H^∞-calculus for the Stokes operator on L_q-spaces

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Abstract

It is proved that the Stokes operator on a bounded domain, an exterior domain, or a perturbed half-space Ω admits a bounded H^∞-calculus on $L_q(\Omega)$ if $q \in (1, \infty)$.

1 Introduction

Let A_Ω be the Stokes operator in the Banach space $L_{q,\sigma}(\Omega)$ of all q-integrable solenoidal vector fields on a domain $\Omega \subset \mathbb{R}^n$. In this article we show that A_Ω admits a bounded H^∞-calculus for a fairly large class of domains Ω and for all $q \in (1, \infty)$. For an arbitrary Banach space X, the class $H^\infty(X)$ of all operators admitting a bounded H^∞-calculus has been studied by many authors [McI86], [CDMY96], [Frö98], [DHP01a]. Since it is contained in $\mathrm{BIP}(X)$, the class of all operators having bounded imaginary powers, it enjoys all properties of this larger class. For further information in this direction see [PS93], [MP97] and [DV87]. For instance, the domain of fractional powers can be determined in terms of a complex interpolation space. Another reason is the maximal L^q-regularity of the associated evolution equation $u_t + Au(t) = f(t)$. However, there are also useful properties which do not hold true for operators in $\mathrm{BIP}(X)$ but which are valid for operators belonging to $H^\infty(X)$. Among those let us mention that $\mathrm{BIP}(X)$ is not stable under small perturbations. In fact, there seem to be only restrictive perturbation results known, [PS93]. However, there is a perturbation result for the class $H^\infty(X)$, whose assumptions can be verified in the particular case of the Stokes operator.

In 1981 Giga [Gig81] investigated the analyticity of the Stokes semigroup. In a subsequent paper [Gig85] he considered domains of fractional powers of the Stokes operator and proved that the Stokes operator on a bounded C^∞-domain has bounded imaginary powers. Consequently, it has maximal L^q-regularity. In [GS91], it has been shown, that one can also obtain global in time $L^q - L^s$ estimates. The paper in hand extends the results of [Gig85] and several ways. By checking the details in Giga's proof one realizes that it is possible to generalize that result to the H^∞-case. This leads to a proof for the bounded H^∞-calculus for such domains. Our approach, however, is more direct and fairly self-contained, whereas Giga's proof makes heavy use of pseudodifferential operators and Seeley's theory on the descripton of fractional powers of an elliptic system [See71]. Moreover, our result includes unbounded domains which might be of independent interest as well as domains with merely C^3 boundary. More precisely, exterior domains and perturbed half-spaces can be handled.

One can also treat the problem of extending the property of having bounded imaginary powers to the allegedly stronger property of admitting a bounded H^∞-calculus by purely functional analytic methods. This has recently be carried out by Kalton and Weis [KW].

It is known that the class of all operators admitting a bounded H^∞-calculus coincides with the (a priori smaller) class of all operators admitting an R-bounded H^∞-calculus if the underlying Banach space has property (α), see [KW01] and [CdPSW00]. Since the space $L_{q,\sigma}(\Omega)$ is known to enjoy this property for any domain Ω and any $q \in [1, \infty]$, we can immediately conclude that A_Ω even admits an R-bounded H^∞-calculus for the domains treated in Section 3. This is relevant for perturbations of the Stokes operator for the following reason: The classical theorem of Dore and Venni [DV87] yields closedness of the operator sum $A + B$ if X is a UMD space, both A and B belong to $\mathrm{BIP}(X)$, the resolvents of A and B commute and the sum of the power angles is less than π. Recently, Kalton and Weis [KW] proved an "assymetric" version of this theorem, where A is merely assumed to be sectorial, but B admits an R-bounded H^∞-calculus.

Our strategy of proving that the Stokes operator A_Ω admits a bounded H^∞-calculus in $L_{q,\sigma}(\Omega)$ is to apply the perturbation result for the bounded H^∞-calculus to the Stokes operator on the bent half-space. Then we localize the original problem on Ω: Cover Ω by finitely many balls and treat each ball separately. Those balls which are entirely contained in Ω turn out to be easy to handle by transforming the problem to $\mathbb{R}^n$. On the other hand, if a ball meets the boundary of Ω, it is possible to reduce the problem to the bent half-space case. It is therefore enough to know that the Stokes operator on the bent half-space admits a bounded H^∞-calculus. Since it is already known [DHP01b] that the Stokes operator on the half-space $\mathbb{R}^n_+$ admits a bounded H^∞-calculus it is quite natural to introduce an invertible transformation which maps the bent half-space onto $\mathbb{R}^n_+$. This change of coordinates leads to a transformation A_T of the corresponding Stokes operator. By choosing the radii of the aforementioned balls small enough, the bending function is as close to zero as we please. This implies that also A_T is close to $A_{\mathbb{R}^n_+}$ in the sense of a recent perturbation result for H^∞-calculus due to Prüss. Therefore A_T must also have a bounded H^∞-calculus which yields the result.

The article is organized as follows. In Section 2 we fix notation and recall some auxillary tools on Stokes operators, interpolation theory and H^∞-calculus that will be needed in subsequent sections. Section 3 contains our main results. We start in Section 3.1 by explaining the transition from the Stokes operator on the bent half-space to the operator A_T mentioned above. The subsequent sections contain the proof of the bounded H^∞-calculus for the Stokes operator on the bent half-space, the bounded domain and the perturbed half-space respectively. Finally, we provide two appendices on regularity properties of the Helmholtz projection and on the domain of fractional powers of the Stokes operator. These appendices contain auxillary material which seems not to be contained in the standard literature.

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2 Preliminaries

2.1 Notation

Throughout the article we assume that $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $m \in \mathbb{N}$. By $C^m(\Omega)$ we denote the space of all m-times continuously differentiable functions and by $C^m_c(\Omega)$ its subspace consisting of all functions in $C^m(\Omega)$ which are compactly supported. Further, let $C^\infty_c(\overline{\Omega}) := \{u \restriction_\Omega : u \in C^\infty_c(\mathbb{R}^n)\}$, and denote by $C^m_b(\Omega)$ the Banach space of all m-times continuously differentiable functions whose derivatives up to order m are bounded. For $q \in [1, \infty]$, $L_q(\Omega)$ denotes the usual Lebesgue space of all q-integrable functions and for $s \in \mathbb{R}$, $W^{s,q}(\Omega)$ is the Sobolev space of order s. If $s = m \in \mathbb{N}$ and $q \in (1, \infty)$, the norm in $W^{s,q}(\Omega)$ is given by $\|u\|_{m,q} := \left(\sum_{j=0}^m \int_\Omega |\nabla^j u|^q \mathrm{d}x\right)^{1/q}$, where ∇^j is the vector of all possible j-th order differentials. Moreover, $W^{s,q}_0(\Omega)$ denotes the closure of $C^\infty_c(\Omega)$ in $W^{s,q}(\Omega)$. We shall further need the homogenous Sobolev space $\hat{W}^{1,q}(\Omega)$ consisting of all functions u having finite Dirichlet energy $\int_\Omega |\nabla u|^q \mathrm{d}x$, modulo constants. It becomes a Banach space when equipped with the norm

$$\|u\|_{\hat{W}^{1,q}(\Omega)} := \left(\int_\Omega |\nabla u|^q \mathrm{d}x\right)^{1/q}.$$

Its dual space $(\hat{W}^{1,q}(\Omega))'$ will occur frequently and is denoted by $\hat{W}^{-1,q'}(\Omega)$, where q' is the Hölder conjugated exponent given by $1/q + 1/q' = 1$ and $\|\cdot\|_{-1,q}$ always denotes the norm in this space. For further properties of these spaces, in particular for the proof of the density of $C^\infty_c(\overline{\Omega})$ in $\hat{W}^{1,q}(\Omega)$, we refer to [FS94]. If $\partial\Omega$ is smooth enough, the trace operator defined by $\gamma(u) := u \restriction_{\partial\Omega}$ maps $W^{s,q}(\Omega)$ continuously into $W^{s-1/q,q}(\partial\Omega)$. Its kernel is exactly the space $W^{s,q}(\Omega) \cap W^{1,q}_0(\Omega)$. See [Ada78], p. 215. For $u \in L_q(\Omega)$ and $v \in L_{q'}(\Omega)$ we use the standard notation $(u, v)_\Omega := \int_\Omega uv \mathrm{d}x$.

Let us remark that we will use the same notations for the corresponding spaces of vector fields on Ω. For a domain $\Omega \subset \mathbb{R}^n$ denote by $L_{q,\sigma}(\Omega)$ the space of all q-integrable solenoidal vector fields on Ω. For the class of domains treated in this article (see Section 2.2 for the precise definition) is well-known that there is a compatible family $(P_{\Omega,q})_{q\in(1,\infty)}$ of continuous projections from $L_q(\Omega)$ onto $L_{q,\sigma}(\Omega)$ such that $P_{\Omega,2}$ is orthogonal. For the proofs, see [FM77], [McC81], [Miy82], [BM88], [ST98]. The operator $P_{\Omega,q}$ is called the *Helmholtz projection*. Since we restrict ourselves to those values of q and q remains fixed throughout the article, we shall write P_Ω for short. Clearly, the range $G_q(\Omega) := (1 - P_\Omega)(L_q(\Omega))$ is also a closed subspace of $L_q(\Omega)$.

If X and Y are Banach spaces, the space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$, and $\mathcal{L}(X)$ is an abbreviation for $\mathcal{L}(X,X)$. For any closed operator A in X, its domain and range are denoted by $\mathrm{dom}(A)$ and $\mathrm{ran}(A)$ respectively. Its resolvent set is denoted by $\rho(A)$ and its spectrum by $\sigma(A)$.

Finally, Δ_Ω denotes the Dirichlet Laplacian in $L_q(\Omega)$, defined on $W_0^{1,q}(\Omega)\cap W^{2,q}(\Omega)$, and $A_\Omega = -P_\Omega\Delta_\Omega$ is the Stokes operator in $L_{q,\sigma}(\Omega)$, defined on $W_0^{1,q}(\Omega)\cap W^{2,q}(\Omega)\cap L_{q,\sigma}(\Omega)$. For details on the Stokes operator and on the Navier-Stokes equation we refer to the textbooks [Gal98] and [Soh01].

2.2 A priori estimates for the generalized Stokes resolvent problem

We will frequently make use of an inequality for the solution (u,p) of the generalized Stokes resolvent problem

$$(SRP)^\Omega_{f,g}\begin{cases} \lambda u - \Delta u + \nabla p &= f \quad \text{on} \quad \Omega,\\ \nabla\cdot u &= g \quad \text{on} \quad \Omega,\\ \gamma u &= 0, \end{cases}$$

where Ω is a C^3-domain which is either bounded, exterior, $\mathbb{R}^n$, a bent half-space or a perturbed half-space. In [FS94], Farwig and Sohr proved the following theorem.

Theorem 2.1 *Let $1<q<\infty$, $0<\theta<\pi$, $n\geq 2$, $\delta>0$. Let $f\in L_q(\Omega)$, $g\in W^{1,q}(\Omega)\cap\hat{W}^{-1,q}(\Omega)$ if Ω is unbounded or $g\in W^{1,q}(\Omega)$ with $\int_\Omega g\,\mathrm{d}x = 0$ if Ω is bounded. Then there is a unique solution $(u,p)\in\mathrm{dom}(\Delta_\Omega)\times\hat{W}^{1,q}(\Omega)$ of $(SRP)^\Omega_{f,g}$ and some constant $C=C(\Omega,q,\theta,\delta)>0$ such that*

$$\|\lambda u\|_q + \|\nabla^2 u\|_q + \|\nabla p\|_q \leq C(\|f\|_q + \|\nabla g\|_q + \|\lambda g\|_{-1,q})$$

and

$$\|\lambda u\|_q + \|-\Delta u + \nabla p\|_q \leq C(\|f\|_q + \|\lambda g\|_{-1,q})$$

for all $\lambda\in\Sigma_{\pi-\theta} := \{z\in\mathbb{C}\setminus\{0\} : |\arg z| < \pi-\theta\}$ with $|\lambda|\geq\delta$. Moreover, if $\Omega=\mathbb{R}^n$ or $\Omega=\mathbb{R}^n_+$ or Ω is bounded, then C is independent of δ.

2.3 An interpolation property for the domain of the Dirichlet Laplacian

We will frequently make use of the following interpolation property for the Dirichlet Laplacian in $L_q(\Omega)$: If $1<q<\infty$, $0<\alpha<1/2q$ and Ω is as in Section 2.2, then

$$[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_\alpha = W^{2\alpha,q}(\Omega), \tag{1}$$

where $[\cdot,\cdot]_\alpha$ denotes complex interpolation of order α. This can be seen as follows: It is well-known, see [Tri78], that $[L_q(\Omega),W_0^{s,q}(\Omega)]_\alpha = W_0^{\alpha s,q}(\Omega)$ and $[L_q(\Omega),W^{s,q}(\Omega)]_\alpha = W^{\alpha s,q}(\Omega)$ for all $\alpha\in[0,1]$ and all $s>0$. The obvious inclusion $W_0^{2,q}(\Omega)\subset\mathrm{dom}(\Delta_\Omega)\subset W^{2,q}(\Omega)$ therefore implies

$$W_0^{1,q}(\Omega) = [L_q(\Omega),W_0^{2,q}(\Omega)]_{1/2}\subset[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_{1/2}\subset[L_q(\Omega),W^{2,q}(\Omega)]_{1/2} = W^{1,q}(\Omega).$$

In particular, the norm in $[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_{1/2}$ is equivalent to $\|\cdot\|_{1,q}$. By [Tri78], Theorem 1.9.3/1 (c), $\mathrm{dom}(\Delta_\Omega)$ is dense in $[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_{1/2}$. Therefore we also have

$$\begin{aligned}[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_{1/2} &= \overline{\mathrm{dom}(\Delta_\Omega)}^{\|\cdot\|_{[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_{1/2}}} = \overline{\mathrm{dom}(\Delta_\Omega)}^{\|\cdot\|_{1,q}}\subset\overline{W_0^{1,q}(\Omega)}^{\|\cdot\|_{1,q}}\\ &= W_0^{1,q}(\Omega),\end{aligned}$$

i.e., we have $[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_{1/2} = W_0^{1,q}(\Omega)$. The reiteration property, [Tri78] Remark 1.9.3/1, gives us

$$[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_\alpha = [L_q(\Omega),[L_q(\Omega),\mathrm{dom}(\Delta_\Omega)]_{1/2}]_{2\alpha} = [L_q(\Omega),W_0^{1,q}(\Omega)]_{2\alpha} = W_0^{2\alpha,q}(\Omega),$$

but $W^{2\alpha,q}(\Omega) = W_0^{2\alpha,q}(\Omega)$ by our assumption on α, see again [Tri78], Theorem 4.3.2/1 (a).

2.4 Operators with bounded H^∞-calculus

Recall that a closed operator A on a complex Banach space X is called *sectorial*, if it satisfies the following two conditions:

(i) A is densely defined, injective and has dense range,

(ii) $(-\infty, 0) \subset \rho(A)$ and there is some $M \geq 0$ such that $\|\lambda(\lambda + A)^{-1}\| \leq M$ for all $\lambda > 0$.

In this case there is some $\phi \in [0, \pi)$ such that the sector

$$\Sigma_{\pi-\phi} := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi - \phi\}.$$

is contained in $\rho(-A)$, and $\sup\{|\lambda(\lambda + A)^{-1}| : \lambda \in \Sigma_{\pi-\phi}\} < \infty$. The smallest such ϕ is called the *spectral angle* of A and is denoted by ϕ_A. Oberserve that $\sigma(A) \setminus \{0\} \subset \Sigma_{\phi_A}$. Moreover, if A is sectorial, and $\phi_A \leq \frac{\pi}{2}$, it generates a bounded and holomorphic C_0-semigroup on X. For instance, the Stokes operator in $L_{q,\sigma}(\Omega)$ generates a bounded and holomorphic semigroup for all domains treated in this article.

