)}\|_{2,\Omega}^2 + C_1\|T\|_{p,Z}^p\big)\Big]^{1/p} + \|T\|_{p,Z}\,. \tag{5.29}$$

The function y must belong to $L^{\gamma+1}(Z)$, which is implied by (5.28), and to $L^p(Z)$, which is implied (5.29), hence $y \in L^{\hat{p}}(Z, \mathcal{S}^3)$, as we stated.

We set $\gamma' = 1 + \frac{1}{\gamma}$. Using inequality (5.28) again and noting (5.24) we obtain

$$\begin{aligned}
\|\eta\|_{q,Z}^q = \|\mathcal{H}_{\tilde{z}^{(0)}}[T]\|_{q,Z}^q &= \int_Z (c_1|T-ky|^r)^{1+\frac{1}{r}}\, d(x,t) = c_1^q \|T-ky\|_{p,Z}^p \\
&\leq 2c_1^{q-1}\Big(C_1\|T\|_{p,Z}^p + \frac{k}{2}\,\|y^{(0)}\|_{2,\Omega}^2\Big)\,, \qquad (5.30)\\
\|\kappa\|_{\gamma',Z}^{\gamma'} &= \int_Z (c_2|ky|^\gamma)^{1+\frac{1}{\gamma}}\, d(x,t) = c_2^{\gamma'}\|ky\|_{\gamma+1,Z}^{\gamma+1} \\
&\leq c_2^{\gamma'-1}\Big(C_1\|T\|_{p,Z}^p + \frac{k}{2}\|y^{(0)}\|_{2,\Omega}^2\Big)\,. \qquad (5.31)
\end{aligned}$$

(5.30), (5.31) and the equation (5.25) imply

$$y_t = \eta - \kappa \in L^{\min(q,\gamma')}(Z,\mathcal{S}^3) = L^{\hat{q}}(Z,\mathcal{S}^3),$$

again, as we claimed above. Moreover, (5.30) implies that the history functional is a mapping $\mathcal{H}_{\tilde{z}^{(0)}} : L^p(Z,\mathcal{S}^3) \to L^q(Z,\mathcal{S}^3)$, and that the estimate from condition (H1) is satisfied.

Next we verify that condition (H2) is fulfilled. Before Lemma 5.3 we showed that the vector field $\tilde{g}$ is monotone, hence the monotonicity of $\mathcal{H}_{\tilde{z}^{(0)}}$ follows from Lemma 5.1. As in Example 1 we use that the hemicontinuity of the history functional is a consequence of the demicontinuity. To prove demicontinuity, we first derive an estimate for the difference of two solutions of the initial value problem (5.25), (5.26):

Thus let $T_1, T_2 \in L^p(Z,\mathcal{S}^3)$, and let y_1, y_2 be the corresponding solutions of (5.25), (5.26). As above we set

$$\eta_i = \mathcal{H}_{\tilde{z}^{(0)}}[T_i] = c_1|T_i - ky_i|^r \frac{T_i - ky_i}{|T_i - ky_i|}\,, \qquad \kappa_i = c_2|ky_i|^\gamma \frac{y_i}{|y_i|}\,,$$

for $i = 1, 2$. Let $\overline{T} = T_1 - T_2$, $\overline{y} = y_1 - y_2$, $\overline{\eta} = \eta_1 - \eta_2$, $\overline{\kappa} = \kappa_1 - \kappa_2$. Then (5.25) yields for $0 \leq t \leq T_e$ that

$$\begin{aligned}
\partial_t \frac{k}{2}\,|\overline{y}(x,t)|^2 &= k\overline{y}\cdot\overline{y}_t = \overline{\eta}\cdot k\overline{y} - \overline{\kappa}\cdot k\overline{y} \qquad (5.32)\\
&= -\overline{\eta}\cdot(\overline{T} - k\overline{y}) - \overline{\kappa}\cdot k\overline{y} + \overline{\eta}\cdot\overline{T} \leq \overline{\eta}\cdot\overline{T} \leq (|\eta_1| + |\eta_2|)\,|\overline{T}|\,.
\end{aligned}$$

Here we used that

$$(\eta,\kappa) = \Big(c_1|T-ky|^r\,\frac{T-ky}{|T-ky|}\,,\, c_2|ky|^\gamma\,\frac{ky}{|ky|}\Big) = (\nabla\varphi_1)(T-ky, ky)$$

with $\varphi_1(z) = \varphi_1(z', z'') = \frac{c_1}{r+1}\,|z'|^{r+1} + \frac{c_2}{\gamma+1}\,|z''|^{\gamma+1}$. Since φ_1 is convex, the vector field $\nabla\varphi_1$ is monotone, which implies that

$$\overline{\eta}\cdot(\overline{T} - k\overline{y}) + \overline{\kappa}\cdot k\overline{y} \geq 0\,.$$

Using that $\overline{y}(0) = 0$, we obtain from (5.32) by integration that

$$\begin{aligned}
&\int_\Omega \frac{k}{2}\,|\overline{y}(t)|^2 dx \leq \int_Z (|\eta_1| + |\eta_2|)|\overline{T}|\, d(x,t) \\
&\leq (\,\|\eta_1\|_{q,Z} + \|\eta_2\|_{q,Z}\,)\,\|\overline{T}\|_{p,Z} \leq C_2\Big(M(T_1,T_2) + 1\Big)^{\frac{1}{q}}\,\|\overline{T}\|_{p,Z}\,,
\end{aligned}$$

with $M(T_1,T_2) = \max\left(\|T_1\|^p_{p,Z}\,,\ \|T_2\|^p_{p,Z}\right)$ and with a suitable constant C_2. To get the last inequality sign we employed (5.30). Therefore

$$\frac{k}{2}\,\|\overline{y}\|^2_{2,Z} = \frac{k}{2}\int_0^{T_e} \|\overline{y}(t)\|^2_{2,\Omega}\,dt \le T_e C_2 \Big(M(T_1,T_2)+1\Big)^{\frac{1}{q}}\,\|\overline{T}\|_{p,Z}\,. \tag{5.33}$$