A special class of sectorial operators on which we will focus throughout the article is the set of operators which admit a bounded H^∞-calculus. Before we can introduce these operators we need to define for $\phi \in (0, \pi)$ the space

$$\mathcal{H}^\infty(\Sigma_\phi) := \{h : \Sigma_\phi \to \mathbb{C} : h \text{ is holomorphic and bounded}\}$$

as well as its subspace $\mathcal{H}_0^\infty(\Sigma_\phi)$ given by

$$\mathcal{H}_0^\infty(\Sigma_\phi) := \{h \in \mathcal{H}^\infty(\Sigma_\phi) : |h(z)| \leq C\frac{|z|^s}{1 + |z|^{2s}} \text{ for some } C \geq 0, s > 0\}. \tag{2}$$

Let A be a sectorial operator on X with spectral angle ϕ_A, and let $\phi \in (\phi_A, \pi)$ and $\theta \in (\phi_A, \phi)$. The path

$$\Gamma : \mathbb{R} \to \mathbb{C}, \quad \Gamma(t) := \begin{cases} -t\mathrm{e}^{i\theta} & , t < 0, \\ t\mathrm{e}^{-i\theta} & , t \geq 0, \end{cases} \tag{3}$$

stays in the resolvent set of A with the only possible exception at $t = 0$. In view of Cauchy's integral formula, for $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$, we may define $h(A)$ by the Bochner integral

$$h(A) := \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - A)^{-1}\mathrm{d}\lambda, \tag{4}$$

which exists according to (2). A is said to *admit a bounded H^∞-calculus*, if there is some $C \geq 0$ with

$$\|h(A)x\| \leq C\|h\|_\infty\|x\| \tag{5}$$

for all $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$ and all $x \in X$. The smallest possible ϕ for which inequality (5) holds is called the *$\mathcal{H}^\infty$-angle* of A and is denoted by ϕ_A^∞. Clearly, we always have $\phi_A^\infty \geq \phi_A$. We denote by $\mathcal{H}^\infty(X)$ the class of all sectorial operators that admit a bounded H^∞-calculus. If $A \in \mathcal{H}^\infty(X)$, we may define $h(A)$ for arbitrary $h \in \mathcal{H}^\infty(\Sigma_\phi)$ by the following method. Put $g(z) = z(1 + z)^{-2}$ and let

$$h(A) = \frac{1}{2\pi i}\left(\int_\Gamma h(\lambda)\frac{\lambda}{(1 + \lambda)^2}(\lambda - A)^{-1}\mathrm{d}\lambda\right)(1 + A)^2A^{-1} = (hg)(A)g(A)^{-1},$$

initially defined on the dense subspace $\mathrm{dom}(A) \cap \mathrm{ran}(A)$ of X. It is known that inequality (5) is still valid for those h. Consequently, $h(A)$ extends to a unique element in $\mathcal{L}(X)$, again denoted by $h(A)$. Moreover, it is easy to see that this definition of $h(A)$ is compatible with the definition (4) in the case $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$.

The following classes of operators are known to admit a bounded H^∞-calculus: Bounded operators, normal sectorial operators in Hilbert spaces (in particular self-adjoint operators) and negative generators of positive contraction semigroups in L_p-spaces. For details see the survey article [DHP01a]. In [DHP01b], it has been proved that also the Stokes operator in $L_{q,\sigma}(\mathbb{R}_+^n)$ admits a bounded H^∞-calculus if $1 < q < \infty$.

Remark 2.2 For Banach spaces X, Y, a densely defined linear operator $A : \mathrm{dom}(A) \to X$ and a continuous isomorhism $J : X \to Y$ the following easy statements are well-known. For details see e.g. [DHP01a], Proposition 2.11.

(i) A generates a bounded holomorphic C_0-semigroup on X, if and only if JAJ^{-1} generates a bounded holomorphic C_0-semigroup on Y.

(ii) $A \in \mathcal{H}^\infty(X)$ if and only if $JAJ^{-1} \in \mathcal{H}^\infty(Y)$. In that case we also have $\phi_A^\infty = \phi_{JAJ^{-1}}^\infty$.

(iii) $A \in \mathcal{H}^\infty(X)$ if and only if $A^{-1} \in \mathcal{H}^\infty(Y)$. If this is true, then $\phi_A^\infty = \phi_{A^{-1}}^\infty$.

3 The main result

This section contains our main result which reads as follows.

Theorem 3.1 *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a C^3-domain which is either bounded, exterior, or a perturbed half-space. Then the Stokes operator A_Ω admits a bounded H^∞-calculus in $L_{q,\sigma}(\Omega)$ if $1 < q < \infty$.*

As already mentioned in the introduction, we get the following slightly stronger assertion for free, because for $1 < q < \infty$, $L_{q,\sigma}(\Omega)$ is a Banach space with property (α). For details on R-boundedness and Banach spaces with property (α) we refer to [CdPSW00] and to [DJT95].

Theorem 3.2 *Under the assumptions of Theorem 3.1, A_Ω admits an R-bounded H^∞-calculus in $L_{q,\sigma}(\Omega)$ if $1 < q < \infty$.*

We shall prove Theorem 3.1 in several steps. First of all, we may assume that $q < 2$, the general case follows by taking adjoints. The Stokes operator A_{H_ω} on the bent half-space H_ω associated with ω is introduced in Section 3.1. It is shown that A_{H_ω} is similar to some perturbation A_T of the Stokes operator $A_{\mathbb{R}^n_+}$ on the half-space $\mathbb{R}^n_+$. In view of Remark 2.2 (ii) A_{H_ω} admits a bounded H^∞-calculus if this is true for A_T, which is proved in Section 3.2. In Sections 3.3 and 3.4 the general case is proved by reducing the problem to the cases already treated before.

3.1 The Stokes operator on bent half-spaces

Given a three times continuously differentiable and compactly supported function $\omega : \mathbb{R}^{n-1} \to [0, \infty)$, let

$$H_\omega := \{x = (x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\}$$

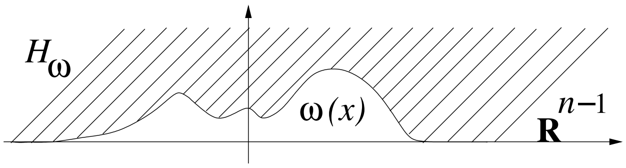

Figure 1: The bent half-space determined by ω

be the bent half-space determined by ω, see Figure 1. The transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\phi(x', x_n) := (x', x_n - \omega(x'))$ maps H_ω onto the half-space $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and satisfies $\det \phi'(x) = 1$ for all $x \in \mathbb{R}^n$. Therefore we may define $\Phi(u) := u \circ \phi^{-1}$ for any function defined on H_ω. Clearly, Φ is a continuous isomorphism from $W^{s,q}(H_\omega)$ to $W^{s,q}(\mathbb{R}^n_+)$ and also from $W^{s,q}_0(H_\omega)$ to $W^{s,q}_0(\mathbb{R}^n_+)$ for $s \in [0, 3]$. In what follows, we shall omit the subscript Ω if $\Omega = \mathbb{R}^n_+$, i.e. we set $P = P_{\mathbb{R}^n_+}$, $\Delta = \Delta_{\mathbb{R}^n_+}$ and $A = A_{\mathbb{R}^n_+}$.

Let $\lambda \in \mathbb{C}$. It is easy to see that a pair (u, p) is a solution of the Stokes resolvent problem

$$(\lambda - \Delta_{H_\omega})u + \nabla p = f, \quad \nabla \cdot u = 0$$

on $L_q(H_\omega)$ if and only if $(\tilde{u}, \tilde{p}) := (u \circ \phi^{-1}, p \circ \phi^{-1})$ solves the equations

$$(\lambda - (\Delta + R_1))\tilde{u} + (\nabla + R_2)\tilde{p} = f \circ \phi^{-1}, \quad (\nabla + R_2) \cdot \tilde{u} = 0 \tag{6}$$

on $L_q(\mathbb{R}^n_+)$, where R_1, R_2 are given by

$$R_1 = |\nabla'\omega|^2 \partial_n^2 - 2(\nabla'\omega, 0) \cdot (\nabla \partial_n) - (\Delta'\omega)\partial_n, \quad R_2 = -\partial_n(\nabla'\omega, 0). \tag{7}$$

Since

$$\Phi(L_q(H_\omega)) = \Phi(L_{q,\sigma}(H_\omega)) \oplus \Phi(G_q(H_\omega))$$

it is natural to introduce the spaces

$$\begin{aligned} C^\infty_{c,\sigma,R}(\mathbb{R}^n_+) &:= \{u \in C^\infty_c(\mathbb{R}^n_+) : (\nabla + R_2)\cdot u = 0\} \\ L^R_{q,\sigma}(\mathbb{R}^n_+) &:= \Phi(L_{q,\sigma}(H_\omega)) = \overline{C^\infty_{c,\sigma,R}(\mathbb{R}^n_+)}^{\|\cdot\|_q} \end{aligned}$$

as well as the projection $P_R u = \Phi P_{H_\omega} \Phi^{-1}$ which maps $L_q(\mathbb{R}^n_+)$ continuously onto $L^R_{q,\sigma}(\mathbb{R}^n_+)$. In terms of this modified Helmholtz projection equation (6) may be rephrased as the operator equation $(\lambda + A_R)\tilde{u} = f \circ \phi^{-1}$, where $A_R = -P_R(\Delta + R_1)$, defined on $W^{2,q}(\mathbb{R}^n_+) \cap W^{1,q}_0(\mathbb{R}^n_+) \cap L^R_{q,\sigma}(\mathbb{R}^n_+)$.

One problem in comparing A_R and $A_{\mathbb{R}^n_+}$ is that these operators act in the different Banach spaces $L^R_{q,\sigma}(\mathbb{R}^n_+)$ and $L_{q,\sigma}(\mathbb{R}^n_+)$. To overcome this problem we introduce the bounded linear operator T in $L_q(\mathbb{R}^n_+)$ by $Tu(x) = (\phi^{-1})'(x)u(x) = (I - S)u(x)$ with

$$Su = (0, \dots, 0, (\nabla'\omega, 0)\cdot u)$$

and I being the identity in $L_q(\mathbb{R}^n_+)$. Note that T is invertible with $T^{-1} = I + S$. Moreover, it is easy to check that T maps $L^R_{q,\sigma}(\mathbb{R}^n_+)$ continously onto $L_{q,\sigma}(\mathbb{R}^n_+)$ as well as $\mathrm{dom}(A_R)$ continuously onto $\mathrm{dom}(A)$.

3.2 H^∞-calculus for the Stokes operator on bent half-spaces

In this section we use the notation of the previous section. Our aim is to prove the following:

Theorem 3.3 *Let $1 < q < \infty$ and let $\omega : \mathbb{R}^{n-1} \to [0,\infty)$ be three times differentiable and compactly supported. The Stokes operator A_{H_ω} admits a bounded H^∞-calculus on $L_{q,\sigma}(H_\omega)$ if $\|\omega\|_{C^1}$ is sufficiently small.*

This result will proved in several steps. We shall use the fact that the Stokes operator $A_{\mathbb{R}^n_+}$ admits a bounded H^∞-calculus which has been proved by Desch, Hieber and Prüss, [DHP01b], by utilizing the symmetry of $\mathbb{R}^n_+$ to obtain an explicit expression for the resolvent of $A_{\mathbb{R}^n_+}$. We shall apply a recent perturbation result due to Prüss [DDH$^+$02] to show that $A_{\mathbb{R}^n_+}$ may be perturbed by a purely second order differential operator without destroying this property, provided the perturbation is relativly bounded with small enough bound. The main ingredients for the treatment of the lower order terms are the inequalities for the generalized Stokes resovent problem that have been stated in Theorem 2.1. We start by recalling the perturbation theorem.

Theorem 3.4 *(Prüss): Let X be a UMD space and let A be a linear operator in X which admits a bounded H^∞-calculus. Let B be a closed linear operator in X satisfying the following conditions.*

(i) $\mathrm{dom}(A) \subset \mathrm{dom}(B)$ *and* $\|Bx\| \le \kappa\|Ax\|$ *for all* $x \in \mathrm{dom}(A)$ *and some constant* $\kappa < 1$,

(ii) there is some $\alpha \in (0,1)$ such that $B(\mathrm{dom}(A^{1+\alpha})) \subset \mathrm{dom}(A^\alpha)$,

(iii) There is a constant C such that $\|A^\alpha Bx\| \le C\|A^{1+\alpha}x\|$ for all $x \in \mathrm{dom}(A^{1+\alpha})$.

Then $A + B$ admits a bounded H^∞-calculus provided that κ is small enough.

Recall that a Banach space X is a UMD space, if and only if the Hilbert transform acts boundedly in $L_q(\mathbb{R}, X)$ for all $\in (1,\infty)$ and note that every $L_q(\Omega)$ space with $q \in (1,\infty)$ and Ω being an open subset of $\mathbb{R}^n$ has this property. In order to apply Theorem 3.4 with A being the Stokes operator in $\mathbb{R}^n_+$ we define $A_T := TA_RT^{-1}$ on $\mathrm{dom}(A)$ as well as $B := A_T - A$. From Remark 2.2 (ii) we get that A_{H_ω} admits a bounded H^∞-calculus if and only if this is true for A_T. However, we can not apply Theorem 3.4 directly to A and B because the inequality $\|Bu\| \le \kappa\|Au\|$ does not hold since Bu contains lower order derivatives. Therefore we decompose B as $B = B_1 + B_2$ where B_2 is purely of second order. First note that on $\mathrm{dom}(B)$

$$\begin{aligned} B &= TA_RT^{-1} - A \\ &= -TP_R(\Delta + R_1)(I + S) + T(I + S)P\Delta \\ &= -TP_RR_1T^{-1} - T(P_R - P)\Delta + TSP\Delta - TP_R\Delta S. \end{aligned}$$

With $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$ we get for $u \in W^{2,q}(\mathbb{R}^n_+)$

$$\begin{aligned} \Delta S u &= e_n \Delta(\nabla' \omega \cdot u') \\ &= e_n \left(\nabla' \Delta' \omega \cdot u' + 2 \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} (\partial_j \partial_k \omega) \partial_k u^j + \nabla' \omega \cdot \Delta u' \right) \end{aligned} \tag{8}$$

and

$$\begin{aligned} R_1 T^{-1} u &= R_1 u + R_1 S u \\ &= |\nabla' \omega|^2 \partial_n^2 u - \Delta' \omega \partial_n u - 2(\nabla' \omega, 0) \cdot \partial_n \nabla u \\ &\quad + e_n \left(|\nabla' \omega|^2 \partial_n^2 \nabla' \omega \cdot u' - \Delta' \omega \partial_n \nabla' \omega \cdot u' - 2(\nabla' \omega, 0) \cdot \partial_n \nabla \nabla' \omega \cdot u' \right) \\ &= |\nabla' \omega|^2 \partial_n^2 u - \Delta' \omega \partial_n u - 2(\nabla' \omega, 0) \cdot \partial_n \nabla u \\ &\quad + e_n \Bigg(|\nabla' \omega|^2 \partial_n^2 \nabla' \omega \cdot u' - \Delta' \omega \partial_n \nabla' \omega \cdot u' - 2(\nabla' \omega, 0) \cdot \sum_{j=1}^{n-1} (\nabla' \partial_j \omega) \partial_n u^j \\ &\quad - 2 \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} (\partial_j \omega)(\partial_k \omega) \partial_k \partial_n u^j \Bigg) . \end{aligned} \tag{9}$$

This yields

$$B = B_1 + B_2$$

where

$$\begin{aligned} B_2 u \;:=\; & -T(P_R - P)\Delta + TSP\Delta - TP_R \left(|\nabla' \omega|^2 \partial_n^2 u + 2(\nabla' \omega, 0) \cdot \partial_n \nabla u \right) \\ & - TP_R e_n \left(\nabla' \omega \cdot \Delta u' - |\nabla' \omega|^2 \nabla' \omega \cdot \partial_n^2 u' + 2 \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} (\partial_j \omega)(\partial_k \omega) \partial_k \partial_n u^j \right) \end{aligned}$$

and

$$B_1 := B - B_2.$$

Since $B_2 u$ contains only second order derivatives of u we may write

$$B_2 = -T(P_R - P)\Delta + TSP\Delta + TP_R \sum_{|\alpha|=2} a_\alpha D^\alpha \tag{10}$$

with certain matrices $a_\alpha \in C_c^2(\mathbb{R}^{n-1})^{n \times n}$. Similarly,

$$B_1 = TP_R \sum_{k=1}^{n} b_k \partial_k + TP_R c$$

with $b_k \in C_c^1(\mathbb{R}^{n-1})^{n \times n}$ and $c \in C_c(\mathbb{R}^{n-1})^{n \times n}$. Due to (8) and (9) we get for $\|\omega\|_{C_b^3(\mathbb{R}^{n-1})} \leq 1$

$$\sum_{|\alpha|=2} \|a_\alpha\|_\infty \leq C \|\omega\|_{C_b^1(\mathbb{R}^{n-1})}, \tag{11}$$

$$\sum_{k=1}^{n} \|b_k\|_\infty \leq C \|\omega\|_{C_b^2(\mathbb{R}^{n-1})}, \tag{12}$$

$$\|c\|_\infty \leq C \|\omega\|_{C_b^3(\mathbb{R}^{n-1})}. \tag{13}$$

In what follows, we will apply the perturbation Theorem 3.4 only to B_2 whereas B_1 will be treated directly. To estimate the first term in (10) we need the following lemma.

Lemma 3.5 *It holds*

$$\|(P_R - P)u\|_q \leq C \|\nabla' \omega\|_\infty \|u\|_q$$

for all $u \in L_q(\mathbb{R}^n_+)$.