Now we can prove the demicontinuity. Let $\{T_m\}_{m=1}^\infty$ be a sequence which converges in $L^p(Z,\mathcal{S}^3)$ to $T\in L^p(Z,\mathcal{S}^3)$. Setting $\overline{T} = T_m - T$ in (5.33), we see that $\{y_m\}_{m=1}^\infty$ converges to y in $L^2(Z,\mathcal{S}^3)$. Consequently, we can select a subsequence $\{T'_m\}_{m=1}^\infty$, which converges pointwise almost everywhere in Z to T, such that $\{y'_m\}_{m=1}^\infty$ converges pointwise almost everywhere in Z to y. Thence, $\mathcal{H}_{\hat{z}^{(0)}}[T_m] = c_1|T_m - ky_m|^r\,\frac{T_m-ky_m}{|T_m-ky_m|}$ converges pointwise almost everywhere to $\mathcal{H}_{\hat{z}^{(0)}}[T]$. From (5.30) we see that the sequence $\{\mathcal{H}_{\hat{z}^{(0)}}[T_m]\}_{m=1}^\infty$ is uniformly bounded in $L^q(Z,\mathcal{S}^3)$. Therefore we can select a subsequence $\{T''_m\}_{m=1}^\infty$ of $\{T'_m\}_{m=1}^\infty$, such that $\{\mathcal{H}_{\hat{z}^{(0)}}[T''_m]\}_{m=1}^\infty$ converges weakly in $L^q(Z,\mathcal{S}^3)$. Since weak and pointwise limit coincide, $\{\mathcal{H}_{\hat{z}^{(0)}}[T''_m]\}_{m=1}^\infty$ converges weakly to $\mathcal{H}_{\hat{z}^{(0)}}[T]$. Moreover, these considerations show that any weakly converging subsequence of $\{\mathcal{H}_{\hat{z}^{(0)}}[T_m]\}_{m=1}^\infty$ must converge to $\mathcal{H}_{\hat{z}^{(0)}}[T]$. Thus, the sequence $\{\mathcal{H}_{\hat{z}^{(0)}}[T_m]\}_{m=1}^\infty$ converges to $\mathcal{H}_{\hat{z}^{(0)}}[T]$ weakly. This implies that $\mathcal{H}_{\hat{z}^{(0)}}$ is demicontinuous, and we conclude that condition (H2) is fulfilled.

To prove that the coercivity conditions (H3) and (H4) are satisfied if $\gamma > r$, note first that (5.25) yields similarly as in (5.27)

$$\begin{aligned} T\cdot\mathcal{H}_{\hat{z}^{(0)}}[T] &= T\cdot\eta = \partial_t\,\frac{k}{2}\,|y|^2 + (T-ky)\cdot\eta + ky\cdot\kappa \\ &= \partial_t\,\frac{k}{2}\,|y|^2 + c_1|T-ky|^{r+1} + c_2|ky|^{\gamma+1}. \end{aligned}$$

Integration yields

$$\begin{aligned} (T,\mathcal{H}_{\hat{z}^{(0)}}[T])_Z &= \frac{k}{2}\|y(T_e)\|^2_{2,\Omega} - \frac{k}{2}\|y^{(0)}\|^2_{2,\Omega} + c_1\|T-ky\|^p_{p,Z} + c_2\|ky\|^{\gamma+1}_{\gamma+1,Z} \\ &\ge c_1\|T-ky\|^p_{p,Z} - \frac{k}{2}\|y^{(0)}\|^2_{2,\Omega}\,. \end{aligned} \tag{5.34}$$

Furthermore, set $\beta = (T_e|\Omega|)^{\frac{\gamma-r}{(\gamma+1)p}}$. Then (5.28) implies for all $T\in L^p(Z,\mathcal{S}^3)$ with $C_1\|T\|^p_{p,Z} \ge \beta_1 = \frac{k}{2}\|y^{(0)}\|^2_{2,\Omega}$ that

$$\begin{aligned} \|ky\|_{p,Z} &\le \beta\|ky\|_{\gamma+1,Z} \\ &\le \beta\left[\frac{1}{c_2}\Big(C_1\|T\|^p_{p,Z}+\beta_1\Big)\right]^{\frac{1}{\gamma+1}} \le \beta\Big(\frac{2C_1}{c_2}\Big)^{\frac{1}{\gamma+1}}\|T\|^{\frac{r+1}{\gamma+1}}_{p,Z}\,. \end{aligned}$$

We set $\alpha = \beta(2C_1/c_2)^{1/(\gamma+1)}$ and obtain with the inverse triangle inequality from the last estimate

$$\|T-ky\|_{p,Z} \ge \|T\|_{p,Z} - \|ky\|_{p,Z} \ge \|T\|_{p,Z} - \alpha\|T\|^{\frac{r+1}{\gamma+1}}_{p,Z} \ge \frac{1}{2}\|T\|_{p,Z}\,, \tag{5.35}$$

for $\|T\|_{p,Z} \ge \max(\omega,(\beta_1/C_1)^{1/p})$, where $\omega > 0$ is the solution of the equation $\frac{1}{2}\omega = \alpha\omega^{\frac{r+1}{\gamma+1}}$. Here we use that $\frac{r+1}{\gamma+1} < 1$. Combination with (5.34) results in

$$\frac{(T,\mathcal{H}_{\hat{z}^{(0)}}[T])_{p,Z}}{\|T\|_{p,Z}} \ge \frac{c_1(\frac{1}{2}\|T\|_{p,Z})^p - \beta_1}{\|T\|_{p,Z}} \to \infty$$

for $\|T\|_{p,Z} \to \infty$, which proves that condition (H4) is satisfied.

To verify (H3) we first observe that (5.30) and (5.35) imply

$$\|\mathcal{H}_{\tilde{z}^{(0)}}[T]\|_{q,Z} = c_1 \|T - ky\|_{p,Z}^{\frac{p}{q}} \geq C_2(\|T\|_{p,Z})^{\frac{p}{q}} \to \infty \tag{5.36}$$

for $\|T\|_{p,Z} \to \infty$. Furthermore, noting again (5.30), we derive from the inequality (5.34) that

$$(T, \mathcal{H}_{\tilde{z}^{(0)}}[T])_Z \geq c_1^{1-q} \, \|\mathcal{H}_{\tilde{z}^{(0)}}[T]\|_{q,Z}^q - \beta_1 \,.$$

Since $q > 1$, this inequality and (5.36) together imply that condition (H3) is satisfied. The proof of the lemma is complete.

Proof of Theorem 1.8: For $\varepsilon_p^{(0)}, \varepsilon_n^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ we have $\tilde{z}^{(0)} = k^{1/2}(\varepsilon_p^{(0)} - \varepsilon_n^{(0)}) \in L^2(\Omega, \mathcal{S}^3)$. Therefore the assumption of Lemma 5.3 is satisfied, and we conclude from this lemma that the history functional $\mathcal{H}_{\tilde{z}^{(0)}}$ defined by the constitutive equations (1.48) – (1.50) satisfies the assumptions of Theorem 1.6. That theorem yields all statements of Theorem 1.8 concerning the functions u, T and ε_p. The statements for ε_n are obtained from Lemma 5.3(i), since $\tilde{z} = k^{1/2}(\varepsilon_p - \varepsilon_n)$.

6 Coercive models - the method based on the Yosida approximation

In this section we present another method to prove existence of global in time solutions for the quasistatic problem (1.10) - (1.13) possessing a positive definite free energy. Namely, we transfer the method used in [CG00, Che01] to investigate the dynamical problem and use the Yosida approximation G_λ of the maximal monotone constitutive multifunction g to verify Theorem 1.4. In the first step of the proof we replace $g(B^T - Lz)$ by a global Lipschitz function f and show that the theory of differential equations in Banach spaces yields global in time solutions of the resulting initial-boundary value problem. This result is formulated in Theorem 6.1. In the second step we use this result to construct a sequence of approximate solutions $\{(u^\lambda, T^\lambda, z^\lambda)\}_{\lambda>0}$ and prove a priori estimates, which allow to pass to the limit $\lambda \to 0^+$. These estimates are formulated in Theorems 6.2 and 6.3.

For brevity we use in this section the notation

$$\varepsilon(u(x,t)) = \frac{1}{2}(\nabla_x u(x,t) + \nabla_x^T u(x,t))\,.$$

Since we study positive definite free energy ψ in this section, the matrix L is assumed to be positive definite.

Let us start our investigation with the following problem of the type (1.10) - (1.13) containing only global Lipschitz nonlinearities

$$\operatorname{div}_x T(x,t) = -b(x,t)\,, \tag{6.1}$$

$$T(x,t) = \mathcal{D}(\varepsilon(x,t) - Bz(x,t))\,, \tag{6.2}$$

$$\varepsilon(x,t) = \frac{1}{2}(\nabla_x u(x,t) + \nabla_x^T u(x,t))\,, \tag{6.3}$$

$$z_t(x,t) = f(\varepsilon(x,t), z(x,t))\,, \tag{6.4}$$

$$z(x,0) = z^{(0)}(x)\,, \tag{6.5}$$

where the nonlinear constitutive function f is everywhere defined and global Lipschitz, which means that

$$\exists\, L > 0 \quad \forall\, (\varepsilon^1, z^1),\ (\varepsilon^2, z^2) \in \mathcal{S}^3 \times \mathbb{R}^N$$
$$|f(\varepsilon^1, z^1) - f(\varepsilon^2, z^2)| \le L(|\varepsilon^1 - \varepsilon^2| + |z^1 - z^2|)\,. \tag{6.6}$$

We are going to prove existence of global in time solutions for the system (6.1) - (6.5) with the boundary conditions (1.25). Let us denote by $B^{\perp}$ the orthogonal projector of $\mathbb{R}^N$ onto the subspace $\{(\varepsilon_p, y) \in \mathbb{R}^N : \varepsilon_p = 0\}$. Then equation (6.4) can be written in the components (ε_p, y)

$$-\mathcal{D}^{-1}T_t(x,t) + \varepsilon(u_t(x,t)) \;=\; B\, f(\varepsilon(u(x,t)), z(x,t))\,, \tag{6.7}$$
$$y_t(x,t) \;=\; B^{\perp} f(\varepsilon(u(x,t)), z(x,t))\,, \tag{6.8}$$

where $z = (\varepsilon(u) - \mathcal{D}^{-1}T, y)$. Hence, this system describes the evolution of the triple (u, T, y). Obviously, in general we cannot expect that the system (6.7) - (6.8) possesses unique solutions. However, if we add to this system the equation of motion (6.1) then we can use the "orthogonality" of the functions T and $\varepsilon(u)$. Let us define some Hilbert spaces in which we will solve our problem:

$$H_{1,\Gamma_1}(\Omega;\mathbb{R}^3) \;=\; \{u \in H_1(\Omega;\mathbb{R}^3) : u_{|\Gamma_1} = 0\},$$
$$L^2_{\Gamma_2,\mathrm{sol}}(\Omega;\mathcal{S}^3) \;=\; \{T \in L^2(\Omega;\mathcal{S}^3) : \operatorname{div} T = 0,\ T\, n_{|\Gamma_2} = 0\}.$$

$H_{1,\Gamma_1}(\Omega;\mathbb{R}^3)$ is the space of vectorial Sobolev functions with traces vanishing on Γ_1, and $L^2_{\Gamma_2,\mathrm{sol}}(\Omega;\mathcal{S}^3)$ is the space of L^2-functions with vanishing divergence and with trace in the normal direction to the boundary vanishing on Γ_2. Note that for functions $T \in L^2(\Omega;\mathcal{S}^3)$ with $\operatorname{div} T \in L^2(\Omega;\mathbb{R}^3)$ the trace in the normal direction to the boundary is well defined and belongs to the space $H^2_{-1/2}(\partial\Omega;\mathbb{R}^3)$ (see for example [Tem83] p. 14). Moreover, let us denote by

$$\mathbb{H}(\Omega) = H_{1,\Gamma_1}(\Omega;\mathbb{R}^3) \times L^2_{\Gamma_2,\mathrm{sol}}(\Omega;\mathcal{S}^3) \times L^2(\Omega;\mathbb{R}^{N-6})$$

the Hilbert space in which we solve the system (6.7) - (6.8). In this space we redefine the scalar product as follows

$$\langle\, (u^1, T^1, y^1),\ (u^2, T^2, y^2)\, \rangle_{\mathbb{H}} = (\mathcal{D}\varepsilon(u^1), \varepsilon(u^2))_{\Omega} + (\mathcal{D}^{-1}T^1, T^2)_{\Omega} + (y^1, y^2)_{\Omega}\,. \tag{6.9}$$

By the assumption $|\Gamma_1| > 0$ the scalar product $\langle\cdot\,,\,\cdot\rangle_{\mathbb{H}}$ is equivalent to the standard scalar product.