Proof. As is well known, see [Gal98] p. 107, we have $Pu = u - \nabla p$ with $p \in \hat{W}^{1,q}(\mathbb{R}^n_+)$ being the unique solution of the weak Neumann problem

$$(\nabla p, \nabla \varphi) = (u, \nabla \varphi), \quad \varphi \in \hat{W}^{1,q'}(\mathbb{R}^n_+), \tag{14}$$

where $(\cdot,\cdot)$ denotes dual pairing. Similarly, $P_R u = u - (\nabla + R_2)p_R$, where p_R solves the following problem:

$$((\nabla + R_2)p_R, (\nabla + R_2)\varphi) = (u, (\nabla + R_2)\varphi), \quad \varphi \in \hat{W}^{1,q'}(\mathbb{R}^n_+) \tag{15}$$

(observe that $\hat{W}^{1,q'}(\mathbb{R}^n_+) = \{p \in L_{q',\mathrm{loc}}(\mathbb{R}^n_+) : (\nabla + R_2)p \in L_{q',\mathrm{loc}}(\mathbb{R}^n_+)\}$ modulo constants, since $\|(\nabla + R_2)\cdot\|_q$ and $\|\nabla\cdot\|_q$ are equivalent norms on $\hat{W}^{1,q}(\mathbb{R}^n_+)$). From (15) we conclude

$$\begin{aligned} (\nabla p_R, \nabla\varphi) &= (u, (\nabla + R_2)\varphi) - ((R_2 p_R, \nabla\varphi) + (\nabla p_R, R_2\varphi) + (R_2 p_R, R_2\varphi)) \\ &= (u, \nabla\varphi) + (u, R_2\varphi) - ((\nabla + R_2)p_R, R_2\varphi) - (R_2 p_R, \nabla\varphi). \end{aligned} \tag{16}$$

Subtracting (14) from (16) yields

$$(\nabla p_R - \nabla p, \nabla\varphi) = (u, R_2\varphi) - ((\nabla + R_2)p_R, R_2\varphi) - (R_2 p_R, \nabla\varphi).$$

Since $\nabla p_R, \nabla p \in G_q$ and $G_q' = G_{q'}$ we get

$$\begin{aligned} &\|\nabla p_R - \nabla p\|_q \\ = \; & \sup_{\phi \in G_{q'}, \|\phi\|_{q'}=1} |(\nabla p_R - \nabla p, \phi)| = \sup_{\varphi \in \hat{W}^{1,q'}, \|\nabla\varphi\|_{q'}=1} |(\nabla p_R - \nabla p, \nabla\varphi)| \\ \leq \; & \sup_{\varphi \in \hat{W}^{1,q'}, \|\nabla\varphi\|_{q'}=1} (\|u\|_q \|R_2\varphi\|_{q'} + \|(\nabla + R_2)p_R\|_q \|R_2\varphi\|_{q'} + \|R_2 p_R\|_q \|\nabla\varphi\|_{q'}) \\ \leq \; & \sup_{\varphi \in \hat{W}^{1,q'}, \|\nabla\varphi\|_{q'}=1} \|\nabla'\omega\|_\infty (\|u\|_q \|\partial_n\varphi\|_{q'} + \|(\nabla + R_2)p_R\|_q \|\partial_n\varphi\|_{q'} + \|\partial_n p_R\|_q \|\nabla\varphi\|_{q'}) \\ \leq \; & \|\nabla'\omega\|_\infty (\|u\|_q + \|(\nabla + R_2)p_R\|_q + \|\partial_n p_R\|_q). \end{aligned}$$

Since

$$\|\partial_n p_R\|_q \leq C\|(\nabla + R_2)p_R\|_q = C\|(1 - P_R)u\|_q \leq C\|u\|_q$$

we obtain the desired estimate. □

With this lemma at hand it is not difficult to verify the first condition of Theorem 3.4.

Proposition 3.6 *Condition (i) of Theorem 3.4 holds true for A being the Stokes operator in $L_{q,\sigma}(\mathbb{R}^n_+)$ and B_2 defined by identity (10), provided that $\|\omega\|_{C_b^1(\mathbb{R}^{n-1})}$ is small enough.*

Proof. First note that $\mathrm{dom}(B) = \mathrm{dom}(A)$ by the definition of B. We will treat the three different terms in (10) separately. Let $u \in \mathrm{dom}(A)$. By the preceeding lemma and Proposition B.1 (b) with $k = 2$, the first term can be estimated as follows.

$$\|T(P_R - P)\Delta u\|_q \leq C\|\nabla'\omega\|_\infty \|\Delta u\|_q \leq C\|\nabla'\omega\|_\infty \|Au\|_q.$$

The corresponding inequality for the second term is trivial:

$$\|TSP\Delta u\|_q \leq C\|\nabla'\omega\|_\infty \|Au\|_q.$$

In view of inequality (11) and Proposition B.1, the third expression in (10) has the following upper bound:

$$\|TP_R \sum_{|\alpha|=2} a_\alpha D^\alpha u\|_q \leq C \sum_{|\alpha|=2} \|a_\alpha\|_\infty \|D^\alpha u\|_q \leq C\|\omega\|_{C_b^1(\mathbb{R}^{n-1})} \|Au\|_q.$$

These inequalities together immediately prove the assertion. □

In order to verify the second and the third hypothesis of the perturbation theorem we need the following lemma which follows easily from Sobolev's inequality. Recall that Sobolev's inequality states that for $n \in \mathbb{N}$ and $q \in (1, n)$

$$\|u\|_{L_{q^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L_q(\mathbb{R}^n)}, \quad u \in W^{1,q}(\mathbb{R}^n),$$

where q^* is the Sobolev-conjugated exponent given by $1/q^* = 1/q - 1/n$.

Lemma 3.7 *Let $n \geq 3$, $q \in (1, n-1)$. For any $a \in C_b^1(\mathbb{R}^{n-1})$ with compact support there is a constant $C > 0$ such that*

$$\|\nabla(au)\|_{L_q(\mathbb{R}^n_+)} \leq C\|\nabla u\|_{L_q(\mathbb{R}^n_+)}$$

for all $u \in W^{1,q}(\mathbb{R}^n_+)$. On the LHS, a has to be regarded as a function of n variables in the obvious way.

Proof. Since $\nabla(au) = a\nabla u + u\nabla a$ it is enough to prove that $\|u\partial_j a\|_{L_q(\mathbb{R}^n_+)} \leq C\|\nabla u\|_{L_q(\mathbb{R}^n_+)}$. With $K := \mathrm{supp}(a)$ we get

$$\begin{aligned}\|u\partial_j a\|^q_{L_q(\mathbb{R}^n_+)} &= \int_0^\infty \|u(\cdot, x_n)\partial_j a(\cdot)\|^q_{L_q(\mathbb{R}^{n-1})}\mathrm{d}x_n = \int_0^\infty \|u(\cdot, x_n)\partial_j a(\cdot)\|^q_{L_q(K)}\mathrm{d}x_n \\ &\leq C\int_0^\infty \|u(\cdot, x_n)\|^q_{L_q(K)}\mathrm{d}x_n.\end{aligned}$$

Denoting by q^* the Sobolev-conjugated exponent, the calculation continues and Sobolev's inequality yields

$$\begin{aligned}\|u\partial_j a\|^q_{L_q(\mathbb{R}^n_+)} &\leq C\int_0^\infty \|u(\cdot, x_n)\|^q_{L_{q^*}(K)}\mathrm{d}x_n \leq C\int_0^\infty \|u(\cdot, x_n)\|^q_{L_{q^*}(\mathbb{R}^{n-1})}\mathrm{d}x_n \\ &\leq C\int_0^\infty \|\nabla u(\cdot, x_n)\|^q_{L_q(\mathbb{R}^{n-1})}\mathrm{d}x_n = C\|\nabla u\|^q_{L_q(\mathbb{R}^n_+)}.\end{aligned}$$

□

For fixed $\lambda > 0$ and any function u defined on $\mathbb{R}^n_+$ we set

$$(J_\lambda u)(x) := u(\lambda x).$$

Observe that J_λ is an isomorphism in each of the spaces $W^{s,q}(\mathbb{R}^n_+)$, $s > 0$, $q \geq 1$ with $J_\lambda^{-1} = J_{1/\lambda}$. Moreover, it is also an isomorphism in $L_{q,\sigma}(\mathbb{R}^n_+)$ and in $\mathrm{dom}(A^\alpha)$ with $\alpha > 0$ because J_λ commutes with the Helmholtz projection P. For any bounded operator K in $L_q(\mathbb{R}^n_+)$, define $K_\lambda \in \mathcal{L}(L_q(\mathbb{R}^n_+))$ by $K_\lambda := J_\lambda^{-1}KJ_\lambda$. Because of

$$\nabla^k J_\lambda = \lambda^k J_\lambda \nabla^k, \qquad k \in \mathbb{N},$$

we have for $u \in W^{k,q}(\mathbb{R}^n_+)$

$$\|J_\lambda u\|_{k,q} = \lambda^{-n/q}\sum_{j=0}^k \lambda^j\|\nabla^j u\|_q. \tag{17}$$

This gives us for $k = 0$ the inequality

$$\|K_\lambda u\|_q = \|J_\lambda^{-1}KJ_\lambda u\|_q = \lambda^{n/q}\|KJ_\lambda u\|_q \leq \lambda^{n/q}\|K\|_{\mathcal{L}(L_q(\mathbb{R}^n_+))}\|J_\lambda u\|_q = \|K\|_{\mathcal{L}(L_q(\mathbb{R}^n_+))}\|u\|_q.$$

By symmetry we also get $\|Ku\|_q \leq \|K_\lambda\|_{\mathcal{L}(L_q(\mathbb{R}^n_+))}\|u\|_q$. Hence we even have

$$\|K_\lambda\|_{\mathcal{L}(L_q(\mathbb{R}^n_+))} = \|K\|_{\mathcal{L}(L_q(\mathbb{R}^n_+))}. \tag{18}$$

We shall further need an expression for the commutator between J_λ and fractional powers of $(A+\mu)$, where $\mu \in \rho(-A)$. Commuting J_λ with the Stokes operator yields

$$(A+\mu)J_\lambda = (-P\Delta + \mu)J_\lambda = (-\lambda^2 PJ_\lambda\Delta + \mu) = J_\lambda(\lambda^2 A + \mu),$$

which implies

$$(A+\mu)^{-1}J_\lambda = J_\lambda(\lambda^2 A + \mu)^{-1}.$$

By induction we deduce

$$(A+\mu)^k J_\lambda = J_\lambda(\lambda^2 A + \mu)^k$$

for all $k \in \mathbb{Z}$ and $\lambda > 0$. Since A admits a bounded H^∞-calculus, so does rA for $r > 0$, see [DHP01a]. By this fact we obtain the same equality for $0 < \alpha < 1$:

$$\begin{aligned}(A+\mu)^{-\alpha}J_\lambda &= \frac{1}{2\pi i}\int_\Gamma (\mu+z)^{-\alpha}(z-A)^{-1}J_\lambda \mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\Gamma (\mu+z)^{-\alpha}J_\lambda(z-\lambda^2 A)^{-1}\mathrm{d}z \\ &= J_\lambda(\lambda^2 A + \mu)^{-\alpha},\end{aligned}$$

where Γ is the contour defined in (3). Writing $s \in \mathbb{R}$ as $s = k - \alpha$ with $k \in \mathbb{Z}$ and $0 < \alpha < 1$ it follows

$$(A+\mu)^s J_\lambda = (A+\mu)^k (A+\mu)^{-\alpha} J_\lambda = J_\lambda(\lambda^2 A+\mu)^k(\lambda^2 A+\mu)^{-\alpha} = J_\lambda(\lambda^2 A+\mu)^s \tag{19}$$

for arbitrary $s \in \mathbb{R}$, $\lambda > 0$ and $\mu \in \rho(-A)$.

With the aid of Lemma 3.7 we can prove the following proposition which establishes the key-estimate for verifying the remaining assumptions of the perturbation Theorem 3.4.

Proposition 3.8 *Let $\lambda > 0$ be fixed and let $1 < q < n-1$. Define $B_{2,\lambda} := J_\lambda^{-1} B_2 J_\lambda$ on* $\mathrm{dom}(A)$. *Then $B_{2,\lambda}(\mathrm{dom}(A)) \subset L_{q,\sigma}(\mathbb{R}^n_+)$ and*

(a) $\|B_{2,\lambda}(A+1)^{-1}u\|_q \le C\lambda^2\|u\|_q$, $u \in L_{q,\sigma}(\mathbb{R}^n_+)$,

(b) $\|B_{2,\lambda}(A+1)^{-1}u\|_{1,q} \le C\lambda^2\|u\|_{\mathrm{dom}(A^{1/2})}$, $u \in \mathrm{dom}(A^{1/2})$.

The constant C does not depend on λ.

Proof. We first rewrite B_2 as

$$\begin{aligned}
B_2 &= -T(P_R - P)\Delta + TSP\Delta + TP_R \sum_{|\alpha|=2} a_\alpha D^\alpha = TP_R(-\Delta + \sum_{|\alpha|=2} a_\alpha D^\alpha) + T(I+S)P\Delta \\
&= TP_R(-\Delta + \sum_{|\alpha|=2} a_\alpha D^\alpha) - A.
\end{aligned}$$

From the last line it can be read off that $B_{2,\lambda}(\mathrm{dom}(A)) \subset L_{q,\sigma}(\mathbb{R}^n_+)$.

(a) According to (18) we get for $u \in L_{q,\sigma}(\mathbb{R}^n_+)$

$$\begin{aligned}
\|B_{2,\lambda}(A+1)^{-1}u\|_q &= \|J_\lambda^{-1}(TP_R(-\Delta + \sum_{|\alpha|=2} a_\alpha D^\alpha) - A)J_\lambda(A+1)^{-1}u\|_q \\
&\le \|J_\lambda^{-1}TP_R J_\lambda J_\lambda^{-1}\Delta J_\lambda(A+1)^{-1}u\|_q \\
&\quad + \|J_\lambda^{-1}TP_R \sum_{|\alpha|=2} a_\alpha J_\lambda J_\lambda^{-1} D^\alpha J_\lambda (A+1)^{-1}u\|_q \\
&\quad + \|J_\lambda^{-1}PJ_\lambda J_\lambda^{-1}\Delta J_\lambda(A+1)^{-1}u\|_q \\
&\le C\left(2\|J_\lambda^{-1}\Delta J_\lambda(A+1)^{-1}u\|_q + \sum_{|\alpha|=2}\|J_\lambda^{-1}D^\alpha J_\lambda(A+1)^{-1}u\|_q\right) \\
&\le C\lambda^2\left(\|\Delta(A+1)^{-1}u\|_q + \sum_{|\alpha|=2}\|D^\alpha(A+1)^{-1}u\|_q\right) \\
&\le C\lambda^2\|u\|_q.
\end{aligned}$$

(b) Since $q \in (1, n-1)$ we may apply Lemma 3.7 to obtain for $u \in W^{1,q}(\mathbb{R}^n_+)$

$$\|\nabla Tu\|_q \le \|\nabla u\|_q + \|\nabla Su\|_q \le \|\nabla u\|_q + C\|\nabla u\|_q \le C\|\nabla u\|_q.$$

The same argument applied to a_α gives us

$$\|\nabla \sum_{|\alpha|=2} a_\alpha D^\alpha u\|_q \le C\|\nabla^3 u\|_q, \qquad u \in W^{3,q}(\mathbb{R}^n_+).$$

Because $\|(\nabla + R_2)\cdot\|_q$ and $\|\nabla\cdot\|_q$ are equivalent norms on $W^{1,q}(\mathbb{R}^n_+)$, it is easy to see that the regularity, proved for P_{H_ω} in Appendix A.3, holds also true for $P_R = \Phi P_{H_\omega}\Phi^{-1}$. This implies together with the above two inequalities

$$\|\nabla B_{2,\lambda}(A+1)^{-1}u\|_q = \|\nabla J_\lambda^{-1}(TP_R(-\Delta + \sum_{|\alpha|=2} a_\alpha D^\alpha) - A)J_\lambda(A+1)^{-1}u\|_q$$

$$
\begin{aligned}
&= \lambda^{n/q}\lambda^{-1}\|\nabla(TP_R(-\Delta+\sum_{|\alpha|=2}a_\alpha D^\alpha)-A)J_\lambda(A+1)^{-1}u\|_q \\
&\le \lambda^{-1+n/q}\Bigg(\|\nabla TP_R\Delta J_\lambda(A+1)^{-1}u\|_q \\
&\qquad +\|\nabla TP_R\sum_{|\alpha|=2}a_\alpha D^\alpha J_\lambda(A+1)^{-1}u\|_q \\
&\qquad +\|\nabla P\Delta J_\lambda(A+1)^{-1}u\|_q\Bigg) \\
&\le C\lambda^{-1+n/q}\,3\|\nabla^3 J_\lambda(A+1)^{-1}u\|_q \\
&= C\lambda^{2+n/q}\|J_\lambda\nabla^3(A+1)^{-1}u\|_q \\
&= C\lambda^2\|\nabla^3(A+1)^{-1}u\|_q.
\end{aligned}
$$

In view of Proposition B.1, we can further estimate this last expression and obtain

$$
\|\nabla B_{2,\lambda}(A+1)^{-1}u\|_q \le C\lambda^2\|A^{3/2}(A+1)^{-1}u\|_q = C\lambda^2\|A(A+1)^{-1}A^{1/2}u\|_q \le C\lambda^2\|A^{1/2}u\|_q.
$$

This together with part (a) implies the assertion of (b). □

Proposition 3.9 *Let $1<q<n-1$ and let $0<\alpha<\frac{1}{2q}$. Then conditions (ii) and (iii) of Theorem 3.4 hold true for A being the Stokes operator in $L_{q,\sigma}(\mathbb{R}^n_+)$ and $B=B_2$.*

Proof. Since A admits a bounded H^∞-calculus it obviously has bounded imaginary powers which implies by [Tri78], Theorem 1.15.3 that

$$
\operatorname{dom}(A^\alpha) = [L_{q,\sigma}(\mathbb{R}^n_+), \operatorname{dom}(A)]_\alpha,
$$

where $[\cdot,\cdot]_\alpha$ denotes complex interpolation of order α. By general properties of interpolation functors (see [Tri78], Theorem 1.17.1.1) we have

$$
[L_{q,\sigma}(\mathbb{R}^n_+), \operatorname{dom}(A)]_\alpha = [L_{q,\sigma}(\mathbb{R}^n_+), \operatorname{dom}(A^{1/2})]_{2\alpha} = [L_q(\mathbb{R}^n_+), \operatorname{dom}(\Delta)]_\alpha \cap L_{q,\sigma}(\mathbb{R}^n_+).
$$

The interpolation space on the right hand side is known to be $W^{2\alpha,q}(\mathbb{R}^n_+)$ by our assumption $0<\alpha<\frac{1}{2q}$, see Section 2.3. Therefore we have

$$
\operatorname{dom}(A^\alpha) = W^{2\alpha,q}(\mathbb{R}^n_+) \cap L_{q,\sigma}(\mathbb{R}^n_+).
$$

By similar arguments we see that

$$
[L_{q,\sigma}(\mathbb{R}^n_+), W^{1,q}(\mathbb{R}^n_+)\cap L_{q,\sigma}(\mathbb{R}^n_+)]_{2\alpha} = W^{2\alpha,q}(\mathbb{R}^n_+)\cap L_{q,\sigma}(\mathbb{R}^n_+) = \operatorname{dom}(A^\alpha).
$$

Proposition 3.8 implies that $B_{2,\lambda}(A+1)^{-1}$ is a bounded operator in $L_{q,\sigma}(\mathbb{R}^n_+)$ and also from $\operatorname{dom}(A^{1/2})$ to $W^{1,q}_\sigma(\mathbb{R}^n_+) := W^{1,q}(\mathbb{R}^n_+)\cap L_{q,\sigma}(\mathbb{R}^n_+)$. Again, by interpolation it is also bounded from $[L_{q,\sigma}(\mathbb{R}^n_+), \operatorname{dom}(A^{1/2})]_{2\alpha}$ to $[L_{q,\sigma}(\mathbb{R}^n_+), W^{1,q}_\sigma(\mathbb{R}^n_+)]_{2\alpha}$, i.e.

$$
B_{2,\lambda}(A+1)^{-1} \in \mathcal{L}(\operatorname{dom}(A^\alpha))
$$

with

$$
\begin{aligned}
\|B_{2,\lambda}(A+1)^{-1}\|_{\mathcal{L}(\operatorname{dom}(A^\alpha))} &\le \|B_{2,\lambda}(A+1)^{-1}\|^{2\alpha}_{\mathcal{L}(L_{q,\sigma}(\mathbb{R}^n_+))}\|B_{2,\lambda}(A+1)^{-1}\|^{1-2\alpha}_{\mathcal{L}(\operatorname{dom}(A^{1/2}),W^{1,q}_\sigma(\mathbb{R}^n_+))} \\
&\le C\lambda^2.
\end{aligned}
$$

By putting $\lambda=1$ we see that

$$
B_2(\operatorname{dom}(A^{1+\alpha})) = B_{2,1}(\operatorname{dom}(A^{1+\alpha})) = B_{2,1}(A+1)^{-1}(\operatorname{dom}(A^\alpha)) \subset \operatorname{dom}(A^\alpha),
$$

proving that condition (ii) of Theorem 3.4 is satisfied. For the proof of condition (iii), we use the scaling method introduced in [McC81] and [BM88]. For $u\in\operatorname{dom}(A^{1+\alpha})$ let $v=(A+1)u$. By using the fact that $\|(A+1)^\alpha\cdot\|_q$ and $\|\cdot\|_{\operatorname{dom}(A^\alpha)}$ are equivalent norms on $\operatorname{dom}(A^\alpha)$, see Proposition B.1, we get

$$
\begin{aligned}
\|(A+1)^\alpha B_{2,\lambda}u\|_q &= \|(A+1)^\alpha B_{2,\lambda}(A+1)^{-1}v\|_q \le C\|B_{2,\lambda}(A+1)^{-1}v\|_{\operatorname{dom}(A^\alpha)} \\
&\le C\lambda^2\|v\|_{\operatorname{dom}(A^\alpha)} \le C\lambda^2\|(A+1)^\alpha v\|_q = C\lambda^2\|(A+1)^{1+\alpha}u\|_q.
\end{aligned}
$$

Next, for arbitrary $w \in \mathrm{dom}(A)$, define $u \in \mathrm{dom}(A^\alpha)$ by $u = J_\lambda^{-1} w$. Then

$$\begin{aligned}
\|(A+\lambda^2)^\alpha B_2 w\|_q &= \lambda^{2\alpha}\lambda^{-n/q}\|J_\lambda^{-1}(\lambda^{-2}A+1)^\alpha B_2 w\|_q \\
&= \lambda^{2\alpha-n/q}\|(A+1)^\alpha J_\lambda^{-1} B_2 J_\lambda J_\lambda^{-1} w\|_q \\
&= \lambda^{2\alpha-n/q}\|(A+1)^\alpha B_{2,\lambda} u\|_q \\
&\leq C\lambda^{-n/q+2\alpha+2}\|(A+1)^{1+\alpha} u\|_q \\
&= C\lambda^{-n/q+2\alpha+2}\|(A+1)^{1+\alpha} J_\lambda^{-1} w\|_q \\
&= C\lambda^{-n/q+2\alpha+2}\|J_\lambda^{-1}(\lambda^{-2}A+1)^{1+\alpha} w\|_q \\
&= C\|\lambda^{2\alpha+2}(\lambda^{-2}A+1)^{1+\alpha} w\|_q \\
&= C\|(A+\lambda^2)^{1+\alpha} w\|_q .
\end{aligned}$$

Passing to the limit $\lambda \to 0$ yields

$$\|A^\alpha B_2 w\|_q \leq C\|A^{1+\alpha} w\|_q,$$

i.e., condition (iii) of Theorem 3.4 is verified. □

Proposition 3.6 und 3.9 now immediately imply the following.

Corollary 3.10 *Let $1 < q < n-1$. The operator $A + B_2$ admits a bounded H^∞-calculus on $L_{q,\sigma}(\mathbb{R}^n_+)$ if $\|\omega\|_{C^1_b(\mathbb{R}^{n-1})}$ is sufficiently small.*

Proof. (of Theorem 3.3). Of course we want to apply Corollary 3.10. Therefore we first assume $q < n-1$. Let $\phi \in (\phi_A^\infty, \pi)$, and fix $\theta \in (\phi_A, \phi)$. By $\Gamma_{r,R}$ we denote the contour

$$\Gamma_{r,R} = \{se^{i\theta} : s \in [r,R]\} \cup \{se^{-i\theta} : s \in [r,R]\}$$

for $0 \leq r < R \leq \infty$. Now we write

$$\frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - A - B)^{-1}\mathrm{d}\lambda = \frac{1}{2\pi i}\int_{\Gamma_{0,1}} h(\lambda)(\lambda - A - B)^{-1}\mathrm{d}\lambda + \frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(\lambda - A - B)^{-1}\mathrm{d}\lambda$$

and start by examining the latter integral on the RHS which turns out be easy to handle: By the resolvent identity we get

$$(\lambda - A - B)^{-1} = (\lambda - A - B_2)^{-1} + (\lambda - A - B_2)^{-1}B_1(\lambda - A - B)^{-1}.$$

It is easily seen that Gagliardo-Nirenberg's inequality (see [Fri69] and Appendix A.2) implies together with Theorem 2.1 that

$$\|\nabla((\lambda - A_{H_\omega})^{-1})\|_{\mathcal{L}(L_{q,\sigma}(H_\omega),L_q(H_\omega))} \leq C|\lambda|^{-1/2}.$$

Therefore

$$\|(\lambda - A - B_2)^{-1}B_1(\lambda - A - B)^{-1}\|_{\mathcal{L}(L_{q,\sigma}(H_\omega),L_q(H_\omega))} \leq C|\lambda|^{-3/2}$$

for all $\lambda \in \mathbb{C} \setminus \Sigma_{\phi_A}$ with $|\lambda| \geq 1$. Therefore we obtain

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(\lambda - A - B)^{-1} f\mathrm{d}\lambda\right\|_{L_q(\mathbb{R}^n_+)} \leq C\|h\|_\infty\|f\|_{L_q(\mathbb{R}^n_+)}, \quad f \in L_{q,\sigma}(\mathbb{R}^n_+), h \in \mathcal{H}^\infty(\Sigma_\phi).$$

This gives us

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(\lambda - A_{H_\omega})^{-1} f\mathrm{d}\lambda\right\|_{L_q(H_\omega)} \leq C\|h\|_\infty\|f\|_{L_q(H_\omega)}$$

for all $f \in L_{q,\sigma}(H_\omega)$ and all $h \in \mathcal{H}^\infty(\Sigma_\phi)$ since we may write $A_{H_\omega} = \Phi^{-1}T^{-1}(A+B)T\Phi$, where T and Φ are isomorphisms.

The case $|\lambda| \leq 1$ is more involved. Here we reduce the bent half-space problem to problems on a half-space and a bounded domain through a localization. Let $R > 0$ such that $H_\omega \setminus B_R(0) = \mathbb{R}^n_+ \setminus B_R(0)$. We choose a cut-off function $\eta_0 \in C_c^\infty(\mathbb{R}^n)$ satisfying $0 \leq \eta_0 \leq 1$, $\eta_0 \equiv 1$ on $B_R(0)$ and $\mathrm{supp}(\eta_0) \subset B_{2R}(0)$ and set $\eta_1 := 1 - \eta_0$. Further, we put $\Omega_1 := \mathbb{R}^n_+$ and choose a bounded domain $\Omega_0 \subset H_\omega$ with $B_{2R}(0) \cap H_\omega \subset \Omega_0$ and such that $\partial\Omega_0$ is C^3. See Figure 2 for an illustration of this construction. For $f \in L_{q,\sigma}(H_\omega)$, let

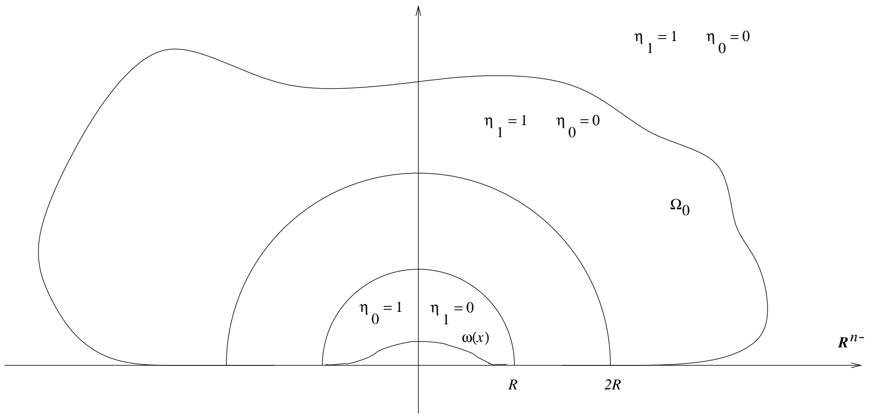

Figure 2: Resolution of the unity subordinate Ω_0, Ω_1

$(u,p) \in \operatorname{dom}(A_{H_\omega}) \times \hat{W}^{1,q}(H_\omega)$ be the unique solution of the Stokes resolvent problem $(SRP)^{H_\omega}_{f,0}$. It is easy to see that the pair $(\eta_j u, \eta_j p)$ solves the generalized Stokes resolvent problem $(SRP)^{\Omega_j}_{f_j,g_j}$, where $f_j = \eta_j f - 2\nabla u \cdot \nabla \eta_j - u\Delta\eta_j + p\nabla\eta_j$ and $g_j = u\nabla\eta_j$. In order to apply previous results on the Stokes operator A_{Ω_j} we have to split the solutions (u_j, p_j) of the above problems in the following way:

$$(\eta_j u, \eta_j p) = (v_j, p_j^v) + (w_j, p_j^w),$$

with $(v_j, p_j^v), (w_j, p_j^w)$ being the unique solutions of $(SRP)^{\Omega_j}_{P_{\Omega_j} f_j, 0}$ and $(SRP)^{\Omega_j}_{(I-P_{\Omega_j})f_j, g_j}$, respectively. Since $(I - P_{\Omega_j})f_j \in G_q(\Omega_j)$, it can be written as the gradient of a function $q \in \hat{W}^{1,q}(\Omega_j)$, i.e.

$$(I - P_{\Omega_j})f_j = \nabla q_j.$$

Hence, w_j can also be regarded as the unique flow of the problem $(SRP)^{\Omega_j}_{0,g_j}$ with pressure $p_j^w - q_j$. For this reason we have to look at the two integrals on the right hand side of

$$\int_{\Gamma_{0,1}} h(\lambda)\eta_j u \mathrm{d}\lambda = \int_{\Gamma_{0,1}} h(\lambda) v_j \mathrm{d}\lambda + \int_{\Gamma_{0,1}} h(\lambda) w_j \mathrm{d}\lambda, \quad j = 0, 1. \tag{20}$$

We begin with the case $j = 0$. Clearly, f_0 satisfies the estimate

$$\|f_0\|_{L_q(\Omega_0)} \le C \left(\|f\|_{L_q(H_\omega)} + \|\nabla u\|_{L_q(H_\omega)} + \|u\|_{L_q(\Omega_0)} + \|p\|_{L_q(\Omega_0)} \right).$$

Since $u \in W_0^{1,q}(H_\omega)$, it follows from Sobolev's inequality

$$\|u\|_{L_q(\Omega_0)} \le C\|u\|_{L_{q^*}(\Omega_0)} \le C\|u\|_{L_{q^*}(H_\omega)} \le C\|\nabla u\|_{L_q(H_\omega)},$$

and by Poincaré's inequality

$$\|p\|_{L_q(\Omega_0)} \le C\|\nabla p\|_{L_q(\Omega_0)}$$

because we may assume $\int_{\Omega_0} p(x)\mathrm{d}x = 0$. In view of Theorem 2.1 we get the estimate

$$\|f_0\|_{L_q(\Omega_0)} \le C \left(\|f\|_{L_q(H_\omega)} + \|\nabla u\|_{L_q(H_\omega)} + \|\nabla p\|_{L_q(H_\omega)} \right) \le C \left(2 + \frac{1}{\sqrt{|\lambda|}} \right) \|f\|_{L_q(H_\omega)},$$

for all $\lambda \in \mathbb{C} \setminus \Sigma_{\phi_A}$ with $|\lambda| \le 1$. Hence, having in mind that $0 \in \rho(A_{\Omega_0})$, we know that $\|(\lambda - A_{\Omega_0})^{-1}\| \le C/(1 + |\lambda|)$. Therefore we obtain for the first integral in (20)

$$\begin{aligned}
\| \int_{\Gamma_{0,1}} h(\lambda) v_0 \mathrm{d}\lambda \|_{L_q(\Omega_0)} &= \| \int_{\Gamma_{0,1}} h(\lambda)(\lambda - A_{\Omega_0})^{-1} P_{\Omega_0} f_0 \mathrm{d}\lambda \|_{L_q(\Omega_0)} \\
&\le C\|h\|_\infty \int_0^1 \frac{1}{|se^{i\theta} + 1|} \|f_0\|_{L_q(\Omega_0)} \mathrm{d}s \\
&\le C\|h\|_\infty \int_0^1 \frac{1}{|se^{i\theta} + 1|} \left(1 + \frac{1}{\sqrt{s}} \right) \mathrm{d}s \|f\|_{L_q(H_\omega)} \\
&\le C\|h\|_\infty \|f\|_{L_q(H_\omega)}
\end{aligned}$$

for all $f \in L_{q,\sigma}(H_\omega)$ and all $h \in \mathcal{H}^\infty(\Sigma_\phi)$. For w_0 we have according to Theorem 2.1 and again Sobolev's inequality the estimate

$$\|w_0\|_{L_q(\Omega_0)} \le C\|g_0\|_{-1,q} \le C\|g_0\|_{L_q(\Omega_0)} \le C\|u\|_{L_{q^*}(\Omega_0)} \le C\|\nabla u\|_{L_q(H_\omega)} \le \frac{C}{\sqrt{|\lambda|}}\|f\|_{L_q(H_\omega)}. \tag{21}$$

This implies for the second integral in (20)

$$\|\int_{\Gamma_{0,1}} h(\lambda)w_0 \mathrm{d}\lambda\|_{L_q(\Omega_0)} \le C\|h\|_\infty \|f\|_{L_q(H_\omega)}$$

for all $f \in L_{q,\sigma}(H_\omega)$ and all $h \in \mathcal{H}^\infty(\Sigma_\phi)$.