The next step in our investigation is the cancellation of the given data (the external force and the boundary data). Hence, let us assume that our data b, γ_D, γ_N have the following regularity

$$b \in C^1(\mathbb{R}_+; L^2(\Omega;\mathbb{R}^3))\,, \tag{6.10}$$

$$\gamma_D \in C^1(\mathbb{R}_+; H^2_{1/2}(\Gamma_1;\mathbb{R}^3))\,, \quad \gamma_N \in C^1(\mathbb{R}_+; H^2_{-1/2}(\Gamma_2;\mathbb{R}^3)) \tag{6.11}$$

and denote by (u^*, T^*) the global in time solution of the linear problem

$$\operatorname{div}_x T^*(x,t) \;=\; -b(x,t)\,, \tag{6.12}$$
$$T^*(x,t) \;=\; \mathcal{D}\varepsilon(u^*(x,t))\,, \tag{6.13}$$
$$u^*(x,t)_{|\Gamma_1} \;=\; \gamma_D(x,t)\,, \quad T^*(x,t)\, n(x)_{|\Gamma_2} = \gamma_N(x,t)\,. \tag{6.14}$$

The solution of this problem has the regularity

$$u^* \in C^1(\mathbb{R}_+; H_1(\Omega;\mathbb{R}^3))\,,\ T^* \in C^1(\mathbb{R}_+; L^2(\Omega;\mathcal{S}^3))\,. \tag{6.15}$$

Thus for the differences $\tilde u = u - u^*$ and $\tilde T = T - T^*$ we obtain the following evolution problem in the space $\mathbb{H}(\Omega)$

$$(\tilde u_t, \tilde T_t, y_t) = \mathcal{A}(t)(\tilde u, \tilde T, y)\,, \tag{6.16}$$

where the operator $\mathcal{A}(t)$ is defined by:

$$\begin{aligned}
&\forall\ (v,S,w) \in \mathbb{H}(\Omega)\\
&\langle\, \mathcal{A}(t)(\tilde u,\tilde T,y),(v,S,w)\,\rangle_{\mathbb{H}} = (\mathcal{D}B\, f(\varepsilon(\tilde u)+\varepsilon(u^*(t)),\tilde z + z^*(t))\,,\, \varepsilon(v)) \qquad (6.17)\\
&-(B\, f(\varepsilon(\tilde u)+\varepsilon(u^*(t)),\tilde z + z^*(t))\,,\, S) + (B^\perp\, f(\varepsilon(\tilde u)+\varepsilon(u^*(t)),\tilde z + z^*(t))\,,\, w)\,,
\end{aligned}$$

where $\tilde z = (\varepsilon(\tilde u) - \mathcal{D}^{-1}\tilde T, y)$ and $z^*(t) = (\varepsilon(u^*(t)) - \mathcal{D}^{-1}T^*(t), 0)$. The initial value for the function y is known and equal to $B^\perp z^{(0)}$, but for the functions $\tilde u$ and $\tilde T$ we have to compute the initial values from the initial value for the plastic strain $\varepsilon_p(0) = B\, z^{(0)}$. Hence, $\tilde u(0)$ is the unique solution of the problem

$$\operatorname{div}_x \mathcal{D}(\varepsilon(\tilde u(0)) - \varepsilon_p(0)) = 0\,, \tag{6.18}$$

$$\tilde u(0)_{|\Gamma_1} = 0\,, \quad \mathcal{D}(\varepsilon(\tilde u(0)) - \varepsilon_p(0))\, n_{|\Gamma_2} = 0, \tag{6.19}$$

and we define $\tilde T(0) = \mathcal{D}(\varepsilon(\tilde u(0)) - \varepsilon_p(0))$.

Theorem 6.1 *For initial data $(\varepsilon_p(0), y(0)) \in L^2(\Omega;\mathbb{R}^N)$ the problem (6.16) possesses a global, unique solution*

$$(\tilde u, \tilde T, y) \in C^1(\mathbb{R}_+; \mathbb{H}(\Omega))\,. \tag{6.20}$$

Proof: Let us choose $t_1, t_2 \in \mathbb{R}_+$ and (u^1,T^1,y^1), (u^2,T^2,y^2), $(v,S,w) \in \mathbb{H}(\Omega)$. The Lipschitz continuity of the function f yields

$$\begin{aligned}
&\langle\, \mathcal{A}(t_1)(u^1,T^1,y^1) - \mathcal{A}(t_2)(u^2,T^2,y^2),(v,S,w)\,\rangle_{\mathbb{H}}\\
&\quad\le C\int_\Omega \Big\{|\varepsilon(u^1)-\varepsilon(u^2)| + |T^1-T^2| + |y^1-y^2| + |\varepsilon(u^*(t_1)) - \varepsilon(u^*(t_2))|\\
&\qquad\qquad + |T^*(t_1) - T^*(t_2)|\Big\}\cdot(|\varepsilon(v)| + |S| + |w|)\, dx\\
&\quad\le L\Big\{\|(u^1,T^1,y^1) - (u^2,T^2,y^2)\|_{\mathbb{H}}\\
&\quad + \sup_{\tau\in(t_1,t_2)} (\|\varepsilon(u_t^*(\tau))\|_{2,\Omega} + \|T_t^*(\tau)\|_{2,\Omega})|t_1-t_2|\Big\}\|(v,S,w)\|_{\mathbb{H}}\,,
\end{aligned}$$

where the constants C and L do not depend on t. Consequently the family of operators $\mathcal{A}(t)$ is continuous with respect to t and global Lipschitz in the space $\mathbb{H}(\Omega)$ with the Lipschitz constant L independent of t. Thus from the theory of differential equations in Banach spaces it follows that the evolution problem (6.16) possesses global in time, unique solutions having the regularity (6.20).