In the second case, $j = 1$, we have to treat the terms of f_1 separately. For each $q \in (1,\infty)$ there exists $\alpha \in (0,1)$ and $q_1 \in (1,q)$ satisfying

$$\frac{1}{q} = \alpha\left(\frac{1}{q_1} - \frac{2}{n}\right) + (1-\alpha)\frac{1}{q_1} = -\frac{2\alpha}{n} + \frac{1}{q_1}.$$

Therefore we may apply Gagliardo-Nirenberg's inequality, see [Fri69], Theorem 9.3 for the $\mathbb{R}^n$ case and Appendix A.2 for the half-space case. Using the fact that P is bounded in each $L_r(\mathbb{R}^n_+)$, $1 < r < \infty$, we obtain

$$\begin{aligned}
&\|(\lambda - A)^{-1}P(\nabla u \cdot \nabla\eta_1)\|_{L_q(\mathbb{R}^n_+)} \\
&\quad \le C\|\nabla^2(\lambda - A)^{-1}P(\nabla u \cdot \nabla\eta_1)\|^{\alpha}_{L_{q_1}(\mathbb{R}^n_+)} \|(\lambda - A)^{-1}P(\nabla u \cdot \nabla\eta_1)\|^{1-\alpha}_{L_{q_1}(\mathbb{R}^n_+)} \\
&\quad \le C|\lambda|^{\alpha-1}\|\nabla u \cdot \nabla\eta_1\|_{L_{q_1}(\mathbb{R}^n_+)}.
\end{aligned}$$

Because of $\mathrm{supp}\nabla\eta_1 \subset \Omega_0$ we further get

$$\|\nabla u \cdot \nabla\eta_1\|_{L_{q_1}(\mathbb{R}^n_+)} \le C\|\nabla u\|_{L_{q_1}(\Omega_0)} \le C\|\nabla u\|_{L_{q^*}(\Omega_0)} \le C\|\nabla^2 u\|_{L_q(H_\omega)}. \tag{22}$$

Consequently,

$$\|(\lambda - A)^{-1}P(\nabla u \cdot \nabla\eta_1)\|_{L_q(\mathbb{R}^n_+)} \le C|\lambda|^{\alpha-1}\|f\|_{L_q(H_\omega)} \tag{23}$$

for all $\lambda \in \mathbb{C} \setminus \Sigma_{\phi_A}$ with $|\lambda| \le 1$. For the terms $(\lambda - A)^{-1}P(u\Delta\eta_1)$, $(\lambda - A)^{-1}P(p\nabla\eta_1)$ one gets in a completely analogous way an inequality like (23). This time, instead of (22), one has to use

$$\|u\Delta\eta_1\|_{L_{q_1}(\mathbb{R}^n_+)} \le C\|u\|_{L_{q^{**}}(H_\omega)} \le C\|\nabla u\|_{L_{q^*}(H_\omega)} \le C\|\nabla^2 u\|_{L_q(H_\omega)}, \tag{24}$$

which we can get by applying Sobolev's inequality on H_ω (see Appendix A.1) and

$$\|p\|_{L_q(\Omega_0)} \le C\|\nabla p\|_{L_q(\Omega_0)},$$

respectively. With these preparations we obtain

$$\begin{aligned}
\|\int_{\Gamma_{0,1}} h(\lambda)v_1 \mathrm{d}\lambda\|_{L_q(\mathbb{R}^n_+)} &= \|\int_{\Gamma_{0,1}} h(\lambda)(\lambda - A)^{-1}Pf_1 \mathrm{d}\lambda\|_{L_q(\mathbb{R}^n_+)} \\
&\le \|\int_{\Gamma_{0,1}} h(\lambda)(\lambda - A)^{-1}P(\eta_1 f)\mathrm{d}\lambda\|_{L_q(\mathbb{R}^n_+)} \\
&+ \|\int_{\Gamma_{0,1}} h(\lambda)(\lambda - A)^{-1}P(2\nabla u \cdot \nabla\eta_1 + u\Delta\eta_1 + p\nabla\eta_1)\mathrm{d}\lambda\|_{L_q(\mathbb{R}^n_+)} \\
&\le C\left(\|h\|_\infty\|f\|_{L_q(H_\omega)} + \|h\|_\infty \int_0^1 s^{\alpha-1}\mathrm{d}s\|f\|_{L_q(H_\omega)}\right) \\
&\le C\|h\|_\infty\|f\|_{L_q(H_\omega)}
\end{aligned}$$

for all $f \in L_{q,\sigma}(H_\omega)$ and all $h \in \mathcal{H}^\infty(\Sigma_\phi)$. The estimate of the w_1-term is completely analogous to the case $j = 0$.

Summarizing, we obtain

$$\begin{aligned}
\|\int_{\Gamma_{0,1}} h(\lambda)(\lambda - A_{H_\omega})^{-1}f\mathrm{d}\lambda\|_{L_q(\mathbb{R}^n_+)} &= \|\int_{\Gamma_{0,1}} h(\lambda)u\mathrm{d}\lambda\|_{L_q(H_\omega)} \\
&\le \sum_{j=0} \|\int_{\Gamma_{0,1}} h(\lambda)\eta_j u\mathrm{d}\lambda\|_{L_q(H_\omega)} \\
&\le C\|h\|_\infty\|f\|_{L_q(H_\omega)}
\end{aligned}$$

for all $f \in L_{q,\sigma}(H_\omega)$ and all $h \in \mathcal{H}^\infty(\Sigma_\phi)$. This proves the assertion for $q \in (1,2)$. The case $q = 2$ is clear because A_{H_ω} is self-adjoint in $L_{2,\sigma}(H_\omega)$. The general case follows from the case $q < 2$ by taking adjoints. $\square$

3.3 H^∞-calculus for the Stokes operator on bounded domains

Let Ω be a bounded C^3-domain. It is well known that in this case $0 \in \rho(A_\Omega)$, which immediately implies

$$\|\frac{1}{2\pi i}\int_{\Gamma_{0,1}} h(\lambda)(\lambda - A_\Omega)^{-1}\mathrm{d}\lambda\|_{\mathcal{L}(L_{q,\sigma}(\Omega))} \leq C\|h\|_\infty \tag{25}$$

for all $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$ and some $\phi \in (0,\pi/2)$. Hence it suffices to consider the case $|\lambda| \geq 1$ to which we want to apply the following localization method which is described in more detail in [SS]. For some $\delta > 0$ to be fixed later, consider the open covering of $\partial\Omega$ consisting of all open balls $B_\delta(x)$ of radius δ, centered at $x \in \partial\Omega$. By assumption, $\partial\Omega$ is compact, so we have

$$\partial\Omega \subset \bigcup_{j=1}^N B_\delta(x_j)$$

for some $N = N(\delta) \in \mathbb{N}$ and certain $x_1, \ldots, x_N \in \partial\Omega$. Choose an open subset Ω_0 of Ω such that $\overline{\Omega_0} \subset \Omega$ and $\Omega \subset \Omega_0 \cup \bigcup_{j=1}^N B_\delta(x_j)$. Put $\Omega_j := B_{2\delta}(x_j) \cap \Omega$, $j = 1, \ldots, N$, and let $\eta_j \in C_c^\infty(\mathbb{R}^n)$, $j = 1, \ldots, N$, be such that $\eta_j \equiv 1$ on $B_\delta(x_j)$ and $\mathrm{supp}(\eta_j) \subset B_{2\delta}(x_j)$ as well as $\eta_0 \equiv 1$ on Ω_0 and $\mathrm{supp}(\eta_0) \subset \Omega$. Next,

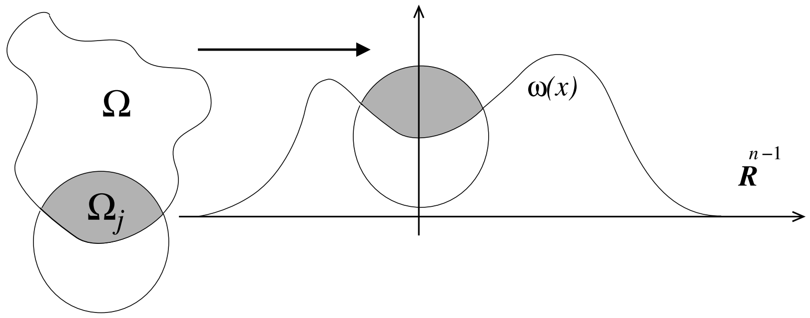

Figure 3: The localization method

for given $f \in L_{q,\sigma}(\Omega)$, let $(u,p) \in \mathrm{dom}(A_\Omega) \times \hat{W}^{1,q}(\Omega)$ be the unique solution of the Stokes resolvent problem $(SRP)^\Omega_{f,0}$. We get the localized equations

$$(SRP)^{\Omega_j}_{f_j,g_j}\begin{cases} \lambda\eta_j u - \Delta\eta_j u + \nabla\eta_j p &= f_j \quad \text{on} \quad \Omega_j, \\ \nabla\cdot\eta_j u &= g_j \quad \text{on} \quad \Omega_j, \\ \gamma\eta_j u &= 0, \end{cases}$$

$j = 0, \ldots, N$, where $f_j = \eta_j f - 2\nabla u \cdot \nabla\eta_j - u\Delta\eta_j + p\nabla\eta_j$ and $g_j = u\nabla\eta_j$, which shall be reduced either to the bent half-space case ($j = 1, \ldots, N$) or to the $\mathbb{R}^n$ case ($j = 0$). To do so we have to rotate and translate the localized problems. However, it is easy to see that such transformations lead to an equivalent Stokes resolvent problem. For example, if U and P solve the Stokes resolvent problem $(SRP)^Q_{F,G}$ on some open subset $Q \subset \mathbb{R}^n$ and $x := V\tilde{x} := \mathcal{O}\tilde{x} + x_0$, where $\mathcal{O}$ is an orthogonal transformation, then $\tilde{U}(\tilde{x}) := \mathcal{O}^t U(V\tilde{x})$ and $\tilde{P}(\tilde{x}) := P(V\tilde{x})$ solve the equivalent Stokes resolvent problem $(SRP)^{V^{-1}Q}_{\tilde{F},\tilde{G}}$ on $V^{-1}Q$ where $\tilde{F}(\tilde{x}) := \mathcal{O}^t F(V\tilde{x})$ and $\tilde{G}(\tilde{x}) := G(V\tilde{x})$. Thus, for simplicity, we shall omit this kind of transformations in the sequel.
Since $\partial\Omega \in C^3$ we can, by choosing δ small enough, for each $j = 1, \ldots, N$ find a function $\omega_j \in C_c^3(\mathbb{R}^{n-1})$ such that (with $H_j = H_{\omega_j}$)

$$\Omega_j \subset H_j, \quad B_{2\delta}(x_j) \cap \partial\Omega \subset \partial H_j$$

and $\|\omega_j\|_{C^1} \leq \kappa$ with κ as in Theorem 3.3. Thus, by extending the localized functions by 0 we can regard every localized equation as Stokes resolvent problem on H_j, where $H_0 := \mathbb{R}^n$. We cannot apply Theorem 3.3 directly, because $\mathrm{div}\eta_j u = g_j \neq 0$ in general. Therefore let L be the solution operator of the problem

$$\left\{\begin{array}{rcll}(1-\Delta)w + \nabla p^w &=& 0 & \text{on} \quad H,\\ \nabla\cdot w &=& g & \text{on} \quad H,\\ \gamma w &=& 0, & \end{array}\right. \tag{26}$$

where H may be any domain in $\mathbb{R}^n$ satisfying the assumptions of Theorem 2.1. According to [FS94] Corollary 1.5 the operator

$$L : \hat{W}^{-1,q}(H) \cap W^{1,q}(H) \to W^{2,q}(H) \cap W_0^{1,q}(H)$$

if H is unbounded or with $L_{q,0}(H) := \{u \in L_q(H) : \int_H u \mathrm{d}x = 0\}$

$$L : L_{q,0}(H) \cap W^{1,q}(H) \to W^{2,q}(H) \cap W_0^{1,q}(H)$$

if H is bounded is continuous and satisfies in any case both of the following estimates:

$$\|Lg\|_q \leq C\|g\|_{-1,q} \quad \text{and} \quad \|Lg\|_{2,q} \leq C(\|g\|_{-1,q} + \|\nabla g\|_q) \tag{27}$$

for all $g \in \mathrm{dom}(L)$. Now we set $w_j := Lg_j$ and $v_j := \eta_j u - w_j$, i.e., we write $\eta_j u$ as

$$\eta_j u = v_j + w_j, \qquad j = 1, \ldots, N.$$

The v_j's satisfy the equations

$$\begin{aligned}(\lambda - \Delta)v_j + \nabla(\eta_j p - p^{w_j}) &= f_j + (1-\lambda)w_j\\ &= P_{H_j}(f_j + (1-\lambda)w_j) + (I - P_{H_j})(f_j + (1-\lambda)w_j).\end{aligned}$$

Now $(I - P_{H_j})(f_j + (1-\lambda)w_j)$ is a gradient field, so it can be written in the form

$$(I - P_{H_j})(f_j + (1-\lambda)w_j) = \nabla q_j$$

for some $q_j \in \hat{W}^{1,q}(H_j)$, $j = 1, \ldots, N$. Thus v_j can also be regarded as the Stokes flow of the unique solution $(v_j, \eta_j p - p^{w_j} - q_j)$ of the generalized Stokes resolvent problem $(SRP)^{H_j}_{P_{H_j}(f_j+(1-\lambda)w_j),0}$. Consequently

$$v_j = (A_{H_j} + \lambda)^{-1} P_{H_j}(f_j + (1-\lambda)w_j)$$

The identity

$$\lambda(A_{H_j} + \lambda)^{-1} P_{H_j} w_j = P_{H_j} w_j - A_{H_j}(A_{H_j} + \lambda)^{-1} P_{H_j} w_j$$

gives us the following formula for $\eta_j u$

$$\begin{aligned}&\eta_j u = v_j + w_j\\ &= (\lambda + A_{H_j})^{-1} P_{H_j} f_j + (\lambda + A_{H_j})^{-1} P_{H_j} w_j + A_{H_j}(\lambda + A_{H_j})^{-1} P_{H_j} w_j + (1 - P_{H_j})w_j,\end{aligned} \tag{28}$$

$j = 1, \ldots, N$. We treat these four addends separately and begin with the second one. Since $\nabla\eta_j$ is compactly supported, we get by (27) and Poincaré's inequality

$$\begin{aligned}\|w_j\|_{L_q(H_j)} &= \|Lg_j\|_q \leq C\|g_j\|_{-1,q} = C\|u \cdot \nabla\eta_j\|_{-1,q}\\ &= \sup_{\psi \in \hat{W}^{1,q'}(H_j), \|\nabla\psi\|_{q'}=1} \left| \int_{\mathrm{supp}(\eta_j)\cap\Omega} u \nabla\eta_j \psi \, \mathrm{d}x \right|\\ &\leq C \sup_{\psi \in \hat{W}^{1,q'}(H_j), \|\nabla\psi\|_{q'}=1} \|u\|_{L_q(\Omega)} \|\psi\|_{L_{q'}(\mathrm{supp}(\eta_j)\cap\Omega)}\\ &\leq C\|u\|_{L_q(\Omega)} \leq C|\lambda|^{-1}\|f\|_{L_q(\Omega)}.\end{aligned}$$

This implies

$$\|(\lambda + A_{H_j})^{-1} P_{H_j} w_j\|_{L_q(H_j)} \leq C\frac{1}{|\lambda|}\|P_{H_j} w_j\|_{L_q(H_j)} \leq C\frac{1}{|\lambda|}\|w_j\|_{L_q(H_j)} \leq C\frac{1}{|\lambda|^2}\|f\|_{L_q(\Omega)},$$

for $\lambda \in \Sigma_{\pi-\theta}$, $|\lambda| \geq 1$. Hence

$$\begin{aligned} \|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(\lambda - A_{H_j})^{-1}P_{H_j}w_j \mathrm{d}\lambda\|_{L_q(H_j)} &= \frac{1}{2\pi}\|\int_{\Gamma_{1,\infty}} h(\lambda)((-\lambda)+A_{H_j})^{-1}P_{H_j}w_j\mathrm{d}\lambda\|_{L_q(H_j)} \\ &\leq C\|h\|_\infty \|f\|_{L_q(\Omega)} \end{aligned} \tag{29}$$

for all $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$, $j = 1, \ldots, N$.

The remaining three addends are more involved. For the first one of (28) we need the following preparations. For a bounded domain $G \subset \mathbb{R}^n$ we use the following identification of the homogenous Sobolev space

$$\hat{W}^{1,q}(G) = W^{1,q}(G) \cap L_{q,0}(G).$$

We want to remark that for an arbitrary $\Omega \subset \mathbb{R}^n$ and $G \subset \Omega$ for every $p \in \hat{W}^{1,q}(\Omega)$ it is always possible to choose a constant $c = c(G,p)$ such that $p_G = p + c \in L_{q,0}(G)$. The next lemma states an extra decay in λ of the pressure of the Stokes resolvent problem.

Lemma 3.11 *Let $\theta \in (\pi/2, \pi)$, $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$ as in Theorem 2.1 and $(u,p) \in \operatorname{dom}(A_\Omega) \times \hat{W}^{1,q}$ the unique solution of the Stokes resolvent problem $(SRP)^\Omega_{f,0}$, where $f \in L_{q,\sigma}(\Omega)$. Then, for each $\alpha \in (0, \frac{1}{2q'})$ and for every bounded domain $G \subset \Omega$ of class $C^{1,1}$ we have*

$$\|p_G\|_{L_{q,0}(G)} \leq C|\lambda|^{-\alpha}\|f\|_{L_{q,\sigma}(\Omega)}, \qquad \lambda \in \Sigma_\theta, \quad |\lambda| \geq 1$$

with some constant $C = C(G,\alpha) > 0$ independent of λ and f.

Proof. It is easy to see that $(L_{q,0}(G))' = L_{q',0}(G)$. We estimate $(p_G, \varphi)_G := \int_G p_G\varphi$ for an arbitrary $\varphi \in L_{q',0}(G)$. According to [Bog79], [Bog80] or [Gal98], for every $\varphi \in L_{q',0}(G)$ there is a solution $\phi \in W_0^{1,q'}(G)$ of the divergence problem

$$\begin{cases} \nabla\cdot\phi = \varphi & \text{on } G, \\ \phi = 0 & \text{on } \partial G, \end{cases}$$

with

$$\|\phi\|_{W^{1,q'}(G)} \leq C\|\varphi\|_{L_{q',0}(G)}. \tag{30}$$

Since $\phi \in W_0^{1,q}(G)$ we may regard ϕ also as an element in $W^{1,q}(\Omega)$. Using

$$\nabla p_G(x) = (I - P_\Omega)\Delta u(x), \qquad x \in \Omega,$$

which can be obtained by recalling $\nabla p_G = \nabla p$ and applying $(I - P_\Omega)$ to the first line of $(SRP)^\Omega_{f,0}$, we may calculate

$$\begin{aligned} (p_G,\varphi)_G &= (p_G, \nabla\cdot\phi)_G = -(\nabla p_G, \phi)_G = -(\nabla p_G,\phi)_\Omega \\ &= -((I-P_\Omega)\Delta_\Omega u, \phi)_\Omega = (-\Delta_\Omega u, (I-P_\Omega)\phi)_\Omega. \end{aligned}$$

Since $-\Delta_\Omega$ has bounded imarinary powers, (see e.g. [PS93]) we get by the interpolation property proved in Section 2.3 that

$$\operatorname{dom}((-\Delta_\Omega)^\alpha) = [L_q(\Omega), \operatorname{dom}(-\Delta_\Omega)]_\alpha = W^{2\alpha,q}(\Omega)$$

for $q \in (1,\infty)$ and $\alpha \in [0, \frac{1}{2q})$. Since $P_\Omega \in \mathcal{L}(W^{1,q'}(\Omega))$, see [Fra00], we have

$$(I-P_\Omega)\phi \in W^{1,q'}(\Omega) \subset W^{2\alpha,q'}(\Omega) = \operatorname{dom}((-\Delta_\Omega)^\alpha).$$

Hence, the above calculation yields together with inequality (30)

$$\begin{aligned} |(p_G,\varphi)_G| &= |((-\Delta_\Omega)^{1-\alpha}u, (-\Delta_\Omega)^\alpha(I-P_\Omega)\phi)_\Omega| \\ &\leq \|(-\Delta_\Omega)^{1-\alpha}u\|_{L_q(\Omega)}\|(-\Delta_\Omega)^\alpha(I-P_\Omega)\phi\|_{L_{q'}(\Omega)} \\ &\leq C\|(-\Delta_\Omega)^{1-\alpha}u\|_{L_q(\Omega)}\|(I-P_\Omega)\phi\|_{W^{1,q'}(\Omega)} \\ &\leq C\|(-\Delta_\Omega)^{1-\alpha}u\|_{L_q(\Omega)}\|\phi\|_{W^{1,q'}(G)} \\ &\leq C\|(-\Delta_\Omega)^{1-\alpha}u\|_{L_q(\Omega)}\|\varphi\|_{L_{q',0}(G)}. \end{aligned}$$

To estimate the term $(-\Delta_\Omega)^{1-\alpha}u$ we write u in the form $u = (\lambda - \Delta_\Omega)^{-1}(f - \nabla p_G)$ and obtain by a simple interpolation argument and Theorem 2.1

$$\|(-\Delta_\Omega)^{1-\alpha}u\|_{L_q(\Omega)} = \|(-\Delta_\Omega)^{1-\alpha}(\lambda - \Delta_\Omega)^{-1}(f - \nabla p_G)\|_{L_q(\Omega)} \leq C|\lambda|^{-\alpha}\|f\|_{L_q(\Omega)}$$

for all $\lambda \in \Sigma_\theta$, $|\lambda| \geq 1$. This gives us

$$|(p_G, \varphi)_G| \leq C|\lambda|^{-\alpha}\|f\|_{L_q(\Omega)}\|\varphi\|_{L_{q'},0(G)}$$

for all $\varphi \in L_{q',0}(G)$. Consequently,

$$\|p_G\|_{L_{q,0}(G)} = \sup_{\varphi \in L_{q',0}(G), \varphi \neq 0} \frac{|(p_G, \varphi)_G|}{\|\varphi\|_{L_{q',0}(G)}} \leq C|\lambda|^{-\alpha}\|f\|_{L_q(\Omega)},$$

and the lemma is proved. $\square$

With the above lemma it is easy to verify the desired estimate for the first addend of (28). We have

$$(\lambda - A_{H_j})^{-1}P_{H_j}f_j = (\lambda - A_{H_j})^{-1}P_{H_j}(\eta_j f - 2\nabla u \cdot \nabla \eta_j - u\Delta\eta_j + p\nabla\eta_j).$$

We may set $p = p_G$ since $p \in \hat{W}^{1,q}(\Omega)$, where $G \subset \Omega$ shall be a bounded domain of class C^2 satisfying $\Omega \cap \operatorname{supp}(\nabla\eta_j) \subset G$ for all $j = 0, \dots, N$ (in the situation here we can choose $G = \Omega$). By using the bounded H^∞-calculus of A_{H_j}, $j = 0, \dots, N$, Theorem 2.1 and Lemma 3.11 we may estimate

$$\begin{aligned}
&\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(\lambda - A_{H_j})^{-1}P_{H_j}f_j \mathrm{d}\lambda\|_{L_q(H_j)} \\
\leq\ &\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(\lambda - A_{H_j})^{-1}P_{H_j}\eta_j f \mathrm{d}\lambda\|_{L_q(H_j)} \\
&+\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(\lambda - A_{H_j})^{-1}P_{H_j}(-2\nabla u \cdot \nabla\eta_j - u\Delta\eta_j + p\nabla\eta_j)\mathrm{d}\lambda\|_{L_q(H_j)} \\
\leq\ &C\left(\|h\|_\infty\|f\|_{L_q(\Omega)} + \|h\|_\infty\int_1^\infty \frac{1}{s}\left(\frac{1}{s} + \frac{1}{s^{1/2}} + \frac{1}{s^\alpha}\right)\|f\|_{L_q(G)}\mathrm{d}s\right) \\
\leq\ &C\|h\|_\infty\|f\|_{L_q(\Omega)}
\end{aligned} \tag{31}$$

for all $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$, $j = 0, \dots, N$, and any fixed $\alpha \in (0, \frac{1}{2q'})$.

For the third addend of (28) we write w_j as

$$w_j = LM_{\nabla\eta_j}u$$

where $M_{\nabla\eta_j}u := \nabla\eta_j \cdot u$. The estimate for the operator $K_j := LM_{\nabla\eta_j}$ stated in the next lemma will be useful.

Lemma 3.12 *Let $1 < q < \infty$, H_j, K_j and $G \subset \Omega$ defined as above. Then for some constant $C = C(G)$ it holds*

$$\|K_j u\|_{W^{1,q}(H_j)} \leq C\|u\|_{L_q(G)}$$

for all $u \in L_q(G)$ and all $j = 0, \dots, N$.

Proof. Set $G_j := \Omega \cap \operatorname{supp}\nabla\eta_j$. For $\psi \in \hat{W}^{1,q}(H_j) \cap C_c^\infty(\overline{H_j})$ with $\int_{G_j}\psi \mathrm{d}x = 0$ we have by Poincaré's inequality

$$\|\nabla\eta_j\psi\|_{W^{1,q}(G_j)} \leq C\|\nabla\psi\|_{L_q(G_j)} \leq C\|\nabla\psi\|_{L_q(H_j)}. \tag{32}$$

This yields

$$\begin{aligned}
\|M_{\nabla\eta_j}u\|_{\hat{W}^{-1,q}(H_j)} &= \sup_{\psi \in C_c^\infty(\overline{H_j})} \frac{|\int_{H_j}(u \cdot \nabla\eta_j)\psi \mathrm{d}x|}{\|\nabla\psi\|_{L_q(H_j)}} \\
&= \sup_{\psi \in C_c^\infty(\overline{H_j})} \frac{|\int_G u \cdot (\nabla\eta_j\psi)\mathrm{d}x|}{\|\nabla\eta_j\psi\|_{W^{1,q}(G)}} \frac{\|\nabla\eta_j\psi\|_{W^{1,q}(G_j)}}{\|\nabla\psi\|_{L_q(H_j)}} \\
&\leq C\|u\|_{(W^{1,q}(G))'}
\end{aligned}$$

for all $u \in W^{1,q}(G) \subset \left(W^{1,q}(G)\right)'$, $j = 0, \dots, N$. Together with (27) this leads to

$$\begin{aligned} \|K_j u\|_{L_q(H_j)} &= \|LM_{\nabla\eta_j} u\|_{L_q(H_j)} \le C\|M_{\nabla\eta_j} u\|_{\hat{W}^{-1,q}(H_j)} \\ &\le C\|u\|_{(W^{1,q}(G))'} \end{aligned}$$

and

$$\begin{aligned} \|K_j u\|_{W^{2,q}(H_j)} &\le C\left(\|M_{\nabla\eta_j} u\|_{\hat{W}^{-1,q}(H_j)} + \|\nabla M_{\nabla\eta_j} u\|_{L_q(H_j)}\right) \\ &\le C\left(\|u\|_{(W^{1,q}(G))'} + \|\nabla(\nabla\eta_j \cdot u)\|_{L_q(H_j)}\right) \\ &\le C\|u\|_{W^{1,q}(G)} \end{aligned}$$

for all $u \in W^{1,q}(G)$, $j = 0, \dots, N$. Since $W^{1,q}(G)$ is a dense subspace of $\left(W^{1,q}(G)\right)'$ the first inequality above implies that K_j can be extended to a bounded operator from $\left(W^{1,q}(G)\right)'$ to $L_q(H_j)$. From the second one we get that K_j is also bounded from $W^{1,q}(G)$ to $W^{2,q}(H_j)$. By interpolation, K_j is also bounded from $L_q(G) = \left[\left(W^{1,q}(G)\right)', W^{1,q}(G)\right]_{1/2}$ to $W^{1,q}(H_j) = [L_q(H_j), W^{2,q}(H_j)]_{1/2}$ for $j = 0, \dots, N$, which yields the assertion. □

Using the fact that $P_{H_j} \in \mathcal{L}(W^{1,q}(H_j))$ and again the identity

$$\mathrm{dom}(A_{H_j}^\alpha) = [L_{q,\sigma}(H_j), \mathrm{dom}(A_{H_j})]_\alpha = W^{2\alpha,q}(H_j), \quad \alpha \in [0, \frac{1}{2q})$$

(see also, [Tri78] and [Fra00]), we deduce, if we set $\alpha := \frac{1}{4q}$, say,

$$P_{H_j} w_j \in W^{1,q}(H_j) \subset W^{2\alpha,q}(H_j) = \mathrm{dom}(A_{H_j}^\alpha).$$

By a simple interpolation argument and Lemma 3.12 we get

$$\begin{aligned} \|A_{H_j}(\lambda - A_{H_j})^{-1} P_{H_j} w_j\|_{L_q(H_j)} &= \|A_{H_j}^{1-\alpha}(\lambda - A_{H_j})^{-1} A_{H_j}^\alpha P_{H_j} w_j\|_{L_q(H_j)} \\ &\le C|\lambda|^{-\alpha}\|A_{H_j}^\alpha P_{H_j} w_j\|_{L_q(H_j)} \le C|\lambda|^{-\alpha}\|P_{H_j} w_j\|_{W^{2\alpha,q}(H_j)} \\ &\le C|\lambda|^{-\alpha}\|P_{H_j} w_j\|_{W^{1,q}(H_j)} \le C|\lambda|^{-\alpha}\|w_j\|_{W^{1,q}(H_j)} \\ &= C|\lambda|^{-\alpha}\|K_j u\|_{W^{1,q}(H_j)} \le C|\lambda|^{-\alpha}\|u\|_{L_q(\Omega)} \\ &\le C|\lambda|^{-1-\alpha}\|f\|_{L_q(\Omega)} \end{aligned}$$

for $|\lambda| \ge 1$. It follows

$$\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda) A_{H_j}(\lambda - A_{H_j})^{-1} P_{H_j} w_j \mathrm{d}\lambda\|_{L_q(H_j)} \le C\|h\|_\infty \|f\|_{L_q(\Omega)} \tag{33}$$

for all $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$, $j = 1, \dots, N$.

The estimate for the fourth addend of (28) will follow from Lemma 3.13 below. Because we will need a similar estimate in the next section, we state this lemma, just as we did with Lemma 3.11, in a more general form as is needed here. Let $\Omega \subset \mathbb{R}^n$ be a domain which fulfills the assumptions of Theorem 2.1. For $f \in L_{q,\sigma}(\Omega)$, let (u, p) the unique solution of $(SRP)_{f,0}^\Omega$ which exists according to this theorem. Further, let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\nabla\varphi$ has compact support, $\mathrm{supp}\nabla\varphi \cap \Omega \neq \emptyset$, and let $Q \subset \mathbb{R}^n$ be a (possibly unbounded) domain such that $\Omega \cap \mathrm{supp}\nabla\varphi \subset Q$ and $\partial\Omega \cap \mathrm{supp}\nabla\varphi \subset \partial Q$.

Lemma 3.13 *Let L be the solution operator of problem (26) on the domain Q. Then, for the trivial extension of $u \cdot \nabla\varphi$ on Q (also denoted by $u \cdot \nabla\varphi$) we have $u \cdot \nabla\varphi \in \mathrm{dom}(L)$ and*

$$\|\frac{1}{2\pi i}\int_\Gamma h(\lambda) L(u \cdot \nabla\varphi) \mathrm{d}\lambda\|_{L_q(Q)} \le C\|h\|_\infty \|f\|_{L_q(\Omega)}$$

for all $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$ with some constant C that may depend on φ but not on f.