We use Theorem 6.1 to construct a sequence of approximate solutions to the problem

(1.10) - (1.13). To this end we replace g by the Yosida approximation $G_\lambda = \lambda^{-1}(I - J_\lambda)$, where $J_\lambda = (I + \lambda g)^{-1}$ and $\lambda > 0$. This yields the following sequence of approximate problems

$$\begin{aligned} \operatorname{div}_x T^\lambda(x,t) &= -b(x,t)\,, & (6.21)\\ T^\lambda(x,t) &= \mathcal{D}(\varepsilon^\lambda(x,t) - Bz^\lambda(x,t))\,, & (6.22)\\ \varepsilon^\lambda(x,t) &= \frac{1}{2}(\nabla_x u^\lambda(x,t) + \nabla_x^T u^\lambda(x,t))\,, & (6.23)\\ z_t^\lambda(x,t) &= G_\lambda\Big(B^T T^\lambda(x,t) - Lz^\lambda(x,t)\Big)\,, & (6.24)\\ z^\lambda(x,0) &= z^{(0)}(x) & (6.25) \end{aligned}$$

with the boundary condition (1.25). Assuming that the external force b and the boundary data γ_D, γ_N have the regularity required in (6.10), (6.11) and noting that G_λ is globally Lipschitz continuous we conclude from Theorem 6.1 that for all initial data $z^{(0)} \in L^2(\Omega;\mathbb{R}^N)$ the system (6.21) - (6.25) possesses global in time solution

$$(u^\lambda, T^\lambda, z^\lambda) = (\tilde{u}^\lambda, \tilde{T}^\lambda, z^\lambda) + (u^*, T^*, 0) \;\text{ with } (\tilde{u}^\lambda, \tilde{T}^\lambda, B^\perp z^\lambda) \in C^1(\mathbb{R}_+;\mathbb{H}(\Omega))\,.$$

Let us assume that $(\varepsilon, z) \in L^2(\Omega;\mathcal{S}^3 \times \mathbb{R}^N)$ and define the total energy associated with the problem (1.10) - (1.13) by

$$\mathcal{E}(\varepsilon,z) = \int_\Omega \rho\psi(\varepsilon,z)\,dx = \frac{1}{2}\int_\Omega \mathcal{D}(\varepsilon - Bz)\cdot(\varepsilon - Bz)\,dx + \frac{1}{2}\int_\Omega Lz\cdot z\,dx\,. \qquad (6.26)$$

Note that the positive definiteness of the matrix L implies that the right-hand side of (6.26) induces in the space $L^2(\Omega;\mathcal{S}^3 \times \mathbb{R}^N)$ a norm equivalent to the standard norm. This norm will be denoted by $\|(\varepsilon,z)\|_\psi$. Moreover, let us denote by $T^{(0)}(x)$ the initial stress obtained as the solution of the problem

$$\begin{aligned} \operatorname{div}_x T^{(0)}(x) &= -b(x,0)\,, & (6.27)\\ T^{(0)}(x) &= \mathcal{D}(\varepsilon(u^{(0)}(x)) - \varepsilon_p(0))\,, & (6.28)\\ u^{(0)}(x)_{|\Gamma_1} &= \gamma_D(x,0)\,, \quad T^{(0)}(x)\,n(x)_{|\Gamma_2} = \gamma_N(x,0)\,. & (6.29) \end{aligned}$$

Also, we set

$$\Delta(g) = \{h \in L^2(\Omega;\mathbb{R}^N)\,|\,\text{there is } \xi \in L^2(\Omega;\mathbb{R}^N) \text{ such that } \xi(x) \in g(h(x))\,, \text{a.e.}\}\,.$$

Theorem 6.2 *Suppose that the external force b and the boundary data γ_D, γ_N satisfy*

$$b \in H_2^\infty(0,T_e;L^2(\Omega;\mathbb{R}^3))\,, \qquad (6.30)$$

$$\gamma_D \in H_3^\infty(0,T_e;H^2_{1/2}(\Gamma_1;\mathbb{R}^3))\,, \quad \gamma_N \in H_2^\infty(0,T_e;H^2_{-1/2}(\Gamma_2;\mathbb{R}^3))\,, \qquad (6.31)$$

for all $T_e > 0$. Moreover, assume for the initial data $z^{(0)} \in L^2(\Omega;\mathbb{R}^N)$ that

$$B^T\,T^{(0)} - Lz^{(0)} \in \Delta(g)\,,$$

where the initial stress $T^{(0)}$ is defined as the solution of the system (6.27) - (6.29). Then for all $T_e > 0$ the solution of the approximate problem (6.21) - (6.25) satisfies the following inequality

$$\sup_{t\in(0,T_e)} \|(\varepsilon_t^\lambda(t), z_t^\lambda(t))\|_\psi \le C(T_e)\,,$$

where the constant $C(T_e)$ does not depend on λ.

Proof: Let us fix $T_e > 0$ and for $h > 0$ let us denote by $(\varepsilon_h^\lambda(t), z_h^\lambda(t))$ the shifted functions $(\varepsilon^\lambda(t+h), z^\lambda(t+h))$. Then for the time differences we obtain

$$\frac{d}{dt}\frac{1}{2}\|(\varepsilon_h^\lambda - \varepsilon^\lambda, z_h^\lambda - z^\lambda)\|_\psi^2 = \int_\Omega \mathcal{D}(\varepsilon_h^\lambda - \varepsilon^\lambda - Bz_h^\lambda + Bz^\lambda)\cdot(\varepsilon_{h,t}^\lambda - \varepsilon_t^\lambda - Bz_{h,t}^\lambda + Bz_t^\lambda)\,dx$$

$$+ \int_\Omega L(z_h^\lambda - z^\lambda)\cdot(z_{h,t}^\lambda - z_t^\lambda)\,dx = \quad \text{(using the notation } v^\lambda = u_t^\lambda)$$

$$= \int_\Omega (v_h^\lambda - v^\lambda)\cdot(b_h - b)\,dx + \int_{\partial\Omega}(v_h^\lambda - v^\lambda)\cdot(T_h^\lambda - T^\lambda)\,n\,d\sigma$$

$$- \int_\Omega \Big\{G_\lambda(B^T T_h^\lambda - Lz_h^\lambda) - G_\lambda(B^T T^\lambda - Lz^\lambda)\Big\}\cdot((B^T T_h^\lambda - Lz_h^\lambda) - (B^T T^\lambda - Lz^\lambda))\,dx$$

$$\leq \text{(by the monotonicity of the function } G_\lambda)$$

$$\leq \int_\Omega (v_h^\lambda - v^\lambda)\cdot(b_h - b)\,dx + \int_{\partial\Omega}(v_h^\lambda - v^\lambda)\cdot(T_h^\lambda - T^\lambda)\,n\,d\sigma\,. \qquad (6.32)$$