Proof. We have $u \cdot \nabla\varphi \in \hat{W}^{-1,q}(Q) \cap W^{1,q}(Q)$ if Q is unbounded, since $u \cdot \nabla\varphi$ has compact support. Assume for the moment that $f \in \mathrm{dom}(A_\Omega)$. Then we may write for $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$

$$\frac{1}{2\pi i}\int_\Gamma h(\lambda) u \cdot \nabla\varphi \mathrm{d}\lambda = \nabla\varphi \cdot h(A_\Omega) f = \nabla\varphi \cdot \frac{1}{2\pi i}\int_\Gamma \frac{h(\lambda)}{1+\lambda}(\lambda - A_\Omega)^{-1} \mathrm{d}\lambda (1 + A_\Omega) f.$$

By this representation it is easy to see that we also have

$$\frac{1}{2\pi i}\int_\Gamma h(\lambda)u\cdot\nabla\varphi \mathrm{d}\lambda \in \hat{W}^{-1,q}(Q)\cap W^{1,q}(Q).$$

If Q is bounded we use

$$u\cdot\nabla\varphi = \nabla\cdot u\varphi$$

to get in view of $u\varphi\restriction_{\partial Q}=0$ and the Gauss Theorem that

$$u\cdot\nabla\varphi, \quad \frac{1}{2\pi i}\int_\Gamma h(\lambda)u\cdot\nabla\varphi \mathrm{d}\lambda \in W^{1,q}(Q)\cap L_{q,0}(Q).$$

The continuity of L implies together with (27) that

$$\|\frac{1}{2\pi i}\int_\Gamma h(\lambda)L(u\cdot\nabla\varphi)\mathrm{d}\lambda\|_{L_q(Q)} \quad = \quad \|L(\nabla\varphi\cdot h(A_\Omega)f)\|_q \leq C\|\nabla\varphi\cdot h(A_\Omega)f\|_{-1,q}.$$

To estimate the norm on the right hand side recall that $\operatorname{supp}(u\cdot\nabla\varphi)\subset\Omega$. By (1) and the identity $\operatorname{dom}((-\Delta_\Omega)^\alpha)=[L_{q'}(\Omega),\operatorname{dom}(-\Delta_\Omega)]_\alpha = W^{2\alpha,q'}(\Omega)$, $\alpha\in(0,\frac{1}{2q'})$, we get for $\psi\in C_c^\infty(\overline{Q})$

$$\begin{aligned}
(\nabla\varphi\cdot h(A_\Omega)f,\psi)_Q &= \left(\nabla\varphi\cdot(1+A_\Omega)\frac{1}{2\pi i}\int_\Gamma\frac{h(\lambda)}{1+\lambda}(\lambda-A_\Omega)^{-1}f\mathrm{d}\lambda,\psi\right)_\Omega\\
&= \left((1-\Delta_\Omega)\frac{1}{2\pi i}\int_\Gamma\frac{h(\lambda)}{1+\lambda}(\lambda-A_\Omega)^{-1}\mathrm{d}\lambda f, P_\Omega\psi\nabla\varphi\right)_\Omega\\
&= \left(\frac{1}{2\pi i}\int_\Gamma\frac{h(\lambda)}{1+\lambda}(1-\Delta_\Omega)^{1-\alpha}(\lambda-A_\Omega)^{-1}\mathrm{d}\lambda f,(1-\Delta_\Omega)^\alpha P_\Omega\psi\nabla\varphi\right)_\Omega.
\end{aligned}$$

Completely analogous to (32) we get

$$\|\psi\nabla\varphi\|_{W^{1,q'}(\Omega)} \quad \leq \quad C\|\nabla\psi\|_{L_{q'}(Q)}.$$

Thus, as in the proof of Lemma 3.11 we obtain

$$\|(1-\Delta_\Omega)^\alpha P_\Omega\psi\nabla\varphi\|_{L_{q'}(\Omega)}\leq C\|\psi\nabla\varphi\|_{W^{1,q'}(\Omega)}\leq C\|\nabla\psi\|_{L_{q'}(\Omega)}$$

and

$$\|(1-\Delta_\Omega)^{1-\alpha}(\lambda-A_\Omega)^{-1}f\|_{L_q(\Omega)}\leq C|\lambda|^{-\alpha}\|f\|_{L_q(\Omega)}.$$

This yields

$$\begin{aligned}
|(\nabla\varphi\cdot h(A_\Omega)f,\psi)_Q| &\leq C\|h\|_\infty\int_0^\infty\frac{1}{(1+s)s^\alpha}\|f\|_{L_q(\Omega)}\mathrm{d}s\|\nabla\psi\|_{L_{q'}(\Omega)}\\
&\leq C\|h\|_\infty\|f\|_{L_q(\Omega)}\|\nabla\psi\|_{L_{q'}(\Omega)}
\end{aligned}$$

for all $h\in\mathcal{H}_0^\infty(\Sigma_\phi)$. Consequently,

$$\begin{aligned}
\|\frac{1}{2\pi i}\int_\Gamma h(\lambda)L(u\cdot\nabla\varphi)\mathrm{d}\lambda\|_{L_q(Q)} &\leq C\|\nabla\varphi\cdot h(A_\Omega)f\|_{-1,q}\\
&= \sup_{\psi\in C_c^\infty(\overline{Q}),\nabla\psi\neq0}\frac{|(\nabla\varphi\cdot h(A_\Omega)f,\psi)_Q|}{\|\nabla\psi\|_{L_{q'}(Q)}}\\
&\leq C\|h\|_\infty\|f\|_{L_q(\Omega)}
\end{aligned}$$

for all $h\in\mathcal{H}_0^\infty(\Sigma_\phi)$ and the assertion follows. □

Similar to (21) we obtain the estimate

$$\|L(u\cdot\nabla\eta_j)\|_{L_q(H_j)}\leq\frac{C}{\sqrt{|\lambda|}}\|f\|_{L_q(\Omega)}.$$

Thus, setting $H_j = Q$ and $\eta_j = \varphi$, $j \in \{0, \dots, N\}$ we obtain by Lemma 3.13 for the fourth addend of (28)

$$
\begin{aligned}
&\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(I - P_{H_j})w_j \mathrm{d}\lambda\|_{L_q(H_j)} \leq \\
\leq\ & C\left(\|\frac{1}{2\pi i}\int_{\Gamma} h(\lambda)L(u\cdot\nabla\eta_j)\mathrm{d}\lambda\|_{L_q(H_j)} + \|\frac{1}{2\pi i}\int_{\Gamma_{0,1}} h(\lambda)L(u\cdot\nabla\eta_j)\mathrm{d}\lambda\|_{L_q(H_j)}\right) \\
\leq\ & C\left(\|h\|_\infty\|f\|_{L_q(\Omega)} + \|h\|_\infty\int_0^1 \|L(u\cdot\nabla\eta_j)\|_{L_q(H_j)}\mathrm{d}s\right) \\
\leq\ & C\left(\|h\|_\infty\|f\|_{L_q(\Omega)} + \|h\|_\infty\int_0^1 \frac{1}{\sqrt{s}}\|f\|_{L_q(\Omega)}\mathrm{d}s\right) \\
\leq\ & C\|h\|_\infty\|f\|_{L_q(\Omega)} \qquad (34)
\end{aligned}
$$

for all $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$. Combining (29), (33), (31) and (34) we get

$$
\begin{aligned}
\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)(\lambda - A_\Omega)^{-1}f\mathrm{d}\lambda\|_{L_q(\Omega)} &= \|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)u\mathrm{d}\lambda\|_{L_q(\Omega)} \\
&\leq \sum_{j=0}^{N}\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)\eta_j u\mathrm{d}\lambda\|_{L_q(\Omega_j)} \\
&\leq \sum_{j=0}^{N}\|\frac{1}{2\pi i}\int_{\Gamma_{1,\infty}} h(\lambda)\eta_j u\mathrm{d}\lambda\|_{L_q(H_j)} \\
&\leq C\|h\|_\infty\|f\|_{L_q(\Omega)}
\end{aligned}
$$

for all $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$. In view of (25) we thus have proved the following theorem.

Theorem 3.14 *Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary is of class C^3. Then the Stokes operator A_Ω admits a bounded H^∞-calculus in $L_{q,\sigma}(\Omega)$.*

3.4 H^∞-calculus for the Stokes operator on exterior domains and on perturbed half-spaces

In this section we consider the Stokes operator A_Ω, where $\Omega \subset \mathbb{R}^n$ is either an exterior domain, i.e. the complement of a compact set, or a perturbed half-space by which we mean that there is a compact set K in $\mathbb{R}^n$ such that $\mathbb{R}^n_+ \setminus K = \Omega \setminus K$, see Figure 4. We will show that the Stokes operator A_Ω on such a domain also admits a bounded H^∞-calculus. This is more or less a consequence of the results in Subsections 3.1 and 3.3. Using the same localization as in the proof of Theorem 3.3 we can reduce the perturbed half-space problem to the case of a bounded domain and the half-space. If Ω is exterior we can reduce the problem to the bounded domain case and to $\mathbb{R}^n$. Instead of repeating large parts of the proofs of Theorem 3.3 and Theorem 3.14, we only explain the essential steps that differ in this situation.

Theorem 3.15 *Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be an exterior domain or a perturbed half-space whose boundary is of class C^3. Then the Stokes operator A_Ω admits a bounded H^∞-calculus in $L_{q,\sigma}(\Omega)$.*

Proof. Let $B_R(0)$ a ball such that $\Omega \setminus B_R(0) = \mathbb{R}^n_+ \setminus B_R(0)$ if Ω is a perturbed half-space or $\Omega \setminus B_R(0) = \mathbb{R}^n \setminus B_R(0)$ if Ω is an exterior domain. In both of the two cases we can use the same construction of $\Omega_0, \Omega_1, \eta_0, \eta_1$ as in the proof of Theorem 3.3 with the only difference that we set $\Omega_0 = B_{2R}(0)$ and $\Omega_1 = \mathbb{R}^n$ if Ω is an exterior domain. As before, we split the H^∞ integral into the two parts $|\lambda| \leq 1$ and $|\lambda| > 1$. For the treatment of the former integral we only have to modify inequality (24) since we applied Sobolev's inequality for H_ω at this point. The remaining parts of the proof can be copied verbatim, because nowhere else we have used the special structure of H_ω again. To obtain an estimate like (24) if Ω is a perturbed half-space or an exterior domain we will apply the following generalization of Poincaré's inequality on Ω_0. If $Q \subset \mathbb{R}^n$ is a bounded Lipschitz domain and V is a closed subspace of $W^{1,q}(Q)$, then there are equivalent:

(i) There is some $u_0 \in V$ and some constant $C_0 \geq 0$ such that $u_0 + \xi \in V$ implies $|\xi| \leq C_0$ for $\xi \in \mathbb{R}^n$.

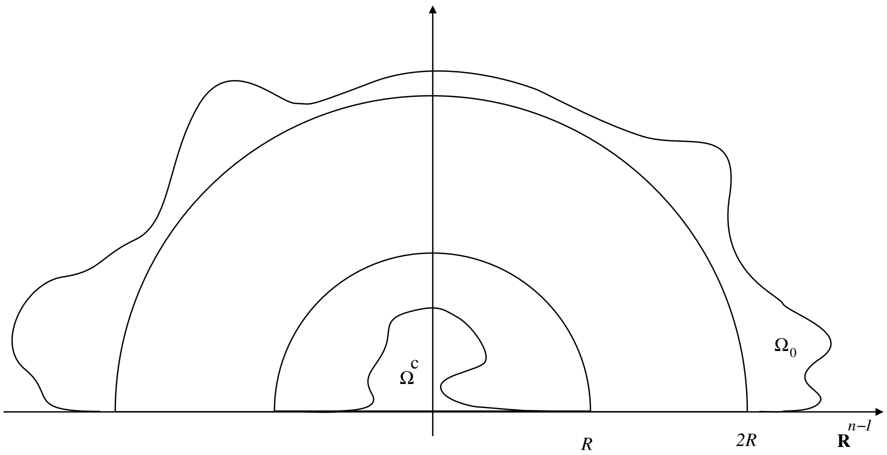

Figure 4: Resolution of the identity for the perturbed half-space

(ii) There is a constant $C > 0$ such that

$$\|u\|_{L_q(Q)} \leq C\|\nabla u\|_{L_q(Q)}, \quad u \in V.$$

A proof of that result can be found e.g. in [Alt99]. If $S \subset \partial Q$ is not a null set with respect to the boundary measure it is easy to see, that $W^{1,q}_{0,S}(Q) := \{u \in W^{1,q}(Q) : \gamma u \restriction_S = 0\}$ is a closed subspace of $W^{1,q}(Q)$, which satisfies condition (i) of the above equivalence. Thus, if we set $S := \partial\Omega_0 \cap \partial\Omega$, we deduce the validity of Poincaré's inequality on $W^{1,q}_{0,S}(\Omega_0)$. This gives us for the Stokes flow $u \in \mathrm{dom}(A_\Omega)$ of the solution (u, p) of $(SRP)^\Omega_{f,0}$, where $f \in L_{q,\sigma}(\Omega)$,

$$\|u\Delta\eta_1\|_{L_{q_1}(\Omega_1)} \leq C\|u\|_{L_q(\Omega_0)} \leq C\|\nabla u\|_{L_q(\Omega_0)}, \tag{35}$$

with q_1 as in Theorem 3.3. To see that we may estimate the last term again by Poincaré's inequality we have to verify (i) for the subspace

$$V := \nabla\left[W^{1,q}_{0,S}(\Omega_0) \cap W^{2,q}(\Omega_0)\right] = \left\{\nabla v : v \in W^{1,q}_{0,S}(\Omega_0) \cap W^{2,q}(\Omega_0)\right\}.$$

of $W^{1,q}(\Omega_0)$. Clearly, (i) follows if we can show that there is no non-trivial constant function in V. If $w = \nabla v \in V$ is constant for some $v \in W^{1,q}_{0,S}(\Omega_0) \cap W^{2,q}(\Omega_0)$, then $v(x) = Mx + b$, where $M \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Hence, the set of zeros for v is an affine subspace of $\mathbb{R}^n$. But the only affine subspace that contains S is $\mathbb{R}^n$, since we may assume that $\partial\Omega$ is not an affine subspace of $\mathbb{R}^n$ (otherwise we are in the half-space case). This implies $v = 0$ which in turn implies $w = 0$. It remains to show that V is closed in $W^{1,q}(\Omega_0)$. This can be seen by direct calculation or by the following argument: We set $X := W^{1,q}_{0,S}(\Omega_0) \cap W^{2,q}(\Omega_0)$ and

$$T : X \to W^{1,q}(\Omega_0), \quad Tu := \nabla u.$$

Since T is injective its inverse is well-defined on $\mathrm{ran}(T) = V$. The boundedness of T implies the closedness of T^{-1}. We will show that T^{-1} is continuous, which immediately yields the closedness of its domain V. By Poincaré's inequality on $W^{1,q}_{0,S}(\Omega_0)$ we obtain

$$\begin{aligned} \|T^{-1}u\|_X &= \|T^{-1}u\|_{2,q} \leq C\left(\|T^{-1}u\|_q + \|\nabla T^{-1}u\|_{1,q}\right) \\ &\leq C\left(\|\nabla T^{-1}u\|_q + \|\nabla T^{-1}u\|_{1,q}\right) \\ &= C\left(\|u\|_q + \|u\|_{1,q}\right) \leq C\|u\|_{1,q} = C\|u\|_V \end{aligned}$$

for all $u \in V$ proving the continuity of T^{-1}. Consequently, Poincaré's inequality is valid on V which gives us together with (35)

$$\|u\Delta\eta_1\|_{L_{q_1}(\Omega_1)} \leq C\|\nabla u\|_{L_q(\Omega_0)} \leq C\|\nabla^2 u\|_{L_q(\Omega_0)} \leq C\|\nabla^2 u\|_{L_q(\Omega)}.$$

So, replacing (24) by the above line the proof for $|\lambda| \leq 1$ is finished.

For $|\lambda| \geq 1$ we can transfer the proof in Theorem 3.14 for that λ's. Instead of the localization used there which reduces the problem on Ω to problems on H_ω and $\mathbb{R}^n$, we take the above localization and reduce it to problems on the bounded domain Ω_0 and the unbounded domain Ω_1 (which is either $\mathbb{R}^n$ or $\mathbb{R}^n_+$). The localized equations remain unchanged as well as formula (28) for the localized functions $\eta_j u$, $j = 0, 1$. This allows us to copy the proof of Theorem 3.14 without any further change. Applying Theorem 3.14 to A_{Ω_0} and using the bounded H^∞-calculus of $A_{\mathbb{R}^n_+}$ and $A_{\mathbb{R}^n}$ complete the proof of Theorem 3.15. $\square$

A Regularity of the Helmholtz projection

Lemma A.1 *Let $\omega \in C_c^{1,1}(\mathbb{R}^{n-1})$ and let H_ω be the bent half-space associated with ω as introduced in Section 3.1. Further, let $1 < q, q^* < \infty$ with $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$. Then the Sobolev inequality is valid for H_ω, i.e. there is a $C > 0$ such that*

$$\|u\|_{q^*} \leq C\|\nabla u\|_q$$

for all $u \in W^{1,q}(H_\omega)$.

Proof. First recall that $\mathbb{R}^n_+$ is a so-called (ε, ∞) domain, i.e. there is some $\varepsilon > 0$ with the following property: For all $x, y \in \mathbb{R}^n_+$ there is a rectifiable arc γ, joining x to y and satisfying $L(\gamma) \leq \frac{1}{\varepsilon}|x - y|$ as well as

$$d(z) \geq \varepsilon \frac{|x - z||y - z|}{|x - y|}, \quad z \in \gamma,$$

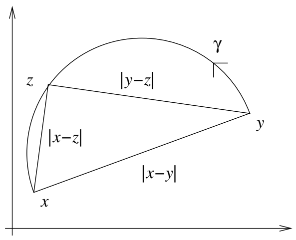

Figure 5: The half-space is an (ε, ∞) domain

where $L(\gamma)$ denotes the length of γ and $d(z) = z_n$ is the distance from z to the boundary of $\mathbb{R}^n_+$. This can be easily seen by taking for γ the upper half of the circle with diameter being the segment connecting x and y, see Figure 5. It is known, see [Jon81] for details, that unbounded (ε, ∞) domains are extension domains for the Dirichlet energy space, i.e. there is a bounded operator $E : \hat{W}^{1,q}(\mathbb{R}^n_+) \to \hat{W}^{1,q}(\mathbb{R}^n)$ with $Ef \restriction_{\mathbb{R}^n_+} = f$ for all $f \in \hat{W}^{1,q}(\mathbb{R}^n_+)$. Since $\phi(x', x_n) = (x', x_n - \omega(x))$ is a C^1-diffeomorphism mapping H_ω to $\mathbb{R}^n_+$, the assertion follows.