The boundary integral is defined in the following sense

$$\int_{\partial\Omega}(v_h^\lambda - v^\lambda)\cdot(T_h^\lambda - T^\lambda)\,n\,d\sigma = \langle\,(T_h^\lambda - T^\lambda)\,n\,,\,(v_h^\lambda - v^\lambda)\,\rangle_{\partial\Omega}\,, \qquad (6.33)$$

where the brackets $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denote the duality form between the spaces $H^2_{1/2}(\partial\Omega;\mathbb{R}^3)$ and $H^2_{-1/2}(\partial\Omega;\mathbb{R}^3)$. Using the boundary conditions we get

$$\frac{d}{dt}\frac{1}{2}\|(\varepsilon_h^\lambda - \varepsilon^\lambda, z_h^\lambda - z^\lambda)\|_\psi^2 \;\leq\; \langle\,(\gamma_{N,h} - \gamma_N)\,,\,(v_h^\lambda - v^\lambda)\,\rangle_{\Gamma_2} \qquad (6.34)$$

$$+\langle\,(T_h^\lambda - T^\lambda)\,n\,,\,(\partial_t\gamma_{D,h} - \partial_t\gamma_D)\,\rangle_{\Gamma_1} + \int_\Omega (v_h^\lambda - v^\lambda)\cdot(b_h - b)\,dx\,.$$

Next we integrate (6.34) with respect to t, divide by h^2 and shift the difference operators from the velocity and from the stress to the given data (for more details see [CG00], Theorem 4). Finally we pass to the limit $h \to 0^+$ and arrive at the inequality

$$\|(\varepsilon_t^\lambda(t), z_t^\lambda(t))\|_\psi^2 \leq \|(\varepsilon_t^\lambda(0), z_t^\lambda(0))\|_\psi^2 + 2\int_0^t \|\gamma_{N,tt}(\tau)\|_{-1/2,2,\Gamma_2}\|v^\lambda(\tau)\|_{1/2,2,\partial\Omega}\,d\tau$$

$$+2\int_0^t \|T^\lambda(\tau)\,n\|_{-1/2,2,\partial\Omega}\|\gamma_{D,ttt}(\tau)\|_{1/2,2,\Gamma_1}\,d\tau + 2\int_0^t \|v^\lambda(\tau)\|_{2,\Omega}\|b_{tt}(\tau)\|_{2,\Omega}\,d\tau$$

$$+C\Big\{\sup_{t\in(0,T_e)}\|\gamma_{N,t}(t)\|_{-1/2,2,\Gamma_2}\sup_{t\in(0,T_e)}\|v^\lambda(t)\|_{1/2,2,\partial\Omega}$$

$$+\sup_{t\in(0,T_e)}\|\gamma_{D,tt}(t)\|_{1/2,2,\Gamma_1}\sup_{t\in(0,T_e)}\|T^\lambda(t)\,n\|_{-1/2,2,\partial\Omega}$$

$$+\sup_{t\in(0,T_e)}\|v^\lambda(t)\|_{2,\Omega}\sup_{t\in(0,T_e)}\|b_t(t)\|_{2,\Omega}\Big\}\,, \qquad (6.35)$$

where the constant $C > 0$ is independent of λ. The boundary norms $\|v^\lambda\|_{1/2,2,\partial\Omega}$ and $\|T^\lambda\,n\|_{-1/2,2,\partial\Omega}$ appearing in the right-hand side of inequality (6.35) can be estimated using the continuity of the trace operator

$$\|v^\lambda\|_{1/2,2,\partial\Omega} \;\leq\; C\,\|v^\lambda\|_{1,2,\Omega}\,, \qquad (6.36)$$

$$\|T^\lambda\,n\|_{-1/2,2,\partial\Omega} \;\leq\; C\,(\|T^\lambda\|_{2,\Omega} + \|\operatorname{div} T^\lambda\|_{2,\Omega})\,, \qquad (6.37)$$

where the constant $C > 0$ depends on the domain Ω only. Inserting (6.36) and (6.37) into (6.35) yields

$$\|(\varepsilon_t^\lambda(t), z_t^\lambda(t))\|_\psi^2 \le \|(\varepsilon_t^\lambda(0), z_t^\lambda(0))\|_\psi^2 + C(T_e,\alpha) + \alpha \sup_{t\in(0,T_e)} \|v^\lambda(t)\|_{1,2,\Omega}^2 + \alpha \sup_{t\in(0,T_e)} \|T^\lambda(t)\|_{2,\Omega}^2 \,, \tag{6.38}$$

where $\alpha > 0$ is any positive number and the constant $C(T_e, \alpha)$ does not depend on λ. Thus we have only to estimate the L^2-norm of the stress and the H_1-norm of the velocity. The first one can be estimated by the time derivative of the stress

$$\|T^\lambda(t)\|_{2,\Omega} \le \|T^\lambda(0)\|_{2,\Omega} + \sqrt{t}\int_0^t \|T_t^\lambda(\tau)\|_{2,\Omega}\, d\tau \,, \tag{6.39}$$

and the second one can be estimated using the ellipticity of the equation of motion

$$\|v^\lambda(t)\|_{1,2,\Omega} \le C\,(\|b_t(t)\|_{-1,2,\Omega} + \|\varepsilon_{p,t}^\lambda(t)\|_{2,\Omega} + \|\gamma_{D,t}(t)\|_{1/2,2,\Gamma_1} + \|\gamma_{N,t}(t)\|_{-1/2,2,\Gamma_2})\,, \tag{6.40}$$

where the constant $C > 0$ depends on the set Ω only. Inserting (6.39) and (6.40) into (6.38) and choosing α sufficiently small we arrive at the inequality

$$\|(\varepsilon_t^\lambda(t), z_t^\lambda(t))\|_\psi^2 \le C(T_e)\,(\|(\varepsilon_t^\lambda(0), z_t^\lambda(0))\|_\psi^2 + 1)\,. \tag{6.41}$$

Finally, the assumption $B^T T^{(0)} - Lz^{(0)} \in \Delta(g)$ and the properties of the Yosida approximation (see [AC84] paragraph 3) imply that the sequence $\|(\varepsilon_t^\lambda(0), z_t^\lambda(0))\|_\psi$ is bounded. The proof is complete.

From Theorem 6.2 we conclude that the sequence $\{(\varepsilon^\lambda, z^\lambda)\}_{\lambda>0}$ is bounded in $H_1^\infty(L^2)$, and this together with (6.22) implies that the sequence $\{T^\lambda\}_{\lambda>0}$ is bounded in $H_1^\infty(L^2)$. Hence, we can pass to the weak limit $\lambda \to 0^+$ in the system (6.21) – (6.24) and obtain that the limit functions satisfy the following system of equations

$$\operatorname{div}_x T(x,t) = -b(x,t), \tag{6.42}$$

$$T(x,t) = \mathcal{D}(\varepsilon(x,t) - Bz(x,t)), \tag{6.43}$$

$$\varepsilon(x,t) = \frac{1}{2}(\nabla_x u(x,t) + \nabla_x^T u(x,t)), \tag{6.44}$$

$$z_t(x,t) = \chi(x,t), \tag{6.45}$$

where $\chi(x,t) = w - \lim_{\lambda\to 0^+} G_\lambda(B^T T^\lambda - Lz^\lambda)$. To end the proof of existence of global solutions to our system (1.10) – (1.13) it remains to show that

$$\chi(x,t) \in g(B^T T(x,t) - Lz(x,t)) \;\text{ for a.e. } (x,t) \in Z\,. \tag{6.46}$$

Note that by the L^2- weak-strong closedness of the graph of the maximal monotone operator g the inclusion (6.46) follows immediately from the strong L^2-convergence of the sequence $\{B^T T^\lambda - Lz^\lambda)\}_{\lambda>0}$.

Theorem 6.3 *Suppose that the boundary data γ_D, γ_N, the external force b and the initial data z^0 satisfy the assumptions from Theorem 6.2. Then for all $\lambda, \mu > 0$ and for all $T_e > 0$ the inequality*

$$\|(\varepsilon^\lambda - \varepsilon^\mu, z^\lambda - z^\mu)\|_\psi^2 \le \frac{1}{2}(\lambda+\mu)C^2(T_e)T_e \tag{6.47}$$

holds, where $C(T_e)$ is the constant from Theorem 6.2.

Proof: The proof of this theorem is standard (see for example the proof of Theorem 3.1, p. 54 in [Bré73]). Nevertheless, for completeness we present it here. Proceeding similarly as in the proof of Theorem 6.2, we obtain for the difference of two approximation steps $(\varepsilon^\lambda - \varepsilon^\mu, z^\lambda - z^\mu)$ that

$$\frac{d}{dt}\frac{1}{2}\|(\varepsilon^\lambda - \varepsilon^\mu, z^\lambda - z^\mu)\|_\psi^2 = \tag{6.48}$$
$$-\int_\Omega \Big\{ G_\lambda(B^T T^\lambda - Lz^\lambda) - G_\mu(B^T T^\mu - Lz^\mu) \Big\} ((B^T T^\lambda - Lz^\lambda) - (B^T T^\mu - Lz^\mu))\, dx\,.$$

Let us denote by $w^i = B^T T^i - Lz^i$ for $i = \lambda, \mu$. Using $G_\lambda = \lambda^{-1}(I - J_\lambda)$ we obtain

$$\begin{aligned}
&-\int_\Omega (w^\lambda - w^\mu)(G_\lambda(w^\lambda) - G_\mu(w^\mu))\, dx \\
&\quad = -\int_\Omega (\lambda G_\lambda(w^\lambda) + J_\lambda(w^\lambda) - \mu G_\mu(w^\mu) - J_\mu(w^\mu)) \cdot (G_\lambda(w^\lambda) - G_\mu(w^\mu))\, dx \\
&\quad \leq \text{(by the inclusion } G_i(w^i) \in g(J_i(w^i))) \qquad (6.49)\\
&\quad \leq -\int_\Omega (\lambda G_\lambda(w^\lambda) - \mu G_\mu(w^\mu)) \cdot (G_\lambda(w^\lambda) - G_\mu(w^\mu))\, dx \\
&\quad \leq \frac{1}{4}(\lambda + \mu)(\|G_\lambda(w^\lambda)\|_{2,\Omega}^2 + \|G_\mu(w^\mu)\|_{2,\Omega}^2)\,.
\end{aligned}$$

We use Theorem 6.2 and equation (6.24) to see that $\|G_i(w^i)\|_{2,\Omega}$ is uniformly bounded for $0 \leq t \leq T_e$. Insertion of (6.49) into (6.48) and integration with respect to t ends the proof.

Proof of Theorem 1.4: From Theorem 6.3 we conclude that the sequences $\{\varepsilon^\lambda\}_{\lambda>0}$ and $\{z^\lambda\}_{\lambda>0}$ are L^2-Cauchy sequences (note that the energy norm $\|\cdot\|_\psi$ is equivalent to the standard L^2 norm). Consequently, the sequence $\{B^T T^\lambda - Lz^\lambda\}_{\lambda>0}$ is a Cauchy sequence in $L^\infty(L^2)$ which implies that the inclusion (6.46) holds. Therefore a solution with the properties stated in Theorem 1.4 exists. Uniqueness follows as in the proof of Theorem 1.3 in Section 3.

Appendix

In this appendix we study the transformation of constitutive equations.