$\square$

Remark A.2 Actually, it can be shown that one has more general extension operators for unbounded (ε, ∞) domains: If Ω is a domain of this type, $N \in \mathbb{N}$, and $q_0, \ldots, q_N \in (1, \infty)$, there is an extension operator

$$E : \bigcap_{j=0}^{N} \hat{W}^{j,q_j}(\Omega) \to \bigcap_{j=0}^{N} \hat{W}^{j,q_j}(\mathbb{R}^n) \quad \text{with} \quad \|\nabla^j Eu\|_{L_{q_j}(\mathbb{R}^n)} \leq C\|\nabla^j u\|_{L_{q_j}(\Omega)}$$

for all $j = 1, \ldots, N$ and all $u \in \bigcap_{j=0}^N \hat{W}^{j,q_j}(\Omega)$. For details concerning extension operators in Sobolev spaces, see [Chu92]. As an easy consequence, Gagliardo-Nirenberg's inequality extends to (ε, ∞) domains. In particular, it holds true for $\mathbb{R}^n_+$ and the bent half-space H_ω with ω as in Lemma A.1.

Let Ω be either a bounded domain or $\Omega = \mathbb{R}^n_+$. It is well-known that the solution of the Neumann problem on Ω associated to the Helmholtz projection admits higher regularity. This implies immediately the regularity of P_Ω, i.e. $P_\Omega \in \mathcal{L}(W^{k,q}(\Omega))$ for $1 < q < \infty$ and $k \in \mathbb{N} \cup \{0\}$. The next proposition shows that this also holds true for $\Omega = H_\omega$.

Proposition A.3 *Let $1 < q < \infty$ and $k \in \mathbb{N} \cup \{0\}$. Then the Helmholtz projection P_{H_ω} is a bounded operator in $W^{k,q}(H_\omega)$. In particular, if $1 < q < n$, then*

$$\|\nabla^k P_{H_\omega} u\|_q \leq C\|\nabla^k u\|_q, \quad u \in W^{k,q}(H_\omega).$$

Proof. Let $\eta_0, \eta_1, \Omega_0, \Omega_1$ as in Theorem 3.3. The case $k = 0$ is well-known, so we only prove the assertion for $k = 1$. The general case then follows by induction. We consider the localized Neumann-Problems

$$(NP) \left\{ \begin{array}{rcll} \Delta(\eta_j p) & = & \eta_j \operatorname{div} u + 2\nabla\eta_j \cdot \nabla p + p\Delta\eta_j & =: f_j \quad \text{on} \quad \Omega_j, \\ \frac{\partial}{\partial\nu}(\eta_j p) & = & (u\eta_j + p\nabla\eta_j) \cdot \nu & =: g_j \quad \text{on} \quad \partial\Omega_j \end{array} \right.$$

for $j=0,1$. From well-known regularity properties for the Neumann problem on $\mathbb{R}^n_+$ (see [Fra00]) we get

$$\begin{aligned}\|\nabla^2\eta_1 p\|_{L_q(\Omega_1)} &\leq C\left(\|f_1\|_{L_q(\Omega_1)}+\|g_1\|_{\hat{W}^{1-1/q,q}(\partial\Omega_1)}\right)\\ &\leq C\left(\|\nabla u\|_{L_q(H_\omega)}+\|p\|_{W^{1,q}(\Omega_0)}+\|u\eta_1+p\nabla\eta_1\|_{\hat{W}^{1-1/q,q}(\partial\Omega_1)}\right),\end{aligned}$$

where $\hat{W}^{1-1/q,q}(\partial\Omega_1)$ is the trace Sobolev space, treated in detail e.g. in [Gal98]. By [Gal98], Theorem II 8.2, we can estimate the latter term on the right hand side which yields

$$\begin{aligned}\|\nabla^2\eta_1 p\|_{L_q(\Omega_1)} &\leq C\left(\|\nabla u\|_{L_q(H_\omega)}+\|p\|_{W^{1,q}(\Omega_0)}+\|u\eta_1+p\nabla\eta_1\|_{\hat{W}^{1,q}(\Omega_1)}\right)\\ &\leq C\left(\|\nabla u\|_{L_q(H_\omega)}+\|u\|_{L_q(\Omega_0)}+\|p\|_{W^{1,q}(\Omega_0)}\right).\end{aligned}$$

By using regularity properties for the Neumann problem on bounded domains, we can treat the case $j=0$ in a similar way which gives us

$$\|\nabla^2\eta_0 p\|_{L_q(\Omega_0)} \leq C\left(\|\nabla u\|_{L_q(H_\omega)}+\|u\|_{L_q(\Omega_0)}+\|p\|_{W^{1,q}(\Omega_0)}\right).$$

It is always possible to choose p such that $\int_{\Omega_0} p(x)\mathrm{d}x=0$. From Poincaré's inequality and $P_{H_\omega}\in\mathcal{L}(L_q(H_\omega))$ we therefore obtain

$$\|p\|_{L_q(\Omega_0)}\leq\|\nabla p\|_{L_q(\Omega_0)}\leq\|\nabla p\|_{L_q(H_\omega)}\leq\|u\|_{L_q(H_\omega)}.$$

Hence, the above two estimates imply

$$\begin{aligned}\|\nabla^2 p\|_{L_q(H_\omega)} &\leq \|\nabla^2\eta_1 p\|_{L_q(\Omega_1)}+\|\nabla^2\eta_0 p\|_{L_q(\Omega_0)}\\ &\leq \|u\|_{W^{1,q}(H_\omega)},\end{aligned}$$

which gives us

$$\|P_{H_\omega}\|_{W^{1,q}(H_\omega)}\leq\|u\|_{W^{1,q}(H_\omega)}.$$

Assume now $1<q<n$. With Lemma A.1 and the boundedness of Ω_0 we may conclude

$$\|u\|_{L_q(\Omega_0)}\leq C\|u\|_{L_{q^*}(\Omega_0)}\leq C\|u\|_{L_{q^*}(H_\omega)}\leq C\|\nabla u\|_{L_q(H_\omega)}.$$

The Helmholtz-Projection P_{H_ω} does not depend on q and is continuous for all $1<q<\infty$. Together with Lemma A.1 this leads to

$$\|p\|_{W^{1,q}(\Omega_0)}\leq\|\nabla p\|_{L_q(\Omega_0)}\leq C\|\nabla p\|_{L_{q^*}(H_\omega)}\leq C\|u\|_{L_{q^*}(H_\omega)}\leq C\|\nabla u\|_{L_q(H_\omega)}.$$

The above two estimates for $\nabla^2\eta_1 p$ and $\nabla^2\eta_0 p$ now imply

$$\|\nabla P_{H_\omega}u\|_{L_q(H_\omega)} \leq \|\nabla u\|_{L_q(H_\omega)}+\|\nabla^2 p\|_{L_q(H_\omega)}\leq C\|\nabla u\|_{L_q(H_\omega)}$$

for $u\in W^{1,q}(H_\omega)$. □

B Sobolev estimates for powers of the Stokes operator on $\mathbb{R}^n_+$

Proposition B.1 *Let $1<q<\infty$ and let A be the Stokes operator in $L_{q,\sigma}(\mathbb{R}^n_+)$. Then*

(a) For each $0<s<1$, the norms $\|(A+1)^s\cdot\|_q$ and $\|\cdot\|_{\mathrm{dom}(A^s)}$ are equivalent,

(b) for each $k\in\mathbb{N}$, the norms $\|A^{k/2}\cdot\|_q$ and $\|\nabla^k\cdot\|_q$ are equivalent.

Proof. To prove (a), note that for $r<0$, $(A+1)^r$ is a bounded operator in $L_{q,\sigma}(\mathbb{R}^n_+)$. From Remark 2.2 (iii) we know that $A^{-1}\in\mathcal{H}^\infty(L_{q,\sigma}(\mathbb{R}^n_+))$ with the same $\mathcal{H}^\infty$-angle which immediately implies that also $(A^{-1}+1)^r$ is bounded on $L_{q,\sigma}(\mathbb{R}^n_+)$ for $r<0$. From

$$(A+1)^r=(A^{-1}+1)^r A^r,$$

valid for all $r \in \mathbb{R}$, we can therefore conclude

$$\begin{aligned} \|u\|_{\mathrm{dom}(A^s)} &= \|u\|_q + \|A^s u\|_q \\ &= \|(A+1)^{-s}(A+1)^s u\|_q + \|(A^{-1}+1)^{-s}(A+1)^s u\|_q \\ &\le C\|(A+1)^s u\|_q \end{aligned}$$

for all $u \in \mathrm{dom}(A^s)$. The converse inequality can be proved by the same arguments:

$$\begin{aligned} \|(A+1)^s u\|_q &= \|(A+1)(A+1)^{s-1}u\|_q \\ &\le C\left(\|A(A+1)^{s-1}u\|_q + \|(A+1)^{s-1}u\|_q\right) \\ &\le C\left(\|A^{1-s}(A+1)^{s-1}A^s u\|_q + \|u\|_q\right) \\ &= C\left(\|(A^{-1}+1)^{s-1}A^s u\|_q + \|u\|_q\right) \\ &\le C\|u\|_{\mathrm{dom}(A^s)}. \end{aligned}$$

To verify (b) we first establish the estimates

$$\|u\|_{k,q} \le C\|(A+1)^{k/2}u\|_q \le C\|u\|_{k,q} \tag{36}$$

for each $k \in \mathbb{N}$ and all $u \in \mathrm{dom}(A^{k/2})$. The equivalence of the norms in question is then obtained from these estimates by the scaling method which was already used in the proof of Proposition 3.9. Since $\|\cdot\|_{\mathrm{dom}(A)}$ and $\|\cdot\|_{2,q}$ are equivalent norms on $\mathrm{dom}(A)$, the resolvent $(A+1)^{-1}$ is a bounded operator from $(L_{q,\sigma}(\mathbb{R}^n_+), \|\cdot\|_q)$ to $(\mathrm{dom}(A), \|\cdot\|_{2,q})$. This implies for $u \in \mathrm{dom}(A)$

$$\|u\|_{2,q} = \|(A+1)^{-1}(A+1)u\|_{2,q} \le C\|(A+1)u\|_q \le C\|u\|_{\mathrm{dom}(A)} \le C\|u\|_{2,q}. \tag{37}$$

Since $A \in \mathcal{H}^\infty(L_{q,\sigma}(\mathbb{R}^n_+))$ we know from [Tri78] and [BM88] that

$$\begin{aligned} \mathrm{dom}(A^{1/2}) &= [L_{q,\sigma}(\mathbb{R}^n_+), \mathrm{dom}(A)]_{1/2} = [L_q(\mathbb{R}^n_+), \mathrm{dom}(\Delta)]_{1/2} \cap L_{q,\sigma}(\mathbb{R}^n_+) \\ &= W^{1,q}_0(\mathbb{R}^n_+) \cap L_{q,\sigma}(\mathbb{R}^n_+). \end{aligned}$$

In particular, the norms $\|\cdot\|_{\mathrm{dom}(A^{1/2})}$ and $\|\cdot\|_{1,q}$ are equivalent on $\mathrm{dom}(A^{1/2})$. By (a), the norm $\|\cdot\|_{\mathrm{dom}(A^{1/2})}$ is also equivalent to $\|(A+1)^{1/2}\cdot\|_q$. This yields

$$\|u\|_{1,q} \le C\|(A+1)^{1/2}u\|_q \le C\|u\|_{1,q} \tag{38}$$

for all $u \in \mathrm{dom}(A^{1/2})$. Consider the Stokes equations on the half-space:

$$\left\{ \begin{aligned} u - \Delta u + \nabla p &= f \quad \text{on} \quad \mathbb{R}^n_+, \\ \nabla \cdot u &= 0 \quad \text{on} \quad \mathbb{R}^n_+, \\ \gamma u &= 0. \end{aligned} \right. \tag{39}$$

For $f \in L_{q,\sigma}(\mathbb{R}^n_+)$ this equation has the unique solution $u = (A+1)^{-1}f \in \mathrm{dom}(A)$ which satisfies

$$\|u\|_{2,q} \le C\|f\|_q, \tag{40}$$

(see e.g. [FS94] or [Sol77]). Moreover, for $k \in \mathbb{N} \cup \{0\}$ and $g \in W^{k,q}(\mathbb{R}^n_+)$ we get from [Gal98] Theorem IV.3.2, that for any solution v of the stationary equation

$$(SSE)^{\mathbb{R}^n_+}_{g,0} \left\{ \begin{aligned} -\Delta v + \nabla p &= g \quad \text{on} \quad \mathbb{R}^n_+, \\ \nabla \cdot v &= 0 \quad \text{on} \quad \mathbb{R}^n_+, \\ \gamma v &= 0, \end{aligned} \right.$$

which satisfies $\nabla^2 v \in L_q(\mathbb{R}^n_+)$ we have

$$\|\nabla^{k+2}v\|_q \le C\|g\|_{k,q}. \tag{41}$$

Next, let $f \in W^{1,q}(\mathbb{R}^n_+) \cap L_{q,\sigma}(\mathbb{R}^n_+)$, u be the solution of (39) and put $g = f - u \in W^{1,q}(\mathbb{R}^n_+)$. Trivially, u is a solution of $(SSE)^{\mathbb{R}^n_+}_{g,0}$ with $\nabla^2 u \in L_q(\mathbb{R}^n_+)$. Hence by (40) and (41) we get that $\nabla^3 u \in L_q(\mathbb{R}^n_+)$ with

$$\|\nabla^3 u\|_q \le C\|g\|_{1,q} \le C(\|f\|_{1,q} + \|u\|_{1,q}) \le C\|f\|_{1,q}.$$

By induction over k we obtain that for every $k \in \mathbb{N}$ and each $f \in W^{k,q}(\mathbb{R}^n_+) \cap L_{q,\sigma}(\mathbb{R}^n_+)$ the solution u of (39) satisfies

$$\|u\|_{k+2,q} \leq C\|f\|_{k,q}.$$

Since $u = (A+1)^{-1}f$, this implies in view of the regularity of the Helmholtz projection (Proposition A.3) that

$$\|u\|_{k+2,q} \leq C\|f\|_{k,q} = C\|(A+1)u\|_{k,q} \leq C\|u\|_{k+2,q} \tag{42}$$

for all $u \in \mathrm{dom}(A) \cap W^{k,q}(\mathbb{R}^n_+)$. We will prove (36) by induction: The inequalities (38) and (37) yield (36) for $k = 1$ and $k = 2$ respectively. Suppose now $u \in \mathrm{dom}(A^{(k+2)/2})$ and that (36) holds true for all $j \leq k+1 \in \mathbb{N}$. This implies $(A+1)u \in \mathrm{dom}(A^{k/2}) \subset W^{k,q}(\mathbb{R}^n_+)$ and with (42) we obtain

$$\begin{aligned}\|u\|_{k+2,q} &\leq C\|(A+1)u\|_{k,q} \leq C\|(A+1)^{k/2}(A+1)u\|_q \\ &= C\|(A+1)^{(k+2)/2}u\|_q.\end{aligned}$$

Conversely, the calculation

$$\begin{aligned}\|(A+1)^{(k+2)/2}u\|_q &= \|(A+1)^{k/2}(A+1)u\|_q \\ &\leq C\|(A+1)u\|_{k,q} \leq C\|u\|_{k+2,q}\end{aligned}$$

shows that (36) is valid for all $k \in \mathbb{N}$.

Now let $w \in \mathrm{dom}(A^{k/2})$ and $\lambda > 0$. As in the proof of Proposition 3.9 we set $u = J_\lambda^{-1}w = w(\frac{1}{\lambda}\,\cdot) \in \mathrm{dom}(A^{k/2})$. By equality (19) we get

$$\|(A+\lambda^2)^{k/2}w\|_q = \lambda^{k-n/q}\|(A+1)^{k/2}u\|_q.$$

Moreover, by (17) we have

$$\lambda^{k-n/q}\|u\|_{k,q} = \lambda^{k-n/q}\|J_\lambda^{-1}w\|_{k,q} = \sum_{j=0}^{k} \lambda^{k-j}\|\nabla^j w\|_q.$$

The above two inequalities imply together with (36) that

$$\sum_{j=0}^{k} \lambda^{k-j}\|\nabla^j w\|_q \leq C\|(A+\lambda^2)^{k/2}w\|_q \leq C\sum_{j=0}^{k} \lambda^{k-j}\|\nabla^j w\|_q$$

for all $w \in \mathrm{dom}(A^{k/2})$ and all $\lambda > 0$ (note that J_λ is an automorphism of this space). Passing to the limit $\lambda \to 0$ yields the assertion. □

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