Lemma A1 *Let*

$$\begin{aligned}
T &= \mathcal{D}(\varepsilon - Bz) && \text{(A1)}\\
z_t &= g(-\rho\nabla_z\psi(\varepsilon, z)) = g(B^T T - Lz) && \text{(A2)}\\
z(0) &= z^{(0)} && \text{(A3)}
\end{aligned}$$

be constitutive equations of monotone type with a linear mapping $B : \mathbb{R}^N \to \mathcal{S}^3$ and with a linear, symmetric, positive semi-definite mapping $L : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\dim(\ker B) = N - 6, \quad \dim(\ker L) = 6, \quad \ker B + \ker L = \mathbb{R}^N. \tag{A4}$$

Then these constitutive equations can be transformed to equations of the form (1.27) – (1.30) *with a monotone vector field $\tilde{g} = (\tilde{g}_1, \tilde{g}_2)$.*

Conversely, equations of the form (1.27) – (1.30) *with a monotone vector field* $\tilde{g}$ *can be transformed to constitutive equations of the form* (A1) – (A3)*, which are of monotone type with mappings* B *and* L *satisfying* (A4).

Proof: We first consider a linear isomorphism $V = (V_1, V_2) : \mathbb{R}^N \to \mathcal{S}^3 \times \mathbb{R}^{N-6}$. The adjoint mapping $V^T : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathbb{R}^N$ is given by

$$V^T(T, y) = V_1^T T + V_2^T y\,,$$

for all $(T, y) \in \mathcal{S}^3 \times \mathbb{R}^{N-6}$. We use this isomorphism to assign to a vector field $g : \mathbb{R}^N \to \mathbb{R}^N$ another vector field $\tilde{g} = (\tilde{g}_1, \tilde{g}_2) : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathcal{S}^3 \times \mathbb{R}^{N-6}$ by

$$\tilde{g} = V \circ g \circ V^T\,.$$

This defines an invertible mapping between vector fields. The inverse is

$$\tilde{g} \mapsto g = V^{-1} \circ \tilde{g} \circ V^{-T}.$$

Clearly, this mapping and the inverse map monotone vector fields to monotone vector fields: g is monotone if and only if $\tilde{g}$ is monotone. Moreover, it is obvious that a function $(\varepsilon, z) : Z \to \mathcal{S}^3 \times \mathbb{R}^N$ satisfies the equations

$$T = \mathcal{D}(\varepsilon - V_1 z), \tag{A5}$$

$$z_t = g(V_1^T T - V_2^T V_2 z), \tag{A6}$$

$$z(0) = z^{(0)}, \tag{A7}$$

if and only if the function $(\varepsilon, Vz) = (\varepsilon, V_1 z, V_2 z) : Z \to \mathcal{S}^3 \times \mathcal{S}^3 \times \mathbb{R}^{N-6}$ satisfies the equations

$$\begin{aligned} T &= \mathcal{D}(\varepsilon - V_1 z)\,, \\ (Vz)_t = Vg(V_1^T T - V_2^T V_2 z) &= Vg(V^T(T, -V_2 z)) = \tilde{g}(T, -V_2 z)\,, \\ Vz(0) &= Vz^{(0)}\,. \end{aligned}$$

With the notation $\varepsilon_p = V_1 z$, $\tilde{z} = V_2 z$ the latter system can be written in the form

$$T = \mathcal{D}(\varepsilon - \varepsilon_p) \tag{A8}$$

$$\varepsilon_{p_t} = \tilde{g}_1(T, -\tilde{z}) \tag{A9}$$

$$\tilde{z}_t = \tilde{g}_2(T, -\tilde{z}) \tag{A10}$$

$$\varepsilon_p(0) = V_1 z^{(0)}, \quad \tilde{z}(0) = V_2 z^{(0)}\,. \tag{A11}$$

Hence, these equations, which coincide with (1.27) – (1.30), are equivalent to the equations (A5) – (A7).

Assume now that g, B and L are given such that (A4) holds and such that (A1), (A2) are constitutive equations of monotone type. We specialize V such that (A5), (A6) become equal to (A1), (A2). Note first that L is symmetric and positive semi-definite. Therefore there exists the symmetric and positive semi-definite square root $L^{1/2}$. Let $C : \mathbb{R}^N \to \mathbb{R}^{N-6}$ be a surjective linear mapping with

$$\ker C = \ker L\,,$$

such that $C|_{R(L)} : R(L) \to \mathbb{R}^{N-6}$ is an isometric isomorphism. With this mapping define $V : \mathbb{R}^N \to \mathcal{S}^3 \times \mathbb{R}^{N-6}$ by

$$Vz = (V_1 z, V_2 z) = (Bz, CL^{1/2}z)\,. \tag{A12}$$

This mapping is invertible, since

$$\ker V = \ker B \cap \ker(CL^{1/2}) = \ker B \cap \ker L^{1/2} = \ker B \cap \ker L = \{0\}\,,$$

by (A4). Hence, V is a linear isomorphism. Moreover, because $\ker C = \ker L = \ker L^{1/2}$ and because $C|_{R(L)} : R(L) \to \mathbb{R}^{N-6}$ is an isometric isomorphism, it follows that $C^T C : \mathbb{R}^N \to \mathbb{R}^N$ is the orthogonal projection onto $(\ker L)^\perp = R(L) = R(L^{1/2})$. Therefore $I - C^T C$ is the projection to $\ker L^{1/2}$, which implies that

$$L = L^{1/2}(C^T C + (I - C^T C))L^{1/2} = L^{1/2} C^T C L^{1/2} = V_2^T V_2\,,$$

by (A12).

We insert this equation into (A6) and use $V_1 = B$ to see that with this definition of V the system (A5) – (A7) takes the form of the equations (A1) – (A3). Hence (A1) – (A3) can be transformed to the equations (A8) – (A11), where $\varepsilon_p = Bz$, $\tilde{z} = CL^{1/2}z$.

Conversely, assume that a monotone vector field $\tilde{g} = (\tilde{g}_1, \tilde{g}_2) : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathcal{S}^3 \times \mathbb{R}^{N-6}$ is given. Then for any isomorphism $V = (V_1, V_2)$ the equations (A8) – (A11) are equivalent to (A5) – (A7) with $g = V^{-1} \circ \tilde{g} \circ V^{-T}$, and (A5) – (A7) take the form of (A1) – (A3) if we define B and L by

$$B = V_1, \quad L = V_2^T V_2\,.$$

Note that L is symmetric and positive semi-definite, and that these mappings satisfy (A4). Therefore the equations (1.27) – (1.30) can always be written in the form of constitutive equations of monotone type with positive semi-definite free energy.

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